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Model-Independent Infinity-Category Theory with Motivation from First-Order Logic

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[**ABSTRACT**]: Model-independent ∞-category theory has had abundant applications in homotopy theory and algebraic topology, with its choices of foundations probed over decades. The first chapter aims to demonstrate the process of defining the notion of ∞-categories from ordinary 1-categories. We start from a topological point of view, by constructing higher-dimensional structures of path homotopy classes of topological spaces from fundamental groupoids. These structures of various dimensions can then assemble into categories of the corresponding dimensions, provided that the definitions for them are well-behaved. We demonstrate details about data in 2-categories, not only because it is the first step to construct higher-dimensional categories, but also for the reason that weak ∞-categories — ∞-categories with weakly invertible morphisms of dimensions greater than 1 — along with ∞-functors and ∞-natural transformations assemble into a Cartesian closed 2-category ∞-CAT. In the second chapter, we shift our focus to choices of foundations for various mathematical subjects. First-order logic and its derived formal set theory perform well for setting up a rigorous foundation for classical mathematics. Nevertheless, limitations of first-order logic as a foundation are prominently reflected when applying to category theory and ∞-category theory. A new foundation called type theory has been proposed and shown its compatibility when dealing with these modern mathematical subjects, which is briefly discussed at the end.

[**Key words**]: category, 2-category, ∞-category, groupoid, homotopy, first-order logic, type theory

[摘要]：独立于模型的无穷范畴理论在同伦论和代数拓扑领域中有着丰富的应用，其构架基础的选择在过去几十年中得到了探索。本文第一章旨在展现从一般的一阶范畴开始定义无穷范畴的过程。我们从拓扑的角度出发，从基本群胚开始建立拓扑空间上道路同伦类的高维结构。这些不同维度的结构可以装配到相应维度的范畴中，前提是这些范畴的定义有良好的基本性质。我们给出定义二阶范畴中各个要素的过程细节，不仅仅是因为这是构建高维范畴的第一步，更因为弱无穷范畴——维度大于一的态射都是弱可逆的无穷范畴——以及无穷函子、无穷自然变化构成了一个笛卡尔闭的二阶范畴。在第二章中，我们将重点转移至各种数学研究主题的构架基础的选择上。一阶逻辑及其导出的规范集合论在为经典数学构架基础上表现良好。然而，当应用于范畴论和无穷范畴论时，一阶逻辑作为构架基础的局限性被显著地反应出来。一个新的称作类型论的构架基础被提出，其在处理这些现代数学的主题时显示出很好的优势。这将在最后简要讨论。

[关键词]：群胚；范畴；同伦；一阶逻辑；类型论

Contents

1. Introduction to ∞-Category Theory

1.1 From Fundamental Groupoids to Higher Constructions

The concept 'groupoid' was originally ignited from analyzing the structure of paths in a topological space. In a topological space, composition of two paths with the same terminal point of the first and original point of the second is generically defined to be the concatenation by reparametrizing with double speed. However, this partial binary operation has a fatal weakness — it is not associative. To deal with this problem, topologists invented the concept of 'homotopy', partitioning the set of paths between two fixed points into equivalence classes of based homotopy called path classes. Composition of path classes with common terminal and original points can then be well-defined as the class containing the composition of any two paths in their corresponding classes respectively, provided that homotopy property is preserved under composition of paths between these points. Besides the associativity of this partially-defined binary operation, each path class also has the class containing the constant loop based at its original point as a right identity and that based at its terminal point as a left identity likewise, while the class containing the inverse path serves as the two-sided inverse to its right and left identity. If we restrict ourselves to a fixed point, the set of path classes consisting of loops based at that point equipped with this operation forms a group, called the fundamental group of the space based at that point[1]. Moreover, for path-connected points, the fundamental groups based at them are the same up to group isomorphism.

Like the concept of group abstracting the structure of the set of path classes starting and ending at a fixed point, how can we extract a similar abstract one to be applicable on the structure of path classes of a topological space? The information of a group is apparently not enough, since we have to describe the relations between path classes based at different points via any path classes connecting them. From category theory, a category with exactly one object admits a monoid consisting of its morphisms, with monoid identity as the identity morphism and monoid multiplication as the composition of morphisms (notice that in such a category, any pair of morphisms are composable). If all the morphisms of this category are invertible, i.e. they are isomorphisms, then such a monoid is meanwhile a group, with group inverse of a morphism as its inverse morphism. Analogously, a category with all morphisms invertible should also have some utilization. This is just an ideal choice for abstracting the structure we have mentioned above and is exactly what a groupoid means[2].

Definition 1.1. *A **groupoid** is a category all of whose morphisms are isomorphisms.*

It is easy to see if G is a groupoid, then for any object X of G, the collection of morphisms $f : X \to X$ is a group, as they form the morphisms of the subcategory with the single object X.

Proposition 1.2. *Let X be a topological space. The collection of path homotopy classes between points of X forms a groupoid, whose objects are all points of X, morphisms are all path classes, with identity morphisms as the classes containing constant loops, composite morphisms as compositions of path classes.*

Proof. It is easy to verify that it is a category. For any morphism $[p] : x \to y$ where x, y are points of X and p is a path from x to y, the inverse path class $[p^{-1}]$ is the inverse morphism of $[p]$. Hence every morphism is invertible. □

Definition 1.3. *The collection of path classes of a topological space equipped with its groupoid structure is called the **fundamental groupoid** of the space.*

Like 1-category theory provides a relatively abstract prototype for studying structures of path homotopy classes of topological spaces, ∞-category theory plays its role in studying the structures of path homotopy classes of all dimensions of topological spaces. We are now going to discuss model-independent ∞-categories and see its power in analyzing these structures in topological spaces.

1.2 Model-Independent ∞-Category Theory

In category theory, there are three basic concepts: category, functor and natural transformation. Analogously, to develop ∞-category theory, we have to define the notions of

∞-category, ∞-functor and ∞-natural transformation. The idea for construction them is unsurprising: first define them for dimension 2[3] and then naturally extend them.

Before starting, let us recall the definitions of dimension 1. In this case, we often omit to write the dimension. In fact, modern mathematicians, especially for those in the area of algebra and topology, already get familiar with these notions, as they have been invented and drastically developed since 1945. We only write the definition of natural transformations here, since it plays an important role in construction higher-dimensional categories. Definitions of categories and functors will be restated in Section 2.2, for verifying the coincidence of them in first-order logic with the ordinary ones.

Definition 1.4. *Let C, D be categories, $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A **natural transformation** φ from F to G is a map from the collection of objects of $\mathcal{C}$ to the collection of morphisms of D that sends each object X in C to a morphism $\varphi(X) : FX \to GX$ such that $\varphi(Y) \circ Ff = Gf \circ \varphi(X)$ whenever $f : X \to Y$ is a morphism in C, denoted as $\varphi : F \to G$ (or $\varphi : F \to G : \mathsf{C} \to \mathsf{D}$).*

Remark 1.5. *It should be aware that the notation $\varphi : F \to G$ has nothing to do with saying that φ is a map from F to G, just like the notation $f : X \to Y$ for a morphism, though the latter one often even makes no sense to be regarded as a map.*

Definition 1.6. *Let C, D be categories, $F : \mathsf{C} \to \mathsf{D}$ be a functor. The **identity (natural) transformation** $1_F : F \to F$ is defined to be the natural transformation from F to itself that sends X to the identity morphism 1_{FX}, i.e. $1_F(X) = 1_{FX} : FX \to FX$.*

It is easy to verify that the identity of a functor is indeed a natural transformation. Moreover, we can define composition between natural transformations.

Definition 1.7. *Let C, D be categories, $F, G, H : \mathsf{C} \to \mathsf{D}$ be functors. Given two natural transformations $\varphi : F \to G$ and $\psi : G \to H$, the **composite (natural) transformation** $\psi.\varphi : F \to H$ is defined by setting $\psi.\varphi(X) = \psi(X) \circ \varphi(X) : FX \to HX$ whenever X is an object of C.*

A natural transformation $\varphi : F \to G : \mathsf{C} \to \mathsf{D}$ can also be combined with functors towards C and away from D to form new natural transformations. This is what whisker composition means.

Definition 1.8. *Let C, D be categories, $F, G : \mathsf{C} \to \mathsf{D}$ be functors, $\varphi : F \to G$ be a natural transformation. Given a category C' and a functor $H : \mathsf{C}' \to \mathsf{C}$, the* ***left whisker composite (natural) transformation*** *$\varphi \cdot H : F \circ H \to G \circ H : \mathsf{C}' \to \mathsf{D}$ is defined by setting $\varphi \cdot H(X') = \varphi(HX')$ whenever X' is an object of C'; analogously, given a category D' and a functor $K : \mathsf{D} \to \mathsf{D}'$, the* ***right whisker composite (natural) transformation*** *$K \cdot \varphi : K \circ F \to K \circ G : \mathsf{C} \to \mathsf{D}'$ is defined by setting $K \cdot \varphi(X) = K\varphi(X)$ whenever X is an object of C; having these definitions, the* ***whisker composite (natural) transformation*** *$K \cdot \varphi \cdot H : K \circ F \circ H \to K \circ G \circ H : \mathsf{C}' \to \mathsf{D}'$ is naturally defined to be $K \cdot (\varphi \cdot H)$ (or equivalently, $(K \cdot \varphi) \cdot H$).*

Compositions between natural transformations and natural transformations, natural transformations and functors, have satisfactory properties as follows. The proofs of them are straightforward and easy.

Proposition 1.9. *Provided with the notions for categories C, D, functors F, G, H, K, H', K', natural transformations φ, ψ, χ and their composites and whisker composites well-defined when occurring, we have*

1) associativity:

$$\chi.(\psi.\varphi) = (\chi.\psi).\varphi,$$

$$K' \cdot (K \cdot \varphi \cdot H) \cdot H' = (K' \circ K) \cdot \varphi \cdot (H \circ H');$$

2) identical property:

$$\varphi.1_F = \varphi = 1_G.\varphi,$$

$$K \cdot 1_F = 1_{K \circ F},$$

$$1_F \cdot H = 1_{F \circ H},$$

$$K \cdot 1_F \cdot H = 1_{K \circ F \circ H},$$

$$\varphi \cdot 1_{\mathsf{C}} = \varphi = 1_{\mathsf{D}} \cdot \varphi.$$

3) distributivity:

$$K \cdot (\psi.\varphi) \cdot H = (K \cdot \psi \cdot H).(K \cdot \varphi \cdot H).$$

Besides, we have an interchange property as follows.

Proposition 1.10. *Let $\mathsf{C}, \mathsf{D}, \mathsf{E}$ be categories, $F, G : \mathsf{C} \to \mathsf{D}$ and $H, K : \mathsf{D} \to \mathsf{E}$ be functors, $\varphi : F \to G$ and $\psi : H \to K$ be natural transformations. Then*

$$(\psi \cdot G).(H \cdot \varphi) = (K \cdot \varphi).(\psi \cdot F).$$

Having the idea of categories, functors and natural transformations, we are next to define 2-categories. It should be applicable to the category of (locally small) categories as a prime example. Thus, like natural transformations define operations between functors, morphisms between morphisms are to be introduced, while Proposition 1.9 and Proposition 1.10 provides rules for them.

Definition 1.11. *A 2-category C is a category such that*
1) for each pair of objects X, Y and each pair of morphisms $f, g : X \to Y$, there exists one specific collection, denoted by $\mathsf{C}_2(f, g)$, whose elements are called ***2-morphisms*** *and written as $\varphi : f \to g$ (or $\varphi : f \to g : X \to Y$);*
2) for each pair of objects X, Y and each morphism $f : X \to Y$, there exists one specific element of $\mathsf{C}_2(f, f)$ called the ***identity 2-morphism*** *of f, denoted by 1_f;*
3) for each pair of objects X, Y, each triple of morphisms $f, g, h : X \to Y$, and each 2-morphisms $\varphi : f \to g$ and $\psi : g \to h$, there exists one specific 2-morphism in $\mathsf{C}_2(f, h)$ called the ***composite 2-morphism of*** *φ* ***and*** *ψ, denoted by $\psi.\varphi$;*
4) for each 4-tuple of objects X, Y, X', Y', each morphisms $h : X' \to X$ and $k : Y \to Y'$, each pair of morphisms $f, g : X \to Y$ and each 2-morphism $\varphi : f \to g$, there exists one specific 2-morphism in $\mathsf{C}_2(k \circ f \circ h, k \circ g \circ h)$ called the ***whisker composite 2-morphisms of*** *f* ***via*** *h* ***and*** *k, denoted by $k \cdot \varphi \cdot h$;*

5) provided with the notions for objects X, Y, morphisms f, g, h, k, h', k', 2-morphisms φ, ψ, χ and and their composites and whisker composites well-defined when occurring, there are

$$\chi.(\psi.\varphi) = (\chi.\psi).\varphi,$$

$$k' \cdot (k \cdot \varphi \cdot h) \cdot h' = (k' \circ k) \cdot \varphi \cdot (h \circ h'),$$

$$\varphi.1_f = \varphi = 1_g.\varphi,$$

$$k \cdot 1_f \cdot h = 1_{k \circ f \circ h},$$

$$1_Y \cdot \varphi \cdot 1_X = \varphi,$$

$$k \cdot (\psi.\varphi) \cdot h = (k \cdot \psi \cdot h).(k \cdot \varphi \cdot h);$$

6) for each triple of objects X, Y, Z, each pair of morphisms $f, g : X \to Y$, each pair of morphisms $h, k : Y \to Z$, each 2-morphisms $\varphi : f \to g$ and $\psi : g \to h$, there is

$$(\psi \cdot g).(h \cdot \varphi) = (k \cdot \varphi).(\psi \cdot f),$$

and the ***whisker composite*** 2***-morphism of*** φ ***and*** ψ*, denoted by $\psi \cdot \varphi$, is defined to be the element $(\psi \cdot g).(h \cdot \varphi)$ in $\mathsf{C}_2(h \circ f, k \circ g)$.*

Remark 1.12. *Whisker compositions of 2-morphisms via two morphisms indeed induce left and right whisker compositions of 2-morphisms via one morphism, by letting $\varphi \cdot h := 1_Y \cdot \varphi \cdot h : f \circ h \to g \circ h : X \to Y'$ and $k \cdot \varphi := k \cdot \varphi \cdot 1_X : k \circ f \to k \circ g : X' \to Y$. Actually whisker compositions of composable 2-morphisms induce whisker compositions of 2-morphisms via two morphisms, as $k \cdot \varphi \cdot h = 1_k \cdot \varphi \cdot 1_h$.*

Proposition 1.13. *Let C be a 2-category. Provided the notions for 2-morphisms φ, ψ, χ in C and their whisker composites well-defined when occurring, we have*

1) associativity:

$$\chi \cdot (\psi \cdot \varphi) = (\chi \cdot \psi) \cdot \varphi,$$

2) identical property:

$$\varphi \cdot 1_{1_X} = \varphi = 1_{1_Y} \cdot \varphi.$$

Proposition 1.14. *Let* C *be a* 2*-category,* X, Y, Z *be objects,* $f, g, h : X \to Y$ *be morphisms,* $r, s, t : Y \to Z$ *be morphisms,* $\varphi : f \to g, \psi : g \to h, \sigma : r \to s, \tau : s \to t$ *be* 2*-morphisms. Then we have the middle-four interchange property as follows:*

$$(\tau.\sigma) \cdot (\psi.\varphi) = (\tau \cdot \psi).(\sigma \cdot \varphi).$$

Therefore, a 2-category is just a category, i.e. a collection of objects and morphisms, in addition to a collection of 2-dimensional morphisms defined between parallel morphisms. And 2-morphisms with adjacent parallel morphisms have compositions analogous to those for morphisms, but there are something more than data in categories — whisker compositions between 2-morphisms with adjacent objects. These compositions are both associative and satisfy other well-behaved properties.

Having giving a formal definition of a 2-category, we can inductively define morphisms of any dimension and then assemble objects, morphisms and higher-dimensional morphisms into a higher-dimensional category. The notion of ∞-categories was first introduced for solving the problem of Grothendieck's homotopy hypothesis, which posits that the fundamental ∞-groupoid construction — higher-dimensional version of fundamental groupoids in topological spaces — defines an equivalence between homotopy types and ∞-groupoids[4]. Homotopy types can be understood as isomorphism classes of objects in the homotopy category of topological spaces, partitioning spaces into homotopy equivalence classes. This structure can actually be assembled into a weak ∞-category, called the weak ∞-category of topological spaces, in which all higher morphisms are weakly invertible. ∞-groupoids, on the other hand, are weak ∞-categories in which all morphisms are weakly invertible. The notion of weak ∞-categories have much more applications in algebraic topology than ∞-categories. Models of weak ∞-categories include, in order of appearance, simplicial categories, quasi-categories, relative categories, Segal categories, complete Segal categories and others as well. They have satisfactory properties in subjects they belong to. However, model-independent ∞-category theory is also meaningful to analyze[5], which has been developed through this section. Though its definition was not explicitly displayed, the idea of its structure is analogous to 2-categories. Moreover, when restricting to weak ∞-categories, we have the following connections with a particular 2-category.

Theorem 1.15. *The collection of weak ∞-categories, weak ∞-functors and weak ∞-transformations assembles into a Cartesian closed 2-category*[6].

The 2-category consisting of weak ∞-categories, weak ∞-functors and weak ∞-transformations is denoted by $\infty\text{-}\mathsf{CAT}$. It is surprising that a large portion of model-independent ∞-category theory can actually be involved in the theory of this 2-category. Therefore the work we have done for 2-categories are not only used for deriving the concept of ∞-categories, but also for exploring properties of them, including those models in various subjects.

Logically speaking, category theory provides an alternative choice of ZFC set theory to conduct mathematical study. To give more precise explanations for this, we are to discuss mathematical logic, among which the most fundamental and easy-approaching one is first-order logic. It serves as a foundation of mathematical world, including formal set theory and category theory.

2. Reimagining the Foundations of Category Theory

Logic, or mathematical logic, stays in the core of mathematics. Like vector spaces arose historically to represent the 3-dimensional physical space we live, mathematical logic serves as a similar role where the particular reality it attempts to represent is just the mathematics itself. It is a prototype, i.e. a reduced model of the universe of mathematics, with which we already get familiar[7]. In other words, we are doing logic by means of knowledge of mathematics we have studied. It easily provides a target for polemic attack that we are using mathematics to demonstrate itself, which is a vicious circle. The fact, necessarily, is not, since we are here using different level of languages, say ’intuitive language’ and ’formal language’. The mathematical world we are facing everyday and which we are going to formalize, is on the intuitive level, whereas the process of formalization is on the formal level. Logicians also use ’meta-language’ and ’object language’ to distinguish them, in the sense that the former one describes the latter. But notice that there is neither ground level nor top level. Whichever level our language stands, there is always an object language that formalizes it, as well as a relatively intuitive meta-language that describes it. It is feasible to build all of mathematics ’ex nihilo’ but should be conducted based on the world we live in.

Modern mathematics, at least the classical parts, are based on set theory (usually ZFC set theory) which plays its role as a foundational framework. As argued previously, formal set theory has intuitive classical mathematics as its meta-linguistic world. To distinguish sets in formal set theory from intuitive sets, we will henceforth always use the word ’collections’ to denote the intuitive ones. Sometimes, however, we also use the same word to represent both the intuitive and formal meanings for an object. For example, positive integers have widespread utilization in the intuitive sense throughout the mathematical context, while we abuse the name and notation to represent the formal ones in set theory as well. Formal set theory is so substantial in classical mathematics that it is often placed within the scope of logic by mathematicians. However, it is to some extent not, since there is a more fundamental subject — first-order predicate calculus — without which even formal set theory would vanish. 'First-order' here means the variables we use as placeholders in quantified sentences have objects as their intended range. By contrast, for instance, second-order version permits properties of objects in this range. First-order predicate calculus is thus the backbone of first-order logic, the one that formalizes all intuitive classical mathematical subjects based on set theory, which of course include set theory itself. We are next then to provide some details about it. Later will we also see its applications on ordinary categories and limitations when encountering these non-classical mathematical objects, including ∞-category theory as well.

Remember again, that all we are going to do is to formalize the intuitive mathematical world!

2.1 First-Order Predicate Calculus

Before starting, we should make it clear that all the concepts we are going to use which formally originate from ZFC set theory, such as integers, cardinality of sets, maps between sets, are in their intuitive meanings.

Definition 2.1. *The collection of* ***logical symbols*** *consists of the following pairwise distinct entities:*

1) the countable collection of ***variables****, often denoted as* $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$*,*

2) the ***closing parenthesis***) *and the* ***opening parenthesis*** (*,*

3) the ***connectives****: the* ***negation*** $\neg$*, the* ***disjunction*** $\vee$*, the* ***conjunction*** $\wedge$*, the* ***implication*** $\Rightarrow$ *and the* ***equivalence*** $\Leftrightarrow$*,*

4) the ***universal quantifier*** $\forall$ *and the* ***existential quantifier*** $\exists$*.*

The logical symbols are certain mathematical objects, whose definitions are, unfortunately, difficult to explicitly expressed. A definition we commonly treat should be formally reducible in the sense that every subject defined could in principle be replaced by the phrase that defines it without affecting the essence of the subject. But this process of reduction must stop eventually, otherwise there would be a ridiculously endless regress. So at the beginning of our exposition, there must be some mathematical objects which we do not define in terms of others but merely take as given: they are called *primitives*. Similarly, mathematical proofs start somewhere as well — there must be some propositions that are not proved but can be used in the proofs they follow: they are called *axioms*[8].

Attitudes towards the understanding of primitives and axioms vary among logicians. They belong to an interesting and dialectic part of the philosophy of mathematics which, however, is too far away from the subject we concentrate on. It is enough at this stage to treat primitives by platonic way — as entities whose meanings are understood in priority in the ordinary world, e.g. the logical symbols. Each symbol is informally defined by its name which suffices to indicate the intended meaning and role it plays. The only one thing that needs commenting is the cardinality of the collection of variables: it is countable since we need and only need arbitrarily finitely many variables in the context. On the other hand, axioms should be regarded as truths (or, sometimes more accurate, assumptions) which we suppose in order to demonstrate the properties of structures that exemplify them, and this is what implicationists often do. We will see their places in Definition 2.12.

After having these logical symbols as primitives, let us then define first-order languages and related concepts.

Definition 2.2. *A **(first-order) language** L is a collection consisting of*

1) the logical symbols,

*2) a collection $\mathcal{C}(L)$ whose elements are called **constants** and which is disjoint from the collection of logical symbols,*

*3) for each $n \in \mathbb{N}^*$, a pairwise disjoint collection $\mathcal{F}_n(L)$ whose elements are called n**-ary functions** (or **functions with arity** n) and which is disjoint from the collections of logical symbols and constants,*

*4) for each $n \in \mathbb{N}^*$, a pairwise disjoint collection $\mathcal{R}_n(L)$ whose elements are called n**-ary relations** (or **relations with arity** n) and which is disjoint from the collections of logical symbols, constants and functions of each arity.*

*A language is said to be **with equality** if an entity $\simeq$ called **equality** is a particular 2-ary* *relation of it. Moreover, the elements that are not logical symbols of a language are called its* ***non-logical symbols****.*

For first-order languages, the unchanged part is logical symbols. Therefore, when referring to a language, it is enough to only speak about its non-logical symbols. In most cases, the collections of constants, functions and relations of each arity is finite, and only finitely many of them is nonempty. Thus we can, for instance, use the description 'let $L = \langle c, R, S, f, g \rangle$ be a language with equality where c is a constant, R is a 2-ary relation, S is a 3-ary relation and f, g are 1-ary relations' to mean that the language L we consider has $\mathcal{C} = \{c\}$, $\mathcal{F}_1 = \{f, g\}$, $\mathcal{F}_n = \emptyset$ for $n \geqslant 2$, $\mathcal{R}_2 = \{R, \simeq\}$, $\mathcal{R}_3 = \{S\}$ and $\mathcal{R}_n = \emptyset$ for $n = 1$ and $n \geqslant 4$. The order of constants, functions and relations we place is not important and can be arbitrary.

Definition 2.3. *Let X be a collection. The collection $\mathcal{W}(X)$ of* ***words*** *of X consists of all finite sequences of X, where we regard any element of X as a 1-sequence. We use the notation $x_1 \ldots x_n$ for a word with n elements $x_1, \ldots, x_n$, respectively.*

Definition 2.4. *Let L be a language. The collection $\mathcal{T}(L)$ of* ***terms*** *of L is the smallest subcollection of $\mathcal{W}(L)$ that contains $\mathcal{V} \cup \mathcal{C}(L)$ and such that $ft_1 \ldots t_n \in \mathcal{T}(L)$ whenever $n \in \mathbb{N}^*$, $f \in \mathcal{F}_n(L)$ and $t_1, \ldots, t_n \in \mathcal{T}(L)$.*

Proposition 2.5. *For a language L, set*

$$\mathcal{T}_0(L) = \mathcal{V} \cup \mathcal{C}(L),$$

$$\mathcal{T}_{k+1}(L) = \mathcal{T}_k(L) \cup \left(\bigcup_{n \in \mathbb{N}^*} \{ft_1 \ldots t_n : f \in \mathcal{F}_n(L), t_1 \in \mathcal{T}_k(L), \ldots, t_n \in \mathcal{T}_k(L)\} \right) \quad (k \in \mathbb{N}),$$

Then $\mathcal{T}(L) = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n(L)$.

Remark 2.6. *Given a language L and a term $t \in \mathcal{T}(L)$, there exists $n \in \mathbb{N}^*$ and pairwise distinct $i_1, \ldots, i_n \in \mathbb{N}$ such that the variables having at least one occurrence in t are among $v_{i_1}, \ldots, v_{i_n}$, which we write t as $t[v_{i_1}, \ldots, v_{i_n}]$ to indicate. There exists a case where no variable has any occurrence in t, which we say t a* ***closed*** *term. Notice that there always exists $m \in \mathbb{N}$ such that $t = t[v_0, \ldots, v_m]$.*

Definition 2.7. *Let L be a language. The collection $\mathcal{A}(L)$ of **atomic formulas** of L consists of all words of the form $Rt_1, \ldots, t_n$ where $n \in \mathbb{N}^*$, $R \in \mathcal{R}_n(L)$ and $t_1, \ldots, t_n \in \mathcal{T}(L)$.*

Remark 2.8. *When writing an atomic formula Rt_1t_2 with a 2-ary relation R, it is sometimes more common to use the notation t_1Rt_2 instead. One typical example is to write $t_1 \simeq t_2$ instead of $\simeq t_1t_2$. We should always be aware that this does not affect the unique readability of the syntax.*

Definition 2.9. *Let L be a language. The collection $\mathcal{F}(L)$ of **formulas** of L is the smallest subcollection of $\mathcal{W}(L)$ that contains $\mathcal{A}(L)$ and such that $\neg F, (F \vee G), (F \wedge G), (F \Rightarrow G), (F \Leftrightarrow G), \forall v_n F, \exists v_n F \in \mathcal{F}(L)$ whenever $F, G \in \mathcal{F}(L)$ and $n \in \mathbb{N}$.*

Proposition 2.10. *For a language L, set*

$$\begin{aligned}
&\mathcal{F}^{(0)}(L) = \mathcal{A}(L),\\
&\mathcal{F}^{(k+1)}(L) = \mathcal{F}^{(k)}(L) \cup \{\neg F : F \in \mathcal{F}^{(k)}(L)\} \cup \{(F\alpha G) : F, G \in \mathcal{F}^{(k)}(L), \alpha \in \{\vee, \wedge, \Rightarrow, \Leftrightarrow\}\}\\
&\qquad\qquad \cup \{Qv_nF : F \in \mathcal{F}^{(k)}(L), n \in \mathbb{N}, Q \in \{\forall, \exists\}\} \quad (k \in \mathbb{N}),
\end{aligned}$$

Then $\mathcal{F}(L) = \bigcup\limits_{n \in \mathbb{N}} \mathcal{F}^{(n)}(L)$. (We use superscripts other that subscripts here just to avoid abuse with the notations for the collections of functions of L with fixed arity.)

Remark 2.11. *Given a language L, a formula $F \in \mathcal{F}(L)$ and $k \in \mathbb{N}$, the occurrence of v_k in F, if any, is called **free** if it is not quantified by $\forall$ or $\exists$, i.e. it is not in any segment of F of the form Qv_kG where $Q \in \{\forall, \exists\}$ and $G \in \mathcal{F}(L)$, and v_k itself is called **free** in F if it has at least one free occurrence in F. Similarly as Remark 2.6 stated, there exists $n \in \mathbb{N}$ and pairwise distinct $i_1, \ldots, i_n \in \mathbb{N}$ such that the free variables in F are among $v_{i_1}, \ldots, v_{i_n}$, which we write F as $F[v_{i_1}, \ldots, v_{i_n}]$ to indicate. There exists a case where F has no free variable, which we say F is a **closed** formula. Notice that there always exists $m \in \mathbb{N}$ such that $F = F[v_0, \ldots, v_m]$.*

Definition 2.12. *Let L be a language. A **theory** T of L is a collection of closed formulas of L, and elements of T are called its **axioms**.*

The word ‘structure’ in mathematics is generally understood as a collection on which certain functions and relations are defined. The languages just defined serve as systems of alphabet and syntax for conducting analysis on structures, while structures provide models for corresponding languages. A language will make no meaningful utilization without certain structure, and a structure needs to be expressed by a language under consideration. We are next to present the definition of a structure of a language and then determine interpretations of terms as well as satisfactions of formulas of that language. To distinguish between intuitive and formal sense, we often use the word ‘model’ for the formal one.

Definition 2.13. *Let L be a language. A **model** (or **structure**) $\mathcal{M}$ of L is a non-empty collection such that*

*1) each constant c has a specified element $\overline{c}^{\mathcal{M}}$ of $\mathcal{M}$ called the **interpretation of** c **in** $\mathcal{M}$,*

*2) each $n \in \mathbb{N}^*$ and each n-ary function f has a map $\overline{f}^{\mathcal{M}}$ from $\mathcal{M}^n$ to $\mathcal{M}$ called the **interpretation of** f **in** $\mathcal{M}$,*

*3) each $n \in \mathbb{N}^*$ and each n-ary relation R has a subcollection $\overline{R}^{\mathcal{M}}$ of $\mathcal{M}^n$ called the **interpretation of** R **in** $\mathcal{M}$.*

In most cases, equality pertains in structures we consider in mathematics. Such a structure(model) has the following definition.

Definition 2.14. *Let L be a language with equality. A model $\mathcal{M}$ of L is said to **respect equality** if the interpretation $\overline{\simeq}^{\mathcal{M}}$ of the equality $\simeq$ in $\mathcal{M}$ is $dia(M) = \{(a, b) \in M^2 : a = b\}$.*

To define a model of a given language, it suffices to determine the underlying collection and interpretations of each non-logical symbols in that collection. For instance, let $L = \langle c, f, R\rangle$ be a language with equality, where c is a constant, f is a 1-ary function and R is a binary relation, then $\mathcal{M} = \mathbb{R}$ is a model of L that respects equality where $\overline{c}^{\mathcal{M}} = \pi$, $\overline{f}^{\mathcal{M}} = \cos$, $\overline{R}^{\mathcal{M}} = \leqslant^{\mathbb{R}}$ and $\overline{\simeq}^{\mathcal{M}} = dia(\mathbb{R})$, which can be written as ‘$\mathcal{M} = \langle \mathbb{R}, \pi, \cos, \leqslant^{\mathbb{R}}\rangle$ respecting equality’. When using this notation, we should always put the underlying collection at the most front and the places of interpretations respectively corresponding to those of the non-logical symbols of the language we write.

Definition 2.15. *Let* L *be a language,* $\mathcal{M}$ *be a model of* L*. For any* $t \in \mathcal{T}(L)$*, suppose* $t = t[v_{i_1}, \ldots, v_{i_n}]$ *for some* $n \in \mathbb{N}^*$ *and pairwise distinct* $i_1, \ldots, i_n \in \mathbb{N}$*. Let* $a_1, \ldots, a_n$ *be elements of* $\mathcal{M}$*. The* ***interpretation of*** t ***in*** $\mathcal{M}$ ***when*** $v_{i_1}, \ldots, v_{i_n}$ ***are interpreted respectively by*** $a_1, \ldots, a_n$ *is an element of* $\mathcal{M}$*, denoted by* $\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$*, defined as follows:*

1) $\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] = a_j$ *whenever* $t = v_{i_j} \in \mathcal{V}$ *for some* $j \in \{1, \ldots, n\}$*,*

2) $\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] = \overline{c}^{\mathcal{M}}$ *whenever* $t = c \in \mathcal{C}(L)$*,*

3) $\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] = \overline{f}^{\mathcal{M}}(\overline{t_1}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n], \ldots, \overline{t_k}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n])$ *whenever* $t = ft_1 \ldots t_k$ *for some* $k \in \mathbb{N}^*$*,* $f \in \mathcal{F}_k(L)$*,* $t_1, \ldots, t_k \in \mathcal{T}(L)$ *and we have defined* $\overline{t_1}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n], \ldots, \overline{t_k}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$*.*

Remark 2.16. *This definition is well-defined. Firstly, it goes through every case of the form of* t *by induction. Secondly, the interpretation of* t *is independent of different expressions of it, i.e. for any* $m \in \mathbb{N}^*$*, pairwise distinct* $i_{n+1}, \ldots, i_{n+m} \in \mathbb{N}$ *which are also distinct from* $i_1, \ldots, i_n$ *and* $b_1, \ldots, b_m \in \mathcal{M}$*, (notice that* $t = t[v_{i_1}, \ldots, v_{i_n}] = t[v_{i_1}, \ldots, v_{i_n}, v_{i_{n+1}}, \ldots, v_{i_{n+m}}]$*)*

$$\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] = \overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n, v_{i_{n+1}} \mid b_1, \ldots, v_{i_{n+m}} \mid b_m],$$

and for any n*-permutation* σ*,*

$$\overline{t}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] = \overline{t}^{\mathcal{M}}[v_{i_{\sigma(1)}} \mid a_{\sigma(1)}, \ldots, v_{i_{\sigma(n)}} \mid a_{\sigma(n)}].$$

Remark 2.17. *When* t *is a closed term of* L*, there is no variable in* t*, thus we can just say about 'the interpretation of* t *in* $\mathcal{M}$*' with no condition, which is denoted by* $\overline{t}^{\mathcal{M}}$*.*

Definition 2.18. *Let* L *be a language,* $\mathcal{M}$ *be a model of* L*. For any* $F \in \mathcal{F}(L)$*, suppose* $F = F[v_{i_1}, \ldots, v_{i_n}]$ *for some* $n \in \mathbb{N}^*$ *and pairwise distinct* $i_1, \ldots, i_n \in \mathbb{N}$*. Let* $a_1, \ldots, a_n$ *be elements of* $\mathcal{M}$*. Then* F *is said to* ***be satisfied in*** $\mathcal{M}$ ***when*** $v_{i_1}, \ldots, v_{i_n}$ ***are interpreted respectively by*** $a_1, \ldots, a_n$*, denoted as* $\mathcal{M} \models F[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$*, if*

1) $(\overline{t_1}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n], \ldots, \overline{t_k}^{\mathcal{M}}[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]) \in \overline{R}^{\mathcal{M}}$ *whenever* $F = Rt_1 \ldots t_k \in \mathcal{A}(L)$ *for some* $k \in \mathbb{N}^*$*,* $R \in \mathcal{R}_k(L)$ *and* $t_1, \ldots, t_k \in \mathcal{T}(L)$*,* *2)* $\mathcal{M} \not\models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = \neg G$ *for some* $G \in \mathcal{F}(L)$*,*

3) $\mathcal{M} \models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *or* $\mathcal{M} \models H[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = (G \vee H)$ *for some* $G, H \in \mathcal{F}(L)$*,*

4) $\mathcal{M} \models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *and* $\mathcal{M} \models H[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = (G \wedge H)$ *for some* $G, H \in \mathcal{F}(L)$*,*

5) $\mathcal{M} \not\models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *or* $\mathcal{M} \models H[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = (G \Rightarrow H)$ *for some* $G, H \in \mathcal{F}(L)$*,*

6) either $\mathcal{M} \models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *and* $\mathcal{M} \models H[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$*, or else* $\mathcal{M} \not\models G[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *and* $\mathcal{M} \not\models H[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = (G \Leftrightarrow H)$ *for some* $G, H \in \mathcal{F}(L)$*,*

7) for every $a \in M$*,* $\mathcal{M} \models G[v_m \mid a, v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = \forall v_m G$ *for some* $G \in \mathcal{F}(L)$ *and* $m \in \mathbb{N} - \{i_1, \ldots, i_n\}$*,*

8) there exists $a \in M$ *such that* $\mathcal{M} \models G[v_m \mid a, v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = \exists v_m G$ *for some* $G \in \mathcal{F}(L)$ *and* $m \in \mathbb{N} - \{i_1, \ldots, i_n\}$*,*

9) for every $a \in M$*,* $\mathcal{M} \models G[v_{i_1} \mid a_1, \ldots, v_{i_j} \mid a, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = \forall v_{i_j} G$ *for some* $G \in \mathcal{F}(L)$ *and* $j \in \{1, \ldots, n\}$*,*

10) there exists $a \in M$ *such that* $\mathcal{M} \models G[v_{i_1} \mid a_1, \ldots, v_{i_j} \mid a, \ldots, v_{i_n} \mid a_n]$ *whenever* $F = \exists v_{i_j} G$ *for some* $G \in \mathcal{F}(L)$ *and* $j \in \{1, \ldots, n\}$*.*

Remark 2.19. *This definition is well-defined. Firstly, it goes through every case of the form of* F *by induction. Secondly, the satisfaction of* F *is independent of different expressions of it, i.e. for any* $m \in \mathbb{N}^*$*, pairwise distinct* $i_{n+1}, \ldots, i_{n+m} \in \mathbb{N}$ *which are also distinct from* $i_1, \ldots, i_n$ *and* $b_1, \ldots, b_m \in \mathcal{M}$*, (notice that* $F = F[v_{i_1}, \ldots, v_{i_n}] = F[v_{i_1}, \ldots, v_{i_n}, v_{i_{n+1}}, \ldots, v_{i_{n+m}}]$*)*

$$\mathcal{M} \models F[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] \text{ if and only if}$$
$$\mathcal{M} \models F[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n, v_{i_{n+1}} \mid b_1, \ldots, v_{i_{n+m}} \mid b_m],$$

and for any n*-permutation* σ*,*

$$\mathcal{M} \models F[v_{i_1} \mid a_1, \ldots, v_{i_n} \mid a_n] \text{ if and only if } \mathcal{M} \models F[v_{i_{\sigma(1)}} \mid a_{\sigma(1)}, \ldots, v_{i_{\sigma(n)}} \mid a_{\sigma(n)}].$$

Moreover, we can see this definition coincides with the meanings of the logical symbols in the intuitive mathematical world.

Remark 2.20. *When F is a closed formula of L, there is no free variable in F, thus we can just say about whether 'F is satisfied in $\mathcal{M}$' or not with no condition, which is denoted as $\mathcal{M} \models F$. This can be generated to theories of L. In fact, the following definitions about theories all apply to closed formulas, as long as we modify a little by regarding a closed formula as a theory with one axiom.*

Definition 2.21. *Let L be a language, $\mathcal{M}$ be a model of L. Given a theory T of L, it is said to be* ***satisfied in*** *$\mathcal{M}$ if each axiom of T is satisfied in $\mathcal{M}$, denoted as $\mathcal{M} \models T$.*

Definition 2.22. *Let L be a language. Given a theory T of L, it is called* ***consistent*** *if it is satisfied in at least one model of L, otherwise it is called* ***contradictory****.*

Definition 2.23. *Let L be a language. Given a theory T of L, it is called* ***universally valid*** *if it is satisfied in any model of L, denoted as $\vdash T$.*

Definition 2.24. *Let L be a language. Given two theories T, S of L, S is called a* ***(semantic) consequence*** *of T if every model of L that satisfies T also satisfies S, denoted as $T \vdash S$.*

Remark 2.25. *We use the word 'semantic' here to distinguish from 'syntactic'. A syntactic consequence means that we can derive(prove) one from another by a formal proof. We will not necessarily introduce any formal proof theory here, thanks to the Gödel's completeness theorem, which states that the notions of semantic consequence and syntactic consequence make no difference. Henceforth, we will ignore the two adjectives and just say 'consequence'.*

Definition 2.26. *Let L be a language. Two theories T, S of L are called* ***universally equivalent*** *if both $T \vdash S$ and $S \vdash T$, denoted as $T \sim S$.*

There are many typical properties about universal validness, universal equivalence and consequence in first-order predicate calculus. However, we will not list them out but rather mention and explain any in need.

Last but not least, we are to introduce maps between models of a language that 'respect' interpretations of non-logical symbols.

Definition 2.27. *Let L be a language, $\mathcal{M}, \mathcal{N}$ be models of L, $\phi : \mathcal{M} \to \mathcal{N}$ be a map. ϕ is called a* ***homomorphism*** *if*

1) for each constant c of L,

$$\phi(\overline{c}^{\mathcal{M}}) = \overline{c}^{\mathcal{N}},$$

2) for each $n \in \mathbb{N}^*$, each n-ary function f of L and each n-tuple of elements $a_1, \ldots, a_n$ of $\mathcal{M}$,

$$\phi(\overline{f}^{\mathcal{M}}(a_1, \ldots, a_n)) = \overline{f}^{\mathcal{N}}(\phi(a_1), \ldots, \phi(a_n)),$$

3) for each $n \in \mathbb{N}^*$, each n-ary relation R of L and each n-tuple of elements $a_1, \ldots, a_n$ of $\mathcal{M}$,

$$(\phi(a_1), \ldots, \phi(a_n)) \in \overline{R}^{\mathcal{N}} \quad whenever \quad (a_1, \ldots, a_n) \in \overline{R}^{\mathcal{M}}.$$

ϕ is called a ***monomorphism*** *if it is a homomorphism and for each $n \in \mathbb{N}^*$, each n-ary relation R of L and each n-tuple elements $a_1, \ldots, a_n$ of $\mathcal{M}$,*

$$(\phi(a_1), \ldots, \phi(a_n)) \in \overline{R}^{\mathcal{N}} \quad if\ and\ only\ if \quad (a_1, \ldots, a_n) \in \overline{R}^{\mathcal{M}}.$$

ϕ is called a ***isomorphism*** *if it is a bijective monomorphism.*

A typical example of isomorphism is the identity map of a model. Obviously, the isomorphic relation with its natural meaning partitions the collection of models of L into equivalence classes.

Proposition 2.28. *Let L be a language with equality, $\mathcal{M}, \mathcal{N}$ be models of L that respect equality. Then every monomorphism from $\mathcal{M}$ to $\mathcal{N}$ is injective.*

Proof. Given a monomorphism $\phi : \mathcal{M} \to \mathcal{N}$, since $\simeq \in L$, for any pair of elements a, b of $\mathcal{M}$,

$$(\phi(a_1), \phi(a_2)) \in \simeq^{\mathcal{N}} \quad if\ and\ only\ if \quad (a_1, a_2) \in \simeq^{\mathcal{M}},$$

i.e. $\phi(a_1) = \phi(a_2)$ if and only if $a_1 = a_2$, which implies that ϕ is injective. □

Provided with enough prerequisites, we can define a 1-category in first-order logic which includes all information provided from the ordinary one.

2.2 Categories in First-Order Logic

In this section, the language we consider, except specially instructing, is always $L = \langle Mor, Obj, \underline{dom}, \underline{cod}, id, \overline{comp}\rangle$ with equality, where Mor, Obj are 1-ary relations, $\underline{dom}, \underline{cod}, id$ are 1-ary functions and $\overline{comp}$ is an 2-ary function.

Definition 2.29. *A **category** C is a model of L that respects equality and satisfies the theory T consisting of the following six closed formulas of L:*

$$\begin{aligned}
&F_1 = \forall v_0(Morv_0 \vee Objv_0),\\
&F_2 = \forall v_0(Morv_0 \Rightarrow (Obj\underline{dom}v_0 \wedge Obj\underline{cod}v_0)),\\
&F_3 = \forall v_0(Objv_0 \Rightarrow (Mor\underline{id}v_0 \wedge \underline{dom}idv_0 \simeq v_0 \wedge \underline{cod}idv_0 \simeq v_0)),\\
&F_4 = \forall v_0\forall v_1((Morv_0 \wedge Morv_1 \wedge \underline{cod}v_0 \simeq \underline{dom}v_1)\\
&\qquad \Rightarrow (Mor\overline{comp}v_0v_1 \wedge \underline{dom}\overline{comp}v_0v_1 \simeq \underline{dom}v_0 \wedge \underline{cod}\overline{comp}v_0v_1 \simeq \underline{cod}v_1)),\\
&F_5 = \forall v_0(Morv_0 \Rightarrow (\overline{comp}v_0id\underline{cod}v_0 \simeq v_0 \wedge \overline{comp}id\underline{dom}v_0v_0 \simeq v_0)),\\
&F_6 = \forall v_0\forall v_1\forall v_2((Morv_0 \wedge Morv_1 \wedge Morv_2 \wedge \underline{cod}v_0 \simeq \underline{dom}v_1 \wedge \underline{cod}v_1 \simeq \underline{dom}v_2)\\
&\qquad \Rightarrow (\overline{comp}\overline{comp}v_0v_1v_2 \simeq \overline{comp}v_0\overline{comp}v_1v_2));
\end{aligned}$$

in other words, $\mathsf{C} \models T$.

Provided with this definition, a category C is a collection consisting of morphisms and objects (following from the axiom F_1) such that

1) each morphism f has two specified objects, i.e. its domain $dom(d)$ and codomain $cod(f)$ (following from the axiom F_2),

2) each object X has one specified morphism whose domain and codomain are both X itself, i.e. its identity morphism (or identity) $1_X : X \to X$ (following from the axiom F_3),

3) each pair of morphisms f, g with $cod(f) = dom(g)$ has one specified morphism whose domain is $dom(f)$ and codomain is $cod(g)$, i.e. their composite morphism (or composite) $g \circ f : dom(f) \to cod(g)$ (following from the axiom F_4),

4) for any morphism f, there are $1_{cod(f)} \circ f = f$ and $f \circ 1_{dom(f)} = f$ (following from the axiom F_5),

5) for any triple of morphisms f, g, h with $cod(f) = dom(g)$ and $cod(g) = dom(h)$, there is $h \circ (g \circ f) = (h \circ g) \circ f$, i.e. the notation $h \circ g \circ f$ is well-defined (following from the axiom F_6).

This corresponds to the ordinary definition of a category.

Definition 2.30. *Let* C, D *be two categories. A* ***functor*** F *from* C *to* D *is a homomorphism* $F : \mathsf{C} \to \mathsf{D}$.

While functors are defined to be specific maps between collections, the parentheses are often omitted unless demanded for notational clarity. Provided with this definition, a functor $F : \mathsf{C} \to \mathsf{D}$ between categories C and D is a map such that

1) F sends each object X in C to an object FX in D (respecting interpretations of Obj),

2) F sends each morphism $f : X \to Y$ in C to a morphism $Ff : FX \to FY$ in D (respecting interpretations of Mor, $\underline{dom}$ and $\underline{cod}$),

3) for each object X in C, F sends 1_X to 1_{FX} (respecting interpretations of id),

4) for each pair of morphisms f, g in C with $cod(f) = dom(g)$, there is $Fg \circ Ff = F(g \circ f)$.

This corresponds to the ordinary definition of a functor between categories. In addition, for each category C, there is an identity functor, denoted by 1_{C}, that serves as the identity of the model itself. Isomorphisms between functors are thus naturally defined to be isomorphisms between models, and the natural isomorphic relation partitions the collection of categories into equivalence classes.

The collection of categories and functors between them assemble into a category, with composition of functors as compostion of homomorphisms. But we should be careful that this may cause a paradox. To clear from that, we often restrict ourselves to locally small categories and speak of the category of locally small categories CAT. Things will be further discussed in the following section.

2.3 Limitations of First-Order Logic as A Foundation

classical mathematical subjects fit well in the foundation of first-order logic, which is sufficient for formalizing them properly. However, it still has some limitations and weaknesses when applying to categories. Firstly, there is more information provided for a 1-category in Definition 2.29: in fact, we have defined two binary operations dom and cod on the category, but only those images of morphisms are meaningful in category theory and are called domains and codomains; similar non-sense interpretations are defined for images of morphisms of id and of non-composable morphisms of $comp$. One way is to re-define the concept of a category by identifying any object with its identity morphism, but problems still exist for the interpretation of $comp$. Secondly, category theory suggests a particular perspective to be used in the study of mathematical objects that pays much attention to the maps between them, i.e. functors between categories, while first-order logic seems not to emphasize homomorphisms between models that much. All these clues suggest category theory induce a novel foundation for mathematics that is more widely applicable. This is what we are going to give a brief overview about in the next section.

One more thing needs to be discussed. In naive set theory and ordinary category theory, there are respectively Russell's paradoxes as 'set of all sets' and 'category of all categories'. This issue seems to be solved by using the vague word 'collection' instead of 'set' when defining a structure(model) in first-order logic. By von Neumann's cumulative hierarchy, there are infinitely many universes in our mathematical world. For a most front universe $\mathcal{U}$, there are just sets. It is in fact a model of the language with equality consisting of a single 2-ary relation $\in$ called membership that respects equality and satisfies the Zermelo-Frankel axioms. Small categories are just those whose objects and morphisms belong to one of this universe, including Cat, the category of all small categories. But we can transfinitely construct the hierarchy and obtain another universe $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{V}$. The notion of collections just include all cases of universes in this hierarchy. However, we should be conscious that Russell's paradoxes do not disappear. Therefore when defining the category of categories, it has to be not too 'large' to include itself as an object, which we guarantee by only considering locally small categories. This is the definition of CAT. In fact, Gödel's second imcompleteness theorem states that any formal axiomatic theory cannot demonstrate its own consistency. This is actually a limitation for all formal axiomatic theory, including type theory which we will briefly discuss. It perfectly solves the problems of category theory mentioned in the previous paragraph.

2.4 An Overview of Type Theory

In first-order logic, interpretations of equality, if exists, pervades throughout any model that respects equality. However, in mathematical usage, many cases happen where speaking about equality does not make sense for all pairs of variables, such as objects and morphisms of various dimension in ∞-category theory. Type theory comes out to divide the mathematical objects we are focusing on into different parts, say types, such that identity information only applies for variables in the context of a common type. Like the way in ZFC set theory to avoid Russell's paradox, this system creates a hierarchy of types by exclusively building larger types, i.e. types in the higher hierarchy, and then assigns each mathematical object to a specific type. For instance, an identity type is of the form

'in the type of two terms $x, y : A$, there is another type $x =_A y$',

where terms in types witness the truth of the statement within a single type. Here we mainly follow the version of Martin-Löf's type theory called dependent type theory[9]. It is a formal system where all constructions are continuous in paths and equivalences between types are well-defined, following from the univalence axiom in the system, with the latter one resulting this system to become the system of univalence foundation, which has a more famous name among categorists: homotopy type theory. Advantages for choosing this as the foundation of category theory and ∞-category theory are not only reflected as being more compatible with the essential idea of categorical notions, but also when comparing proofs of some significant theorems, such as Yoneda lemma with the ordinary ones. Type theory indeed provides an alternative choice of foundation that are more suitable than first-order logic in some cases.

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