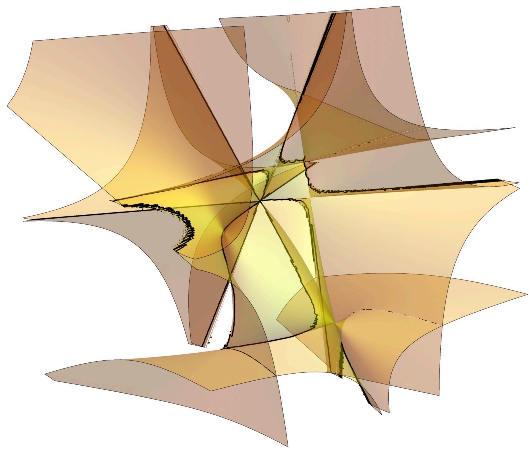
22nd Westlake Math Colloquium | Yifei Zhu: Topology of stratified singular moduli spaces for gapless quantum mechanical systems

Time: 16:00-17:00, Friday, Nov 11, 2022
Venue: E4-233, Yungu Campus & ZOOM
ZOOM ID: 863 6606 6486
PASSCODE: 738489

Host: Dr. Xing Gu, Institute for Theoretical Sciences, Westlake University
Speaker： Dr. Yifei Zhu, an assistant professor at the Department of Mathematics of the Southern University of Science and Technology. His research interests are in algebraic topology and related fields, particularly in its connections to algebraic geometry and number theory via objects such as formal groups, elliptic curves, and modular forms.

Title： Topology of stratified singular moduli spaces for gapless quantum mechanical systems
Abstract: This talk presents an external application of the algebraic topology of moduli spaces. In condensed matter physics, the Hamiltonian of a quantum mechanical system takes a mathematical form of a square matrix, with parameters functions on the 3D momentum space. Such a matrix satisfies the Hermitian symmetry, so that its eigenvalues are real and represent observed energies. We will discuss this space of parameters for Hamiltonians, especially its degeneracy locus where eigenvalues occur with multiplicities. Such a locus gives rise to exceptional properties in the larger scale, with applications to the design of sensing and absorbing devices. We focus on certain non-Hermitian Hamiltonians, the imaginary parts of whose eigenvalues model energy exchange of open systems. Their parameter space possesses intriguing topology, with a stratification of non-isolated singularities, which affords interesting phenomena such as the so-called bulk–edge correspondence. The associated algebraic invariants enable classifications and predictions for phases of matter. This work is in collaboration with C. T. Chan, Jing Hu, Hongwei Jia, Xiaoping Ouyang, Yixiao Wang, Yixin Xiao, Ruo-Yang Zhang, and Zhao-Qing Zhang.

Topology of stratified singular moduli spaces for gapless quantum mechanical systems



Yifei Zhu (SUSTech)

Westlake Math Colloquium, November 2022

Phases of matter: a motivating example

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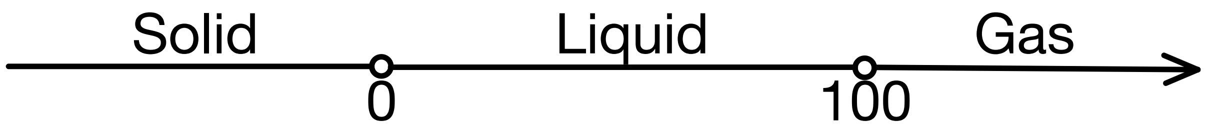

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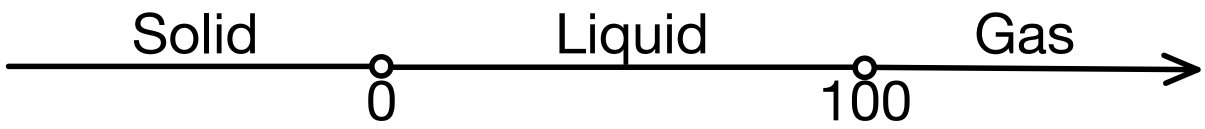

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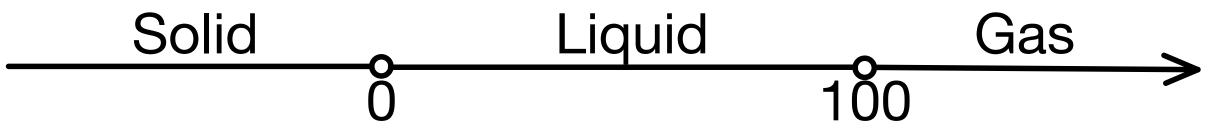

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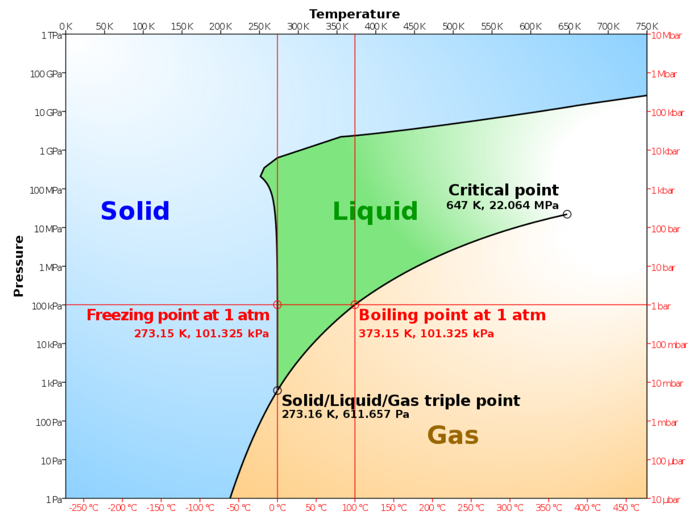

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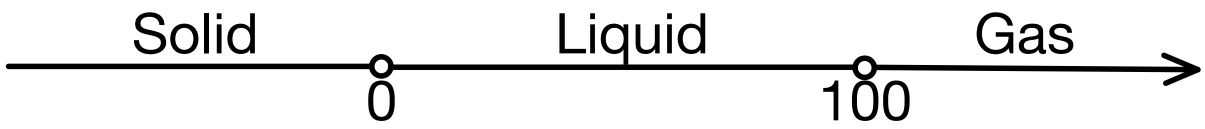

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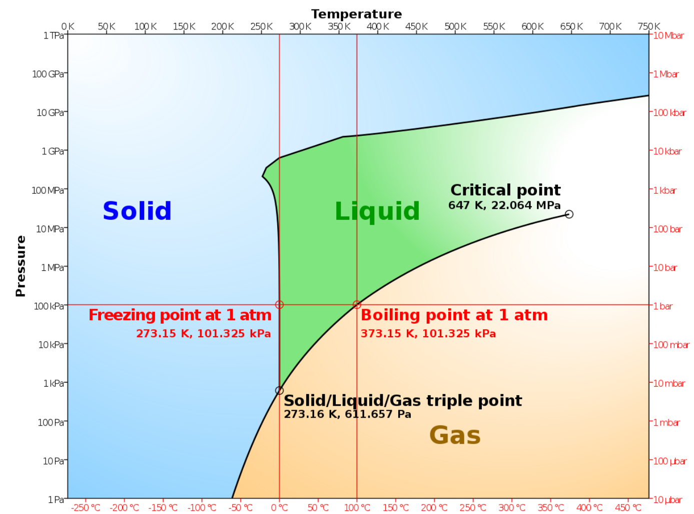

A mathematical framework

There is a space $\mathscr{M}$ of "systems" with a "singular" locus $\Delta \subset \mathscr{M}$, and we are interested in $\pi_0(\mathscr{M} - \Delta)$ or, more generally, the homotopy type of $\mathscr{M} - \Delta$.

Moduli problems: a basic example

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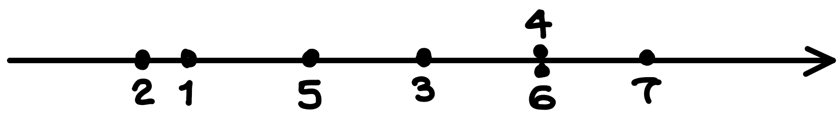

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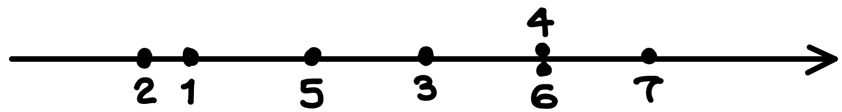

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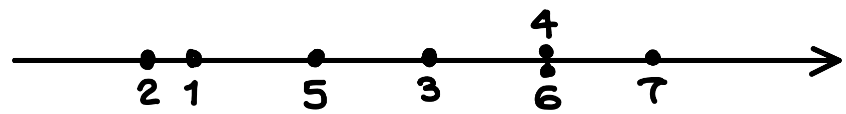

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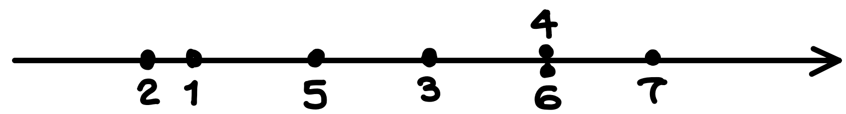

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Let $\Delta \subset \mathscr{M}_n$ be the locus of n-tuples $x = (x^1, \ldots, x^n)$ in which not all x^i are distinct. Configurations in $\mathscr{M}_n - \Delta$ satisfy a "gap condition," and now there *is* nontrivial topology: $\mathscr{M}_n - \Delta$ has $n!$ contractible components.

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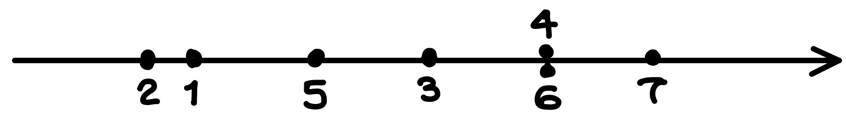

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A gapped configuration $x \in \mathscr{M}_n - \Delta$ determines a permutation $\sigma(x) \in \mathrm{Sym}_n$. In fact, σ induces an isomorphism $\pi_0(\mathscr{M}_n - \Delta) \cong \mathrm{Sym}_n$ of *groups*.

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  Such a complete invariant is not present in all situations.

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Design materials that can “do wonders”, which cannot be found in nature, e.g., invisibility cloaks.

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$$H_2(\overrightarrow{k}) = f_0(\overrightarrow{k})\mathbf{1} + \overrightarrow{f}(\overrightarrow{k}) \cdot \overrightarrow{\sigma}$$

where $\mathbf{1}$ is the identity matrix and $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

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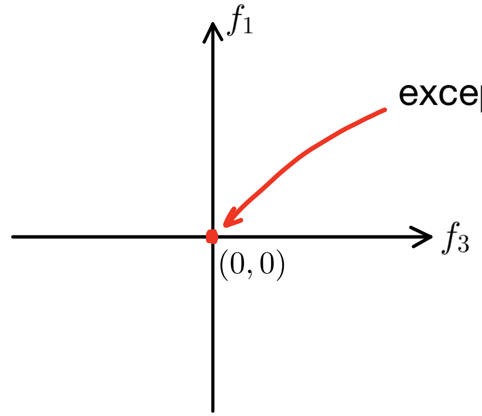

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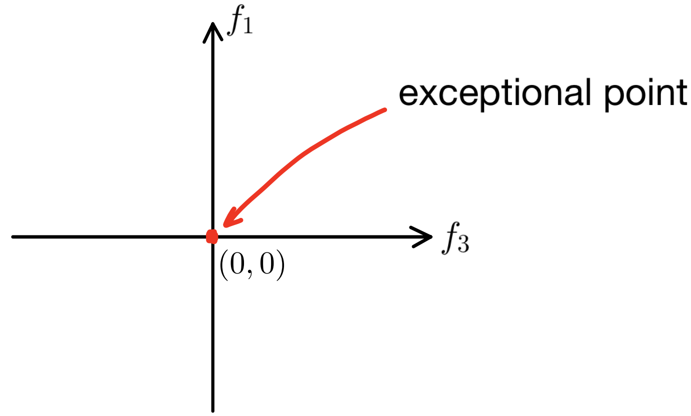

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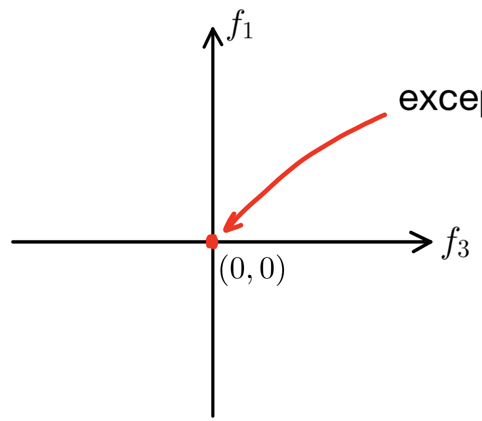

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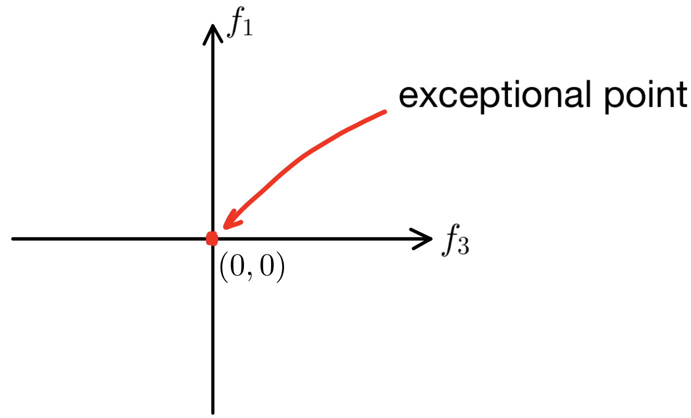

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$$\xrightarrow{\text{normalize}}$$

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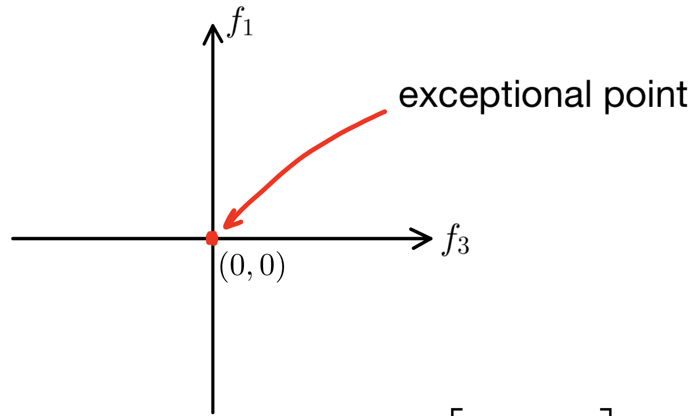

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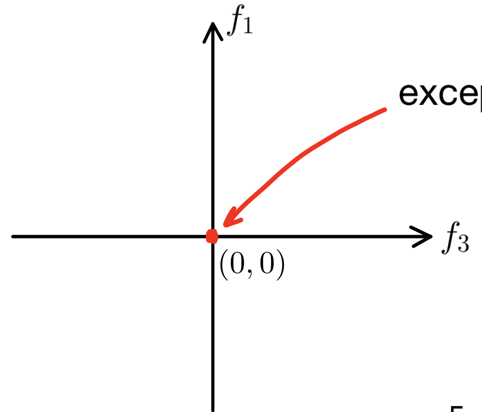

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$$\implies H_2 = \mathbf{1} - 2|\phi_-\rangle\langle\phi_-|$$

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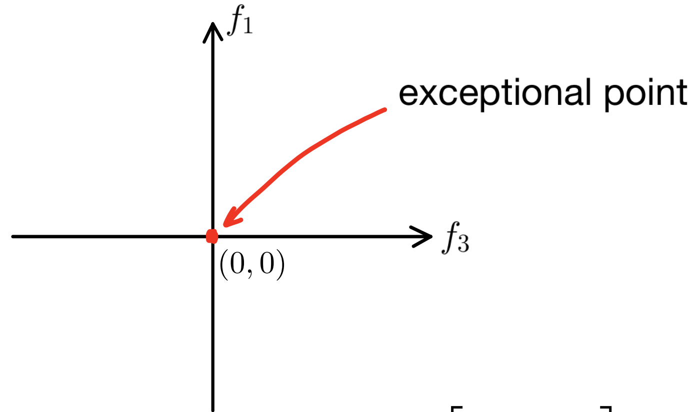

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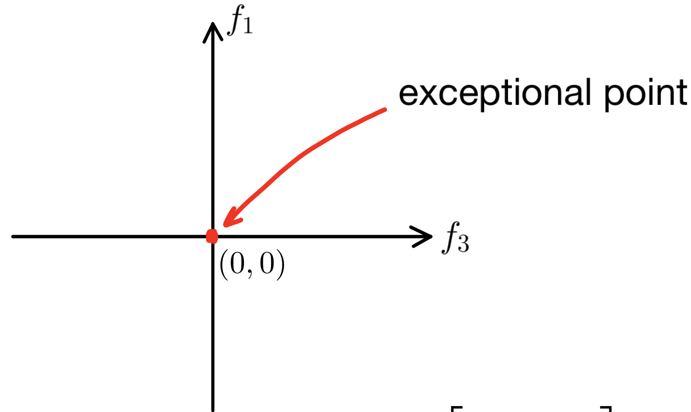
$$\implies H_2 = \mathbf{1} - 2|\phi_-\rangle\langle\phi_-|$$

Observe that $|\phi_-\rangle \mapsto -|\phi_-\rangle$ does not change H_2.

Moduli spaces for quantum mechanical systems

We compute for which values of parameters H_2 has a doubled eigenvalue.
For this purpose, we may assume $f_0 = 0$ and get the characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2 .$$



This gives the moduli space (not quite).
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Thus the moduli space $\mathscr{M}_2 = SO(2)/\mathbb{Z}_2 \cong S^1$ and its “topological charge” (a homotopy invariant) is $\pi_1(\mathscr{M}_2) \cong \mathbb{Z}$.

Moduli spaces for quantum mechanical systems

Taking

$$f_1(\overrightarrow{k}) = k_x k_z$$
$$f_3(\overrightarrow{k}) = \pm k_x^2 + k_y^2 \pm k_z^2 - 4$$

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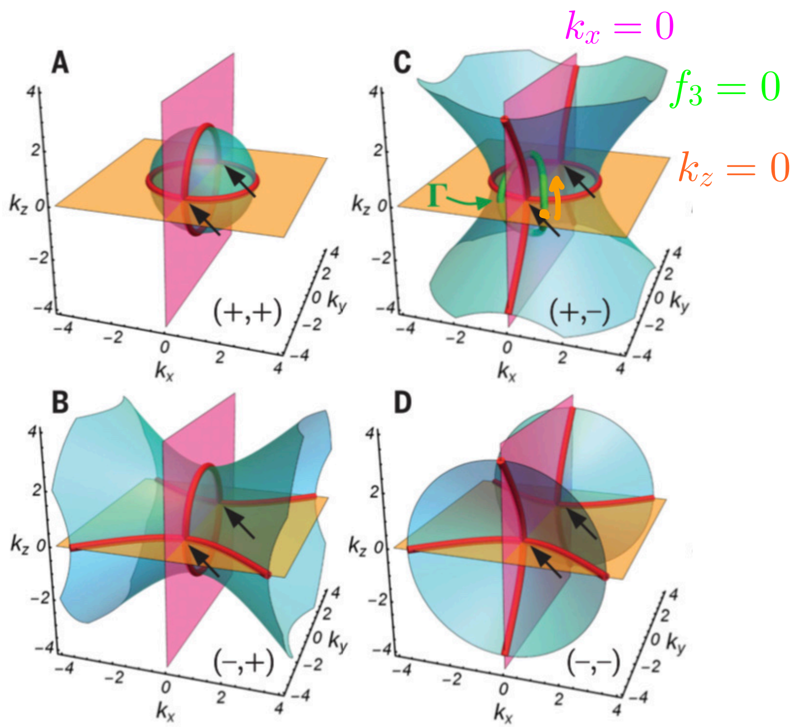

Wu et al., *Science* **365**, 1273—1277 (2019)

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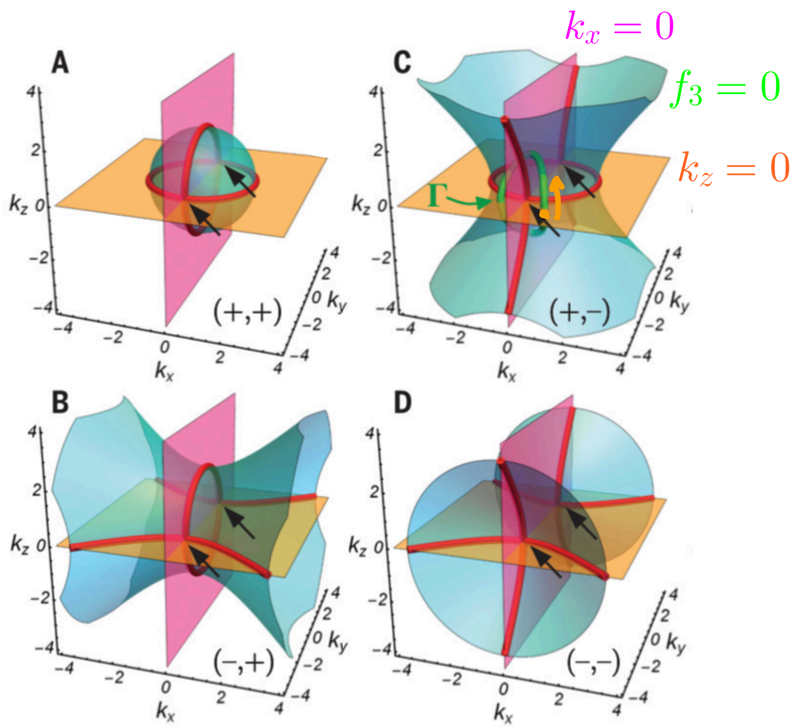

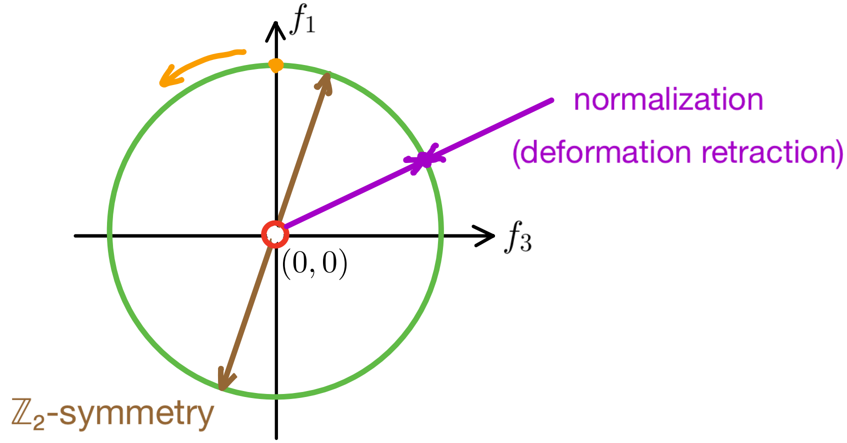

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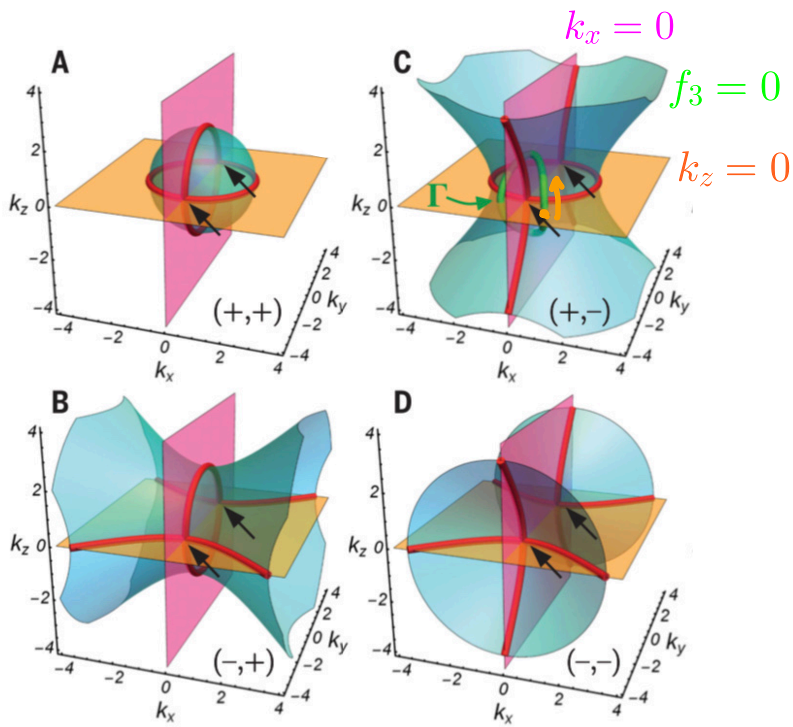

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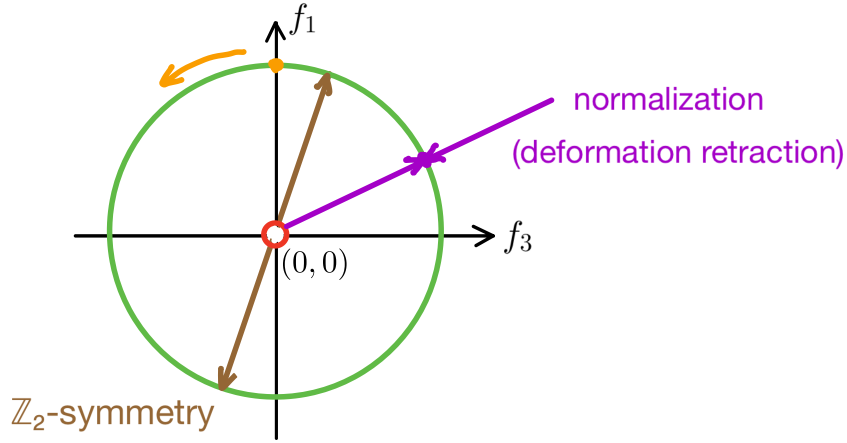
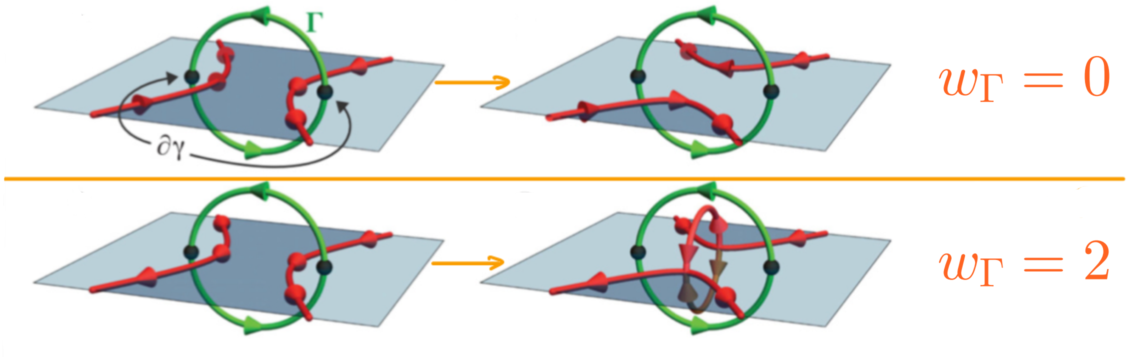


The winding number w_Γ of the loop Γ equals 2.

Moduli spaces for quantum mechanical systems

The winding number serves as a topological classifier:



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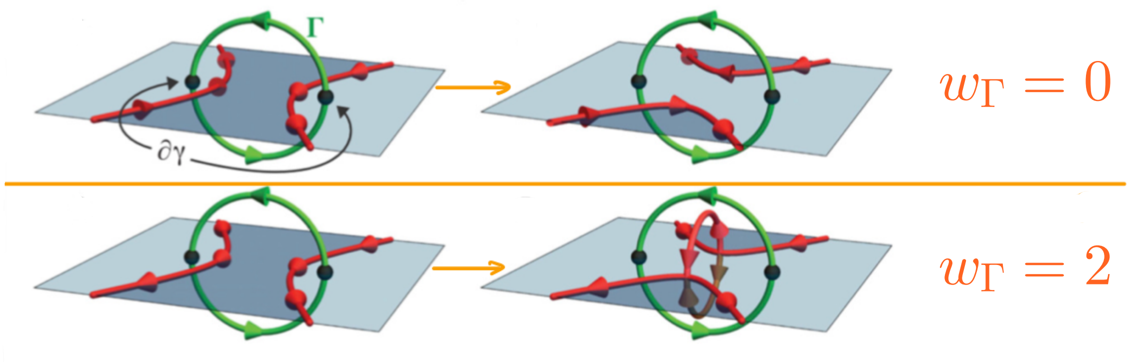

More generally, for all $n \geq 2$, Wu et al. computed topological charges for n-band Hermitian Hamiltonians with PT symmetry and found that $\pi_1(\mathscr{M}_n)$ is non-Abelian when $n \geq 3$.

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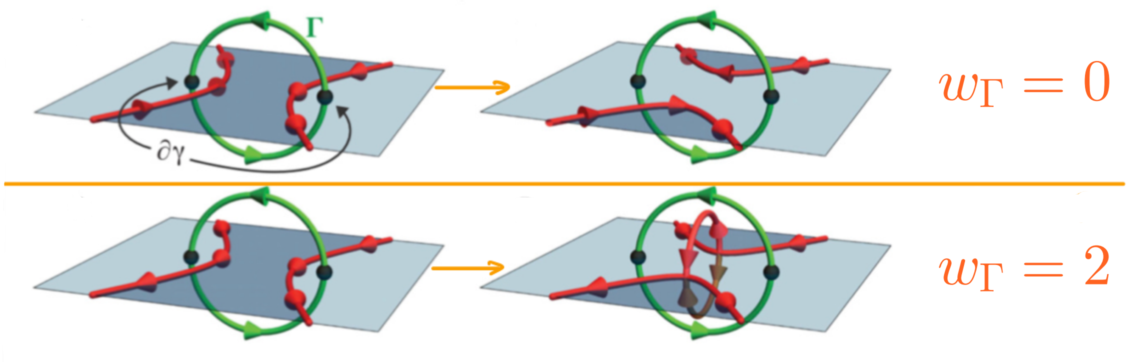

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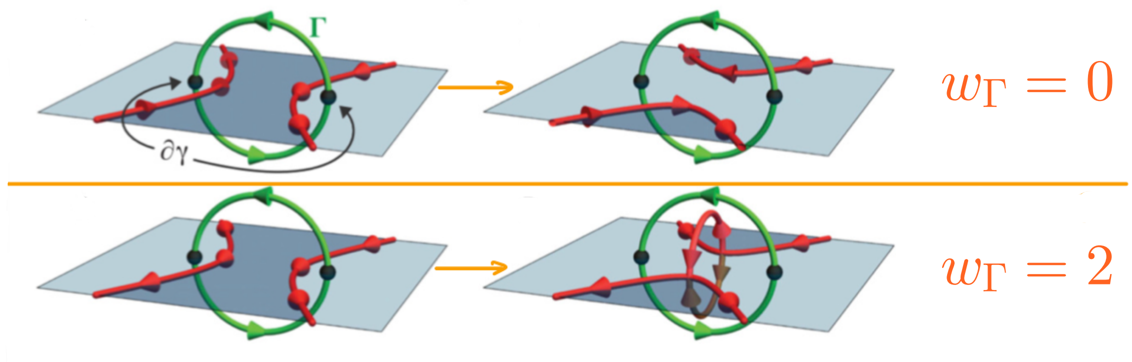

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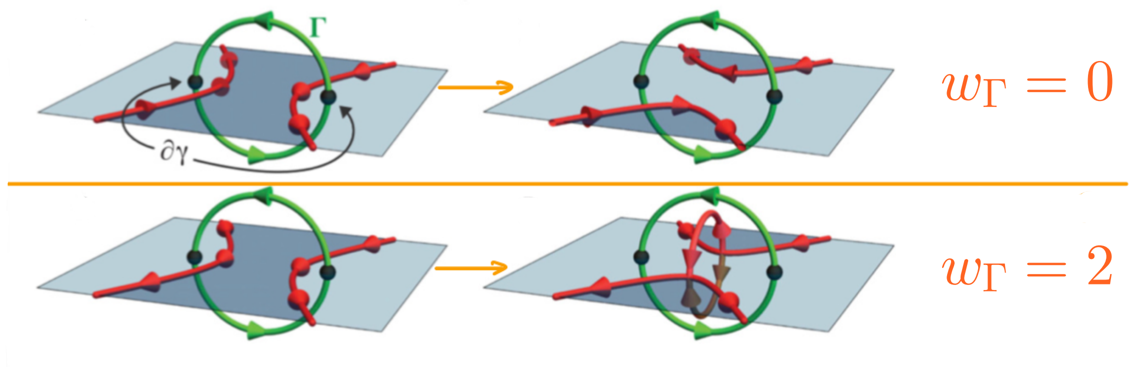

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(We have $\pi_1(\mathrm{SO}(3)) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$, $\mathrm{SU}(2) \cong S^3$ its 2-fold universal cover.)

Moduli spaces for non-Hermitian Hamiltonians

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$$\eta H \eta^{-1} = \overline{H^t} \qquad \text{and} \qquad [PT, H] = 0$$

pseudo-Hermiticity — parity–time symmetry

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Here emerge non-isolated, stratified singular loci, making our systems gapless and their topology much intriguing.

Moduli spaces for non-Hermitian Hamiltonians: 2-band systems

In the generic 2-band case, we give complete invariants.

$$H_2 = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix} \quad \text{(Recall Hermitian} \quad \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix} \quad \text{)}$$

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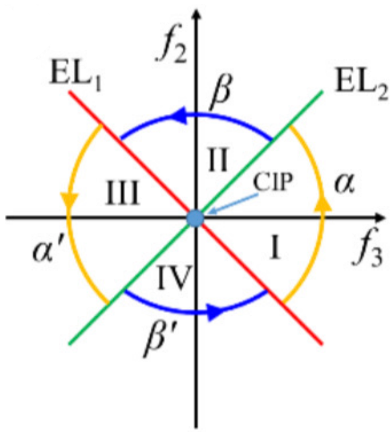

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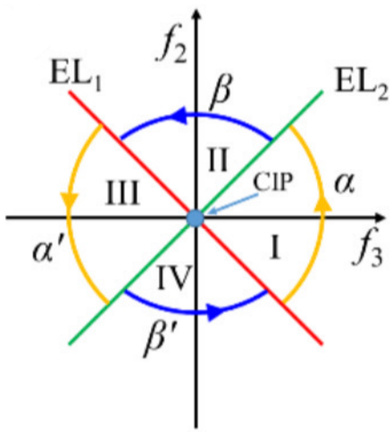

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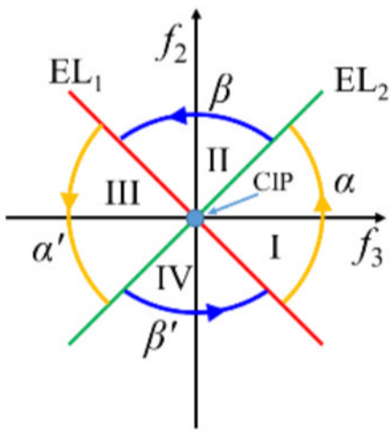

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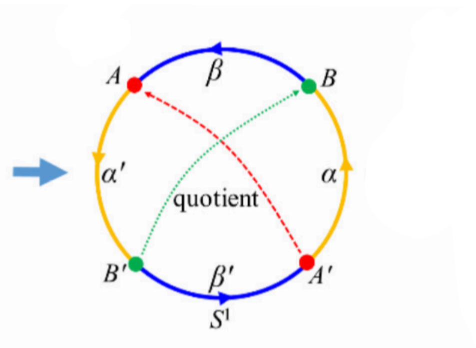

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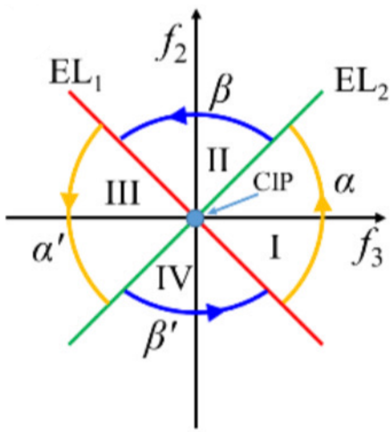

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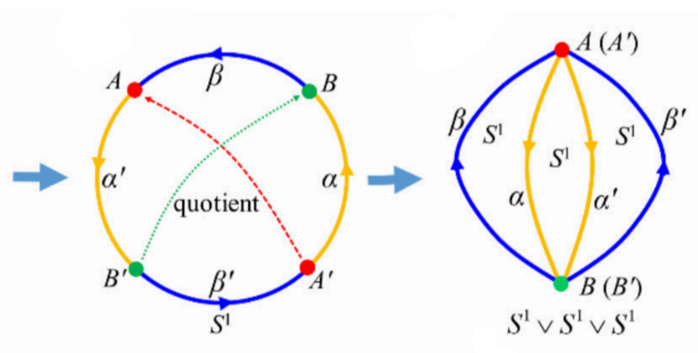

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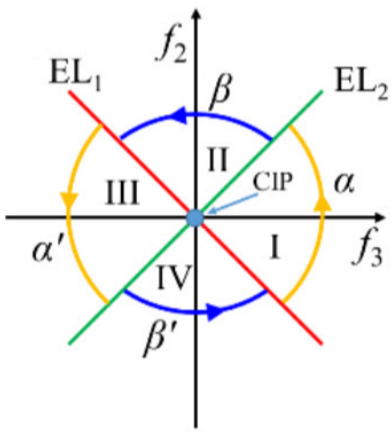

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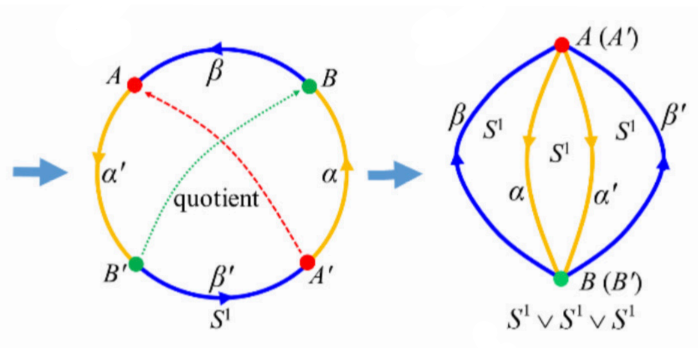
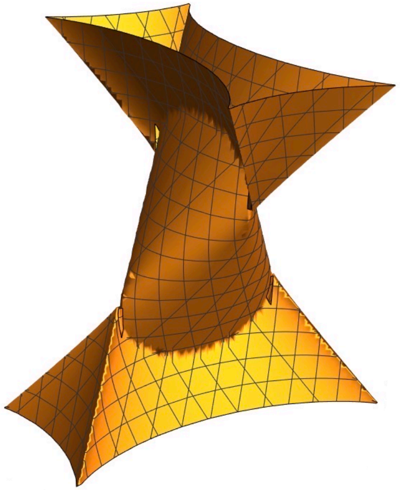


The moduli space $\mathscr{M}_2$ can be identified as $S^1 \vee S^1 \vee S^1$ doubly covering $S^1 \vee S^1$. Thus $\pi_1(\mathscr{M}_2)$ is a free subgroup of $F(\alpha, \beta)$ on 3 generators. This gives the gapless system a physically meaningful, non-Abelian topological charge.

Moduli spaces for non-Hermitian Hamiltonians: 3-band systems

The 3-band case is more complex and exotic. Here is a sample of portraits for the stratified singular moduli spaces.



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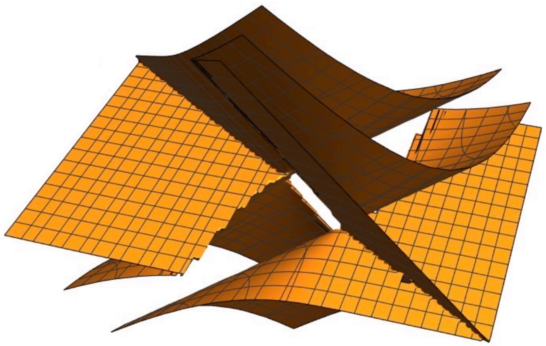

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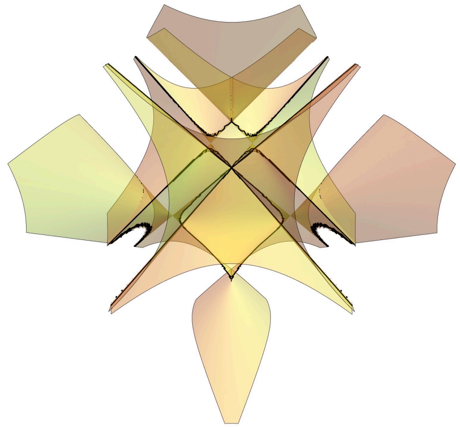

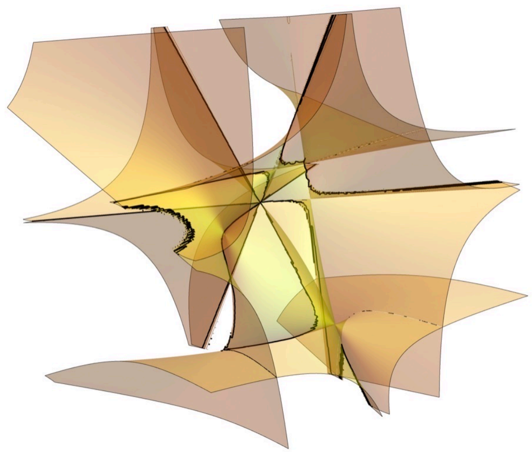

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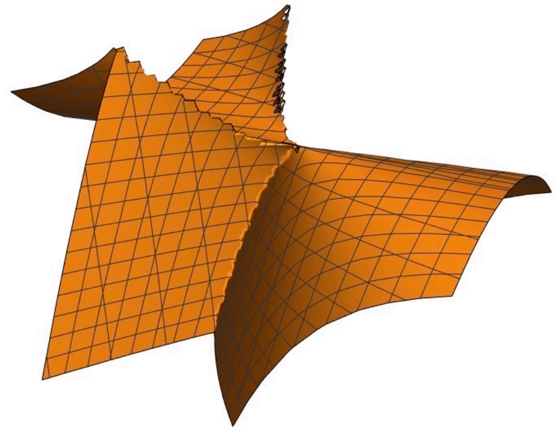

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A mechanical wave system

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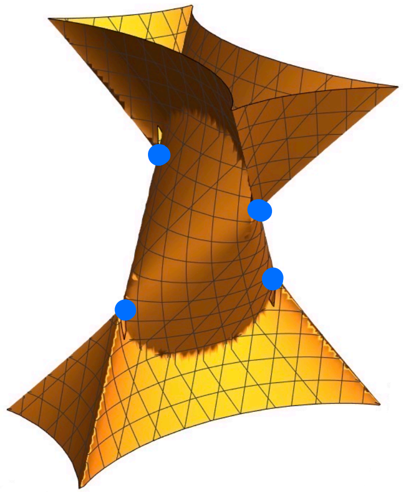

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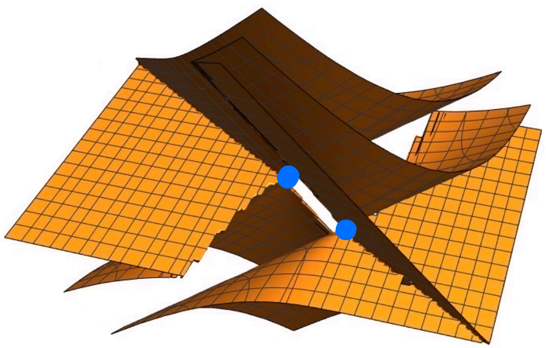

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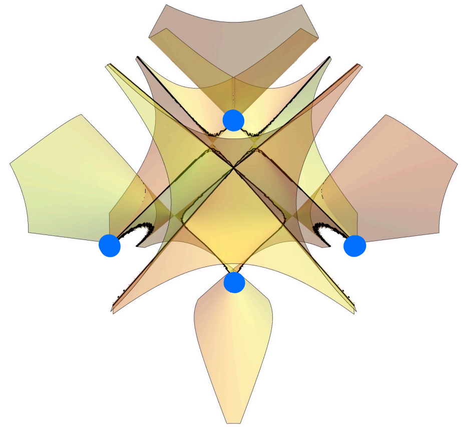

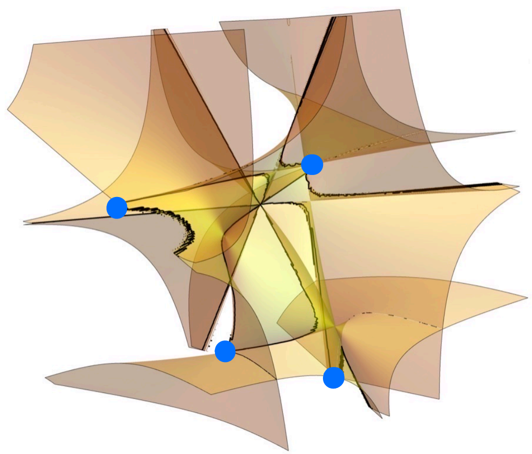

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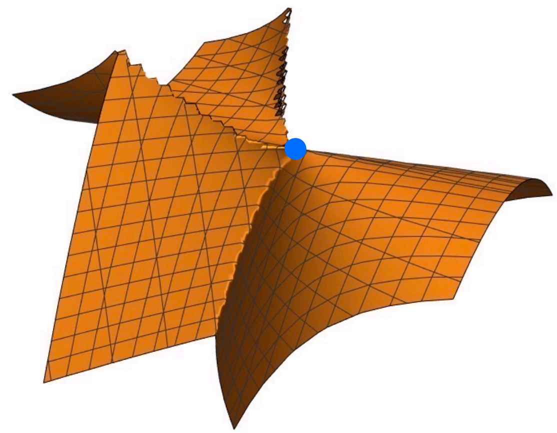

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A mechanical wave system

The central figure within these configurations is the so-called swallowtail catastrophe.

Moduli spaces for non-Hermitian Hamiltonians: 3-band systems

The singularity of a swallowtail arises in the discriminant surface of a generic degree-4 polynomial. Here, it is the characteristic polynomial of $H(f_1, f_2, f_3)$.

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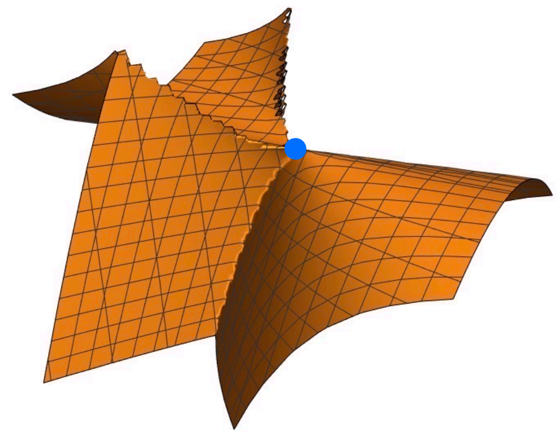

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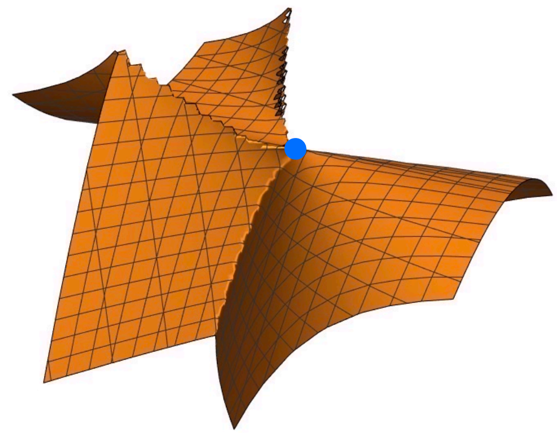
https://yifeizhu.github.io/swtl.mp4



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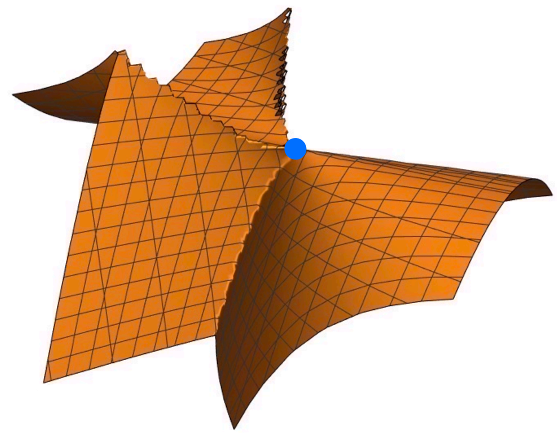

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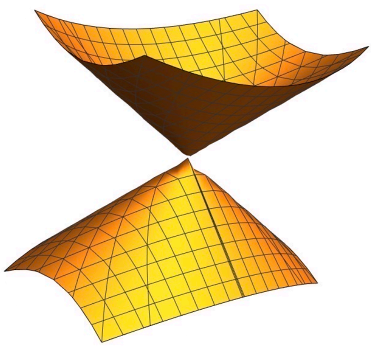
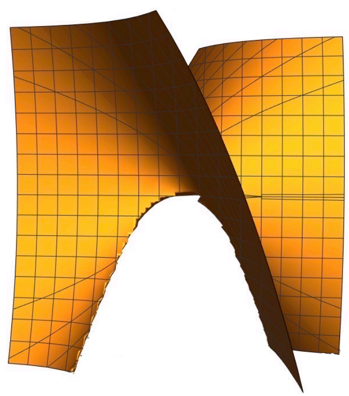
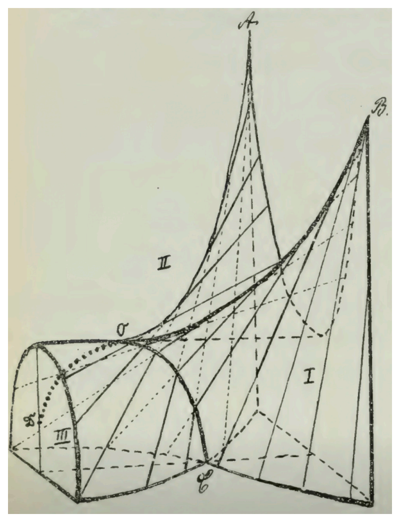
Discriminant surfaces are ruled (in fact, developable).







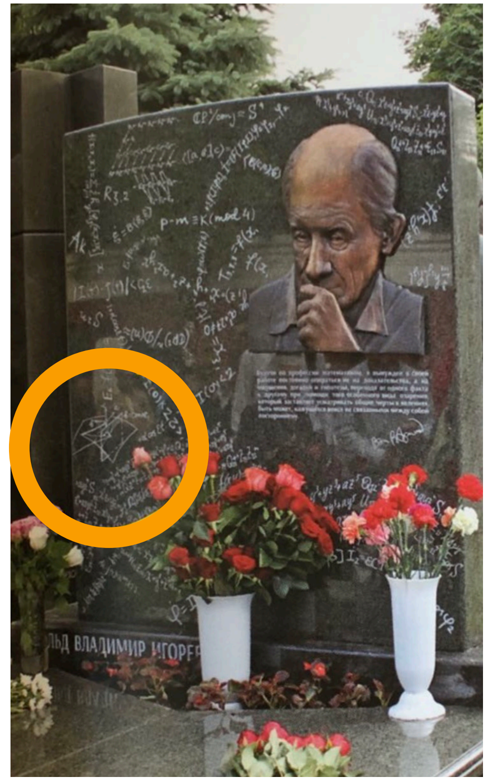
1908 quadratic ($\Delta = b^2 - 4ac$)

cubic

Illustration from Klein's lecture notes by Hellinger.

Moduli spaces for non-Hermitian Hamiltonians: 3-band systems

As the singular loci of moduli spaces for polynomials, swallowtail and other catastrophes are important and well-studied objects in dynamical systems and algebraic geometry. Arnold famously related their complements to braid groups and computed their cohomology, establishing a connection to topology as well.



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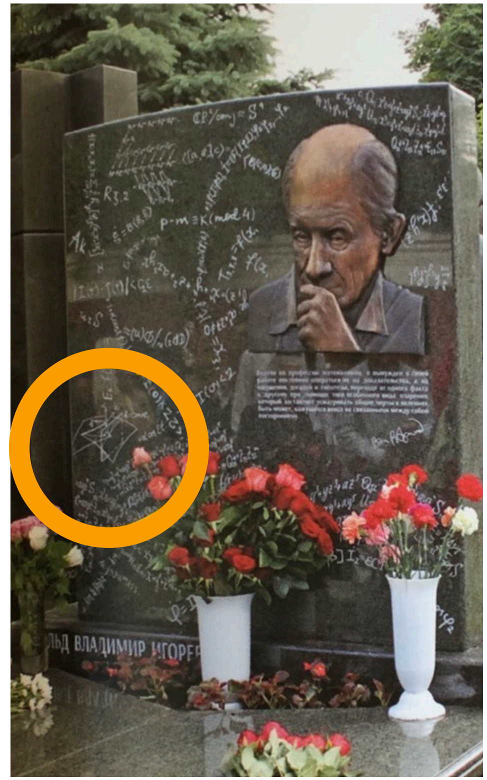

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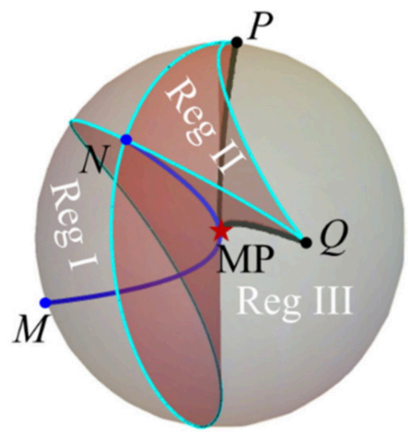
However, moduli spaces for Hamiltonians carry additional structures and are more complex:

- Physicists desire classifications for the behavior of eigenstates along a loop across/encircling the stratified non-isolated singularity (e.g., Berry phase of adiabatic transformation, close and open of gaps).



Reg I and Reg II: PT-exact phases
Reg III: PT-broken phase

- Over the reals, we know less even on the mathematical side.

Moduli spaces for non-Hermitian Hamiltonians: 3-band systems

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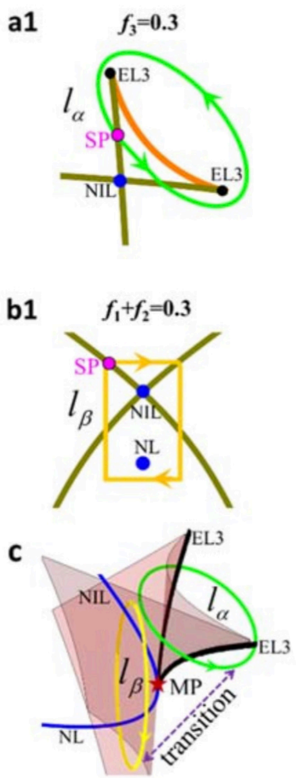

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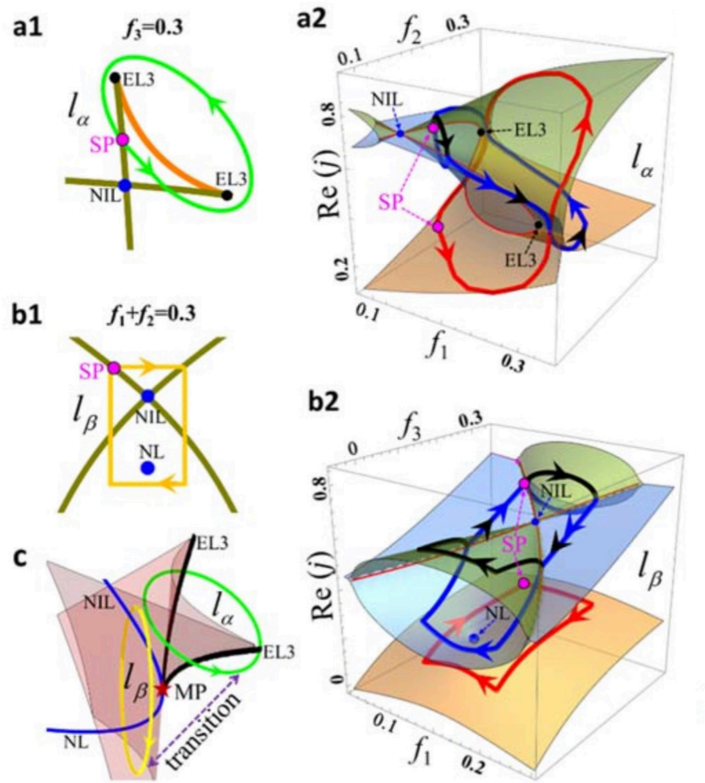

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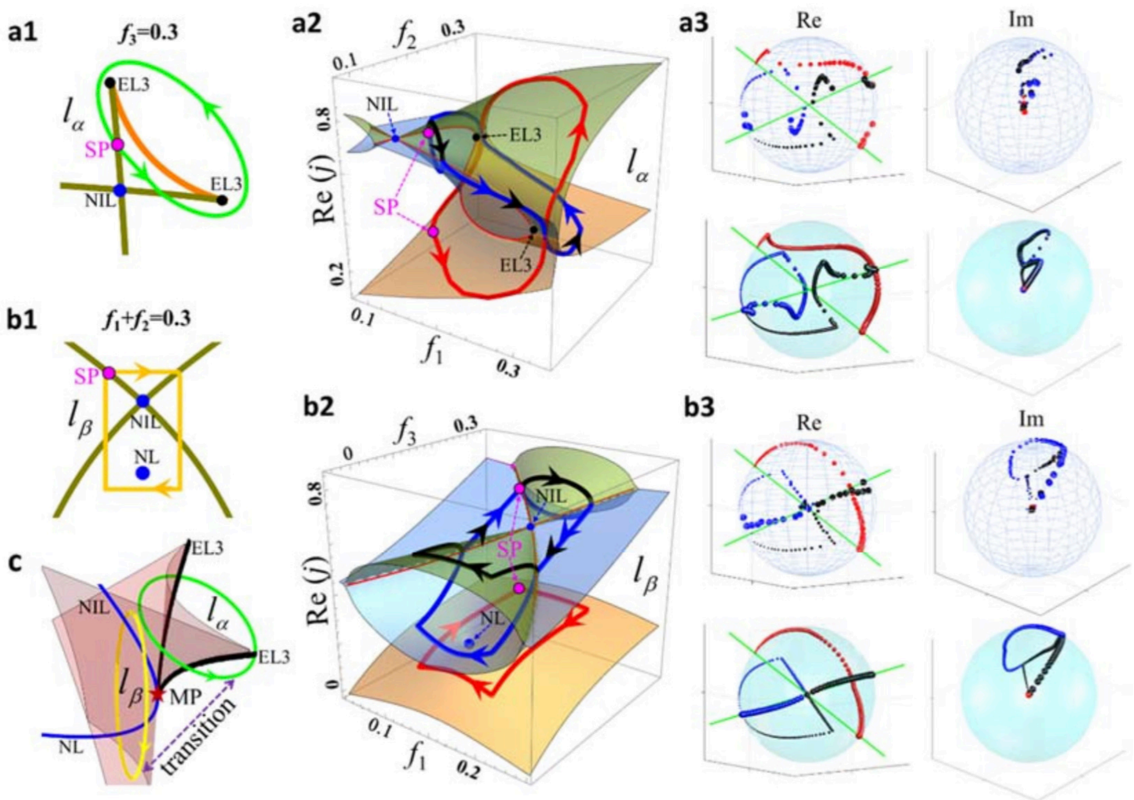

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a3, b3: trajectories of eigenframe, experimental (above) and theoretical (below)

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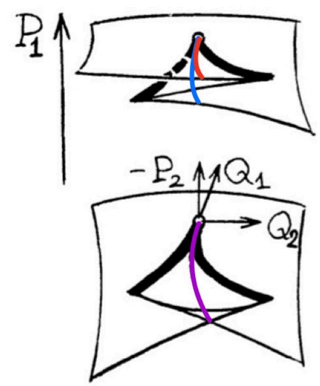

The unfurled swallowtail over the ordinary swallowtail

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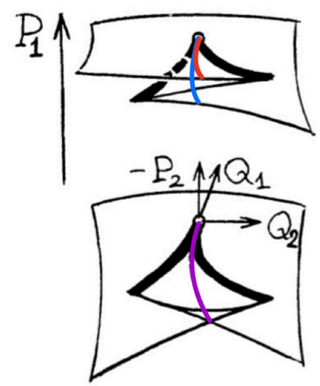

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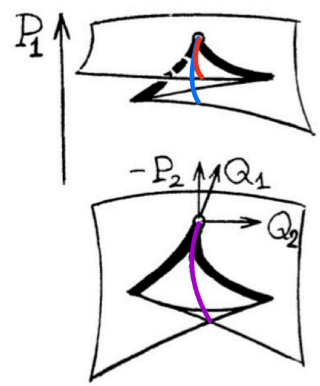

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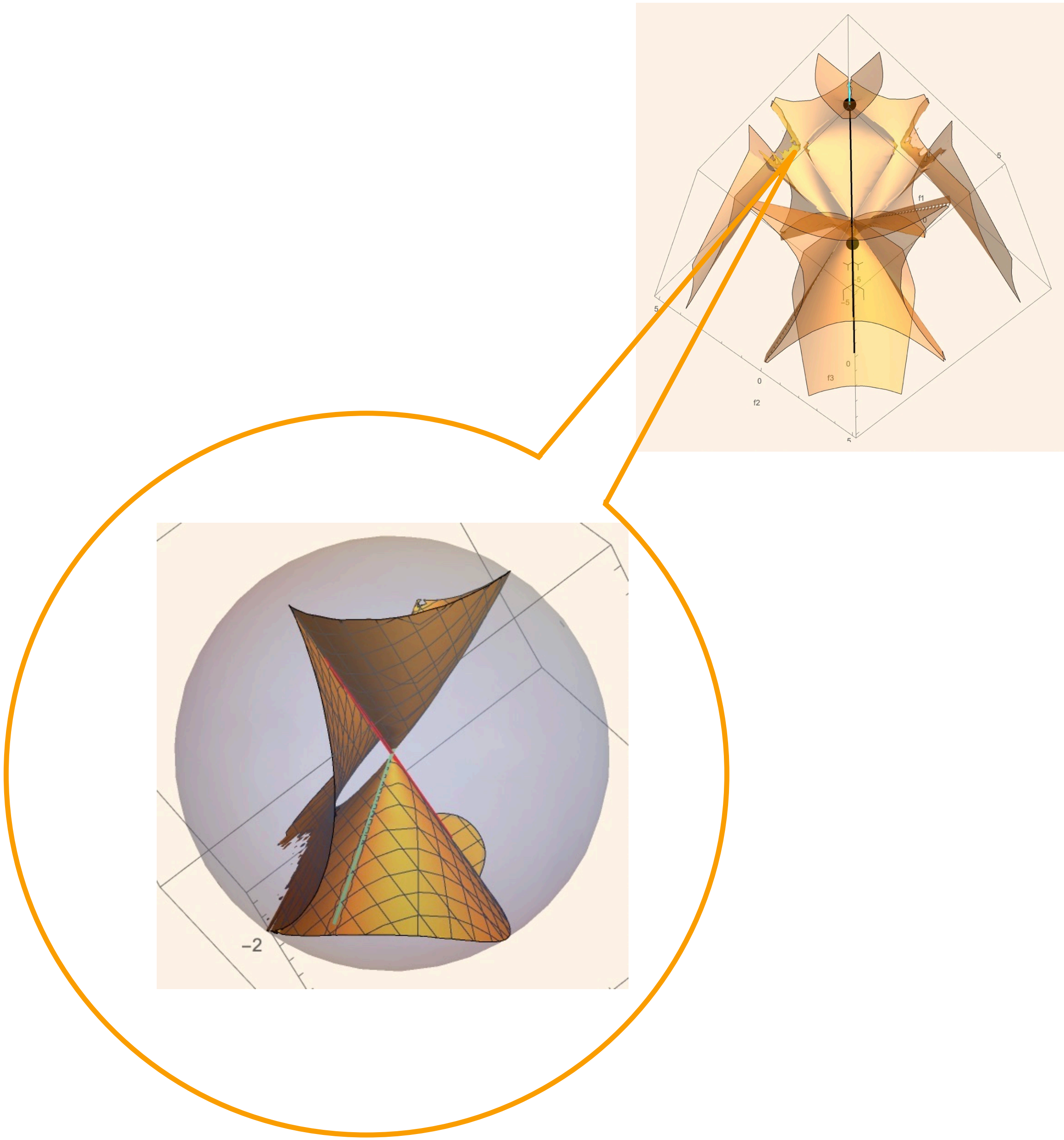
- We have been zooming in to local details of combinations of swallowtails (and other basic types of singularities) in order to pass from local invariants to global (and complete) invariants via fuller power of algebraic topology.



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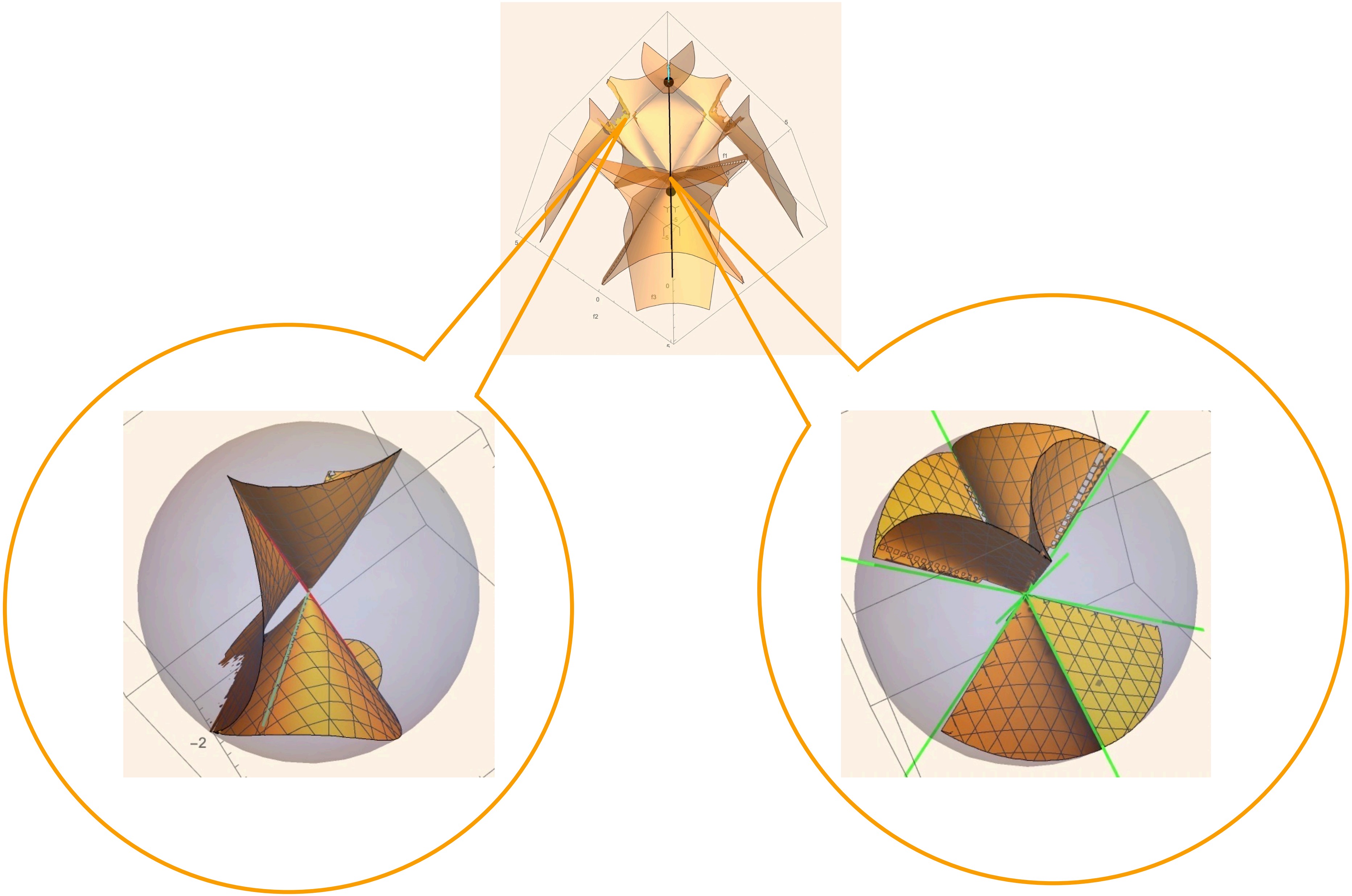
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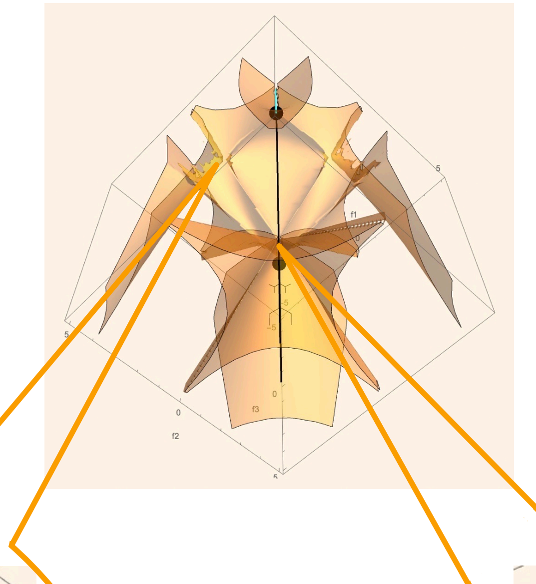

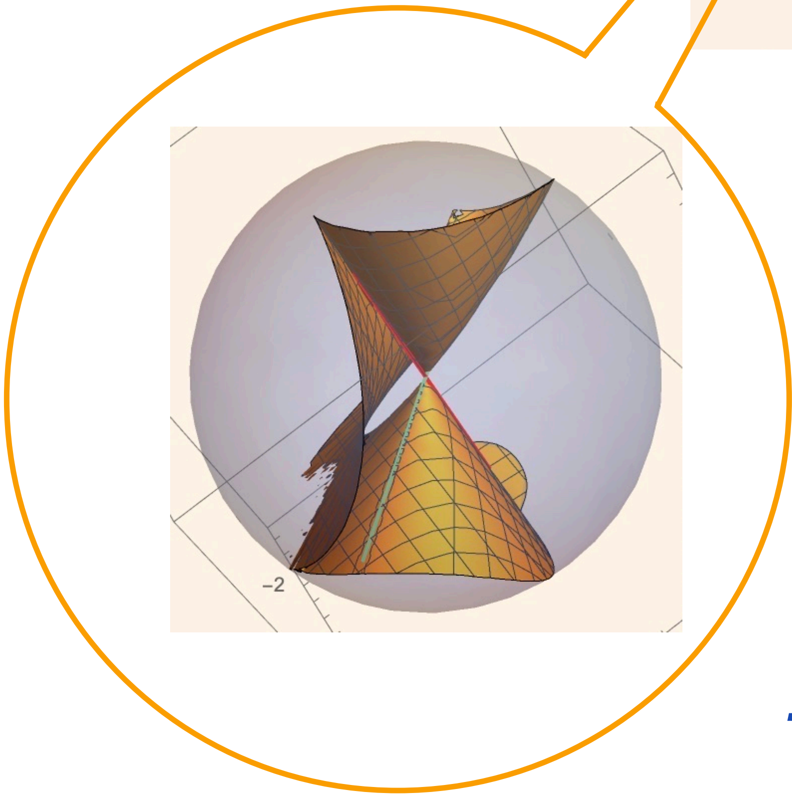

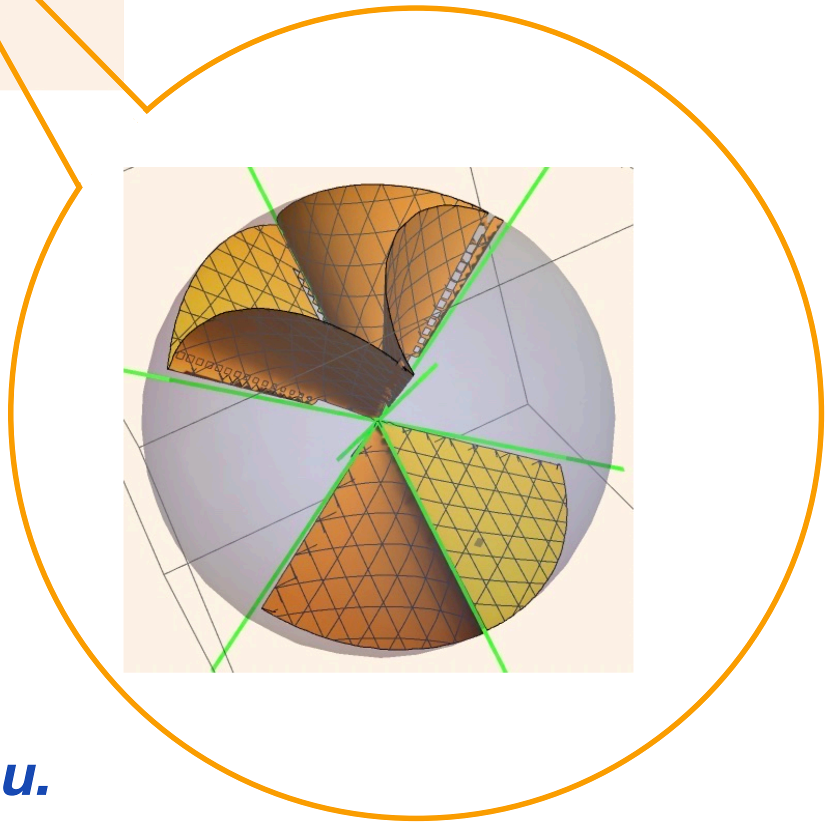

Thank you.

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