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Computational Methods in Algebraic Topology
（代数拓扑学中的计算方法）

朱雅熹

（数学系 指导教师：朱一飞）

[摘要]：本文是一篇基于《Algebraic Topology》（Allen Hatcher）与《同调论》（姜伯驹）的读书笔记，其中介绍了同调代数、奇异同调群、奇异上同调环、向量丛等基本概念和代数拓扑帮助解决实向量丛分类问题的过程。同时文章也包含了胞腔同调、同调群的系数变化和万有系数定理这三种计算方法，并在结尾即计算方法部分通过所介绍的方法计算了若干常见拓扑空间的带系数同调、上同调群，并分析了系数变化作为判断映射是否零伦的方法的优势和盲区。作者希望这篇读书笔记也能为想快速了解这部分代数拓扑学的读者提供帮助。

[关键词]：代数拓扑；计算方法；奇异同调；奇异上同调；向量丛分类

[**Abstract**]: This paper is a reading report that mainly follows "Algebraic Topology" by Allen Hatcher and "Homology Theory" by Jiang Boju. It introduces basic concepts in homological algebra, singular (co)homology theory, ring structure in cohomology theory, vector bundle theory and the process of how algebraic topology helps with classifying vector bundles. It also includes three computational methods: cellular homology, varying coefficients in (co)homology and the Universal coefficients theorem. Towards the end, we present a set of examples that illustrates computations of (co)homology for certain well-known spaces. Then we analyze the advantages and blindness of varying coefficient groups as a method to detect whether a map is null-homotopic. The author also hopes this report serves as a crash course for those who are interested in learning this part of algebraic topology.

[**Keywords**]: Algebraic topology, Computational methods, Singular homology, Singular cohomology, Classifying vector bundles

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1. Introduction

Topology studies properties of topological spaces under continuous transformations. A well-known example is that a donut is topologically equivalent to a mug. Algebraic topology uses algebra to study topology and it helps with solving problems (such as if $n \neq m$, then $\mathbb{R}^n \ncong \mathbb{R}^m$) that are difficult to answer with traditional topology arguments.

This is a reading report on algebraic topology which mainly follows "Homology Theory" by Jiang Boju and "Algebraic Topology" by Allen Hatcher. The purpose is to understand basic concepts in homology and cohomology as well as algebraic topology methods for classifying vector bundles. Also, we introduce basic computational methods and their applications.

The paper is organized as follows. In Chapter 2, we introduce the categorical language. We mainly follow "Homology Theory" by Jiang to present singular homology and related concepts in Chapter 3. Then in Chapter 4 with the help of "An Introduction to Algebraic Topology" by Joseph J. Rotman we give a quick overview of singular cohomology and its ring structure. How algebraic topology helps with classifying real vector bundles is presented in Chapter 5 and this section is based on "Vector Bundles and K-Theory" by Hatcher. Finally, in Chapter 6 we follow "Algebraic Topology" by Hatcher and "An Introduction to Algebraic Topology" by Rotman to study some computational methods as well as their applications, such as computing (co)homology groups of certain topological spaces and detecting null-homotopic maps.

2. Category and Functor

2.1 Category and Functor

Category theory as an abstraction of a lot of mathematical concepts, offers a more general and more concise method to describe many mathematical situations. It plays an important role in modern mathematics.

Definition 2.1. A ***category*** contains three ingredients: a class $\mathcal{C}$ of objects; sets of morphisms $Hom(A, B)$, each of which corresponds to an ordered pair of objects A, B in $\mathcal{C}$; composition $Hom(A, B) \times Hom(B, C) \rightarrow Hom(A, C)$, denoted by $(f, g) \rightarrow gf$ for every object A, B, C in $\mathcal{C}$ such that the followings hold:

(1) $Hom(A, B)$s are disjoint for different ordered pairs of objects.

(2) Associative law holds for composition when defined. i.e. $(fg)h = f(gh)$ for appropriate morphisms f, g, h.

(3) For each object A in $\mathcal{C}$, there exists an identity $1_A \in Hom(A, A)$ such that for all objects B,C of $\mathcal{C}$ and $f \in Hom(A, B)$ and $g \in Hom(C, A)$ we have $f1_A = f$ and $1_A g = g$.

Example 2.2. $\mathcal{C} = \mathbf{Top}$ the class of all topological spaces.

$Hom(A, B)$= {all continuous functions from A to B} with the usual composition for continuous functions.

Example 2.3. $\mathcal{C} = \mathbf{Top^2}$ the class of all ordered pairs of topological spaces (X, A), where $A \subset X$ a subspace. $Hom\big((X, A), (Y, B)\big)$ ={(f, f')| $f: X \rightarrow Y$ continuous and $f(A) \subset B$} with coordinatewise composition.

Definition 2.4. A ***congruence*** on a category $\mathcal{C}$ is an equivalence relation $\sim$ defined on the class of all morphisms in $\mathcal{C}$ such that the followings hold:

(1) $f \in Hom(A, B)$ and $f' \sim f$, then $f' \in Hom(A, B)$.

(2) $f \sim f'$, $g \sim g'$ implies $gf \sim g'f'$.

Definition 2.5. A ***quotient category*** $\mathcal{C}'$ of a category $\mathcal{C}$ is defined on $\mathcal{C}$ with a congruence $\sim$ such that the followings hold:

(1) Objects of $\mathcal{C}$ and $\mathcal{C}'$ are the same.

(2) $Hom_{\mathcal{C}'}(A, B) = \{[f] | f \in Hom_{\mathcal{C}}(A, B)\}$ ($[f]$ denotes the equivalence class of f under the equivalence relation $\sim$).

(3) $[g][f] = [gf]$ for $f \in Hom_{\mathcal{C}'}(A, B)$ and $g \in Hom_{\mathcal{C}'}(B, C)$ where C is an

object in $\mathcal{C}$.

Example 2.6. The quotient category **hTop** of the category **Top** with the congruence $\simeq$ homotopy (details in 3.3). The equivalence class of f is $[f] = \{g \in Hom(X, Y) | g \simeq f\}$, called the homotopy class of f.

Definition 2.7. A ***covariant functor*** $\mathrm{F}: \mathcal{A} \to \mathcal{C}$ is a function between two categories $\mathcal{A}$ and $\mathcal{C}$ such that:

(1) $A \in \mathcal{A}$ implies $\mathrm{F}(A) \in \mathcal{C}$.

(2) Given $f \in Hom(A, A')$ a morphism in $\mathcal{A}$, then $\mathrm{F}(f) \in Hom(\mathrm{F}(A), \mathrm{F}(A'))$ a morphism in $\mathcal{C}$ such that:

(a) For $f \in Hom(A, A')$, $g \in Hom(A', A'')$ such that gf is defined, then $\mathrm{F}(gf) = \mathrm{F}(g)\mathrm{F}(f) \in Hom(\mathrm{F}(A), \mathrm{F}(A)'')$.

(b) For identity map $1_A \in Hom(A, A)$ in $\mathcal{A}$,

$\mathrm{F}(1_A) = 1_{F(A)} \in Hom(\mathrm{F}(A), \mathrm{F}(A))$.

Example 2.8. The ***forgetful functor*** F: **Top** $\to$ **Sets** maps topological spaces to their underlying sets and continuous functions to itself without continuity.

Example 2.9. The ***identity functor*** $\mathrm{I}: \mathcal{C} \to \mathcal{C}$ maps any object A in $\mathcal{C}$ to A and f to f.

Definition 2.10. A ***contravariant functor*** $\mathrm{F}: \mathcal{A} \to \mathcal{C}$ is a function between two categories $\mathcal{A}$ and $\mathcal{C}$ such that:

(1) $A \in \mathcal{A}$ implies $\mathrm{F}(A) \in \mathcal{C}$.

(2) Given $f \in Hom(A, A')$ a morphism in $\mathcal{A}$, $\mathrm{F}(f) \in Hom(\mathrm{F}(A'), \mathrm{F}(A))$ a morphism in $\mathcal{C}$ such that:

(a) For $f \in Hom(A, A')$, $g \in Hom(A', A'')$ such that gf is defined, then $\mathrm{F}(gf) = \mathrm{F}(g)\mathrm{F}(f) \in Hom(\mathrm{F}(A''), \mathrm{F}(A))$.

(b) For identity map $1_A \in Hom(A, A)$ in $\mathcal{A}$,

$\mathrm{F}(1_\mathrm{A}) = 1_{\mathrm{F(A)}} \in Hom(\mathrm{F}(A), \mathrm{F}(A))$.

Note that the contravariant functor changes the "direction" of morphisms.

Definition 2.11. An ***equivalence*** in a category $\mathcal{C}$ is a morphism $f \in Hom(A, B)$ and there exists a morphism $g \in Hom(B, A)$ such that $fg = 1_B$ and $gf = 1_A$.

Theorem 2.12. A functor (of either variance) $\mathrm{T}: \mathcal{A} \to \mathcal{C}$ for two categories maps equivalence to equivalence.

Proof: Here we only prove the covariant case! By the definition of equivalence we have a morphism $f \in Hom(A, B)$ in $\mathcal{A}$ and there exists a morphism $g \in Hom(B, A)$ such that $fg = 1_B$ and $gf = 1_A$. We apply our functor T then by the definition of functor we get $\mathrm{T}(fg) = 1_{\mathrm{T}(B)} = \mathrm{T}(f)\mathrm{T}(g)$ and $\mathrm{T}(gf) = 1_{\mathrm{T}(A)} = \mathrm{T}(g)\mathrm{T}(f)$. Hence $\mathrm{T}(f) \in Hom(\mathrm{T}(A), \mathrm{T}(B))$ is an equivalence in $\mathcal{C}$.

■

3. Singular Homology

3.1 The Category of Chain Complexes

Definition 3.1. A ***graded group*** is a collection of abelian groups, which is denoted by $G_* = \{G_q | q \in \mathbb{Z}\}$.

Definition 3.2. A ***homomorphism*** $\boldsymbol{\varphi_*: G_* \to G'_*}$ ***between two graded groups*** are a collection of homomorphisms $\{\varphi_q: G_q \to G'_q\}$.

Definition 3.3. All graded groups together with graded group homomorphisms form a category **GradedG**. The composition of graded groups homomorphism is defined as

$\varphi_*\theta_* := \{\varphi_q\theta_q\}$.

We sometimes abuse the language and does not specify the "dimension" of one graded group homomorphism. For example if we take an arbitrary element $s_q \in S_q$ of $S_* = \{S_q | q \in \mathbb{Z}\}$, then $\varphi_*(s_q)$ actually means $\varphi_q(s_q)$.

Definition 3.4. A ***chain complex*** $C = \{C_q, \partial_q\}$ is a sequence of abelian groups C_q and homomorphisms (also called qth boundary map) $\partial_q: C_q \to C_{q-1}$ such that $\partial_q\partial_{q+1} = 0$ for each $q \in \mathbb{Z}$.

$$\cdots \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots$$

Note that the condition $\partial_q\partial_{q+1} = 0$ is equivalent to $im\partial_{q+1} \subseteq ker\partial_q$. And a chain complex is indeed a graded group equipped with the boundary maps.

Definition 3.5. Given a chain complex $C = \{C_q, \partial_q\}$. $Z_q(C) := ker\partial_q$ is called the group of ***q-cycles***; $B_q(C) := im\partial_{q+1}$ is called the group of ***q-boundaries***. The quotient group $H_q(C) := Z_q(C)/B_q(C)$ is called the *qth* ***homology group*** of C. The elements of $H_q(C)$ is called ***homology classes***. The homology class of a q-cycle z_q is $[z_q] := z_q + B_q(C)$. We usually put homology groups of all dimensions together as a graded group $H_*(C) = \{H_q(C)\}$.

Note that the definition of quotient group $H_q(C)$ makes sense since C_q is abelian hence its subgroups are normal and by requirement of our qth boundary map we have $im\partial_{q+1} \subseteq ker\partial_q$.

To make chain complexes a category we need to define the morphisms between two objects.

Definition 3.6. Given two chain complexes $C = \{C_q, \partial_q\}$, $D = \{D_q, \partial_q\}$, a ***chain map***

$f: C \to D$ is a sequence of homomorphisms $\{f_q: C_q \to D_q\}$ such that $\partial_q f_q = f_{q-1}\partial_q$ for each $q \in \mathbb{Z}$. i.e. the diagram commutes.

$$\cdots \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots$$

$$\downarrow f_{q+1} \quad \downarrow f_q \quad \downarrow f_{q-1}$$

$$\cdots \to D_{q+1} \xrightarrow{\partial_{q+1}} D_q \xrightarrow{\partial_q} D_{q-1} \to \cdots$$

Note that actually we should denote D by $D = \{D_q, \partial'_q\}$ as it might have different boundary maps than $C = \{C_q, \partial_q\}$. Here we "ignore" the difference for convenience but one should not forget about it.

Lemma 3.7. A chain map $f: C \to D$ induces a homomorphism between homology groups $f_* = H_*(C) \to H_*(D)$, $f_*([z_q]) := [f_q(z_q)]$ for $[z_q] \in H_q(C)$.

Proof: By definition of a chain map, we have $\partial_q f_q = f_{q-1}\partial_q$ for each $q \in \mathbb{Z}$. Take $z_q \in Z_q(C)$, we know $\partial_q(z_q) = 0$ since $Z_q(C) = ker\partial_q$. Hence $\partial_q f_q(z_q) = f_{q-1}\partial_q(z_q) = 0$ implies $f_q(z_q) \in Z_q(D)$. Take $b_q \in B_q(C) := im\partial_{q+1}$, we know there exists some $b_{q+1} \in C_{q+1}$ such that $\partial_{q+1}(b_{q+1}) = b_q$. Hence $f_q(b_q) = f_q\partial_{q+1}(b_{q+1}) = \partial_{q+1}f_{q+1}(b_{q+1}) \in im\partial_{q+1} = B_q(D))$. So $f_q\left(Z_q(C)\right) \subset Z_q(D)$ and $f_q\left(B_q(C)\right) \subset B_q(D)$. We may define $f_* = H_*(C) \to H_*(D)$, $f_*([z_q]) := [f_q(z_q)]$ for $[z_q] \in H_q(C)$.

Firstly, we check if it is well-defined. i.e. independent of the choice of representatives. If $b_q \in B_q(C)$, $f_*([z_q + b_q]) = [f_q(z_q + b_q)] = [f_q(z_q) + f_q(b_q)] = [f_q(z_q)]$ since $f_q\left(B_q(C)\right) \subset B_q(D)$. So it is well-defined.

Secondly, let's verify that it is a homomorphism. For any $[z_q^1], [z_q^2] \in H_q(C)$, $f_*([z_q^1] + [z_q^2]) = f_*([z_q^1 + z_q^2]) = [f_q(z_q^1 + z_q^2)] = [f_q(z_q^1)] + [f_q(z_q^2)] = f_*([z_q^1]) + f_*([z_q^2])$.

So we can define the induced homomorphism $f_*: H_*(C) \to H_*(D)$.

■

Definition 3.8. All chain complexes together with chain maps form a category **Comp**. The composition of chain maps is defined by $\{g_q\}\{f_q\} = \{g_q f_q\}$.

Theorem 3.9. We have a covariant functor $H_*: \mathbf{Comp} \to \mathbf{GradedG}$ (the category of all graded groups) with $H_*(f) = f_*$.

Proof. By Definition 3.5 and Lemma 3.7.

■

Definition 3.10. Given a chain complex $C = \{C_q, \partial_q\}$, we can define its ***subcomplex*** $C' = \{C'_q, \partial'_q\}$ with each C'_q a subgroup of C_q and each $\partial'_q = \partial_q|_{C'_q}$.

Definition 3.11. Given a chain complex $C = \{C_q, \partial_q\}$ and its subcomplex $C' = \{C'_q, \partial'_q\}$. We can define their ***quotient complex*** $C/C' := \{C_q/C'_q, \overline{\partial_q}\}$ where $\overline{\partial_q}$: $c_q + C'_q \to \partial_q(c_q) + C'_{q-1} \in C_{q-1}/C'_{q-1}$.

Note that the map $\overline{\partial_q}$ is well-defined since $\partial_q(C'_q) \subset C'_{q-1}$ by the definition of subcomplex. Actually when dealing with chain complexes we can imagine that we are dealing with abelian groups.

Definition 3.12. Given a family of chain complexes $\{C_i | i \in I\}$ where $C_i = \{C_{iq}, \partial_{iq}\}$ we can define their ***direct sum*** $\bigoplus_{i \in I} C_i := \{\bigoplus_{i \in I} C_{iq}, \bigoplus_{i \in I} \partial_{iq}\}$ ($\bigoplus_{i \in I} C_{iq}$ is direct sum of abelian groups and $\bigoplus_{i \in I} \partial_{iq}$ is direct sum of group homomorphisms), which is also a chain complex.

Theorem 3.13. Given a family of chain complexes $\{C_i | i \in I\}$ where $C_i = \{C_{iq}, \partial_{iq}\}$.

The homology groups of their direct sum $\bigoplus_{\mathrm{i}\in\mathrm{I}} C_i$ has the property $H_*(\bigoplus_{\mathrm{i}\in\mathrm{I}} C_i) \cong \bigoplus_{\mathrm{i}\in\mathrm{I}} H_*(C_i)$.

Proof: For each dimension q. Take an element $c_{iq} \in \bigoplus_{\mathrm{i}\in\mathrm{I}} C_{iq}$. By the definition of direct sum we can write c as $(c_{iq})_{i\in I}$ with finitely many of c_{iq}'s nonzero and we denote such nonzero terms by (c_{iq}). Hence we have

$$\left(\bigoplus_{\mathrm{i}\in\mathrm{I}} \partial_{iq}\right)(c_{iq}) = \bigoplus_{i\in I} (\partial_{iq} c_{iq})$$

So we can see that $\left(\bigoplus_{\mathrm{i}\in\mathrm{I}} \partial_{iq}\right)(c_{iq}) = 0$ if and only if $\bigoplus_{i\in I} (\partial_{iq} c_{iq}) = 0$.

We now define $\Phi: H_q(\bigoplus_{\mathrm{i}\in\mathrm{I}} C_i) \to \bigoplus_{\mathrm{i}\in\mathrm{I}} H_q(C_i)$ by $\Phi([(c_{iq})]) := ([c_{iq}])$.

And Ψ: $\bigoplus_{\mathrm{i}\in\mathrm{I}} H_q(C_i) \to H_q(\bigoplus_{\mathrm{i}\in\mathrm{I}} C_i)$ by $\Psi([c_{iq}]) = ([(c_{iq})])$.

Now we want to check if these functions are well-defined:

If $[(c_{iq})] = [(c'_{iq})]$, then it means $[(c_{iq})] - [(c'_{iq})] = [(c_{iq}) - (c'_{iq})] = [(c_{iq} - c'_{iq})] = 0$, which means there exists $(b_{i(q+1)}) \in \bigoplus_{\mathrm{i}\in\mathrm{I}} C_{i(q+1)}$ such that $\left(\bigoplus_{\mathrm{i}\in\mathrm{I}} \partial_{i(q+1)}\right)(b_{i(q+1)}) = (c_{iq} - c'_{iq})$. The equation holds if and only if $c_{iq} = c'_{iq} + \partial_{i(q+1)} c_{i(q+1)}$ for each $i \in I$, which implies $([c_{iq}]) = ([c'_{iq}])$ if and only if $\Phi([(c_{iq})]) = ([c_{iq}]) = ([c'_{iq}]) = \Phi([(c'_{iq})])$. Hence Φ is well-defined.

Similarly we can prove Ψ is also well-defined. And because both of them are clearly homomorphisms between abelian groups and either of them is indeed an inverse function of another. Hence we have $H_*(\bigoplus_{\mathrm{i}\in\mathrm{I}} C_i) \cong \bigoplus_{\mathrm{i}\in\mathrm{I}} H_*(C_i)$.

■

Definition 3.14. Two chain maps $f, g: C \to D$ are called ***chain homotopic*** if there is a sequence of homomorphisms $T = \{T_q: C_q \to D_{q+1}\}$ such that for all $q \in \mathbb{Z}$ we have $\partial_{q+1} T_q + T_{q-1} \partial_q = g_q - f_q$. We call such $T = \{T_q\}$ a chain homotopy, denoted by $f \simeq g: C \to D$.

Theorem 3.15. Given two homotopic chain maps $f \simeq g: C \to D$, they induce the same homomorphisms $H_*(f) = H_*(g) = f_* = g_*: H_*(C) \to H_*(D)$.

Proof: Since $g_*([z_q]) - f_*([z_q]) = [g_q(z_q)] - [f_q(z_q)]$

$$= [g_q(z_q) - f_q(z_q)]$$

$$= [(g_q - f_q)(z_q)]$$

(By Definition 3.14) $= [\partial_{q+1}T_q(z_q) + T_{q-1}\partial_q(z_q)]$

$(\partial_q(z_q) = 0) = [\partial_{q+1}T_q(z_q)] \in B_q(D) = 0$

Hence $g_* = f_*$

■

Proposition 3.16. The relation of homotopy is an equivalence relation on the set of all chain maps from C *to* D.

Proof: Suppose $T: f \simeq g: C \to D$ and $F: g \simeq h: C \to D$.

(1) Reflexive: $T: f \simeq f$ by setting $T = 0$.

(2) Symmetric: $T: f \simeq g$ such that $\partial_{q+1}T_q + T_{q-1}\partial_q = g_q - f_q$.

So $-T: f \simeq g$ such that $\partial_{q+1}(-T_q) + (-T_{q-1})\partial_q = f_q - g_q$.

(3) Transitive: $T: f \simeq g: C \to D$ and $F: g \simeq h: C \to D$.

$$\begin{aligned} h_q - f_q &= h_q - g_q + g_q - f_q \\ &= \partial_{q+1}F_q + F_{q+1}\partial_q + \partial_{q+1}T_q + T_{q-1}\partial_q \\ &= \partial_{q+1}(F_q + T_q) + (F_{q-1} + T_{q-1})\partial_q \end{aligned}$$

■

Definition 3.17. A chain map $f: C \to D$ is called a ***chain equivalence*** if there exists a chain map $g: D \to C$ such that $gf \simeq 1_C: C \to C$ and $fg \simeq 1_D: D \to D$. And two chain complexes are called chain ***equivalent*** if there exists a chain equivalence between them, denoted by $C \simeq D$.

Proposition 3.18. The relation of chain equivalent is an equivalence relation on the class of all chain complexes.

Proof: We first prove the claim:

$$\text{if } T: f \simeq f': C \to D, F: g \simeq g': D \to E,\ gf \simeq g'f': C \to E.$$

Since $T: f \simeq f': C \to D$, we have $f_q' - f_q = \partial_{q+1} T_q + T_{q-1}\partial_q$. Compose with g on the left side we get $g_q(f_q' - f_q) = g_q(\partial_{q+1} T_q + T_{q-1}\partial_q)$. Since g is a chain map, we have $\partial_q g_q = g_{q-1}\partial_q$. Hence the equation above becomes

$$\partial_{q+1} g_{q+1} T_q + g_q T_{q-1} \partial_q$$

Hence $g_{q+1}T_q: gf' \simeq gf: C \to E$.

Similarly, since $F: g \simeq g': D \to E$, then $g_q' - g_q = \partial_{q+1}F_q + F_{q-1}\partial_q$. Compose with f' on the right side and by the fact that f' is a chain map. We have

$$F_q f_q': gf' \simeq g'f': C \to E$$

By Proposition 3.16 we conclude $gf \simeq g'f'$.

Now we prove the three conditions for equivalence relation:

(1) Reflexive: $C \simeq C$ by the identity chain map 1_C.

(2) Symmetric: $C \simeq D$ implies $D \simeq C$ by definition.

(3) Transitive: If $C \simeq D, D \simeq E$, we have

$$f: C \to D, g: D \to C \text{ such that } fg = 1_D, gf = 1_C$$

$$p: D \to E, q: E \to D \text{ such that } pq = 1_E, qp = 1_D$$

Hence by the claim we just proved we have $qp \simeq 1_D$. Since $f \simeq f$ we get $qpf \simeq f$. Because $g \simeq g$ we also get $gqpf \simeq gf \simeq 1_C$. So pf is an equivalence with inverse gq.

■

3.2 Singular Homology with Coefficient $\mathbb{Z}$

In this section we firstly construct a functor S_* from the category of topological spaces **Top** to the category of chain complexes **Comp**. Then similar to Theorem 3.9 we construct a functor from **Comp** to the category of graded groups **GradedG**. And take their composition we will have the "homology functor" H_*: **Top** $\to$ **GradedG**.

Definition 3.19. A ***standard q-simplex*** $\Delta^q = [x_0, \cdots, x_q]$ is a subset in $\mathbb{R}^{q+1}$ of the form $\Delta^q := \{(x_0, \cdots, x_q) \in \mathbb{R}^{q+1} | \sum_{i=0}^{q} x_i = 1, x_i \in [0,1]\}$.

Definition 3.20. A ***singular q-simplex*** in a topological space X is a continuous map $\sigma_q: \Delta^q \to X$, where Δ^q is the standard q-simplex.

Since $\Delta^1 \approx I$ (homeomorphic to), a singular *1*-simplex in X can be regarded as a path in X. Since Δ^0 is a one-point set, a singular *0*-simplex in X can be regarded as a point in X.

Definition 3.21. Given a topological space X, define the ***singular q-chain group*** $\boldsymbol{S_q(X)}$ as the free abelian group with basis all singular q-simplexes in X. The elements of $S_q(X)$ are called ***singular q-chains*** in X. $S_q(X) = 0$ when $q < 0$.

By the definition of free abelian group, a singular q-chain c_q has a unique expression $c_q = k_1\sigma_q^{(1)} + \cdots + k_n\sigma_q^{(n)}, k_i \in \mathbb{Z},\ \sigma_q^{(i)}: \Delta^q \to X$.

Definition 3.22. Given a singular q-simplex $\sigma_q: \Delta^q \to X$, its ***boundary*** is defined as $\partial_q \sigma_q = \sum_{i=0}^{q} (-1)^i \sigma_q \varepsilon_i^q \in S_{q-1}(X)$, where the *i*th face map

$\varepsilon_i^q: \Delta^{q-1} \to \Delta^q, (x_0, \cdots, x_{q-1}) \to (x_0, \cdots, x_i = 0, \cdots, x_q)$. And $\partial_0 = 0$.

Lemma 3.23. Given a free abelian group F with basis B. If G is an abelian group and $\varphi: B \to G$ is a function. Then the extending by linearity of φ is a unique homomorphism $\tilde{\varphi}: F \to G$ such that $\tilde{\varphi}(b) = \varphi(b)$ for all $b \in B$.

Proof: Take an arbitrary element $x \in F$, we have $x = \sum k_b b$ ($k_b \in \mathbb{Z},\ b \in B$ and all but finitely many of k_bs nonzero). Define $\tilde{\varphi}(x) = \tilde{\varphi}(\sum k_b b) = \sum k_b \varphi(b)$. Now we need to verify if $\tilde{\varphi}$ is well-defined and unique. But since the expression of each x is unique, we have $\tilde{\varphi}$ is well-defined. And because $\tilde{\varphi}$ is defined on basis *B*, it is also unique.

∎

So with the boundary of a singular q-simplex defined, we can extend it by linearity since $S_q(X)$ is a free abelian group generated by singular q-simplexes.

Theorem 3.24. For each $q \in \mathbb{N}$, we have the ***qth boundary operator***, which is a unique homomorphism $\partial_q: S_q(X) \to S_{q-1}(X)$ defined by $\partial_q \sigma_q = \sum_{i=0}^{q} (-1)^i \sigma_q \varepsilon_i^q$ for every singular q-simplex in X.

Proof: For each $q \in \mathbb{N}$, by definition we have the boundary $\partial_q \sigma_q \in S_{q-1}(X)$ for any singular q-simplex in X. Then by Lemma 3.23 we can extend our boundary map ∂_q by linearity (regard F as $S_q(X)$, B as the class of all singular q-simplexes. i.e. the basis of F, and G as $S_{q-1}(X)$). Hence we get the unique homomorphism and denote it by ∂_q for convenience.

∎

Lemma 3.25. $\varepsilon_j^{q+1} \varepsilon_k^q = \varepsilon_k^{q+1} \varepsilon_{j-1}^q : \Delta^{n-1} \to \Delta^{n+1}$ if $k < j$.

Proof: This can be directly shown by computation.

∎

Theorem 3.26. For all $q \in \mathbb{N}$, we have $\partial_q \partial_{q+1} = 0$.

Proof: We prove it by verifying such equation holds for an arbitrary singular (q+1)-simplex σ.

$$\partial_q \partial_{q+1} \sigma = \partial_q \left(\sum_{i=0}^{q+1} (-1)^i \sigma \varepsilon_i^{q+1} \right) = \sum_{j=0}^{q} \sum_{i=0}^{q+1} (-1)^{i+j} \sigma \varepsilon_i^{q+1} \varepsilon_j^q$$

$$= \sum_{j \geq i} (-1)^{i+j} \sigma \varepsilon_i^{q+1} \varepsilon_j^q + \sum_{j < i} (-1)^{i+j} \sigma \varepsilon_i^{q+1} \varepsilon_j^q$$

$$= \sum_{j \geq i} (-1)^{i+j} \sigma \varepsilon_i^{q+1} \varepsilon_j^q + \sum_{j < i} (-1)^{i+j} \sigma \varepsilon_j^{q+1} \varepsilon_{i-1}^q$$

Note that we got the last equation by Lemma 3.25. Now let $m = j, n = i - 1$ in the second sum. $j < i$ implies $j \leq i-1$. Hence the equation becomes:

$$\sum_{j \geq i} (-1)^{i+j} \sigma \varepsilon_i^{q+1} \varepsilon_j^q + \sum_{m \leq n} (-1)^{m+n+1} \sigma \varepsilon_m^{q+1} \varepsilon_n^q$$

Which equals to 0 since $(-1)^{i+j}$ and $(-1)^{m+n+1}$ cancelled each other.

So $\partial_q \partial_{q+1} = 0$ for all $q \in \mathbb{N}$.

■

So far we actually have constructed a chain complex from a topological space X.

Definition 3.27. Given a topological space X, we can define the ***singular chain complex*** $S_*(X) := \{S_q(X), \partial_q\}$. And $\partial_q \partial_{q+1} = 0$ for all $q \in \mathbb{N}$ as proved in Theorem 3.26.

$$\cdots \to S_{q+1}(X) \xrightarrow{\partial_{q+1}} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \to \cdots$$

Now for an object X in the category **Top** we have an object $S_*(X)$ in the category **Comp**. To get a functor between them we need to observe what happens to elements in $Hom(X,Y)$ in **Top**: Given a continuous $f: X \to Y \in Hom(X,Y)$ and a singular q-simplex σ in X. Their composition $f\sigma: \Delta^q \to Y$ is a singular q-simplex in Y. Moreover, if we extend such f by linearity as in Lemma 3.23 (B the class of all singular q-simplex in X; F denotes $S_q(X)$; G denotes $S_q(Y)$; $\varphi = f: B \to S_q(Y)$) we get a homomorphism $f_\#: S_q(X) \to S_q(Y)$ (**$f_\#$ does depend on q!**), which is defined as $f_\#(\sum k_\sigma \sigma) = \sum k_\sigma (f\sigma), k \in \mathbb{Z}$.

Now the last step is to show that $f_\#$ is a chain map.

Lemma 3.28. The $f_\#: S_q(X) \to S_q(Y)$ is a chain map. i.e. the following equation holds: $\partial_q f_\# = f_\# \partial_q$.

Proof: We prove it by considering the basis elements $\sigma: \Delta^q \to X \in S_q(X)$.

$f_\# \partial_q(\sigma) = f_\#\left(\sum_{i=0}^q (-1)^i \sigma \varepsilon_i^q\right) = \left(\sum_{i=0}^q (-1)^i f\left(\sigma \varepsilon_i^q\right)\right)$ and

$\partial_q f_\#(\sigma) = \sum_{i=0}^{q}(-1)^i (f\sigma)\varepsilon_i^q$

Hence the two maps coincide.

■

Theorem 3.29. S_* is a functor from **Top** to **Comp** which assigns a topological space X a chain complex $S_*(X) := \{S_q(X), \partial_q\}$ and a continuous map $f: X \to Y \in Hom(X, Y)$ a chain map $f_\#: S_q(X) \to S_q(Y) \in Hom(S_*(X), S_*(Y))$.

Proof: A routine.

■

Definition 3.30. Given a topological space X and the singular chain complex $S_*(X) := \{S_q(X), \partial_q\}$. $Z_q(X) := ker\partial_q$ is called ***the group of singular q-cycles*** in X; $B_q(X) := im\partial_{q+1}$ is called ***the group of singular q-boundaries*** in X; Their quotient group $Z_q(X)/B_q(X)$, denoted by $H_q(X)$ is called ***the qth singular homology group*** of X. The elements of $H_q(X)$ are called ***singular homology classes***. The singular homology class of a singular q-cycle z_q is $[z_q] := z_q + B_q(X) \in H_q(X)$. We put singular homology groups of all dimensions together as $H_*(X) := H_*(S_*(X)) = \{H_q(X)\}$.

Note that $H_q(\emptyset) = 0$ for $q \in \mathbb{N}$ since the free abelian group generated by the empty basis is nothing but the trivial group.

Now we want to construct a functor from **Comp** to **GradedG** by imitating Theorem 3.9.

Lemma 3.31. Given a chain map $f_\#: S_*(X) \to S_*(Y)$ induced by $f: X \to Y$. We have an induced homomorphism $f_*: H_*(X) \to H_*(Y)$ between graded groups and $f_*([z_q]) := [f_\#(z_q)] \in H_q(Y)$ for $[z_q] \in H_q(X)$.

Proof: Similar to Lemma 3.7.

■

Theorem 3.32. We have a covariant functor $H_*: \mathbf{Comp} \to \mathbf{GradedG}$ with $H_*\big(S_*(X)\big) = H_*(X)$ and $H_*(f_\#) = (f_\#)_*$.

Proof: A routine.

■

So we take the composition of our two functors $S_*: \mathbf{Top} \to \mathbf{Comp}$ and $H_*: \mathbf{Comp} \to \mathbf{GradedG}$ then get a new functor from $\mathbf{Top}$ to $\mathbf{GradedG}$. For convenience we still denote it by H_*. And $H_*(X) = H_*(S_*(X))$ for a topological space X; $H_*(f) = (f_\#)_* = f_*$ for a continuous function f.

Corollary 3.33. Given two homeomorphic topological spaces X and Y, we have two isomorphic singular homology groups $H_*(X)$ and $H_*(Y)$.

Proof. Theorem 2.12 as there is an equivalence between two homeomorphic spaces.

■

Theorem 3.34 (Singular Homology of a one-point space/Dimension Axiom). Given a one-point space $\{pt\}$, its singular homology group for each $q \in \mathbb{N}$ is

$$H_q(\{pt\}) \cong \begin{cases} 0 & q > 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

Proof: The key observation is that since $\{pt\}$ contains only one point. So for each $q \in \mathbb{N}$, there is only one singular q-simplex $\sigma_q: \Delta^q \to X$. So $S_q(\{pt\}) \cong \mathbb{Z}$ as a free abelian group generated by one element. Then we can calculate $\partial_q \sigma_q$ by definition.

$$\partial_q \sigma_q = \sum_{i=0}^{q} (-1)^i \sigma_q \varepsilon_q^i = \left(\sum_{i=0}^{q} (-1)^i\right) \sigma_{q-1} = \begin{cases} 0 & q > 0 \; odd \; or \; q = 0 \\ \sigma_{q-1} & q > 0 \;\; even \end{cases}$$

Note that the second equation holds because there is only one singular simplex for each dimension! And $\sigma_q \varepsilon_q^i$ is a *(q-1)*-simplex.

Hence $\partial_q = 0$ for all odd $q \in \mathbb{N}$ and $q = 0$. ∂_q is an isomorphism for even $q \in \mathbb{N}$.

The singular chain is illustrated below:

$$\cdots \to S_3(\{pt\}) \cong \mathbb{Z} \xrightarrow{0} S_2(\{pt\}) \cong \mathbb{Z} \xrightarrow{\cong} S_1(\{pt\}) \cong \mathbb{Z} \xrightarrow{0} S_0(\{pt\}) \cong \mathbb{Z} \xrightarrow{0} 0$$

Now we compute the homology group of each dimension.

$$Z_q(\{pt\}) = ker\partial_q \cong \begin{cases} \mathbb{Z} & q > 0 \; odd \; or \; q = 0 \\ 0 & q > 0 \; even \end{cases}$$

$$B_q(\{pt\}) = im\partial_{q+1} \cong \begin{cases} \mathbb{Z} & q > 0 \; odd \\ 0 & q > 0 \; even \; or \; q = 0 \end{cases}$$

So we have $H_q(\{pt\}) = Z_q(\{pt\})/B_q(\{pt\}) \cong \begin{cases} 0 & q \in \mathbb{N} \\ \mathbb{Z} & q = 0 \end{cases}$

■

Definition 3.35. Given a singular *0*-chain $c_0 \in S_0(X)$ in a topological space X, we have $c_0 = k_1 a_1 + \cdots + k_r a_r$ where $a_i \in X, k_i \in \mathbb{Z}$ since each *0*-simplex is a point in X. ***The Kronecker index*** of c_0 is defined as $\varepsilon(c_0) := k_1 + \cdots + k_r$.

So the Kronecker index actually defines a function ε between two abelian groups $S_0(X)$ and $\mathbb{Z}$. It is natural to ask if ε is a group homomorphism.

Lemma 3.36. $\varepsilon: S_0(X) \to \mathbb{Z}$ is a homomorphism.

Proof: Take two elements $c_0^{(1)} = k_1 a_1 + \cdots + k_r a_r, c_0^{(2)} = p_1 a_1 + \cdots + p_q a_q$ in $S_0(X)$. Since

$$\begin{aligned} \varepsilon\left(c_0^{(1)} + c_0^{(2)}\right) &= \varepsilon(k_1 a_1 + \cdots + k_r a_r + p_1 a_1 + \cdots + p_q a_q) \\ &= k_1 + \cdots + k_r + p_1 + \cdots + p_q \\ &= (k_1 + \cdots + k_r) + (p_1 + \cdots + p_q) \\ &= \varepsilon\left(c_0^{(1)}\right) + \varepsilon\left(c_0^{(2)}\right) \end{aligned}$$

Hence ε is indeed a homomorphism.

■

How do we compute the homology groups of a certain topological space X? Actually it is kind of hard for us to compute $H_q(X)$ even for a path-connected space as one can imagine. But we can always do it in the $q = 0$ case!

Theorem 3.37. Given a non-empty, path-connected topological space X, $H_0(X) \cong \mathbb{Z}$.

Proof: Since $\partial_0 = 0$ we have $ker\partial_0 = Z_0(X) = S_0(X)$. Now by Lemma 3.36 we have a homomorphism $\varepsilon: S_0(X) = Z_0(X) \to \mathbb{Z}$ in hand. To prove $H_0(X) = Z_0(X)/B_0(X) \cong \mathbb{Z}$, we just need to prove $ker\varepsilon = B_0(X)$ and we will get what we want by First Isomorphism Theorem of Groups.

Firstly we prove $B_0(X) \subset ker\varepsilon$: Since $B_0(X) = im\partial_1$, we take an arbitrary singular *1*-simplex $\sigma_1: \Delta^1 \to X$ then $\partial_1\sigma_1 = x_1 - x_0 \in B_0(X)$ if we denote σ_1 by $[x_0, x_1]$. So $\varepsilon(\partial_1\sigma_1) = 0$ implies $B_0(X) \subset ker\varepsilon$.

Secondly we prove $ker\varepsilon \subset B_0(X)$: Take an arbitrary singular *0*-chain $c_0 = k_1a_1 + \cdots + k_ra_r \in S_0(X)$ in X, where $a_1, \dots, a_r$ are points in X. Select a point $b \in X$ as our base point. By the fact that X is path-connected there is a path from b to a for any point $a \in X$. Hence there is a path (singular *1*-simplex) $\sigma_a: \Delta^1 \to X$ such that $\partial_1\sigma_a = a - b$. So for our c_0, we have $c_0 - \varepsilon(c_0)b = \sum_{i=0}^{r} k_i(a_i - b) \in B_0(X)$ as the boundary of $\sum_{i=0}^{r} k_i\sigma_{a_i} \in S_1(X)$. Finally restrict our choice of c_0 in $\ker\varepsilon \subset S_0(X)$ we get $\ker\varepsilon \subset B_0(X)$.

■

There is a more general case.

Theorem 3.38. Given a set of path components $\{X_i \subset X | i \in I\}$ of a topological space X. We have the direct sum decomposition of the homology groups of X:

$$H_*(X) = \bigoplus_{i\in I} H_*(X_i)$$

Proof: Since the image of a singular simplex lies in a unique path-component of X, then with the help of Theorem 3.13 we can prove it. Refer to Theorem 4.13, page 69, An Introduction to Algebraic Topology[8].

■

By last Theorem we have $H_*(S^0) = H_*(\{a, b\}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = 0 \\ 0 & q > 0 \end{cases}$.

Corollary 3.39. A topological space X is path-connected if and only if $H_0(X) \cong \mathbb{Z}$.

Proof: ($\Rightarrow$) Theorem 3.37.

($\Leftarrow$) Theorem 3.38.

■

3.3 Homotopy Invariance

In last section we constructed the homology functor $H_*: \mathbf{Top} \to \mathbf{GradedG}$ so two homeomorphic topological spaces have isomorphic homology groups. But actually, two topological spaces with the same homotopy type, which is a weaker condition than being homeomorphic, have isomorphic homology groups too.

We start with an introduction to category **hTop**, which is a quotient category of **Top**.

Lemma 3.40. Homotopy is an equivalence relation on the set of all continuous functions from X to Y. i.e. $Hom(X,Y)$ in category **Top**.

Proof: (1) Reflexive: Given $f \in Hom(X,Y)$, we define $F: X \times I \to Y$ by $F(x,t) = f(x)$ for all $x \in X, t \in I$. Hence $f \simeq f$.

(2) Symmetric: Given $f \simeq g \in Hom(X,Y)$. By the definition of homotopy we have a continuous $F: X \times I \to Y$ such that $F(x,0) = f(x), F(x,1) = g(x)$ for all $x \in X$. Hence if we define $G: X \times I \to X$ as $G(x,t) = F(x, 1-t)$, then $g \simeq f$.

(3) Transitive: Given $F: f \simeq g$ and $G: g \simeq h$. We define $H: X \times I \to Y$ as

$$H(x,t) = \begin{cases} F(x,2t) & t \in [0,\frac{1}{2}] \\ G(x,2t-1) & t \in [\frac{1}{2},1] \end{cases}$$

So we have $f \simeq h$ since H is continuous by gluing lemma.

■

We denote the family of all homotopy classes from X to Y by $[X,Y]$.

Lemma 3.41. Given three topological spaces X,Y,Z and $f,f' \in Hom(X,Y)$ and $g,g' \in Hom(Y,Z)$. If $f \simeq f', g \simeq g'$, then $gf \simeq g'f' \in Hom(X,Z)$, which means

$[g][f] = [gf]$.

Proof: We first show that $gf \simeq g'f$ then show $g'f \simeq g'f'$: By the definition we have two continuous maps $F: f \simeq f', G: g \simeq g'$. Define $H: X \times I \to Z$ by $H(x,t) = G(f(x),t)$, which implies $gf \simeq g'f$. Then define $P: X \times I \to Z$ by $P(x,t) = g'F(x,t)$, which shows $g'f \simeq g'f'$. By Lemma 3.40 the $\simeq$ is an equivalence relation hence $gf \simeq g'f \simeq g'f'$.

■

Theorem 3.42. Homotopy is a congruence on **Top**.

Proof: By Lemma 3.40. and Lemma 3.41.

■

Definition 3.43. With the congruence homotopy on **Top** we consider its quotient category **hTop** whose objects are topological spaces and $Hom(X,Y) = [X,Y]$ with composition $[g][f] = [gf]$.

So we can rephrase the definition of a homotopy equivalence as: $f: X \to Y$ is a homotopy equivalence if and only if $[f] \in [X,Y]$ is an equivalence in **hTop**.

Theorem 3.44. Given two homotopic maps $f \simeq g: X \to Y$, we have $H_*(f) = f_* = g_* = H_*(g): H_*(X) \to H_*(Y)$.

Proof: Refer to Corollary 2.11, page 111-113 in Algebraic Topology[3].

■

Now we can see that the homology functor induces a functor H_q: **hTop** $\to$ **Ab** (the category of all abelian groups) since each $H_q(X)$ is abelian.

Corollary 3.45. Given two topological spaces X, Y such that $X \simeq Y$, we have $H_*(X) \cong H_*(Y)$.

Proof: By Theorem 2.12.

∎

One may wonder how we visualize two topological spaces with the same homotopy type. In page 3 of Algebraic Topology there is a statement " two spaces X and Y have are homotopy equivalent if and only if there exists a third space Z containing both X and Y as deformation retracts"[3] . The proof is in Algebraic Topology[3], Corollary 0.21, page 16-17.

Recall that a topological space X is called contractible if it has the same homotopy type as a one-point space $\{pt\}$.

Corollary 3.46. Given a contractible topological space X, we have

$$H_*(X) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q > 0 \end{cases}$$

Proof: By Theorem 3.34.

∎

3.4 Basic Homological Algebra

Definition 3.47. Given a sequence consists of abelian groups and group homomorphisms $C \xrightarrow{f} D \xrightarrow{g} E$, it is called ***exact at D*** if $imf = kerg$. An infinite sequence of abelian groups and group homomorphisms

$$\cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots$$

is an ***exact sequence*** if it is exact at each A_n.

Definition 3.48. Given a sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, it is called a ***short exact sequence*** if it is an exact sequence, where f is a monomorphism and g is an epimorphism by exactness at A and C.

Note that given a chain complex $C = \{C_q, \partial_q\}$. It is an exact sequence if and only if $im\partial_{q+1} = ker\partial_q$, which is equivalent to $H_q(C) = ker\partial_q / im\partial_{q+1} = 0$. So we have its homology groups $H_*(C) = 0$ and such chain complexes are called acyclic complexes.

Definition 3.49. A sequence of chain complexes and chain maps $C \xrightarrow{f} D \xrightarrow{g} E$ is called ***exact*** at D if $C_q \xrightarrow{f_q} D_q \xrightarrow{g_q} E_q$ is exact at D_q for every dimension.

Similarly, we define an exact sequence of chain complexes and chain maps. One should remember that the "0" on the two ends are 0 complexes.

Lemma 3.50. Given an exact sequence of chain complexes and chain maps

$$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

Then for each dimension q there is a homomorphism called connecting homomorphism $\partial_q^*: H_q(E) \to H_{q-1}(C)$, $[e_q] \mapsto [f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)]$ for $e_q \in Z_q(E)$.

Proof: Let $C = \{C_q, \partial_q^C\}, D = \{D_q, \partial_q^D\}, E = \{E_q, \partial_q^E\}$. Take an element $e_q \in Z_q(E)$, we first have $\partial_q^E(e_q) = 0$. Since we have the short exact sequence for each dimension q: $0 \to C_q \xrightarrow{f_q} D_q \xrightarrow{g_q} E_q \to 0$. Each g_q is an epimorphism, hence there exists a $d_q \in D_q$ such that $g_q(d_q) = e_q$ so $g_q^{-1}(e_q)$ makes sense.

By the fact that f, g are chain maps hence commute with boundary maps we get $g_{q-1}\partial_q^D(d_q) = \partial_q^E g_q(d_q) = 0$, which implies that $\partial_q^D(d_q) \in ker g_{q-1} = im f_{q-1}$ by the exactness of the row of dimension $q-1$. This also means that $f_{q-1}^{-1}\partial_q(d_q)$ makes sense.

Then there exists a unique $c_{q-1} \in C_{q-1}$ such that $f_{q-1}(c_{q-1}) = \partial_q^D(d_q)$ since f_{q-1} is a monomorphism. Now with the diagram chasing above we look back and see what will happen if we get a different $d_q' \in D_q$. Suppose we have another $d_q' \in D_q$ such that we get a unique $c_{q-1}' \in C_{q-1}$ by the same process. But one can see that the two elements satisfy $g_q(d_q - d_q') = 0$, which implies $(d_q - d_q') \in ker g_q = im f_q$. Similarly, this offers us a unique $c_q \in C_q$ such that $f_q(c_q) = (d_q - d_q')$. Hence by commutativity we have

$$f_{q-1}\partial_q^C(c_q) = \partial_q^D f_q(c_q) = \partial_q^D(d_q - d_q') = f_{q-1}(c_{q-1} - c_{q-1}')$$

By the injectivity of f_{q-1} we conclude $c_{q-1} - c'_{q-1} = \partial_q^C(c_q)$, which shows that $c_{q-1} - c'_{q-1} \in im\partial_q^C = B_{q-1}(C)$. i.e. $[c_{q-1}] = [c'_{q-1}] \in H_{q-1}(C)$. So there is a homomorphism from $Z_q(E)$ to $C_{q-1}/B_{q-1}(C)$ that maps e_q to $[f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)]$. Clearly this homomorphism maps boundaries in E_q to boundaries in C_{q-1} and cycles to cycles by the property of chain maps. Hence it also provides us a homomorphism $\partial_q^*: H_q(E) \to H_{q-1}(C)$, $[e_q] \mapsto [f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)]$ for $e_q \in Z_q(E)$.

■

Theorem 3.51. Given a short exact sequence of chain complexes and chain maps $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$. We have a long exact sequence

$$\cdots \to H_{q+1}(E) \xrightarrow{\partial_{q+1}^*} H_q(C) \xrightarrow{f_*} H_q(D) \xrightarrow{g_*} H_q(E) \xrightarrow{\partial_q^*} H_{q-1}(C) \to \cdots$$

Proof: We first prove the exactness at $H_q(D)$: $(imf_* \subset kerg_*)$ Since for each dimension we have $g_q f_q = 0$ and $g_* f_* = (gf)_*$ by definition. Hence $g_* f_* = 0$ implies what we need $(kerg_* \subset imf_*)$. Take an element $d_q + B_q(D) \in H_q(D)$ such that $g_*\left(d_q + B_q(D)\right) = [g_q(d_q)] = [0]$, so $g_q(d_q) \in B_q(E) = im\partial_{q+1}^E$, which means that there exists a $e_{q+1} \in E_{q+1}$ such that $\partial_{q+1}^E(e_{q+1}) = g_q(d_q)$. Because g_{q+1} is an epimorphism, there actually exists a d_{q+1} such that $g_{q+1}(d_{q+1}) = e_{q+1}$. So by commutativity

$$\partial_{q+1}^E(e_{q+1}) = \partial_{q+1}^E g_{q+1}(d_{q+1}) = g_q \partial_{q+1}^D(d_{q+1}) = g_q(d_q)$$

This means that $g_q\left(d_q - \partial_{q+1}^D(d_{q+1})\right) = 0$, which by exactness also implies that $d_q - \partial_{q+1}^D(d_{q+1}) \in kerg_q = imf_q$. Hence we can find a unique $c_q \in C_q$ such that $f_q(c_q) = d_q - \partial_{q+1}^D(d_{q+1})$. Now we apply ∂_q^D and commutativity will provides us $\partial_q^D f_q(c_q) = f_{q-1}\partial_q^C(c_q) = \partial_q^D\left(d_q - \partial_{q+1}^D(d_{q+1})\right) = 0$ because $d_q \in Z_q(D)$ and $\partial_q^D \partial_{q+1}^D = 0$. So $f_*([c_q]) = [f_q(c_q)] = [d_q - \partial_{q+1}^D(d_{q+1})] = [0]$ as proved.

Now we prove the exactness at $H_q(E)$: $(img_* \subset ker\partial_q^*)$ We take an element $[e_q] \in img_*$ hence get some element $[d_q] \in H_q(D)$ such that $g_*([d_q]) = [e_q]$. Notice that if the choice of d_q is in $B_q(D)$ we immediately prove what we need. So we may assume that $d_q \in Z_q(D)$. Since $\partial_q^* g_*([d_q]) = \partial_q^*[g_q(d_q)] = [f_{q-1}^{-1}\partial_q^D g_q^{-1} g_q(d_q)] = [f_{q-1}^{-1}\partial_q^D(d_q)] = [f_{q-1}^{-1}(0)] = [0]$ implies that $[e_q] \in ker\partial_q^*$. $(ker\partial_q^* \subset img_*)$ Select a $e_q \in Z_q(E)$ such that $\partial_q^*([e_q]) = [f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)] = [0]$. So there exists a $c_q \in C_q$ such that $\partial_q^C(c_q) = f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)$. We apply f_{q-1} then $f_{q-1}\partial_q^C(c_q) = \partial_q^D f_q(c_q) = \partial_q^D g_q^{-1}(e_q)$ so $\partial_q^D\left(g_q^{-1}(e_q) - f_q(c_q)\right) = 0$, which means $g_q^{-1}(e_q) - f_q(c_q) \in Z_q(D)$. So

$$g_*([g_q^{-1}(e_q) - f_q(c_q)]) = \left[g_q\left(g_q^{-1}(e_q) - f_q(c_q)\right)\right] = [e_q]$$

As proved.

Finally we prove the exactness at $H_q(C)$: $(im\partial_{q+1}^* \subset kerf_*)$ Similarly select a $[c_q] \in im\partial_{q+1}^*$, we can find an $[e_q] \in H_{q+1}(E)$ such that $\partial_{q+1}^*([e_q]) = [c_q]$. If the $e_q \in B_{q+1}(E)$ we naturally prove it. So take such $e_q \in Z_q(E)$. We directly compute $f_*\partial_{q+1}^*([e_q]) = f_*[f_{q-1}^{-1}\partial_q^D g_q^{-1}(e_q)] = [\partial_q^D g_q^{-1}(e_q)] = [0]$, which proves the claim. $(kerf_* \subset im\partial_{q+1}^*)$ Take $[c_q] \in H_q(C)$ such that $f_*([c_q]) = [0]$. So there exists some $d_{q+1} \in D_{q+1}$ such that $\partial_{q+1}^D(d_{q+1}) = f_q(c_q)$. Then we make use of the commutativity $g_q\partial_{q+1}^D(d_{q+1}) = \partial_{q+1}^E g_{q+1}(d_{q+1}) = g_q f_q(c_q) = 0$, which implies $g_{q+1}(d_{q+1}) \in Z_{q+1}(E)$. But

$$\partial_{q+1}^*[g_{q+1}(d_{q+1})] = \left[f_q^{-1}\partial_{q+1}^D g_{q+1}^{-1}\left(g_{q+1}(d_{q+1})\right)\right] = [c_q]$$

As proved.

■

Theorem 3.52. Given a commutative diagram of two short exact sequences of chain complexes and chain maps

$$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

$$\alpha \downarrow \ \beta \downarrow \ \downarrow \gamma$$

$$0 \to C' \xrightarrow{f'} D' \xrightarrow{g'} E' \to 0$$

We have a commutative diagram of two exact homology sequences

$$\cdots \to H_q(C) \xrightarrow{f_*} H_q(D) \xrightarrow{g_*} H_q(E) \xrightarrow{\partial_q^*} H_{q-1}(C) \to \cdots$$

$$\alpha_* \downarrow \quad \beta_* \downarrow \quad \gamma_* \downarrow \quad \alpha_* \downarrow$$

$$\cdots \to H_q(C') \xrightarrow{f_*'} H_q(D') \xrightarrow{g_*'} H_q(E') \xrightarrow{\partial_q^{*\prime}} H_{q-1}(C') \to \cdots$$

Proof: The commutativity of first two squares are due to the homology functor. We just need to show the commutativity of the last square.

$$\alpha_*\partial_q^*([e_q]) = \alpha_*\partial_q^*([g_q(d_q)]) = \alpha_*\left[f_{q-1}^{-1}\partial_q^D g_q^{-1}\left(g_q(d_q)\right)\right] = \alpha_*[f_{q-1}^{-1}\partial_q^D(d_q)]$$

$$= [\alpha f_{q-1}^{-1}\partial_q^D(d_q)] = [{f_{q-1}'}^{-1}\beta\partial_q^D(d_q)] = [{f_{q-1}'}^{-1}\partial_q^{D'}\beta(d_q)]$$

$$= [{f_{q-1}'}^{-1}\partial_q^{D'}{g_q'}^{-1}g_q'\beta(d_q)] = \partial_q^{*\prime}([g_q'\beta(d_q)]) = \partial_q^{*\prime}([\gamma g_q(d_q)])$$

$$= \partial_q^{*\prime}\gamma_*[g_q(d_q)] = \partial_q^{*\prime}\gamma_*[e_q]$$

The first equation holds since g_q is an epimorphism. The fifth equation holds because $\beta f_{q-1} = f_{q-1}'\alpha$.

■

When working with such diagram we can imagine the 3-dimensional space of chains where the *z*-axis of each object is the corresponding chain complex and each *xy*-plane is the diagram of the same dimension.

The question is how are we supposed to get such short exact sequences of chain complexes and chain maps then make use of the theorem? It is answered in the section about relative homology.

Definition 3.53. Given an exact sequence of abelian groups and homomorphisms

$C \xrightarrow{f} D \xrightarrow{g} E$. It is called a ***split exact sequence*** if $D \cong f(C) \oplus G$ for a subgroup G of D.

Lemma 3.54 (The Five Lemma). Given a commutative diagram of two exact sequences of abelian groups and homomorphisms

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5$$

$$\alpha_1 \downarrow \ \alpha_2 \downarrow \alpha_3 \downarrow \ \alpha_4 \downarrow \ \alpha_5 \downarrow$$

$$B_1 \to B_2 \to B_3 \to B_4 \to B\_5$$

If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are all isomorphisms, then so is α_3.

Proof: By the method of diagram chasing.

■

We call two short exact sequence isomorphic if there is a commutative diagram

$$0 \to A \to B \to C \to 0$$

$$f \downarrow \ g \downarrow \ h \downarrow$$

$$0 \to A' \to B' \to C' \to 0$$

such that f, g, h are isomorphisms.

Proposition 3.55. Given a short exact sequence of abelian groups and homomorphisms $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$. It is a split exact sequence if and only if it satisfies one of the following equivalent conditions:

(1). There is a homomorphism $h: E \to D$ such that $gh = 1_E$.

(2). There is a homomorphism $k: D \to C$ such that $kf = 1_C$.

(3). The given sequence is isomorphic (with identity maps on C, E) to the direct sum short exact sequence $0 \to C \xrightarrow{i} C \oplus E \xrightarrow{p} E \to 0$; in particular $D \cong C \oplus E$.

Proof: Refer to Theorem 1.18, page 177, Algebra[5].

■

Corollary 3.56. Given a short exact sequence of abelian groups and homomorphisms

$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$. It is a split exact sequence if the abelian group E is free abelian.

Proof: Refer to Corollary 9.2, page 234, An Introduction to Algebraic Topology[8].

■

3.5 Relative Homology

The relative homology is defined similarly to singular homology. With a topological space X we have its singular chain complex $S_*(X)$. Suppose we have its subspace $A \subset X$. Because singular q-chain group $S_q(A)$ of A is generated by all singular q-simplexes in A, i.e. continuous maps from Δ^q to $A \subset X$. So such simplexes are also generators of $S_q(X)$ and $S_q(A) \subset S_q(X)$ a subgroup. Then with the singular chain complex $S_*(X) = \{S_q(X), \partial_q\}$ we define its subcomplex $S_*(A) = \{S_q(A), \partial_q'\}$, which satisfies the definition of a subcomplex. i.e. $S_q(A)$ a subgroup of $S_q(X)$ and $\partial_q' = \partial_q|_{S_q(A)}$. Naturally we can define their quotient complex $= \{S_q(X)/S_q(A), \overline{\partial}_q\}$, where $\overline{\partial}_q\left(s_q + S_q(A)\right) = \left(\partial_q(s_q) + S_{q-1}(A)\right) \in S_{q-1}(X)/S_{q-1}(A)$.

We firstly will construct a functor $S_*\colon \mathbf{Top^2} \to \mathbf{GradedG}$.

Definition 3.57. Given a pair of topological spaces (X, A). Its ***singular q-chain group*** is defined to be the quotient group $S_q(X)/S_q(A)$. With all the discussion at the beginning we actually already have a corresponding quotient chain complex called ***relative chain complex*** $S_*(X, A) := S_*(X)/S_*(A) = \{S_q(X, A), \overline{\partial}_q\}$ for a topological space X. The definition of ***the group of relative singular q-cycles, q-boundaries*** are similar and we define ***the relative homology groups*** by $H_*(X, A) := H_*\left(S_*(X, A)\right)$.

Actually the singular homology group $H_q(X)$ can be a special case of relative homology group $H_q(X, A) := H_*\left(S_*(X, A)\right)$ by letting $A = \emptyset$ since $S_*(\emptyset) = 0$.

Because $S_q(X)/S_q(A)$ is the free abelian group with basis all cosets of q-simplexes in X whose images are not in A. So the elements of $S_q(X)/S_q(A)$ are those of $S_q(X)$ with their coefficients of simplexes in A ignored. For a more formal statement refer to Exercise II. 1.6, page 75, Algebra[5].

To construct the functor we need to observe the morphism $f: (X, A) \to (Y, B) \in Hom((X, A), (Y, B))$ in the category $\mathbf{Top^2}$. Recall that such f is continuous and satisfies $f(A) \subset B$. So given a singular q-simplex $\sigma_q^{(A)}$ in A we have $f\sigma_q^{(A)}$ a singular q-simplex in B.

Definition 3.58. Given two objects (X, A), (Y, B) and a morphism between them $f: (X, A) \to (Y, B) \in Hom((X, A), (Y, B))$ in the category $\mathbf{Top^2}$. We define the *induced* ***relative chain map*** $f_\#: S_*(X, A) \to S_*(Y, B)$ (the notations are the same for convenience) from the map $f_\#: S_*(X) \to S_*(Y)$ by $f_\#([s_q]) = [f_\#(s_q)]$.

The map is well-defined since it maps $S_*(A)$ to $S_*(B)$. It is a chain map because $f_\#\bar{\partial}_q([s_q]) = f_\#([\partial_q(s_q)]) = [f_\#\partial_q(s_q)] = [\partial_q f_\#(s_q)] = \bar{\partial}_q f_\#([s_q])$ by the commutativity of $f_\#: S_*(X) \to S_*(Y)$.

Now we define the relative chain functor $S_*: \mathbf{Top^2} \to \mathbf{Comp}$ by assigning a topological pair (X, A) the chain complex $S_*(X, A) = \{S_q(X)/S_q(A), \bar{\partial}_q\}$ and a morphism $f: (X, A) \to (Y, B)$ between two pairs the relative chain map $f_\#: S_*(X, A) \to S_*(Y, B)$ between two complexes.

Similarly, we can define a functor from **Comp** to **GradedG**.

Definition 3.59. The ***homomorphism between relative homology groups*** $f_*: H_*(X, A) \to H_*(Y, B)$ induced by $f: (X, A) \to (Y, B)$ is defined by $[[s_q]] \mapsto [[f_\#(s_q)]]$, which is defined as the induced homomorphism by the chain map $f_\#: S_*(X, A) \to S_*(Y, B)$.

So it is logical to define the relative homology functor $H_*: \mathbf{Comp} \to \mathbf{GradedG}$ by assigning a relative chain complex $S_*(X, A)$ the relative homology group $H_*(S_*(X, A))$

and a chain map $f_{\#}: S_*(X, A) \to S_*(Y, B)$ the induced homomorphism $f_*: H_*(S_*(X, A)) \to H_*(S_*(Y, B))$.

Now we will see how to make use of exact sequences by relative homology.

Theorem 3.60. Given a topological pair (X, A), there are two inclusions $i: (A, \emptyset) \to (X, \emptyset)$ and $j: (X, \emptyset) \to (X, A)$ whose inducing chain maps $i_{\#}$ and $j_{\#}$ together with the corresponding relative chain complexes give a short exact sequence $0 \to S_*(A, \emptyset) \xrightarrow{i_{\#}} S_*(X, \emptyset) \xrightarrow{j_{\#}} S_*(X, A) \to 0$ of chain complexes and chain maps. Hence we have an exact sequence of relative homology groups:

$$\cdots \xrightarrow{\partial_{q+1}^*} H_q(A, \emptyset) \xrightarrow{i_*} H_q(X, \emptyset) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_q^*} H_{q-1}(A, \emptyset) \xrightarrow{i_*} \cdots$$

Moreover, if we have a $f: (X, A) \to (Y, B)$, which gives a commutative diagram of two exact sequences of chain complexes and chain maps

$$0 \to S_*(A, \emptyset) \xrightarrow{i_{\#}} S_*(X, \emptyset) \xrightarrow{j_{\#}} S_*(X, A) \to 0$$

$$f_{\#} \downarrow \quad f_{\#} \downarrow \quad f_{\#} \downarrow$$

$$0 \to S_*(B, \emptyset) \xrightarrow{i_{\#}} S_*(Y, \emptyset) \xrightarrow{j_{\#}} S_*(Y, B) \to 0$$

Hence we have a commutative diagram of two exact sequences of homology groups

$$\cdots \to H_q(A, \emptyset) \xrightarrow{i_*} H_q(X, \emptyset) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_q^*} H_{q-1}(A, \emptyset) \to \cdots$$

$$f_* \downarrow \quad f_* \downarrow \quad f_* \downarrow \quad f_* \downarrow$$

$$\cdots \to H_q(B, \emptyset) \xrightarrow{i_*'} H_q(Y, \emptyset) \xrightarrow{j_*'} H_q(Y, B) \xrightarrow{\partial_q^{*\prime}} H_{q-1}(B, \emptyset) \to \cdots$$

Proof: It can be proved by applying Theorem 3.51. to the short exact sequence. Note that the exactness of $0 \to S_*(A, \emptyset) \xrightarrow{i_{\#}} S_*(X, \emptyset) \xrightarrow{j_{\#}} S_*(X, A) \to 0$ comes from the Third Isomorphism Theorem since $S_*(X, A) \cong S_*(X, \emptyset)/S_*(A, \emptyset)$.

■

Now we focus on the homotopy invariance of relative homology.

Definition 3.61. Given two morphisms $f, g: (X, A) \to (Y, B)$ between two topological pairs. We have $f \simeq g \ mod \ A$ if there is a continuous function $F: (X \times I, A \times I) \to (Y, B)$ such that $F\big((x,0),(a,0)\big) = f, F\big((x,1),(a,1)\big) = g$.

Theorem 3.62. Given two homotopic mod A maps $f \simeq g: (X, A) \to (Y, B)$, we have $H_*(f) = f_* = g_* = H_*(g): H_*(X, A) \to H_*(Y, B)$.

Proof: Refer to Proposition 2.19, page 118, Algebraic Topology[3].

■

Corollary 3.63. Given two topological pairs $(X, A), (Y, B)$ such that $(X, A) \simeq (Y, B)$, we have $H_*(X, A) \cong H_*(Y, B)$.

Proof: Apply last theorem.

■

Finally we introduce the exact sequence of the topological triple (X, A, B) with $B \subset A \subset X$ are subspaces. It will be useful together with our homological algebra tools.

Definition 3.64. Given a topological space X with two subspaces $B \subset A \subset X$. The triple (X, A, B) is called a ***topological triple***. A map $f: (X, A, B) \to (X', A', B')$ between two triples means the map $f: X \to X'$ satisfies $f(A) \subset A', f(B) \subset B'$.

A topological pair (X, A) can be regarded as a topological triple $(X, A, \emptyset)$. Now if we have a triple (X, A, B), then we have natural inclusions $i: (A, B) \to (X, B), j: (X, B) \to (X, A)$ hence a short exact sequence of chain complexes

$$0 \to S_*(A, B) \overset{i_\#}{\to} S_*(X, B) \overset{j_\#}{\to} S_*(X, A) \to 0$$

Note that the exactness comes from Third Isomorphism Theorem by letting $S_*(X, A) \cong S_*(X, B)/S_*(A, B)$. Moreover, if we have a map $f: (X, A, B) \to (X', A', B')$ we will get a commutative diagram of chain complexes and chain maps:

$$0 \to S_*(A, B) \overset{i_\#}{\to} S_*(X, B) \overset{j_\#}{\to} S_*(X, A) \to 0$$

$$f_\# \downarrow \quad f_\# \downarrow \quad f_\# \downarrow$$

$$0 \to S_*(A',B') \overset{i'_\#}{\to} S_*(X',B') \overset{j'_\#}{\to} S_*(X',A') \to 0$$

It is very natural for us to apply Theorem 3.60. and get the result below.

Theorem 3.65. Given a topological triple (X, A, B), we have a long exact sequence of homology groups:

$$\cdots \xrightarrow{\partial^*_{q+1}} H_q(A,B) \overset{i_*}{\to} H_q(X,B) \overset{j_*}{\to} H_q(X,A) \overset{\partial^*_q}{\to} H_{q-1}(A,B) \overset{i_*}{\to} \cdots$$

Moreover, if we have a map $f: (X, A, B) \to (X', A', B')$, which gives a commutative diagram of topological pairs:

$$(A,B) \overset{i}{\to} (X,B) \overset{j}{\to} (X,A)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$(A',B') \overset{i'}{\to} (X',B') \overset{j'}{\to} (X',A')$$

There is a commutative diagram of exact rows:

$$\cdots \to H_q(A,B) \overset{i_*}{\to} H_q(X,B) \overset{j_*}{\to} H_q(X,A) \overset{\partial^*_q}{\to} H_{q-1}(A,B) \to \cdots$$

$$f_* \downarrow \quad f_* \downarrow \quad f_* \downarrow \quad f_* \downarrow$$

$$\cdots \to H_q(A',B') \overset{i'_*}{\to} H_q(X',B') \overset{j'_*}{\to} H_q(X',A') \xrightarrow{\partial^{*\prime}_q} H_{q-1}(A',B') \to \cdots$$

Proof: Apply Theorem 3.60. to our earlier discussion.

■

3.6 Augmented Singular Homology

In this section we study a new chain complex called augmented singular chain complex, which is constructed with the help of the Kronecker index. Then with this new chain complex we apply the homology functor and get a new homology group called reduced homology group. The goal of such construction is to simplify the algebraic calculation in ordinary (co)homology theory.

Definition 3.66. Given a topological space X, we define its ***augmented singular chain complex*** $\tilde{S}_*(X) := \{\tilde{S}_q(X), \tilde{\partial}_q\}$ as

$$\tilde{S}_q(X) = \begin{cases} S_q(X) & q \geq 0 \\ \mathbb{Z} & q = -1 \end{cases}, \qquad \tilde{\partial}_q = \begin{cases} \partial_q & q > 0 \\ \varepsilon & q = 0 \end{cases}$$

ε is the Kronecker index defined in Definition 3.35.

The augmented singular chain complex is indeed a chain complex as one can verify that $\tilde{\partial}_q\tilde{\partial}_{q+1} = 0$: We only need to verify the case when $q = 0$ because the part when $q > 0$ is indeed a chain complex by definition. Take an arbitrary singular *1*-chain $c_1 = k_1\sigma_1^{(1)} + \cdots + k_r\sigma_1^{(r)} \in S_1(X)$, we have $\tilde{\partial}_1(c_1) = \partial_1(c_1) = \partial_1\left(\sum_{i=1}^r k_i c_1^{(i)}\right) = \sum_{i=1}^r k_i\left(\partial_1\sigma_1^{(i)}\right) = \sum_{i=1}^r k_i\left(\sum_{n=0}^1 (-1)^n \sigma_1^{(i)}\varepsilon_n^1\right) = \sum_{i=1}^r k_i\left(\sigma_1^{(i)}\varepsilon_0^1 - \sigma_1^{(i)}\varepsilon_1^1\right)$. Hence $\tilde{\partial}_0\tilde{\partial}_1(c_1) = \varepsilon\left(\sum_{i=1}^r k_i\left(\sigma_1^{(i)}\varepsilon_0^1 - \sigma_1^{(i)}\varepsilon_1^1\right)\right) = 0$, which means $\tilde{\partial}_0\tilde{\partial}_1 = 0$.

Definition 3.67. Given a topological space X, its ***reduced singular homology groups*** are defined as $\tilde{H}_*(X) := H_*\left(\tilde{S}_*(X)\right)$. i.e. $\tilde{H}_*(X) = \{\tilde{H}_q(X)\} = \{H_q(\tilde{S}_*(X)\}$ the singular homology groups of the augmented singular chain complex.

Given a continuous map $f: X \to Y$. In order to have its induced homomorphism between reduced singular homology groups we want a chain map $\tilde{f}_\#: \tilde{S}_*(X) \to \tilde{S}_*(Y)$ induced by f between two augmented chain complexes.

Given a continuous map $f: X \to Y$, it induces a chain map $f_\#: S_q(X) \to S_q(Y)$ for every $q \in \mathbb{N}^*$ as we verified. We now need to make such map commute with the augmented boundary map $\tilde{\partial}_0 = \varepsilon$ of dimension 0. To see how our $f_\#$ should behave in dimension -1 we apply it first on a singular *0*-chain in X to get a *0*-chain in Y. Then we apply the Kronecker index ε:

Take an arbitrary singular *0*-chain $c_0 = k_1a_1 + \cdots + k_ra_r \in \tilde{S}_0(X) = S_0(X)$. $f_\#(c_0) = f_\#(k_1a_1 + \cdots + k_ra_r) = k_1f(a_1) + \cdots + k_rf(a_r)$. So we have observed that if we apply ε on $f_\#(c_0)$ we will get the same result as after applying ε on c_0, which tells us once we let $f_\# = 1_\mathbb{Z}$ in dimension -1 then we will get a "augmented" chain map that satisfies the definition i.e. commutes with boundary map for all dimensions! So we may define

$$\tilde{f}_\# = \begin{cases} f_\# & q > -1 \\ 1_\mathbb{Z} & q = -1 \end{cases}$$

Now not only can we define the induced homomorphism between reduced homology groups by a continuous map as we did before, but also we can have a reduced homology functor $\widetilde{H}_*$.

Lemma 3.68. Given a chain map $\tilde{f}_\#: \tilde{S}_*(X) \to \tilde{S}_*(Y)$ induced by $f: X \to Y$. We have an induced homomorphism $f_*: \widetilde{H}_*(X) \to \widetilde{H}_*(Y)$ between graded groups and $f_*([z_q])$ $:= [\tilde{f}_\#(z_q)] \in H_q(Y)$ for $[z_q] \in \widetilde{H}_q(X)$.

Proof: Imitate the earlier ordinary singular homology case.

■

Theorem 3.69. We have a covariant functor $\widetilde{H}_*$ with $\widetilde{H}_*(X) = H_*(\tilde{S}_*(X))$ and $\widetilde{H}_*(f) = (\tilde{f}_\#)_*$ for some good category to $\mathbf{GradedG}$.

Proof: Refer to page 110, A Concise Course in Algebraic Topology[7]. The "good" category is called the category of nondegenerately based spaces, which we will not give formal introductions.

■

Here we can make use of our knowledge in homological algebra to understand the reduced homology group!

Theorem 3.70. Given a nonempty topological space X, we have

$$H_q(X) \cong \begin{cases} \widetilde{H}_q(X) & q > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & q = 0 \end{cases}$$

Proof: Since in the case when $q > 0$ it is not hard to find $\widetilde{H}_q(X) = H_q(X)$. We look at the short exact sequence for $q = 0$:

$$0 \to \widetilde{H}_0(X) \xrightarrow{i} H_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Where i is the inclusion ($im\partial_1 \subset ker\varepsilon \subset S_0(X)$ so $\widetilde{H}_0(X) = ker\varepsilon/im\partial_1$ is a subgroup of $H_0(X) = S_0(X)/im\partial_1$) and ε is the induced epimorphism, which implies the exactness. Since $\mathbb{Z}$ is free abelian, by Corollary 3.56. we have

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

■

It is worth mentioning that $\widetilde{H}_*(\emptyset) = \begin{cases} \mathbb{Z} & q = -1 \\ 0 & q > -1 \end{cases}$

Corollary 3.71. A topological space X is path-connected if and only if $\widetilde{H}_0(X) = 0$.

Proof: Apply Theorem 3.70 to Corollary 3.39.

■

Theorem 3.72 (Reduced Singular Homology of a one-point space).

Given a one-point space $X = \{pt\}$, its reduced singular homology group of each dimension $q = -1,0,1,...$ is 0. i.e. $\widetilde{H}_*(\{pt\}) = 0$.

Proof: Those cases when $q > 0$ are no different from those in Theorem 3.34. So we only need to pay attention to the case when $q = 0, -1$. Since X is a one-point space. For each q there is only one singular q-simplex $\sigma_q: \Delta^q \to X$. When $q = 0$, ($ker\varepsilon \subset im\partial_1$): $ker\varepsilon = \{c_0 = k\sigma_0 \in S_0(X) \mid k = 0\} = 0 \subset im\partial_1$. ($im\partial_1 \subset ker\varepsilon$): $im\partial_1 = \{\partial_1 k\sigma_1 | \sigma_1 \in S_1(X)\} = \{k(\sigma_1\varepsilon_0^1 - \sigma_1\varepsilon_1^1) | \sigma_1 \in S_1(X)\} \subset ker\varepsilon$. Hence $\widetilde{H}_0(X) = 0$. When $q = -1$, since $\tilde{\partial}_{-1} = 0$ we have $Z_{-1}(X) = \mathbb{Z}$. And the fact that the Kronecker index $\varepsilon: S_0(X) \to \mathbb{Z}$ is an epimorphism tells us $im\tilde{\partial}_0 = im\varepsilon = \mathbb{Z}$. So we know $\widetilde{H}_{-1}(X) = 0$.

If we use our last theorem the case $q = 0$ can be easier.

■

We should not forget that the reduced homology is connected to relative homology!

The reduced relative homology groups $\widetilde{H}_*(X,A) := H_*\big(\tilde{S}_*(X)/\tilde{S}_*(A)\big)$ are defined on quotient complex of two augmented chain complexes. Since we will get 0 by taking the quotient $\mathbb{Z}/\mathbb{Z}$, so it tells us $\tilde{S}_*(X)/\tilde{S}_*(A) \cong S_*(X)/S_*(A)$, from which it can be concluded that $\widetilde{H}_*(X,A) \cong H_*(X,A)$. Hence we have a similar theorem here.

Theorem 3.73. Given a topological pair (X,A), there are two inclusions $i:(A,\emptyset) \to (X,\emptyset)$ and $j:(X,\emptyset) \to (X,A)$ whose inducing chain maps $i_\#$ and $j_\#$ together with the corresponding relative chain complexes give a short exact sequence $0 \to \tilde{S}_*(A,\emptyset) \xrightarrow{i_\#} \tilde{S}_*(X,\emptyset) \xrightarrow{j_\#} \tilde{S}_*(X,A) \to 0$ of chain complexes and chain maps. Hence we have an exact sequence of relative homology groups:

$$\cdots \xrightarrow{\partial_{q+1}^*} \widetilde{H}_q(A,\emptyset) \xrightarrow{i_*} \widetilde{H}_q(X,\emptyset) \xrightarrow{j_*} \widetilde{H}_q(X,A) \xrightarrow{\partial_q^*} \widetilde{H}_{q-1}(A,\emptyset) \xrightarrow{i_*} \cdots$$

Proof: By the discussion above and Theorem 3.51.

■

Now we fix a basepoint $x_0 \in X$ as we did when studying relative homology groups. With our discussion earlier we will have many interesting results!

Theorem 3.74. Given a topological pair (X,x_0), we have $\widetilde{H}_*(X) \cong H_*(X,x_0)$.

Proof: With such pair we have the short exact sequence

$$0 \to \tilde{S}_*(x_0,\emptyset) \xrightarrow{i} \tilde{S}_*(X,\emptyset) \xrightarrow{j} \tilde{S}_*(X,x_0) \to 0$$

Then by Theorem 3.51. there is a long exact sequence:

$$\cdots \xrightarrow{\partial_{q+1}^*} \widetilde{H}_q(x_0,\emptyset) \xrightarrow{i_*} \widetilde{H}_q(X,\emptyset) \xrightarrow{j_*} \widetilde{H}_q(X,x_0) \xrightarrow{\partial_q^*} \widetilde{H}_{q-1}(x_0,\emptyset) \xrightarrow{i_*} \cdots$$

Note that $\widetilde{H}_q(x_0,\emptyset)$ is nothing but $\widetilde{H}_q(x_0)$ and similar for $\widetilde{H}_q(X,\emptyset)$. By Theorem 3.72 $\widetilde{H}_q(x_0,\emptyset) = 0$. So we actually have such exact sequence:

$$\cdots \xrightarrow{\partial_{q+1}^*} 0 \xrightarrow{i_*} \widetilde{H}_q(X,\emptyset) \xrightarrow{j_*} \widetilde{H}_q(X,x_0) \xrightarrow{\partial_q^*} 0 \xrightarrow{i_*} \cdots$$

The exactness at $\widetilde{H}_q(X, \emptyset)$ implies the injectivity of j_* and the exactness at $\widetilde{H}_q(X, x_0)$ gives the surjectivity of j_*. So j_* is an isomorphism, which proves the result since $\widetilde{H}_*(X, x_0) \cong H_*(X, x_0)$ as we discussed earlier.

■

3.7 Excision and Mayer-Vietoris

In this section we will understand that when building homology groups, cutting off some special part of topological spaces does not influence the homology groups. The Mayer-Vietoris sequence serves as a powerful computational tool.

Excision 1. Given $U \subset A$ subspaces of X and $\overline{U} \subset A^{\circ}$. Then we have $i_*: H_q(X - U, A - U) \cong H_q(X, A)$, where i_* is the induced homomorphism from the natural inclusion $i: (X - U, A - U) \to (X, A)$.

Excision 2. Given two subspaces $X_1, X_2 \subset X$ and $X = X_1^{\circ} \cup X_2^{\circ}$. Then we have $j_*: H_q(X_1, X_1 \cap X_2) \cong H_q(X, X_2)$, where j_* is the induced homomorphism from the natural inclusion $j: (X_1, X_1 \cap X_2) \to (X, X_2)$.

The Excision 1 tells us that the relative homology group $H_*(X, A)$ depends only on $X - A$. But how can we understand it better? If we consider the quotient space X/A, which is visually obtained by collapsing the subspace A to a point. Since the group $H_*(X, A)$ should depend on $X - A$. When will the map $f: (X, A) \to (X/A, \{pt\})$ gives us an isomorphism? Actually it does induce an isomorphism $f_*: H_*(X, A) \cong H_*(X/A, \{pt\})$ when there is a subspace $B \subset X$ and the following two conditions are satisfied: (1). $\bar{A} \subset B^{\circ}$ (2) There is a deformation retraction $F: B \to A$.

Proof: Refer to page 111-119, An Introduction to Algebraic Topology[8].

■

Theorem 3.75. Given a subspace $A \subset X$. We have $H_*(X, A) \cong H_*(X/A, \{pt\})$ if there is another subspace $B \subset X$ such that (1). $\bar{A} \subset B^{\circ}$ (2) There is a deformation retraction $F: B \to A$.

Proof: We consider the diagram of pairs

$$\begin{array}{ccccc} (X,A) & \overset{i}{\rightarrow} & (X,B) & \overset{j}{\leftarrow} & (X-A,B-A) \\ \downarrow & & \downarrow & & \downarrow k \\ (X/A,\{pt\}) & \overset{\bar{\imath}}{\rightarrow} & (X/A,B/A) & \overset{\bar{\jmath}}{\leftarrow} & ((X/A)-\{pt\},(B/A)-\{pt\}) \end{array}$$

First the map k is a homeomorphism as one can verify by point-set topology. From our condition (1) we have j acts similarly as the one in Excision 1 hence induces an isomorphism $H_*(X-A,B-A) \cong H_*(X,B)$. From condition (2) we have the isomorphism $H_*(X,B) \cong H_*(X,A)$. Moreover, $\bar{\imath}$ induces an isomorphism $H_*(X/A,\{pt\}) \cong H_*(X/A,B/A)$ also because of one deformation retraction. Finally $\bar{\jmath}$ induces the isomorphism $H_*((X/A)-\{pt\},(B/A)-\{pt\}) \cong H_*(X/A,B/A)$ since $\overline{\{pt\}} \subset B/A$. So $H_*(X,A) \cong H_*(X/A,\{pt\})$ by transitivity.

■

Lemma 3.76 (Barratt-Whitehead). Given a commutative diagram with exact rows of abelian groups and homomorphisms, where h_ns are isomorphisms:

$$\begin{array}{c} \cdots \rightarrow A_n \overset{i_n}{\rightarrow} B_n \overset{p_n}{\rightarrow} C_n \overset{d_n}{\rightarrow} A_{n-1} \rightarrow \cdots \\ f_n \downarrow \ g_n \downarrow \ h_n \downarrow \ f_{n-1} \downarrow \\ \cdots \rightarrow A'_n \underset{j_n}{\rightarrow} B'_n \underset{q_n}{\rightarrow} C'_n \underset{s_n}{\rightarrow} A'_{n-1} \rightarrow \cdots \end{array}$$

We have an exact sequence

$$\cdots \rightarrow A_n \xrightarrow{(i_n,f_n)} B_n \oplus A'_n \xrightarrow{g_n - j_n} B'_n \xrightarrow{d_n h_n^{-1} q_n} A_{n-1} \rightarrow \cdots$$

Proof: Refer to Lemma 6.2, page 107, An Introduction to Algebraic Topology[8].

■

Theorem 3.77 (Mayer-Vietoris). Given two subspaces $X_1, X_2 \subset X$ and $X = X_1^\circ \cup X_2^\circ$. We have an exact sequence of homology groups

$$\cdots \rightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1*},i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \overset{D}{\rightarrow} H_{n-1}(X_1 \cap X_2) \rightarrow \cdots$$

Where $i_1: (X_1 \cap X_2, \emptyset) \rightarrow (X_1, \emptyset)$, $i_2: (X_1 \cap X_2, \emptyset) \rightarrow (X_2, \emptyset)$, $g: (X_1, \emptyset) \rightarrow (X, \emptyset)$,

$j: (X_2, \emptyset) \to (X, \emptyset)$ are inclusions and $D = dh_*^{-1}q_*$ ($h: (X_1, X_1 \cap X_2) \to (X, X_2)$, $q: (X, \emptyset) \to (X, X_2)$ are inclusions, $d: H_q(X_1, X_1 \cap X_2) \to H_{q-1}(X_1 \cap X_2, \emptyset)$ is the connecting homomorphism of $(X_1, X_1 \cap X_2)$).

Proof: It is an immediate result by applying the earlier lemma and Excision 2 to the commutative diagram of topological pairs:

$$\begin{array}{ccccc} (X_1 \cap X_2, \emptyset) & \xrightarrow{i_1} & (X_1, \emptyset) & \xrightarrow{p} & (X_1, X_1 \cap X_2) \\ i_2 \downarrow & & g \downarrow & & \downarrow h \\ (X_2, \emptyset) & \xrightarrow{j} & (X, \emptyset) & \xrightarrow{q} & (X, X_2) \end{array}$$

By applying Theorem 3.65. we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \to & H_n(X_1 \cap X_2) & \xrightarrow{i_{1_*}} & H_n(X_1) & \xrightarrow{p_*} & H_n(X_1, X_1 \cap X_2) & \xrightarrow{d} & H_{n-1}(X_1 \cap X_2) & \to \cdots \\ & i_{2_*} \downarrow & & g_* \downarrow & & h_* \downarrow & & i_{2_*} \downarrow & \\ \cdots \to & H_n(X_2) & \xrightarrow[j_*]{} & H_n(X) & \xrightarrow[q_*]{} & H_n(X, X_2) & \xrightarrow[s_*]{} & H_n(X_2) & \to \cdots \end{array}$$

Where h_* is an isomorphism by Excision 2. Then it is proved by applying previous Lemma.

■

Theorem 3.78 (Mayer-Vietoris for reduced homology). Given two subspaces $X_1, X_2 \subset X$ and $X = X_1^\circ \cup X_2^\circ$ with the property $X_1 \cap X_2 \neq \emptyset$. We have an exact sequence of reduced homology groups

$$\cdots \to \widetilde{H}_n(X_1 \cap X_2) \xrightarrow{(i_{1_*}, i_{2_*})} \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \xrightarrow{g_* - j_*} \widetilde{H}_n(X) \xrightarrow{D} \widetilde{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

The maps are those induced by maps in last theorem. And the sequence ends

$$\cdots \to \widetilde{H}_0(X) \oplus \widetilde{H}_0(X_2) \to H_0(X) \to 0$$

Proof: The reason why we require $X_1 \cap X_2 \neq \emptyset$ is because we need to take a $x_0 \in X_1 \cap X_2$. Then consider the commutative diagram:

$$\begin{array}{ccccc} (X_1 \cap X_2, x_0) & \xrightarrow{i_1} & (X_1, x_0) & \xrightarrow{p} & (X_1, X_1 \cap X_2) \\ i_2 \downarrow & & g \downarrow & & \downarrow h \end{array}$$

$$(X_2, x_0) \xrightarrow{j} (X, x_0) \xrightarrow{q} (X, X_2)$$

Then apply Theorem 3.65. We have

$$\begin{array}{ccccccccc}
\cdots \to H_n(X_1 \cap X_2, x_0) & \xrightarrow{i_{1*}} & H_n(X_1, x_0) & \xrightarrow{p_*} & H_n(X_1, X_1 \cap X_2) & \xrightarrow{d} & H_{n-1}(X_1 \cap X_2) \to \cdots \\
i_{2*} \downarrow & & g_* \downarrow & & h_* \downarrow & & i_{2*} \downarrow \\
\cdots \to H_n(X_2, x_0) & \underset{j_*}{\to} & H_n(X, x_0) & \underset{q_*}{\to} & H_n(X, X_2) & \underset{s_*}{\to} & H_n(X_2) \to \cdots
\end{array}$$

Apply the Lemma 3.76. and by Theorem 3.74. $\widetilde{H}_*(X) \cong H_*(X, x_0)$ we get an exact sequence:

$$\cdots \to \widetilde{H}_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \xrightarrow{g_* - j_*} \widetilde{H}_n(X) \xrightarrow{D} \widetilde{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

■

Now we compute few examples

Example 3.79 ((Reduced) Homology groups of S^n). For $n > 0$, we have

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & q = 0 \text{ or } q = n \\ 0 & \text{otherwise} \end{cases}$$

And for all $n \in \mathbb{N}$, we have the reduced homology groups of S^n

$$\widetilde{H}_q(S^n) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Proof: We prove the reduced case by induction. When $n = 0$ we have

$$H_q(S^0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = 0 \\ 0 & q > 0 \end{cases}$$

So we have

$$\widetilde{H}_q(S^0) \cong \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Then consider the general case for an $n > 0$, we have two subsets $X_1 = S^n - \{N\}, X_2 = S^n - \{S\}$, where $\{N\}, \{S\}$ are the north and the south pole. And clearly they satisfy $X_1^\circ \cup X_2^\circ = S^n$ and $X_1 \cap X_2 \neq \emptyset$. So we may apply Theorem 3.78. and get the following exact homology sequence:

$$\cdots \to \widetilde{H}_q(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} \widetilde{H}_q(X_1) \oplus \widetilde{H}_q(X_2) \xrightarrow{g_* - j_*} \widetilde{H}_q(S^n) \xrightarrow{D} \widetilde{H}_{q-1}(X_1 \cap X_2) \to \cdots$$

Because both X_1 and X_2 are homeomorphic to $\mathbb{R}^n$, which is contractible. Hence we have $\widetilde{H}_q(X_1) \cong \widetilde{H}_q(X_2) \cong 0$. Actually $X_1 \cap X_2 \simeq S^{n-1}$ because one can regard $X_1 \cap X_2 \approx \mathbb{R}^n - \{0\}$, which can be retracted to S^{n-1} by the map $r: x \mapsto \frac{x}{\|x\|}$. Define the deformation retraction $F(x,t) = tr(x) + (1-t)x$ we get $1_{\mathbb{R}-\{0\}} \simeq r$. So $\mathbb{R}^n - \{0\} \simeq S^{n-1}$ and the homology sequence above can be rewritten as

$$\cdots \to \widetilde{H}_q(S^{n-1}) \xrightarrow{(i_{1*}, i_{2*})} 0 \xrightarrow{g_* - j_*} \widetilde{H}_q(S^n) \xrightarrow{D} \widetilde{H}_{q-1}(S^{n-1}) \to 0 \to \cdots$$

The exactness implies that $\widetilde{H}_q(S^n) \cong \widetilde{H}_{q-1}(S^{n-1})$, which completes the proof.

■

The theorem above also helps us to solve the classic problem in point-set topology.

Corollary 3.80. Given $m \neq n$, we have (1) S^m, S^n are not homeomorphic. (2) $\mathbb{R}^m, \mathbb{R}^n$ are not homeomorphic.

Proof: (1) If $S^m \approx S^n$ then $H_*(S^m) \approx H_*(S^n)$, which contradicts with Example 3.79. (2) If $\mathbb{R}^m \approx \mathbb{R}^n$, then we have $\mathbb{R}^m - \{0\} \approx \mathbb{R}^n - \{0\}$. Since we know that $S^{n-1} \simeq \mathbb{R}^n - \{0\}$ as in the proof of last theorem. So $S^{n-1} \simeq S^{m-1}$ and by the homotopy invariance they share the same homology groups, which contradicts with our theorem.

■

Corollary 3.81. Given $n \in \mathbb{N}$, (1) S^n is not a retract of D^{n+1}. (2) S^n is not contractible.

Proof: (1) If so, there exists a continuous map $r: D^{n+1} \to S^n$ such that $ri = 1|_{S^n}$ for inclusion $i: S^n \to D^{n+1}$. So $(ri)_* = 1|_{\widetilde{H}_*(S^n)}$. Since $D^n \simeq \{pt\}$, we know that $\widetilde{H}_*(D^{n+1}) = 0$, which contradicts with the sequence:

$$H_*(S^n) \xrightarrow{i_*} H_*(D^{n+1}) \xrightarrow{r_*} H_*(S^n)$$

(2) If so, then $H_*(S^n) \cong H_*(\{pt\})$, which contradicts with the fact.

■

3.8 Singular Homology with Coefficient and Functor $- \otimes G$

In this section G is a fixed abelian group. We start from **Comp** as usual to define our relative singular homology as a more general version of singular homology. Moreover, those discussions about singular homology naturally appear here.

Definition 3.82. Given a free abelian group A with basis $\{a_i\}$. Define the abelian group ***tensor product of A and G*** $A \otimes G$ as

$A \otimes G = \{\sum g_i a_i | \ g_i \in G \text{ and finitely many of } g_i \neq 0\}$. If $\{a'_{i'}\}$ is another basis of A and $a'_{i'} = \sum_i k_{ii'} a_i, k_{ii'} \in \mathbb{Z}$. Then $\sum_{i'} g'_i a'_i = \sum_i g_i a_i$ is equivalent to $\sum_{i'} k_{ii'} g'_{i'}$ for any i. The "addition" in $A \otimes G$ is defined under the same basis as $\sum g_i a_i + \sum g'_i a_i = \sum (g_i + g'_i) a_i$.

Definition 3.83. Given two free abelian groups A, B with basis $\{a_i\}$, $\{b_i\}$ and a homomorphism $f: A \to B$ such that $f(a_i) = \sum_j F_{ij} b_j$ where $F_{ij} \in \mathbb{Z}$. We define a homomorphism $f \otimes 1_G: A \otimes G \to B \otimes G$ as

$$(f \otimes 1_G)(\textstyle\sum_i g_i a_i) = (\sum_j (\sum_i F_{ij} g_i) b_j).$$

Theorem 3.84. $- \otimes G$ is an additive functor from category of free abelian groups **F** to the category of abelian groups **Ab** which maps $A \mapsto A \otimes G$ and $f: A \to B \mapsto f \otimes 1_G: A \otimes G \to B \otimes G$. It is called tensor product functor. The composition is given by $(g \otimes 1_G)(f \otimes 1_G) = (gf \otimes 1_G)$.

Proof: Refer to Corollary 9.27, page 255, An Introduction to Algebraic Topology[8].

■

How does the functor influence different kinds of sequences?

Proposition 3.85. Given abelian groups and homomorphisms, the tensor functor $- \otimes G$ satisfies the following properties:

(1) (Split exact property). Given a split exact sequence $0 \to A \xrightarrow{f} A' \xrightarrow{g} A'' \to 0$. We have a split exact sequence $0 \to A \otimes G \xrightarrow{f \otimes 1_G} A' \otimes G \xrightarrow{g \otimes 1_G} A'' \otimes G \to 0$.

(2) (Half-exact property). Given an exact sequence $A \xrightarrow{f} A' \xrightarrow{f'} A'' \to 0$. We have an exact sequence $A \otimes G \xrightarrow{f \otimes 1_G} A' \otimes G \xrightarrow{f' \otimes 1_G} A'' \otimes G \to 0$.

(3) Given an exact sequence $0 \to A \xrightarrow{f} A' \xrightarrow{f'} A'' \to 0$, when G is torsion-free(no elements of finite order). We have an exact sequence

$$0 \to A \otimes G \xrightarrow{f \otimes 1_G} A' \otimes G \xrightarrow{f' \otimes 1_G} A'' \otimes G \to 0$$

Proof: (1) can be proved by applying Proposition 3.55. (2) Refer to Theorem 9.29, page 257-258, An Introduction to Algebraic Topology[8]. (3) By Property (2) of the functor Tor introduced in 6.2.

■

Corollary 3.86. (1). Given $m \in \mathbb{N}^*$, we have $(\mathbb{Z}/m\mathbb{Z}) \otimes G = G/mG$. (2). Given $m, n \in \mathbb{N}^*$ with $(m, n) = d$, we have $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$.

Proof: (1). We apply the functor to the "right" part of the short exact sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{p} (\mathbb{Z}/m\mathbb{Z}) \to 0$ where $m: 1 \mapsto m$ and by half-exact property to get the exact sequence $G \xrightarrow{m} G \xrightarrow{p \otimes 1_G} (\mathbb{Z}/m\mathbb{Z}) \otimes G$. Here the $m: G \to G$ still "multiplies" the elements by m as one can verify by definition (Suppose $\{a_i\}$ is the basis and $f(a_i) = ma_i = \sum_j F_{ij} a_j$ implies that $F_{ii} = m$ and $F_{ij} = 0$ for $i \neq j$. Hence by the definition of the map $(f \otimes 1_G)(\sum_i g_i a_i) = (\sum_j (\sum_i F_{ij} g_i) a_j) = \sum_i m g_i a_i = m(\sum_i g_i a_i)$). By First Isomorphism Theorem $G/ker(p \otimes 1_G) \cong im(p \otimes 1_G)$ where by exactness $ker(p \otimes 1_G) = im(m) = mG$, $im(p \otimes 1_G) = (\mathbb{Z}/m\mathbb{Z}) \otimes G$. So we have $G/mG \cong (\mathbb{Z}/m\mathbb{Z}) \otimes G$.

(2). Consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{p} (\mathbb{Z}/m\mathbb{Z}) \to 0$. Similar to above we apply $\otimes \mathbb{Z}/n\mathbb{Z}$ and get $\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \xrightarrow{p \otimes 1_{(\mathbb{Z}/n\mathbb{Z})}} (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$. By First Isomorphism Theorem $(\mathbb{Z}/n\mathbb{Z})/ker\left(p \otimes 1_{(\mathbb{Z}/n\mathbb{Z})}\right) \cong im\left(p \otimes 1_{(\mathbb{Z}/n\mathbb{Z})}\right)$ and the fact that $ker\left(p \otimes 1_{(\mathbb{Z}/n\mathbb{Z})}\right) = im(m)$ we finished the proof.

■

Here we illustrate an example to show that the functor does not preserve exactness. Consider the short exact sequence $0 \to \mathbb{Z} \overset{2}{\to} \mathbb{Z} \overset{p}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$. Now we apply the functor with $G = \mathbb{Z}/2\mathbb{Z}$ and get

$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\otimes 1_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{p\otimes 1_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to 0$, which cannot be exact.

So given a chain complex $C = \{C_q, \partial_q\}$ we apply the tensor functor $\_ \otimes G$ then get a new chain complex $C \otimes G = \{C_q \otimes G, \partial_q \otimes 1_G\}$ as one can verify that the composition $(\partial_q \otimes 1_G)(\partial_{q+1} \otimes 1_G) = \partial_q \partial_{q+1} \otimes 1_G = 0 \otimes 1_G = 0$ by the additivity of the functor. Hence we define the relative singular chain complex in this way.

Definition 3.87. Given an abelian group G, a topological pair (X, A). We have its relative singular chain complex $S_*(X, A) = \{S_q(X)/S_q(A), \overline{\partial}_q\}$. The ***relative singular chain complex with coefficient G*** is the chain complex

$$S_*(X, A; G) := \{S_q(X, A) \otimes G, \overline{\partial}_q \otimes 1_G\} = \{(S_q(X)/S_q(A)) \otimes G, \overline{\partial}_q \otimes 1_G\}$$

Moreover, the ***relative singular homology groups*** are the homology groups

$$H_*(X, A; G) := \{H_q(S_*(X, A; G))\}$$

Similarly, given a map $f: (X, A) \to (Y, B)$, we have the ***induced relative chain map*** $f_\# \otimes 1_G: S_*(X, A; G) \to S_*(Y, B; G)$ is defined from the chain map $f_\#: S_*(X; G) \to S_*(Y; G)$. And it induces ***the homomorphism between relative homology groups with coefficients G*** $f_*: H_*(X, A; G) \to H_*(Y, B; G)$.

Note that $S_q(X, A) \otimes G \cong (S_q(X) \otimes G)/(S_q(A) \otimes G)$ from the split exact sequence $0 \to S_q(A) \otimes G \to S_q(X) \otimes G \to S_q(X, A) \otimes G$ by applying the tensor functor and property 1 on the split exact sequence $0 \to S_q(A) \to S_q(X) \to S_q(X, A) \to 0$. So actually $S_q(X, A; G) \cong S_q(X; G)/S_q(A; G)$.

Definition 3.88. The ***relative homology functor with coefficient G*** $H_*(-; G)$ assigns a topological pair (X, A) a graded group $H_*(X, A; G)$ and a morphism $f: (X, A) \to (Y, B)$ a graded group homomorphism $f_*: H_*(X, A; G) \to H_*(Y, B; G)$.

It is a more general situation because $H_*(X) = H_*(X, \emptyset; \mathbb{Z})$. The reduced homology group is defined for nonrelative case because the augmented relative chain is the same as the ordinary relative chain.

Definition 3.89. Given a singular chain complex $S_*(X) = \{S_q(X), \partial_q\}$ of a topological space X. Its ***augmented chain complex with coefficient G*** is defined as $\tilde{S}_*(X; G) := \{\tilde{S}_q(X) \otimes G, \tilde{\partial}_q \otimes 1_G\}$. The ***reduced homology group with coefficient G*** is $\widetilde{H}_*(X; G) := H_*\left(\tilde{S}_*(X; G)\right)$.

Note that $\tilde{\partial}_0 \otimes 1_G$ still takes the sum of coefficients of 0-chain and is denoted by ε. Similarly, given a map $f: X \to Y$, we have the induced chain map $\tilde{f}_\#: \tilde{S}_q(X; G) \to \tilde{S}_q(Y; G)$ which equals to $f_\#: S_q(X; G) \to S_q(Y; G)$ when $q \geq 0$ and 1_G when $q = -1$. Moreover, it induces a homomorphism between reduced homology group with coefficient G $f_*: H_*(X; G) \to H_*(Y; G)$, which is defined to be $f_* = H_*(\tilde{f}_\#)$.

Naturally one can have the dimension axiom.

Theorem 3.90 (Singular Homology of a one-point space/Dimension Axiom). Given a one-point space $X = \{pt\}$, its singular homology group with coefficient G for each $q \in \mathbb{N}$ is

$$H_q(\{pt\}; G) = \begin{cases} 0 & q > 0 \\ G & q = 0 \end{cases}$$

And its reduced singular homology group with coefficient G is

$$\widetilde{H}_*(\{pt\}; G) = 0$$

Proof: Imitate Theorem 3.72.

■

3.9 Axioms of Homology

"The ultimate goal of the axiomatic method is the profound clarity of mathematics, which cannot be reached by logical formalism. As the origin of the experimental method is the transcendental belief in the eternity of the laws of nature, the origin of the axiomatic method is the belief that mathematics is neither a casually developed collection of products of syllogism nor a bunch of "wise" techniques which are created by lucky combinations. Superficial observers are only able to see few different theories, in which there will be one theory offering surprising supports to other theories with the help of mathematical geniuses. Now the axiomatic method is teaching us to seek for the deeper reason for such surprising supports and to grasp the common ideas hidden under the details of each theory so that we can put those ideas in the places they should be in." [1]

Axiom of homology. A homology theory is a covariant functor from $\mathbf{Top^2}$ to **GradedG** which consists of three functions. For each $q \in \mathbb{N}$ it assigns a topological pair (X, A) an abelian group $H_q(X, A; G)$, a continuous $f: (X, A) \to (Y, B)$ a homomorphism $(f_*)_q: H_q(X, A; G) \to H_q(Y, B; G)$ and a topological pair (X, A) a homomorphism $\partial_q^*: H_q(X, A) \to H_{q-1}(A)$, which satisfy the following properties:

(1) (Identity Law). Given an identity map $1_{(X,A)}$, we have an identity homomorphism $1_{H_*(X,A;G)}$.

(2) (Composition Law). $(gf)_* = g_* f_*$ for defined composition.

(3) (Naturality). Given $f: (X, A) \to (Y, B)$, we have a commutative diagram

$$\begin{array}{ccc} H_q(X,A;G) & \xrightarrow{\partial_q^*} & H_{q-1}(A;G) \\ f_* \downarrow & & (f|_A)_* \downarrow \\ H_q(Y,B;G) & \xrightarrow{\partial_q^*} & H_{q-1}(B;G) \end{array}$$

(4) (Exactness Axiom). We have the long exact sequence

$$\cdots \xrightarrow{\partial_{q+1}^*} H_q(A;G) \xrightarrow{i_*} H_q(X;G) \xrightarrow{j_*} H_q(X,A;G) \xrightarrow{\partial_q^*} H_{q-1}(A;G) \xrightarrow{i_*} \cdots$$

(5) (Homotopy Axiom). Given two homotopic maps $f \simeq g: (X, A) \to (Y, B)$, we have $f_* = g_*$.

(6) (Excision Axiom). Given an open subspace $W \subset X$ with $\overline{W} \subset A^{\circ}$ and an inclusion map $i: (X - W, A - W) \to (X, A)$. Then the inclusion induces an isomorphism $i_*: H_*(X - W, A - W) \xrightarrow{\cong} H_*(X, A)$.

(7) (Dimension Axiom). The homology groups of a one-point space $\{pt\}$ are

$$H_q(\{pt\}; G) = \begin{cases} G & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Axiom of reduced homology. A reduced homology theory is a covariant functor from an appropriate category of topology spaces to **GradedG** consists of three functions. For each $q \in \mathbb{N}$ it assigns a topological space X an abelian group $\widetilde{H}_q(X; G)$, a continuous $f: X \to Y$ a homomorphism $(f_*)_q: \widetilde{H}_q(X; G) \to \widetilde{H}_q(Y; G)$ and a topological space X an isomorphism $(S_*)_q: \widetilde{H}_q(X) \xrightarrow{\cong} \widetilde{H}_{q+1}(SX)$, which satisfy the following properties:

(1) (Identity Law). Given an identity map 1_X, we have an identity homomorphism $1_{\widetilde{H}_*(X;G)}$.

(2) (Composition Law). $(gf)_* = g_* f_*$ for defined composition.

(3) (Naturality). Given $f: X \to Y$, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_q(X; G) & \xrightarrow{S_*} & \widetilde{H}_{q+1}(SX; G) \\ f_* \downarrow & & (Sf)_* \downarrow \\ \widetilde{H}_q(Y; G) & \xrightarrow{S_*} & \widetilde{H}_{q+1}(SY; G) \end{array}$$

where $Sf: SX \to SY$ is the quotient map of $f \times 1_I: X \times I \to Y \times I$.

(4) (Exactness Axiom). Given a map $f: X \to Y$, we have the exact sequence

$$\widetilde{H}_q(X; G) \xrightarrow{f_*} \widetilde{H}_q(Y; G) \xrightarrow{i_*} \widetilde{H}_q(Cf; G)$$

where $i: Y \to Cf = Y \cup_f CX$ is the inclusion map.

(5) (Homotopy Axiom). Given two homotopic maps $f \simeq g: (X, A) \to (Y, B)$, we have $f_* = g_*$.

(6) (Dimension Axiom). The homology groups of S^0 are

$$\widetilde{H}_q(S^0; G) = \begin{cases} G & q = 0 \\ 0 & q \neq 0 \end{cases}$$

4. Singular Cohomology

4.1 Basic Definitions in Cohomology

Why do we need cohomology if homology is relatively easier to compute? The answer here is, there is a ring structure in cohomology theory which plays an important role when classifying topological spaces. But now in this section we mainly tell how cohomology groups are defined and formalize everything in the language of category theory.

To introduce basic definitions we begin by defining a contravariant functor $Hom(-, \mathrm{G})$ for a fix abelian group G.

Definition 4.1. Given two abelian groups A, B. We define the set $\boldsymbol{Hom(A, B)}$ $:= \{f | f: A \to B \text{ homomorphisms}\}$.

Actually the definition coincides with the situation in category **Ab**. Now what we want to do is to give a group structure on this set.

Proposition 4.2. $Hom(A, B)$ is an abelian group under the addition $(f + g)(a)$ $:= f(a) + g(a), \forall a \in A$. The identity element is $0: a \mapsto 0$.

Proof: A routine.

■

Definition 4.3. Given three abelian groups A, B, C. A function $f: A \times B \to C$ is called ***bilinear*** if $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$ and $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$.

The bilinearity means the linearity on both coordinates.

Definition 4.4. If we consider $\Phi(a) :=< \Phi, a >$ in a form that is similar to inner product for $\Phi \in Hom(A, B)$, then we have a bilinear function called **Kronecker product**, which is defined by

$$< -,- >: Hom(A,B) \times A \to B;\ < \Phi, a > \mapsto \Phi(a), \forall \Phi \in Hom(A,B),\ \forall a \in A.$$

So once we fixed such G, we have an abelian group $Hom(A,G)$ for an abelian group A. To make the $Hom(-,G)$ a functor we need to pay attention to morphisms.

Definition 4.5. Given a homomorphism $f: A \to A'$, its ***dual homomorphism*** is defined as $f^{\#}: Hom(A',G) \to Hom(A,G); g \mapsto gf$.

Similarly we may write $< f^{\#}(g), a > = < g, f(a) >$.

Theorem 4.6. $Hom(-,G)$ is an additive contravariant functor from **Ab** to **Ab** for a fixed abelian group G.

Proof: A routine.

■

We need to know how does the functor influence different exact sequences.

Theorem 4.7. Given abelian groups and homomorphisms, the hom functor $Hom(-,G)$ satisfies the following properties:

(1) (Split exact property). Given a split exact sequence $0 \to A \xrightarrow{f} A' \xrightarrow{f'} A'' \to 0$. We have $0 \leftarrow Hom(A,G) \xleftarrow{f^{\#}} Hom(A',G) \xleftarrow{f'^{\#}} Hom(A'',G) \leftarrow 0$, which is also split exact.

(2) (Half exact property). Given an exact sequence $A \xrightarrow{f} A' \xrightarrow{f'} A'' \to 0$. We have $Hom(A,G) \xleftarrow{f^{\#}} Hom(A',G) \xleftarrow{f'^{\#}} Hom(A'',G) \leftarrow 0$, which is also an exact sequence.

Proof: (1) can be proved by Proposition 3.55. (2) Refer to Lemma 12.5, page 380, An Introduction to Algebraic Topology[8].

■

Naturally we wonder if applying the functor to a chain complex gives us a new chain complex. Given a singular chain complex $S_*(X) = \{S_q(X), \partial_q\}$ on a topological space X, we apply the hom functor with a fix abelian group G and get $0 \to Hom(S_0(X), G) \overset{\partial_1^\#}{\to} Hom(S_1(X), G) \overset{\partial_2^\#}{\to} Hom(S_2(X), G) \to \cdots$. Moreover, we can see that $\partial_{q+1}^\# \partial_q^\# = (\partial_q \partial_{q+1})^\# = 0^\# = 0$ for $q \in \mathbb{N}$. Hence we do get a new "chain complex" denoted by $S^*(X; G) := Hom(S_*(X), G)$. As the homomorphism defined on a free abelian group is determined by the basis of the free abelian group, we can say that the elements in $Hom(S_q(X), G)$ are G-valued functions defined on the basis of $S_q(X)$, i.e. all singular q-simplexes on X. But the new complex is a little bit different from ordinary chain complex as the "boundary maps" increases the "dimension" due to the usage of contravariant functor $Hom(-, G)$. This problem can be fixed by changing notation $Hom(S_q(X), G)$ to A_{-q} and $\partial_{q+1}^\#$ to d_{-q}. Then we take a look at our new chain complex A_*: $0 \to A_0 \overset{d_0}{\to} A_{-1} \overset{d_{-1}}{\longrightarrow} A_{-2} \overset{d_{-2}}{\longrightarrow} \cdots$ and define homology groups naturally by $H_{-q}(Hom(S_*(X), G) = H_{-q}(A_*) := ker d_{-q}/im d_{-q+1} = ker \partial_{q+1}^\# / im \partial_q^\#$. So one can see that after applying the functor we actually get something that is intrinsically a chain complex. We call it a cochain complex and actually cochain complexes and cochain maps form a category as chain complexes and chain maps do since the two definitions are just formally different. For simplicity we denote $Hom(S_q(X), G)$ by A^q and $\partial_{q+1}^\#$ by δ^q and define similar definitions from the following cochain complex:

$$0 \to A^0 \overset{\delta^0}{\to} A^1 \overset{\delta^1}{\to} A^2 \overset{\delta^2}{\to} \cdots$$

Definition 4.8. Given a topological space X and its singular chain complex $S_*(X) = \{S_q(X), \partial_q\}$. For a fixed abelian group G and $q \in \mathbb{N}$.

$S^q(X; G) := Hom(S_q(X), G)$ is called ***the group of singular q-chains in X with***

coefficient $\boldsymbol{G}$. $Z^q(X;G) := ker\delta^q = ker\partial_{q+1}^{\#}$ is called ***the group of q-cocycles*** in X; $B^q(X;G) := im\delta^{q-1} = im\partial_q^{\#}$ is called ***the group of q-coboundaries*** in X; Their quotient group $Z^q(X;G)/B^q(X;G)$, denoted by $H^q(X;G)$ is called ***the q-th cohomology group of X with coefficients G***. The elements of $H^q(X;G)$ are called ***cohomology classes***. The cohomology class of a q-cocycle z^q is $[z^q] := z^q + B^q(X;G)$. We put cohomology groups of all dimensions together as $H^*(X;G) := \{H^q(X;G)\} = \{H^q(Hom(S_*(X),G))\}$.

Now we already have a target in **GradedG** for an object in **Top**. To construct a functor we need to pay attention to morphisms between two objects. Given a continuous function $f: X \to Y$, we have the induced chain map $f_{\#}: S_*(X) \to S_*(Y)$. Applying the functor $Hom(-,G)$ to the commutative diagram

$$\cdots \to S_{q+1}(X) \xrightarrow{\partial_{q+1}} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \to \cdots$$

$$f_{\#} \downarrow \qquad f_{\#} \downarrow \qquad f_{\#} \downarrow$$

$$\cdots \to S_{q+1}(Y) \xrightarrow{\partial'_{q+1}} S_q(Y) \xrightarrow{\partial'_q} S_{q-1}(Y) \to \cdots$$

We will get a new commutative diagram

$$\cdots \leftarrow Hom(S_{q+1}(X),G) \xleftarrow{\partial_{q+1}^{\#}} Hom(S_q(X),G) \xleftarrow{\partial_q^{\#}} Hom(S_{q-1}(X),G) \leftarrow \cdots$$

$$f_{\#}^{\#} \uparrow \qquad f_{\#}^{\#} \uparrow \qquad f_{\#}^{\#} \uparrow$$

$$\cdots \leftarrow Hom(S_{q+1}(Y),G) \xleftarrow{{\partial'_{q+1}}^{\#}} Hom(S_q(Y),G) \xleftarrow{{\partial'_q}^{\#}} Hom(S_{q-1}(Y),G) \leftarrow \cdots$$

Which can be simplified as

$$\cdots \leftarrow S^{q+1}(X;G) \xleftarrow{\delta^q} S^q(X;G) \xleftarrow{\delta^{q-1}} S^{q-1}(X;G) \leftarrow \cdots$$

$$f_{\#}^{\#} \uparrow \qquad f_{\#}^{\#} \uparrow \qquad f_{\#}^{\#} \uparrow$$

$$\cdots \leftarrow S^{q+1}(Y;G) \xleftarrow{\delta'^q} S^q(Y;G) \xleftarrow{\delta'^{q-1}} S^{q-1}(Y;G) \leftarrow \cdots$$

According to the definition $f_{\#}^{\#}: S^q(Y;G) \to S^q(X;G), h \mapsto hf_{\#}$ for an arbitrary $h \in S^q(Y;G) = Hom(S_q(Y),G)$ i.e. $h: S_q(Y) \to G$. Let $z^q \in Z^q(Y;G) = ker\delta'^q = ker{\partial'_{q+1}}^{\#}$, we have by commutativity $0 = f_{\#}^{\#}{\partial'_{q+1}}^{\#}(z^q) = \partial_{q+1}^{\#}f_{\#}^{\#}(z^q)$, which implies

$f_{\#}^{\#}(Z^q(Y;G)) \subset Z^q(X;G)$. Let $b^q \in B^q(Y;G) = im\delta'^{q-1} = im\partial_q'^{\#}$, we get a $b^{q-1} \in S^{q-1}(Y;G)$ such that $\partial_q'^{\#}(b^{q-1}) = b^q$. Again by commutativity we have $f_{\#}^{\#}(b^q) = f_{\#}^{\#}\partial_q'^{\#}(b^{q-1}) = \partial_q^{\#} f_{\#}^{\#}(b^{q-1})$, which tells us $f_{\#}^{\#}(B^q(Y;G)) \subset B^q(X;G)$. Hence we may define the induced homomorphism $f^*: H^q(Y;G) \to H^q(X;G)$ by $[z^q] \mapsto [f_{\#}^{\#} z^q] = [z^q f_{\#}] \in H^q(X;G)$ for $z^q \in Z^q(Y;G)$. Eventually after collecting such homomorphisms of all dimensions together we have a graded group homomorphism (morphism in **GradedG**) $f^*: H^*(Y;G) \to H^*(X;G)$ for a continuous function $f: X \to Y$ and we are ready to define the cohomology functor.

Definition 4.9. $H^*(-;G)$ is a contravariant functor from **Top** to **GradedG** called ***singular cohomology functor with coefficient G***, which assigns a topological space X a graded group $H^*(X;G)$ and a continuous map $f: X \to Y$ a graded group homomorphism $H^*(f;G) = f^*: H^*(Y;G) \to H^*(X;G)$.

Note that when the coefficient G is ignored we mean the coefficient $\mathbb{Z}$ and such cohomology is called integral cohomology. Now we do not go through every discussion as we did in singular homology and will summarize them axiomatically.

Definition 4.10. Given a topological space X. Define its ***augmented singular cochain complex with coefficient G*** by

$\tilde{S}^*(X;G) := Hom\left(\tilde{S}^*(X;G)\right) = \{Hom(\tilde{S}_q(X), G), \tilde{\partial}_{q+1}^{\#}\} = \{\tilde{S}^q(X;G), \tilde{\delta}^q\}$.

Note that for a augmented singular chain complex, the *0*-dimensional boundary map $\tilde{\partial}_0 = \varepsilon$ adds up the coefficients of points (*0*-complexes) of a *0*-chain. Now the map $\tilde{\partial}_0^{\#} = \varepsilon^{\#}: Hom(\mathbb{Z}, G) \to Hom(\tilde{S}_0(X), G)$. As the homomorphisms defined on $\mathbb{Z}$ is determined by its value at $1 \in \mathbb{Z}$, we define the isomorphism $F: Hom(\mathbb{Z}, G) \to G$ by $f \mapsto f(1)$ so $Hom(\mathbb{Z}, G) \cong G$. Hence the map $\tilde{\partial}_0^{\#}$ can be described as $\tilde{\partial}_0^{\#}(f_g) = f_g \tilde{\partial}_0$

for a homomorphism $f_g: \mathbb{Z} \to G, f_g(1) = g$ for some $g \in G$, which maps a *0*-chain $\sigma_0 = k_1 a_1 + \cdots k_r a_r$ to $k_1 g + \cdots k_r g \in G$. Similarly given a continuous function $f: X \to Y$ we have the induced augmented chain map

$$\tilde{f}_{\#}^{\#} := \begin{cases} f_{\#}^{\#} & q \in \mathbb{N} \\ 1_G & q = -1 \end{cases}$$

Definition 4.11. Given a topological space X. The ***reduced singular cohomology groups with coefficient*** $\boldsymbol{G}$ is defined by $\widetilde{H}^*(X; G) := \left\{H^*\left(Hom(\tilde{S}_*(X), G)\right)\right\}$. Given a continuous $f: X \to Y$ between two topological spaces, the ***induced homomorphism between reduced cohomology groups*** f^* is defined by the induced homomorphism from the augmented cochain map $\tilde{f}_{\#}^{\#}: \tilde{S}^*(Y; G) \to \tilde{S}^*(X; G)$.

Definition 4.12. Given a topological space X and a subspace $A \subset X$. For a fixed abelian group G, we define the ***relative cohomology groups with coefficients*** $\boldsymbol{G}$ by $H^*(X, A; G) := H^*(Hom(S_*(X, A), G))$. Given a morphism $f: (X, A) \to (Y, B)$ between two objects in $\mathbf{Top^2}$ the ***induced homomorphism between cohomology groups*** $f^*: H^*(Y, B; G) \to H^*(X, A; G)$ is defined by the induced homomorphism from the relative chain map $f_{\#}^{\#}: S^*(Y, B; G) \to S^*(X, A; G)$.

When $A = \emptyset$ we have $H^*(X, \emptyset; G) = H^*(X; G)$.

Note that for each dimension q, $H^q(X, A; G) = H_{-q}(Hom(S_*(X, A), G))$. Since we defined the relative cohomology in a way that is similar to relative homology, in which there is a split exact sequence of complexes

$$0 \to S_*(A, \emptyset) \overset{i_\#}{\to} S_*(X, \emptyset) \overset{j_\#}{\to} S_*(X, A) \to 0$$

Since the hom functor preserves the split exact property, applying the functor on the sequence we will get a split exact sequence

$$0 \leftarrow Hom(S_*(A, \emptyset), G) \overset{i_\#^\#}{\leftarrow} Hom(S_*(X, \emptyset), G) \overset{j_\#^\#}{\leftarrow} Hom(S_*(X, A), G) \leftarrow 0$$

Then from Theorem 3.60 there is a long exact sequence

$$0 \to H_0(X,A;G) \xrightarrow{i^*} H_0(X;G) \xrightarrow{j^*} H_0(A;G) \xrightarrow{d^0} H_1(X,A;G) \xrightarrow{i^*} \cdots$$

And $d^q: H_q(A;G) \to H_{q+1}(X,A;G)$ is the connecting homomorphism.

Definition 4.13. $H^*(-;G): \mathbf{Top^2} \to \mathbf{GradedG}$ is a contravariant functor called the ***cohomology functor*** for topological pairs.

Look back to the Kronecker product $< -,- >: Hom(F,B) \times F \to B; < \Phi, a > \mapsto \Phi(a), \forall \Phi \in Hom(F,B), \ \forall a \in F$. If we replace F, B by $S_q(X,A), G$ respectively, we have $< -,- >: S^q(X,A;G) \times S_q(X,A) \to G; < \varphi, c_q > \mapsto \varphi(c_q), \forall \varphi \in S^q(X,A;G)$, $\forall c_q \in S_q(X,A)$. Note that the homomorphism on a free abelian group is determined by its values on base elements. We actually have a commutative diagram

$$S^{q+1}(X,A;G) \times S_{q+1}(X,A) \xrightarrow{<-,->} G$$

$$\uparrow \delta^q \qquad \partial_q \downarrow \qquad 1_G \downarrow$$

$$S^q(X,A;G) \times S_q(X,A) \xrightarrow{<-,->} G$$

i.e. $< \delta^q(c^q), c_{q+1} > = < c^q, \partial_q(c_{q+1}) >, \forall c^q \in S^q(X,A;G), c_q \in S_q(X,A)$

Moreover, given a continuous $f: X \to Y$. For $\forall \varphi \in S^q(Y;G), c_q \in S_q(X)$ we have $< f_\#^\#(\varphi), c_q > = < \varphi, f_\#(c_q) >$. When $c_q = \sigma_q: \Delta^q \to X$, we have in particular $< f_\#^\#(\varphi), \sigma_q > = < \varphi, f(\sigma_q) >$

$$S^q(Y;G) \times S_q(Y) \xrightarrow{<-,->} G$$

$$\downarrow f_\#^\# \quad f_\# \uparrow \quad 1_G \downarrow$$

$$S^q(X;G) \times S_q(X) \xrightarrow{<-,->} G$$

4.2 Axiom of Cohomology

Axiom of cohomology. A cohomology theory is a contravariant functor from $\mathbf{Top^2}$ to **GradedG** consists of three functions. For each $q \in \mathbb{N}$ it assigns a topological pair (X,A) an abelian group $H^q(X,A;G)$, a continuous $f: (X,A) \to (Y,B)$ a homomorphism

$(f^*)^q: H^q(Y,B;G) \to H^q(X,A;G)$ and a topological pair (X,A) a coboundary homomorphism $d^q: H^q(A) \to H^{q+1}(X,A)$, which satisfy the following properties:

(1) (Identity Law). Given an identity map $1_{(X,A)}$, we have an identity homomorphism $1^*_{H^*(X,A;G)}$.

(2) (Composition Law). $(gf)^* = f^*g^*$ for defined composition.

(3) (Naturality). Given $f: (X,A) \to (Y,B)$, we have a commutative diagram

$$H^{q+1}(X,A;G) \overset{d^q}{\leftarrow} H^q(A;G)$$

$$f^* \uparrow \qquad (f|_A)^* \uparrow$$

$$H^{q+1}(Y,B;G) \overset{d^q}{\leftarrow} H^q(B;G)$$

(4) (Exactness Axiom). We have the long exact sequence

$$\cdots \to H_q(X,A;G) \overset{i^*}{\to} H_q(X;G) \overset{j^*}{\to} H_q(A;G) \overset{d^q}{\to} H_{q+1}(X,A;G) \overset{i^*}{\to} \cdots$$

(5) (Homotopy Axiom). Given two homotopic maps $f \simeq g: (X,A) \to (Y,B)$, we have $f^* = g^*$.

(6) (Excision Axiom). Given an open subspace $W \subset X$ with $\overline{W} \subset A^\circ$ and an inclusion map $i: (X-W, A-W) \to (X,A)$. Then the inclusion induces an isomorphism $i^*: H^*(X,A) \overset{\cong}{\to} H^*(X-W, A-W)$.

(7) (Dimension Axiom). The cohomology group of a one-point space $\{pt\}$ is

$$H^q(\{pt\};G) = \begin{cases} G & q=0 \\ 0 & q \neq 0 \end{cases}$$

4.3 Cup Product and Ring Structure

Definition 4.14. A ***graded ring*** is a ring R with additive subgroups $R^n (n \in \mathbb{N})$ such that $R \cong \bigoplus_{n\in\mathbb{N}} R^n$ and $R^nR^m \subset R^{n+m}$ for $n,m \in \mathbb{N}$.

The second condition means $xy \in R^{n+m}$ for $x \in R^n, y \in R^m$.

Definition 4.15. Given a graded ring $R = \bigoplus_{n\in\mathbb{N}} R^n$. An element $x \in R$ has ***degree n*** if $x \in R^n$ and such elements are called ***homogeneous***.

So the 0 element has degree *n* for every $n \in \mathbb{N}$ since $0r_k = 0 \in R^k\ \forall k \in \mathbb{N}$. The identity element 1 has degree 0 since $1 = e_0 + \cdots + e_k, e_i \in R^i$ and $1a_n = a_n = e_0a_n + \cdots + e_ka_n \in R^n \cap (R^n \oplus \cdots \oplus R^{n+k}) = R^n$ for $a_n \in R^n$. So we have $a_n = e_0a_n$ and $e_ia_n = 0, i > 0$, which implies $a = e_0a, \forall a \in R$. Similarly, we exchange the position and get $a = ae_0, \forall a \in R$ so $e_0 = 1 \in R^0$.

Definition 4.16. Given $i = 0,1,2,\cdots,d \in \mathbb{N}$, we define continuous functions ***front face*** λ_i and ***back face*** μ_i from Δ^i to Δ^d by $\lambda_i\colon (e_0,\cdots,e_i) \mapsto (e_0,\cdots,e_i,0,\cdots,0)$ and $\mu_i\colon (e_0,\cdots,e_i) \mapsto (0,\cdots,0,e_0,\cdots,e_i)$.

Lemma 4.17.

(1) Given the *i*th face map $\varepsilon_i^{d+1}\colon \Delta^d \to \Delta^{d+1}$ (recall that the face map $\varepsilon_i^{d+1}(e_0,\cdots,e_d) \mapsto (e_0,\cdots e_{i-1},0,e_i,\cdots,e_d)$), we have $\mu_d^{d+1} = \varepsilon_0^{d+1}$ and $\lambda_d^{d+1} = \varepsilon_{d+1}^{d+1}$.

(2) $\mu_{m+k}^d\mu_k^{m+k} = \mu_k^d$, $\lambda_{n+m}^d\lambda_n^{n+m} = \lambda_n^d$ and $\mu_{m+k}^{n+m+k}\lambda_m^{m+k} = \lambda_{n+m}^{n+m+k}\mu_m^{n+m}$.

(3) $\varepsilon_i^{d+1}\lambda_p^d = \begin{cases} \lambda_{p+1}^{d+1}\varepsilon_i^{p+1} & i \le p \\ \lambda_p^{d+1} & i \ge p+1 \end{cases}$, $\varepsilon_i^{d+1}\mu_q^d = \begin{cases} \mu_q^{d+1} & i \le d-q \\ \mu_{q+1}^{d+1}\varepsilon_{i+q-d}^{q+1} & i \ge d-q+1 \end{cases}$

Proof: It can be verified by calculation.

■

We can remember the formula in a way that is similar to quotient!

So since $S^q(X;G) = Hom(S_q(X),G)$ is the group of homomorphisms from $S_q(X)$ to G. An element $\varphi \in S^q(X;G)$ is a homomorphism defined on the free abelian group $S_q(X)$ whose basis is the class of all singular *q*-complexes in X. So φ is determined by its values $<\varphi,\sigma>$ on base elements $\sigma\colon \Delta^q \to X$.

Definition 4.18. Given a topological space X and a commutative ring R. If we have $\varphi \in S^n(X;R) = Hom(S_n(X), R)$ and $\theta \in S^m(X;R) = Hom(S_m(X), R)$, then define their ***cup product*** $\varphi \cup \theta \in S^{n+m}(X;R)$ by

$$< \varphi \cup \theta, \sigma > = < \varphi, \sigma\lambda_n^{n+m} >< \theta, \sigma\mu_m^{n+m} >, \forall\sigma \in S_{n+m}(X)$$

Theorem 4.19. Given a topological space X and a commutative ring R, we have $S^*(X;R) = \bigoplus_{n\in\mathbb{N}} S^n(X;R)$ a graded ring under cup product.

Proof: The (left)distributivity: $\varphi \in S^n(X;R), \theta, \Psi \in S^m(X;R), \sigma: \Delta^{n+m} \to X$.

$$\begin{aligned} < \varphi \cup (\theta + \Psi), \sigma > &= < \varphi, \sigma\lambda_n^{n+m} >< \theta + \Psi, \sigma\mu_m^{n+m} > \\ &= < \varphi, \sigma\lambda_n^{n+m} > (< \theta, \sigma\mu_m^{n+m} > + < \Psi, \sigma\mu_m^{n+m} >) \\ &= < \varphi \cup \theta, \sigma > + < \varphi \cup \Psi, \sigma > \end{aligned}$$

The associativity: $\varphi \in S^n(X;R), \theta \in S^m(X;R), \Psi \in S^k(X;R), \sigma: \Delta^{n+m+k} \to X$.

$$\begin{aligned} &< \varphi \cup (\theta \cup \Psi), \sigma > \\ &= < \varphi, \sigma\lambda_n^{n+m+k} >< \theta \cup \Psi, \sigma\mu_{m+k}^{n+m+k} > \\ &= < \varphi, \sigma\lambda_n^{n+m+k} >< \theta, \sigma\mu_{m+k}^{n+m+k}\lambda_m^{m+k} >< \Psi, \sigma\mu_{m+k}^{n+m+k}\mu_k^{m+k} > \\ &= < \varphi, \sigma\lambda_{n+m}^{n+m+k}\lambda_n^{n+m} >< \theta, \sigma\lambda_{n+m}^{n+m+k}\mu_m^{n+m} >< \Psi, \sigma\mu_k^{n+m+k} > \\ &= < (\varphi \cup \theta) \cup \Psi, \sigma > \end{aligned}$$

The identity element $e \in S^0(X;R)$ is defined by $< e, x > = 1, \forall x \in S_0(X)$ i.e. points in X since the basis of $S_0(X)$ are points in X. So by the cup product $< e \cup f, \sigma > = < e, \sigma\lambda_0^q >< f, \sigma\mu_q^q > = < f, \sigma >$ it is a left identity. Similarly, it is a right identity hence a two-sided identity.

■

Actually for a fixed commutative ring, $S^*(-;R)$ is a contravariant functor from **Top** to **GradedR** (the category of graded rings and ring homomorphisms).

Lemma 4.20. Given a continuous map $f: X \to Y$, then we have the identity $f_\#^\#(\varphi \cup \theta) = f_\#^\#(\varphi) \cup f_\#^\#(\theta), \forall\varphi \in S^p(Y;R), \theta \in S^q(Y;R)$. And $f_\#^\#$ maps the unit e_y in $S^*(Y;R)$ to unit e_x in $S^*(X;R)$.

Proof: For every $\sigma_{p+q} \in \Delta^{p+q} \to X$,

$$< f_{\#}^{\#}(\varphi \cup \theta), \sigma >=< \varphi \cup \theta, f(\sigma_{p+q}) >$$

$$=< \varphi, f(\sigma_{p+q})\lambda_p^{p+q} >< \theta, f(\sigma_{p+q})\mu_q^{p+q} >$$

$$=< f_{\#}^{\#}(\varphi), \sigma\lambda_p^{p+q} >< f_{\#}^{\#}(\theta), \sigma_{p+q}\mu_q^{p+q} >$$

$$=< f_{\#}^{\#}(\varphi) \cup f_{\#}^{\#}(\theta), \sigma >$$

And $< f_{\#}^{\#}(e_y), x >=< e_y, f(x) >= 1$.

■

Theorem 4.21. For a fixed commutative ring R, $S^*(-;R)$ is a contravariant functor from **Top** to **GradedR**.

Proof: From Theorem 4.19 and Lemma 4.20.

■

However, we normally consider another kind of commutative graded rings which inherits the algebraic structure from $S^*(X;R) = \bigoplus_{n\in\mathbb{N}} S^n(X;R)$ and satisfies the homotopy axiom.

Lemma 4.22. Given a topological space X and a fixed commutative ring R, we have $\delta^{p+q}(\varphi \cup \theta) = \delta^p(\varphi) \cup \theta + (-1)^p \varphi \cup \delta^q(\theta), \forall \varphi \in S^p(X;R), \theta \in S^q(X;R)$.

Proof: Refer to Lemma 12.22, page 394, An Introduction to Algebraic Topology[8].

■

Theorem 4.23. Given a commutative ring R, $H^*(-;R) = \bigoplus_{q\in\mathbb{N}} H^q(-;R)$ is a contravariant functor from **hTop** to **GradedR**.

Proof: Refer to Theorem 12.23, page 395, An Introduction to Algebraic Topology[8].

■

5. Algebraic Topology in Classifying Real Vector Bundles

5.1 Basic Definitions

The ultimate goal of this chapter is to see how to classify all the n-dimensional vector bundles over the same base space. The classifying problem seems purely geometric but somehow can be connected to homotopy classes, which is a concept in topology. The content in this chapter mainly follows "Vector Bundles and K-Theory"[4].

Definition 5.1. An ***n-dimensional real vector bundle*** (E, p, B) is a continuous function $p: E \to B$ between two topological spaces E, B such that for each $b \in B$ the inverse image $p^{-1}(b) \subset E$ has an algebraic structure as a vector space and the property of local triviality holds: There is an open cover $\{U_\alpha\}$ of B such that for each of the open set U_α there is a homeomorphism $h_\alpha: p^{-1}(U_\alpha) \approx U_\alpha \times \mathbb{R}^n$ assigning $b \in U_\alpha$ a vector space $h_\alpha(p^{-1}(b)) \cong \{b\} \times \mathbb{R}^n$ by an vector space isomorphism.

We call such h_α ***local trivialization*** of the vector bundle. The topological space E is called ***total space*** and B is called ***base space***. Vector bundle is actually a special case of fiber bundle by replacing the fiber by the real vector space. That is the reason why the vector spaces $p^{-1}(b)$ are called ***fibers***. The vector bundle $E = B \times \mathbb{R}^n$ is called the ***trivial bundle*** and the map $p: E \to B$ simply maps elements in E to its first coordinate.

Example 5.2. The Mobius strip is also a 1-dimensional real vector bundle.

Definition 5.3. Given two n-dimensional real vector bundles $(E, p, B), (E', p', B')$, a ***vector bundle map*** f between them is a pair of continuous functions (f_1, f_2), $f_1: E \to E', f_2: B \to B'$ with the commutative property $f_2 p = p' f_1$ such that $p_1|_{p^{-1}(b)}: p^{-1}(b) \to p'^{-1}(f_2(b))$ is an isomorphism between vector spaces.

Definition 5.4. Given two n-dimensional real vector bundles $(E, p, B), (E', p', B')$. They are called ***isomorphic*** when we can find vector bundle maps

$f = (f_1, f_2): (E, p, B) \to (E', p', B')$ and $g = (g_1, g_2): (E', p', B') \to (E, p, B)$ such that $g_1 f_1 = 1_E$ and $f_1 g_1 = 1_{E'}$. We denote two isomorphic vector bundles by $(E, p, B) \cong (E', p', B')$.

Note that we call two vector bundles over the same base space B isomorphic if the conditions above are satisfied when we take f_2, g_2 to be identities. And the set of isomorphism classes of n-dimensional real vector bundles is denoted by $Vect_{\mathbb{R}}^n(B)$.

Proposition 5.5. Given an n-dimensional real vector bundle (E, p, B). Let A be any topological space and $f: A \to B$ an arbitrary continuous map. We naturally constructed a vector bundle (D, ρ, B) called the ***induced bundle by f*** or ***the pull-back of (E, p, B) by f*** and a vector bundle map $\hat{f}: (D, g, B) \to (E, p, B)$ as illustrated: $D = \{(a, e) \in A \times E | f(a) = p(e)\}, \rho(a, e) = a;\ \hat{f} = (\tilde{f}, f)$ where $\tilde{f}(a, e) = e$. We denote the constructed bundle by $f^*(E, p, B)$ and it is unique up to isomorphism.

Proof: Refer to Proposition 1.5, page 18, Vector Bundles and K-Theory[4].

■

Before we get to the next theorem we need to introduce a definition called ***paracompact***. A Hausdorff space X is called paracompact if for each open cover $\{U_\alpha\}$ there is a set of continuous functions $\{\varphi_\alpha: X \to I\}$ such that $supp(\varphi_\alpha) = \overline{\{x \in X | \varphi_\alpha(x) \neq 0\}} \subset U_\alpha$ and for every point $x \in X$ there is a neighborhood of it where only finite number of φ_αs are nonzero and $\sum_\alpha \varphi_\alpha = 1$.

Theorem 5.6. Given a vector bundle (E, p, B) and two homotopic maps $f_1 \simeq f_2: A \to B$. The pull-back bundles are isomorphic $f_1^*(E, p, B) \cong f_2^*(E, p, B)$ if A is compact Hausdorff or more generally paracompact.

Proof: Refer to Theorem 1.6, page 20, Vector Bundles and K-Theory[4].

■

5.2 Grassmann Manifolds and Conclusion

We all have encountered the *n*-dimensional real projective space, which is the space of all lines through the origin in $\mathbb{R}^{n+1}$. The Grassmann manifolds can be regarded as a generalization of real projective spaces as the ideas are very similar. And the reason why "manifold" appears is because it can be equipped with the structure of an *n(n-k)*-dimensional smooth manifold. Eventually, we will see the final conclusion that there is a bijection between $[X, BO(n)]$ and $Vect_{\mathbb{R}}^{n}(X)$.

Definition 5.7. Given two numbers $n, k \in \mathbb{N}^*$ and $n > k$. The Grassmann manifold is defined as $G_k(\mathbb{R}^n) := \{k\text{-dimensional linear subspaces of } \mathbb{R}^n\}$.

From the definition we can see that $\mathbb{R}\mathrm{P}^n$ is exactly $G_1(\mathbb{R}^{n+1})$. Now actually from Grassmann manifolds we can construct a kind of vector bundles which has the property that all vector bundles over paracompact spaces can be regarded as pull-backs of such bundles.

Definition 5.8. Given a Grassmann manifold $G_k(\mathbb{R}^n)$. We define the ***canonical bundles over Grassmann manifolds***, which is a subset of the product space $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ by $E_k(\mathbb{R}^n) := \{(l, v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n | v \in l\}$.

One can think of $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$. Those subsets of $\mathbb{R}^n$ can also be regarded as subsets of $\mathbb{R}^{n+1}$. We have the following inclusion sequence of spaces: $G_k(\mathbb{R}^n) \subset G_k(\mathbb{R}^{n+1}) \subset G_k(\mathbb{R}^{n+2}) \subset G_k(\mathbb{R}^{n+3}) \subset \cdots$ and define ***the classifying space*** $G_k(\mathbb{R}^\infty) = \cup_n G_k(\mathbb{R}^n)$. Moreover we also have a similar sequence of spaces $E_k(\mathbb{R}^n) \subset E_k(\mathbb{R}^{n+1}) \subset E_k(\mathbb{R}^{n+2}) \subset E_k(\mathbb{R}^{n+3}) \subset \cdots$ and define ***the universal bundle*** $E_k(\mathbb{R}^\infty) = \cup_n E_k(\mathbb{R}^n)$ (We do not denote it by $(E_k(\mathbb{R}^\infty), p, G_k(\mathbb{R}^\infty))$ for convenience).

Proposition 5.9. The projection $p: E_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), p(l, v) = l$ defines a vector bundle $(E_k(\mathbb{R}^n), p, G_k(\mathbb{R}^n))$ for both finite and infinite n.

Proof: Refer to Lemma 1.15, page 28, Vector Bundles and K-Theory[4].

■

Theorem 5.10. Given a paracompact space X, we have a bijection $[X, G_n(\mathbb{R}^\infty)] \to Vect^n_{\mathbb{R}}(X), f \mapsto f^*(E_k(\mathbb{R}^\infty), p, G_n(\mathbb{R}^\infty))$. i.e. a one-to-one correspondence between the set of all homotopy classes of continuous functions from X to $G_n(\mathbb{R}^\infty)$ and the set of all isomorphism classes of *n*-dimensional real vector bundles over X.

Proof: Refer to Theorem 1.16, page 29, Vector Bundles and K-Theory[4].

■

Because of the theorem above, $G_n(\mathbb{R}^\infty)$ is called the classifying space and $E_k(\mathbb{R}^n)$ is called the universal bundle. There is also a notation denotes $G_n(\mathbb{R}^\infty)$ as $BO(n)$ $:= \lim_{N\to\infty} G_n(\mathbb{R}^{n+N})$, which is related to Lie groups and will not be introduced formally due to the limited knowledge of the writer. So $[X, G_n(\mathbb{R}^\infty)]$ can be written as $[X, BO(n)]$. Now the problem of classifying all *n*-dimensional real vector bundles over X has been transformed to the problem of understanding the set of all homotopy classes of continuous functions from X to $BO(n)$. When making such huge step from geometry to algebra we should ask ourselves "Have we simplified the problem?" and "Have we lost any information?". The answers should be "Note that in this translation we have not lost any information, but nor we have made our problem much easier."[2] Now we will see how cohomology will lead us.

Recall that a continuous function $f: X \to BO(n)$ will induce a homomorphism between cohomology groups $f^*: H^*(BO(n); G) \to H^*(X; G)$ for an arbitrary convenient coefficient group G. Now in order to classify the bundles we need to understand $[X, BO(n)]$, as the constant map and maps that are homotopic to constant map all induce the 0 homomorphism and a constant map corresponds to the trivial bundle, we have the trivial part of it. So how can we detect whether a map is null-homotopic or not? Of course if a homomorphism is not trivial $f^*(c) \neq 0 \in H^*(X; G)$ for some $c \in H^*(BO(n); G)$ we know such $f: X \to BO(n)$ is not trivial and not null-homotopic and so is the pull-back bundle of it. Actually we have an invariant depends only on the homotopy class of f called the total Stiefel-Whitney class, which is an element in the mod 2 cohomology ring $H^*(X; G)$.

5.3 Stiefel-Whitney Classes

As our question is whether a map is null-homotopic, which is equivalent to whether the corresponding vector bundle is trivial. With the help of Stiefel-Whitney classes we can see that the null-homotopic maps will give us the trivial Stiefel-Whitney classes. We approach the Stiefel-Whitney classes in an axiomatic way that speeds up a bit but at the price of its constructions.

Axioms 5.11. There is a unique sequence of functions $w_1, w_2, \cdots$ that gives every real vector bundle (E, p, B) a class $w_i(E, p, B) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$, which depends on the isomorphism type of (E, p, B) such that

(1). $w_i(f^*(E)) = f^*(w_i(E))$ for a pull-back bundle $f^*(E, p, B)$.

(2). $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$ where $w = 1 + w_1 + w_2 + \cdots \in H^*(B; \mathbb{Z}/2\mathbb{Z})$.

(3). $w_i(E) = 0$ if $i > dimE$.

(4). For the canonical line bundle $E \to \mathbb{R}\mathrm{P}^\infty$, $w_1(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$.

Note that ***the total Stiefel-Whitney class*** is the sum

$$w(E, p, B) = 1 + w_1(E, p, B) + w_2(E, p, B) \cdots$$

So given a vector bundle (E, π, X) and if the map $f: X \to BO(n)$ to be detected is actually null-homotopic. We have $(E, \pi, X) = f^*(E_n(\mathbb{R}^\infty), p, BO(n))$ then by the axiom 1 we have

$$w_i(E, \pi, X) = w_i\left(f^*(E_n(\mathbb{R}^\infty), p, BO(n))\right) = f^*\left(w_i(E_n(\mathbb{R}^\infty), p, BO(n))\right)$$

Since $f^*: H^*(BO(n); \mathbb{Z}/2\mathbb{Z}) \to H^*(X; \mathbb{Z}/2\mathbb{Z})$ is trivial as an induced homomorphism from a null-homotopic map $f: X \to BO(n)$, we have $w_i(E, \pi, X)$ trivial in the mod 2 cohomology group. Moreover, knowing that $w_i(E, \pi, X)$ trivial for each i we can also conclude that f is null-homotopic because of the fact that $H^*(BO(n), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})\left[w_1\left(E_n(\mathbb{R}^\infty), p, BO(n)\right), \cdots, w_n\left(E_n(\mathbb{R}^\infty), p, BO(n)\right)\right]$ the polynomial ring.

6. Computational Methods

6.1 Cell Complexes and Cellular Homology

In this section we study the basic definitions related to the CW-complex and construction of spaces. Then cellular homology will be introduced as a computational tool for homology groups of CW-complexes.

Definition 6.1. A ***CW-complex*** X is a topological space with a ***CW-decomposition***, which is an ascending chains of closed subspaces $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots, \bigcup_{q=0}^{\infty} X^q = X$ such that (1) X^0 is a discrete topological space. (2) X^q (***the q-skeleton of X***) is formed by attaching ***q-cells*** $e_i^q \approx (D^q)^\circ$ (topological spaces homeomorphic to the q-dimensional open disk $D^q - S^{q-1} = \{x \in R^q \,|\, \|x\| < 1\}$) to the *(q-1)*-skeleton X^{q-1} via the ***attaching maps*** $\varphi_{i\in I}^q : S_{i\in I}^{q-1}(S^{q-1} \approx \partial D^q) \to X^{q-1}$. So X^n is the quotient space $X^{q-1} \coprod_{i\in I} D_i^q \Big/ {x \sim \varphi_i^q(x)}, x \in S_i^{q-1} \approx \partial D_i^q$. As a set it is the disjoint union $X^{q-1} \coprod_{i\in I} e_i^q$.

Note that for each q-cell e_i^q there is a ***characteristic map*** $\phi_i^q : D_i^q \to X^q$ such that $\phi_i^q|_{e_i^q}$ is a homeomorphism and $\phi_i^q|_{S_i^{q-1}} = \varphi_i^q$. (3)$X$ is given the ***weak topology***. i.e. a subset $A \subset X$ is closed if and only if $A \cap X^q$ is closed in X^q for each $q \in \mathbb{N}$. (4) If $X = X^q$ or $q < \infty$ then we call it a ***finite-dimensional CW-complex***, otherwise an ***infinite-dimensional CW-complex.***

Note that (1) CW-decomposition is not unique. (2) when $q < \infty$ the CW-complex is compact Hausdorff (for any two distinct points there are disjoint neighborhoods of them).

(3) a CW-complex of any dimension is always normal (for any two disjoint closed subsets there are disjoint neighborhoods of them) and any compact subset only intersects with finite many of cells (so the closure $\overline{e_i^q}$ of a cell only intersects with finite many of cells. This property is called closure finite property, together with the Weak topology property they form the name CW-complex). (4) a q-cell of a q-dimensional CW-complex is open in X^q since it intersects $e^q \cap X^k = \emptyset, k < q$. And $e^q \cap X^q = e^q$ with the openness preserved by homeomorphism.

Definition 6.2. Given two CW-complexes X, Y with cells $\{e_i^m\}, \{e_j^n\}$. Their ***product space*** $X \times Y$ has cells $\{e_i^m \times e_j^n\}$, skeletons $(X \times Y)^r = \bigcup_{n+m=r} X^n \times X^m$. The characteristic map for a cell $e_i^m \times e_j^n$ is $\phi_i^m \times \phi_j^n: D_i^m \times D_j^n \to X \times Y$

When at least one of X, Y is a finite-dimensional CW-complex their product complex has the product topology.

Definition 6.3. Given a CW-complex X and its closed subspace $A \subset X$. A is called a ***subcomplex*** of X if it is a union of cells of X. It is CW decomposition is described by $A^k = A \cap X^k$.

Note that since A is closed, the closure of each cells in A is still in A.

Definition 6.4. Given a CW-complex X and its subcomplex $A \subset X$. We call (X, A) a ***CW pair***.

Example 6.5. We can describe S^n with two CW decompositions. One is to consider S^n as gluing the boundary of e^n to a single point e^0 by the constant map $\partial e^n \to e^0$ hence $S^n = e^0 \cup e^n$. But in this way we cannot find the corresponding natural inclusion $S^0 \subset S^1 \subset S^2$ in subcomplex language. So in another decomposition S^n is obtained inductively by gluing two n-cells to the equator S^{n-1}. In this way $S^n = 2e^0 \cup 2e^1 \cup$

$2e^2 \cup \cdots \cup 2e^n$. And in this way we have $S^k (k \leq n)$ a subcomplex of S^n.

Example 6.6. The real projective plane $\mathbb{R}\mathrm{P}^{\mathrm{n}}$ can be described as the quotient space of S^n with antipodal points identified, which is exactly $D^n/\sim$ with the antipodal points on ∂D^n identified (since every point on the upper hemisphere of $S^n/\sim$ is identified with a point on the lower hemisphere and the upper hemisphere is homeomorphic to D^n. Then identifying points on equator is the same as doing it for ∂D^n). And identifying antipodal points of $\partial D^n = S^{n-1}$ is exactly $\mathbb{R}\mathrm{P}^{\mathrm{n-1}}$. So by attaching an *n*-cell to $\mathbb{R}\mathrm{P}^{\mathrm{n-1}}$ by the quotient map $\varphi^n : S^{n-1} \to S^{n-1}/\sim$ we get $\mathbb{R}\mathrm{P}^{\mathrm{n}}$. In this way $\mathbb{R}\mathrm{P}^{\mathrm{n}} = e^0 \cup e^1 \cup \cdots \cup e^n$.

Example 6.7. The complex projective plane $\mathbb{C}\mathrm{P}^{\mathrm{n}}$ can be described as the quotient space of S^{2n+1} under the equivalence relation $v \sim \lambda v, \lambda \in \mathbb{C}, |\lambda| = 1$, which is the quotient space of the disk D^{2n} under the identification $v \sim \lambda v, \lambda \in \mathbb{C}, |\lambda| = 1$ for all $v \in \partial D^{2n} \approx S^{2n-1}$ (since for a point $z = (z_1, \cdots, z_{n+1}) \in S^{2n+1}$ with $z_{n+1} \neq 0$ we can always multiply it by a $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $\lambda z = (\lambda z_1, \cdots, \lambda z_{n+1})$ with $\lambda z_{n+1} > 0$ and such points are in the same equivalence class, whose representative element $w = (w_1, \cdots, w_{n+1})$ with the last coordinate larger than 0 actually is a unique point $w = (w_1, \cdots, w_{n+1}) \in \mathbb{C}^n \times \mathbb{C}$ such that $w_{n+1} = \sqrt{1 - \sum_{i=1}^{n} w_i}$. And $(w_1, \cdots, w_n, 0) \in D^{2n}$ since $\sum_{i=1}^{n} w_i < 1$. For those equivalence classes with the last coordinate zero, for example $[z'] = \{z = (z_1, \cdots, z_n, 0) \in S^{2n+1} | z \sim z'\}$, we have $(z_1, \cdots, z_n, 0) \in \partial D^{2n}$ since $\sum_{i=1}^{n} z_i = 1$). Moreover, $S^{2n-1} \subset \mathbb{C}^n$ with such equivalence relation is $\mathbb{C}\mathrm{P}^{\mathrm{n-1}}$. Hence we have a CW decomposition $\mathbb{C}\mathrm{P}^{\mathrm{n}} = e^0 \cup e^2 \cup \cdots \cup e^{2n}$.

Definition 6.8. Given a CW pair (X, A) we define their ***quotient*** X/A with cells of $X - A$ together with a *0*-cell (collapsing A to a point). For a cell e_α^q in $X - A$ we have its attaching map $\varphi_\alpha^q : S_\alpha^{q-1} \to X^{q-1}$, now its attaching map in X/A is the composition of

$$S^{q-1} \xrightarrow{\varphi_\alpha^q} X^{q-1} \xrightarrow{p} X^{q-1}/A^{q-1}.$$

Definition 6.9. Given a topological space X, we define its ***suspension*** $SX = X \times I/\sim$ with the equivalence relation $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for all $x_1, x_2 \in X$. Intuitively we collapse $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Lemma 6.10. Given a CW-complex X, we have

(1) $H_k(X^q, X^{q-1}) = 0$ for $k \neq q$ and is free abelian if $k = q$ with a basis in one-to-one correspondence with the q-cells of X.

(2) $H_k(X^q) = 0$ for $k > q$ (so if X is finite-dimensional we have $H_k(X) = 0$ for all $k > \dim(X)$).

(3) There is an isomorphism $i^*: H_k(X^q) \xrightarrow{\cong} H_k(X)$ induced by the inclusion map $i: X^q \to X$ for $k < q$. The induced homomorphism is an epimorphism if $k = q$.

Proof: Refer to Lemma 2.34, page 137-139, Algebraic Topology[3].

■

Recall that the long exact sequence for the pairs $(X^{q+1}, X^q), (X^q, X^{q-1})$ and (X^{q-1}, X^{q-2}) are respectively

$$\cdots \to H_{q+1}(X^{q+1}, X^q) \xrightarrow{\partial_{q+1}^*} H_q(X^q) \xrightarrow{i_q} H_q(X^{q+1}) \to H_q(X^{q+1}, X^q) = 0 \to \cdots$$

$$\cdots \to 0 \to H_q(X^q) \xrightarrow{j_q} H_q(X^q, X^{q-1}) \xrightarrow{\partial_q^*} H_{q-1}(X^{q-1}) \to \cdots$$

$$\cdots \to 0 \to H_{q-1}(X^{q-1}) \xrightarrow{j_{q-1}} H_{q-1}(X^{q-1}, X^{q-2}) \to \cdots$$

We define $d_{q+1} = j_q \partial_{q+1}^*$ and $d_q = j_{q-1} \partial_q^*$ and get a sequence of homology groups

$$\cdots \to H_{q+1}(X^{q+1}, X^q) \xrightarrow{d_{q+1}} H_q(X^q, X^{q-1}) \xrightarrow{d_q} H_{q-1}(X^{q-1}, X^{q-2}) \to \cdots$$

And it is actually a chain complex $\{H_q(X^q, X^{q-1}), d_q\}$ called ***cellular chain complex*** since $d_q d_{q+1} = d_{q+1} = j_{q-1} \partial_q^* j_q \partial_{q+1}^* = 0$ since $\partial_q^* j_q = 0$ in the long exact sequence of (X^q, X^{q-1}). So one can have $H_q(X^{q+1}) \cong H_q(X)$ by (3) in previous lemma then by the long exact sequence of (X^{q+1}, X^q) we know $im\partial_{q+1}^* = keri_q$ and the fact that i_q is surjective so we are able to conclude that $H_q(X) \cong H_q(X^q)/im\partial_{q+1}^*$ by $H_q(X^q)/keri_q \cong imi_q$ (First Theorem of Isomorphism). Moreover, the map j_q also induces an isomorphism from $H_q(X^q)/im\partial_{q+1}^*$ to $kerd_q/imd_{q+1}$ because it is injective (from the long exact sequence) and maps $im\partial_q^*$ isomorphically onto $im(j_q\partial_{q+1}^*) = imd_{q+1}$ and $H_q(X^q)$ isomorphically onto $imj_q = ker\partial_q^*$. And j_{q-1} as an injection has the property that $kerd_q = ker(j_{q-1}\partial_q^*) = ker\partial_q^*$.

Definition 6.11. The homology groups of the cellular chain complex are called cellular homology groups of X and denoted by $H_n^{CW}(X)$ and by discussion above we have $H_n^{CW}(X) \cong H_n(X)$.

So we can have few quick applications: (1) $H_q(X) = 0$ if X has no q-cells. (2) If X has no two of its cells in adjacent dimensions, then $H_q(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X.

To compute the cellular boundary map d_n we first identify e_α^n with the basis element of the corresponding cellular homology group. There is a formula called Cellular Boundary Formula which states the relationship between boundary map and degree of the composition of some maps. The proof of the formula is written on page 140-141, Algebraic Topology[3].

Cellular Boundary Formula. $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$. Here $d_{\alpha\beta}$ means the degree of the map $S_\alpha^{n-1} \xrightarrow{\varphi_\alpha^n} X^{n-1} \xrightarrow{q} S_\beta^{n-1}$, where φ_α^n is the attaching map and q is the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

Example 6.12. We can apply the application (2) to the $\mathbb{C}P^n$ which has CW decomposition $e^0 \cup e^2 \cup \cdots \cup e^{2n}$ in Example 6.7. There are no two of its cells in adjacent dimensions so $H_q(\mathbb{C}P^n)$ is free abelian with basis corresponding to a q-cell so $H_q(X) \cong \mathbb{Z}$ when there is a q-cell. So the homology groups of $\mathbb{C}P^n$ are

$$H_q(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & q = 0,2,4,\cdots,2n \\ 0 & otherwise \end{cases}$$

Definition 6.13. Given a continuous function $f: S^n \to S^n, n \in \mathbb{N}^*$. The induced homomorphism $f_*: H_n(S^n) \to H_n(S^n)$ must be defined by $\alpha \mapsto d\alpha$ for some $d \in \mathbb{Z}$ since $H_n(S^n) \cong \mathbb{Z}$. We call this d the ***degree*** of f and denote it by $\deg f$

Here are some basic properties of degree:

(1) $\deg 1_{S^n} = 1$ since $(1_{S^n})_* = 1_{H_*(S^n)}$.

(2) If f is not surjective, then $\deg f = 0$ (since f is not surjective we may choose a point $x_0 \in S^n - f(S^n)$ then factor f as the composition $S^n \to S^n - \{x_0\} \xrightarrow{i} S^n$ where i is the inclusion. Then by the fact that $S^n - \{x_0\}$ is contractible so $H_n(S^n - \{x_0\}) = 0$ hence $f_* = 0$).

(3) If $f \simeq g$, then $f_* = g_*$ so $\deg f = \deg g$. Moreover, if $\deg f = \deg g$, then $f \simeq g$.

(4) $\deg fg = \deg f \deg g$ since $(fg)_* = f_* g_*$. Hence if f is a homotopy equivalence we have $\deg f = \pm 1$.

(5) $\deg f = -1$ if f is a reflection which switches two complementary hemispheres with points on a chosen S^{n-1} fixed.

(6) The antipodal map $\alpha: S^n \to S^n, x \mapsto -x$ has degree $\deg \alpha = (-1)^{n+1}$ since it can be factored as a composition of $n+1$ reflections, each of which changes the sign of one coordinate in $\mathbb{R}^{n+1}$.

(7) If f has no fixed points, then $\deg f = (-1)^{n+1}$ (if $f(x) \neq x$ the line segment connecting $f(x)$ and $-x$ is defined by $t \mapsto (1-t)f(x) - tx$ does not pass through the origin. So there is a homotopy $F: S^n \times I \to S^n$ from f to the antipodal map defined by $F(x,t) = \frac{[(1-t)f(x)-tx]}{|(1-t)f(x)-tx|}$).

Example 6.14. As illustrated in example 6.6 we know $\mathbb{R}\mathrm{P}^n = e^0 \cup e^1 \cup \cdots \cup e^n$ and attaching map $\varphi^n: S^{n-1} \to S^{n-1}/\sim$. By lemma 6.10. (1) we have the cellular chain complex

$$0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \mathbb{Z} \xrightarrow{d_0} 0$$

So in order to compute d_k we need the degree of the map $q\varphi^n: S^{k-1} \xrightarrow{\varphi^n} \mathbb{R}\mathrm{P}^n \xrightarrow{q} S^{k-1} \approx \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$. First we can see $(\varphi^n)^{-1}(\mathbb{R}\mathrm{P}^{k-2}) \approx S^{k-2} \subset S^{k-1}$. Then we have $S^{k-1} - S^{k-2} \approx e^{k-1} \cup e^{k-1}$, which are mapped to $\mathbb{R}\mathrm{P}^{k-1} - \mathbb{R}\mathrm{P}^{k-2}$ homeomorphically. So the map $q\varphi^n$ is a homeomorphism when restricted to each e^{k-1} (one antipodal and one identity). Finally we can conclude that $d_k = \deg\varphi = \deg 1_{S^{k-1}} + \deg\alpha = 1 + (-1)^k$. So the cellular chain complex when n is an odd number is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0$$

When n is an even number:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{0} \mathbb{Z} \to 0$$

So the homology groups of $\mathbb{R}\mathrm{P}^n$:

$$H_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} & k = 0 \; or \; n \; is \; odd, k = n \\ \mathbb{Z}/2\mathbb{Z} & k \; is \; odd, 0 < k < n \\ 0 & otherwise \end{cases}$$

6.2 Universal Coefficient Theorem

It is very natural to ask the relationship between $H_q(X;G)$ and $H_q(X) \otimes G$. As one may mix them up in an optimistic attitude. One can see that actually it is the case when G is free abelian. Moreover, the homology groups of a topological space and the chosen coefficient group determine the corresponding cohomology groups. The properties of Tor and Ext are taken from "An Introduction to Algebraic Topology"[8].

Definition 6.15. Given an abelian group A and an arbitrary short exact sequence $0 \to R \xrightarrow{i} F \to A \to 0$ where F is free abelian (so is R as a subgroup of it). Then for an arbitrary abelian group B we define $\boldsymbol{Tor(A,B)} := ker(i \otimes 1_B)$.

The following theorem tells the relationship between homology groups with coefficient and ordinary homology groups.

Theorem 6.16 (Universal Coefficient Theorem). Given an arbitrary topological space X and an abelian group G, we can find such exact sequences for all $q \geq 0$:

$0 \to H_q(X) \otimes G \xrightarrow{f} H_q(X; G) \to Tor\big(H_{q-1}(X), G\big) \to 0$, where the map f is defined by $f: [z_q] \otimes g \mapsto [z_q \otimes g]$. And the sequence splits. i.e.

$$H_q(X; G) \cong \big(H_q(X) \otimes G\big) \oplus Tor\big(H_{q-1}(X), G\big)$$

Proof: Refer to Theorem 9.32, page 261-264, An Introduction to Algebraic Topology[8].

■

Properties 6.17. Here are some properties of $Tor(-,-)$:

(1) Fix an abelian group G, $Tor(-, G)$ is an additive covariant functor from **Ab** to **Ab**.

(2) Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ and a fixed abelian group G, we have an exact sequence

$$0 \to Tor(A', G) \to Tor(A, G) \to Tor(A'', G) \to A' \otimes G \to A \otimes G \to A'' \otimes G \to 0$$

(3) Given a torsion-free abelian group A(no nontrivial element with finite order), then $Tor(A, G) = 0$ for any abelian group G.

(4) $Tor(\bigoplus_i A_i, G) \cong \bigoplus_i Tor(A_i, G)$ and $Tor\big(A, \bigoplus_j G_j\big) \cong \bigoplus_j Tor\big(A, G_j\big)$.

(5) $Tor(\mathbb{Z}/m\mathbb{Z}, G) \cong G[m] = \{g \in G | mg = 0\}$

(6) $Tor(A, G) \cong Tor(G, A)$ for all A and G.

Actually by property (2) we can interpret Tor as something to measure the failure of exactness of the tensor functor.

Example 6.18. To compute the homology group of $S^n, n \in \mathbb{N}$ with coefficient $\mathbb{Z}/2\mathbb{Z}$, we use Theorem 6.16. so we have for $q \in \mathbb{N}^*$:

$$H_q(S^n;\mathbb{Z}/2\mathbb{Z}) \cong \left(H_q(S^n) \otimes \mathbb{Z}/2\mathbb{Z}\right) \oplus Tor\left(H_{q-1}(S^n), \mathbb{Z}/2\mathbb{Z}\right)$$

Since the homology groups of S^n are $\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}$ and 0, which are all torsion-free abelian groups. Hence by Property (3) we know $Tor\left(H_{q-1}(S^n), \mathbb{Z}/2\mathbb{Z}\right) = 0$

$$H_q(S^n;\mathbb{Z}/2\mathbb{Z}) \cong \left(H_q(S^n) \otimes \mathbb{Z}/2\mathbb{Z}\right)$$

Hence for $n = 0$, $H_q(S^n;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) & q = 0 \\ 0 & otherwise \end{cases}$

For $n \in \mathbb{N}^*$, $H_q(S^n;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & q = 0 \text{ or } q = n \\ 0 & otherwise \end{cases}$

Definition 6.19. Given an abelian group A and an arbitrary short exact sequence $0 \to R \xrightarrow{i} F \to A \to 0$ where F is free abelian (so is R as a subgroup of it). Then for an arbitrary abelian group B we define

$$\boldsymbol{Ext(A,B)} := coker(i^\#) = Hom(R,G)/i^\#\left(Hom(F,G)\right)$$

Definition 6.20. Given an abelian group G, it is called a divisible group if for every $g \in G$ and every $n \in \mathbb{N}^*$, there exists $y \in G$ such that $g = ny$.

Example 6.21. $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are all divisible groups.

Properties 6.22. Here are some properties of $Ext(-,-)$:

(1) Given a short exact sequence $0 \to A' \to A \to A'' \to$ and a fixed abelian group G, we have an exact sequence

$0 \to Hom(A'',G) \to Hom(A,G) \to Hom(A',G) \to Ext(A'',G) \to Ext(A,G) \to Ext(A',G) \to 0$

(2) Given a short exact sequence $0 \to G' \to G \to G'' \to$, we have an exact sequence

$0 \to Hom(A,G') \to Hom(A,G) \to Hom(A,G'') \to Ext(A,G') \to Ext(A,G) \to Ext(A,G'') \to 0$

(3) Given a free abelian group F, then $Ext(F,G) = 0$.

(4) Given a divisible group D, then $Ext(A,D) = 0$.

(5) $Ext(Z/m\mathbb{Z}, G) \cong G/mG$.

Similarly, we can interpret Ext as a measurement for the failure of exactness of hom functor.

Theorem 6.23. Given a topological space X and an abelian group G, there are exact sequences for all $q \in \mathbb{N}$:

$0 \to Ext(H_{q-1}(X), G) \to H^q(X; G) \xrightarrow{\beta} Hom(H_q(X), G) \to 0$, where

$\beta: H^q(Hom(S_*(X), G)) \to Hom(H_q(S_*(X)), G); [\varphi] \mapsto \varphi' \left(\varphi'([z_q]) = \varphi(z_q)\right)$. And the sequence splits. i.e.

$$H^q(X; G) \cong Hom(H_q(X), G) \oplus Ext(H_{q-1}(X), G)$$

Proof: Refer to Theorem 12.11, page 385, An Introduction to Algebraic Topology[8].

■

This theorem reveals the relation between homology groups and cohomology groups.

Example 6.24. The cohomology groups of $S^n, n \in \mathbb{N}^*$ are determined by homology groups of S^n in the way the theorem described

$H^q(S^n; G) \cong Hom(H_q(S^n), G) \oplus Ext(H_{q-1}(S^n), G)$ and since $H_q(S^n)$ is either $\mathbb{Z}$ or 0, which are both free abelian. Then by property (3) we conclude

$$H^q(S^n; G) \cong \begin{cases} G & q = n \; or \; q = 0 \\ 0 & otherwise \end{cases}$$

Example 6.25. The cohomology groups of $\mathbb{R}\mathrm{P}^n$ when $n \in \mathbb{N}^*$ is even: since the homology groups of $\mathbb{R}\mathrm{P}^n$ with coefficient $\mathbb{Z}$ are

$$H_q(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q \; is \; odd, 0 < q < n \\ 0 & otherwise \end{cases}$$

So by Theorem we have

$$H^q(\mathbb{R}\mathrm{P}^n; G) \cong Hom(H_q(\mathbb{R}\mathrm{P}^n), G) \oplus Ext(H_{q-1}(\mathbb{R}\mathrm{P}^n), G)$$

So for $q\ is\ even, 0 < q \leq n$, we have $q - 1 > 0$ and is odd. So $Ext(H_{q-1}(\mathbb{R}P^n), G) = Ext(\mathbb{Z}/2\mathbb{Z}, G) \cong G/2G$ by property (5) and $Hom(H_q(\mathbb{R}P^n), G) = Hom(0, G) = 0$.

When $q > 0$ is odd, we have $q - 1 \geq 0$ is even. So $Ext(H_{q-1}(\mathbb{R}P^n), G) = 0$ and $Hom(H_q(\mathbb{R}P^n), G) = Hom(\mathbb{Z}/2\mathbb{Z}, G) \cong \{g \in G | (2n)g = 0, n \in \mathbb{N}\}$ by observation on the value of [1]. We denote it by $G[2]$.

When $q = 0$, $Hom(H_q(\mathbb{R}P^n), G) = G$ and $Ext(H_{q-1}(\mathbb{R}P^n), G) = 0$.

Hence the cohomology groups of $\mathbb{R}P^n$ with coefficient G when n is even are:

$$H^q(\mathbb{R}P^n; G) \cong \begin{cases} G/2G & q\ is\ even, 0 < q \leq n \\ G[2] & q\ is\ odd \\ G & q = 0 \\ 0 & otherwise \end{cases}$$

6.3 Coefficient as probe for null-homotopic maps

In either homology or cohomology theories, one has a functor from **Top** to **GradedG** with a fixed abelian group G. By homotopy invariance if a map is null-homotopic we have its induced homomorphism trivial. So by varying the coefficient group G one is able to detect whether a map is null-homotopic or not and the choice of the coefficient group sometimes eases the problem. But there might be blindness as we will study in this section, that is, one may have a map that is not null-homotopic but with a trivial induced homomorphism.

Lemma 6.26. Given a map $f: S^k \to S^k$ of degree m, then $f_*: H_k(S^k; G) \to H_k(S^k; G)$ is multiplication by m.

Proof: Given a homomorphism $\varphi: G_1 \to G_2$, it induces chain maps $\varphi_\#: S_q(X, A; G_1) \to S_q(X, A; G_2)$, which induces homomorphisms $\varphi_*: H_q(X, A; G_1) \to H_q(X, A; G_2)$(it is indeed a chain map since $\varphi_\# \tilde{\partial}_q \left(\sum_i g_1^{(i)} \left[z_q^{(i)}\right]\right) = \varphi_\# \left(\sum_i g_1^{(i)} \left[\partial_q\left(z_q^{(i)}\right)\right]\right) = \sum_i \varphi\left(g_1^{(i)}\right) \left[\partial_q\left(z_q^{(i)}\right)\right] =$

$\tilde{\partial}_q \varphi_\# \left(\sum_i g_1^{(i)}\left[z_q^{(i)}\right]\right) = \tilde{\partial}_q \left(\sum_i \varphi\left(g_1^{(i)}\right)\left[z_q^{(i)}\right]\right) = \sum_i \varphi\left(g_1^{(i)}\right)\left[\partial_q\left(z_q^{(i)}\right)\right]$). Similarly the homomorphism φ_* also commutes with the homomorphisms f_* induced by $f:(X,A) \to (Y,B)$. Now suppose $f: S^k \to S^k, \deg f = m$ and let $\varphi: \mathbb{Z} \to G, 1 \mapsto g$ for some $g \in G$. We have a commutative diagram

$$\mathbb{Z} \cong \widetilde{H}_k(S^k;\mathbb{Z}) \xrightarrow{f_*} \widetilde{H}_k(S^k;\mathbb{Z}) \cong \mathbb{Z}$$

$$\varphi \downarrow \quad \varphi_* \downarrow \quad\quad \varphi_* \downarrow \quad\quad \downarrow \varphi$$

$$G \cong \widetilde{H}_k(S^k;G) \xrightarrow{f_*} \widetilde{H}_k(S^k;G) \cong G$$

So for an arbitrary $\alpha \in \widetilde{H}_k(S^k;\mathbb{Z})$, $\varphi_* f_*(\alpha) = \varphi_*(m\alpha) = mg\alpha = f_*\varphi_*(\alpha) = f_*(g\alpha)$ implies the f_* below is multiplication by m as well.

■

Example 6.27. By the Universal coefficient theorem the homology groups of $\mathbb{R}\mathrm{P}^n$ (n is odd) with coefficient an abelian group G are

$$H_q(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} G & q = 0 \; or \; q = n \\ G/2G & q > 0, q \; is \; odd \\ G[2] & q > 0, q \; is \; even \\ 0 & otherwise \end{cases}$$

When n is even we have

$$H_q(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} G & q = 0 \\ G/2G & q > 0, q \; is \; odd \\ G[2] & 0 < q \leq n, q \; is \; even \\ 0 & otherwise \end{cases}$$

Actually choosing a proper abelian group G can simplify the homology groups!

Example 6.28. By the example above the homology groups of $\mathbb{R}\mathrm{P}^n$ with coefficient $\mathbb{Z}/2\mathbb{Z}$ are

$$H_q(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 \leq q \leq n \\ 0 & otherwise \end{cases}$$

Now an example will show that how varying coefficient groups helps with detecting if a map is null-homotopic.

Example 6.29. Given a Moore space $X = M(\mathbb{Z}/m\mathbb{Z}, n)$ $(H_n(X) \cong \mathbb{Z}/m\mathbb{Z}$ and $\widetilde{H}_q(X) = 0, \forall q \neq n)$ obtained by attaching an *(n+1)*-cell e^{n+1} to S^n via a map of degree m. Now consider the quotient map $f: X \to X/S^n \approx S^{n+1}$. It induces trivial homomorphisms $f_*: \widetilde{H}_*(X) \to \widetilde{H}_*(S^{n+1})$ since $\widetilde{H}_q(S^{n+1}) = \mathbb{Z}, q = n+1$ but $\widetilde{H}_q(X) = \mathbb{Z}/m\mathbb{Z}, q = n$. So by using integral homology we are not able to see if f is null-homotopic or not. However, it will be different once we set the coefficient group as $\mathbb{Z}/m\mathbb{Z}$. The part of the long exact sequence

$$0 = \widetilde{H}_{n+1}(S^n; \mathbb{Z}/m\mathbb{Z}) \to \widetilde{H}_{n+1}(X; \mathbb{Z}/m\mathbb{Z}) \xrightarrow{f_*} \widetilde{H}_{n+1}(X/S^n; \mathbb{Z}/m\mathbb{Z})$$

shows that f_* is injective hence nontrivial because $\widetilde{H}_{n+1}(X; \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. Hence the map f is not null-homotopic, which cannot be checked by integral homology.

However, there can be blindness too as one may find trivial induced homomorphism with a map that is not null-homotopic.

Example 6.30. Consider the map $f: S^1 \to S^1$ of degree 2. By Lemma 6.26 we have its induced homomorphism $f_*: H_q(S^1) \to H_q(S^1)$ is multiplication by 2. And it is nontrivial since $H_0(S^1) \cong H_1(S^1) \cong \mathbb{Z}$. But if we change the coefficient group to $\mathbb{Z}/2\mathbb{Z}$, by Lemma 6.26 we have $f_*: H_q(S^1; \mathbb{Z}/2\mathbb{Z}) \to H_q(S^1; \mathbb{Z}/2\mathbb{Z})$ is multiplication by 2 hence trivial.

7. Conclusion

This paper firstly introduces basic concepts in singular homology, then its axiomatic approach. Next, we study basic definitions of cohomology, cohomology axioms and then the ring structure. With the ring structure we begin the study of classifying vector bundles, which ends up in the Stiefel-Whitney class. In the end, we present cellular complex, the Universal coefficients theorem and then compute the (co)homology groups of certain topological spaces. Moreover, we tell the advantages and blindness of using different coefficient groups to detect if a map is null-homotopic.

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