CLC____________ Number__________

UDC___________ Available for reference Yes ☐No ☐

Undergraduate Thesis

Thesis Title： A reading report on Riemann--Hilbert correspondences

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A reading report on Riemann--Hilbert correspondence

[Abstract] The early version of Riemann--Hilbert correspondence was first stated in Hilbert's 21st problem, which was solved and generalized by Deligne. Later, Kashiwara gave a vast generalization of the work of Deligne. Now the Riemann--Hilbert correspondence has become a crucial tool in algebraic geometry, arithmetic geometry, and number theory. This thesis aims to review the different versions of the Riemann--Hilbert correspondence and the basic concepts behind them and also expand some details in Deligne's proof of a relative version of Riemann--Hilbert correspondence

[摘要] 早期版本的黎曼--希尔伯特对应被陈述在希尔伯特第二十一问题当中。德利涅解决了这个问题并且将其推广到更一般的形式。后来Kashiwara给出了更加一般的推广。使得黎曼希尔伯特对应成为了代数几何与数论当中的重要工具。本文主要介绍不同版本的黎曼--希尔伯特对应并回顾其涉及到的基本概念，同时展开了德利涅的一个证明的细节。

[关键词] 黎曼-希尔伯特对应，D模，全纯联络

[Key Words] Riemann--Hilbert-correspondence; D-module; holomorphic connection

Contents

1 Introduction

The Riemann–Hilbert correspondence was first stated for Riemann surfaces. Let X be a compact Riemann surface. Let U be an open subset on X such that $X \setminus U$ consists of finitely many points z_1, ..., z_m. We consider a differential equation locally of the following form:

$$\left((\frac{d}{dz})^n + \sum_{k=1}^{n} a_{n-k}(z)(\frac{d}{dz})^{n-k} \right) f(z) = 0 \tag{1}$$

where each $a_k(z)$ is holomorphic.

For convenience, we consider X to be the unit disk D in $\mathbf{C}$, and U to be the punctured unit disk. Also, we assume (1) is defined globally on U. When each $a_{n-k}(z)$ has at most a pole of order k at 0, we say the equation has a regular singularity at 0.

Let's consider the solutions $u_1(t)$, ... , $u_n(t)$ of (1) . Due to the non-trivial fundamental group of U, these solutions may not be well defined on the whole punctured disk. However, we can consider the polar coordinate $z = e^{2\pi t}$ around 0 and the universal covering of U parameterized by t. Then we can consider the lift of solutions denoted by $\widetilde{u_1}(t)$, ... , $\widetilde{u_n}(t)$ on the universal covering. Let $\widetilde{u}(t)$ be the complex vector $(\widetilde{u_k}(t))_{k=1,\ldots,n}$.

Considering the uniqueness of solutions, there exists a matrix $M \in GL(n, \mathbf{C})$ such that

$$\widetilde{u}(t+1) = M\widetilde{u}(t)$$

for each t.

The matrix M is usually called the matrix of monodromy. It gives a complex representation of $\pi_1(U)$.

This example inspires us to relate the representations of $\pi_1(U)$ with differential equations with regular singularities on X(In fact there is some ambiguity in this description). Hilbert's 21^{st} problem asks if we can obtain all finite dimensional complex representations of $\pi_1(U)$ from the monodromies of differential equations with regular singularities.

There is also another view of the representations of $\pi_1(X)$, that is, the locally constant sheaves, in other words, local systems on X. In fact, we have the following theorem from [1]:

Theorem 1.1. *Let X be a connected and locally simply connected topological space, and let x be a point in x. The category of locally constant sheaves of sets on x is equivalent to the category of sets endowed with a left action of $\pi_1(X, x)$.*

Here, we consider the equivalence between the category of locally constant sheaves of finite dimensional complex vector spaces and complex representations of $\pi_1(X, x)$.

On the other hand, differential equations can be obtained from holomorphic connections on complex vector bundles of X. This will be explained in the following sections of this thesis.

So, we can roughly talk about the Riemann–Hilbert correspondence. A classical and easier version of Riemann-Hilbert correspondence refers to the equivalence between the category of integrable connections and the locally constant sheaf of complex vector spaces on a complex manifold. The definition of an integrable connection will be given in the next sections. Deligne related the regularity of differential equations mentioned earlier in the introduction of this thesis and integrable connections in his famous work[2]. The definition of a connection with regular singularities for varieties may be due to Deligne. Roughly speaking Deligne's Riemann—Hilbert correspondence is the equivalence between integrable connections with regular singularities along a divisor on a smooth algebraic variety and local systems on the complement open subvariety. Deligne also showed that a holomorphic integrable connection must be regular under his settings. Later generalizations replaced integrable connections with regular holonomic D-modules and locally constant sheaves with perverse sheaves.

This thesis mainly provides a review of some basic concepts for the Riemann-Hilbert correspondence, a proof of the relative Riemann–Hilbert correspondence without singularities, and a little introduction to Deligne's Riemann–Hilbert correspondence for regular singularities and kashiwara's Riemann–Hilbert singularities for regular holonomic D-modules.

2 Basic concepts

2.1 Analytic space and sheaves

Definition 2.1 (local model space). *A local model space X is the vanishing set of some analytic functions f_1,..., f_m on an open set V of $\mathbf{C}^n$ for some $n \in \mathbf{Z}$, which the structure sheaf $\mathcal{O}_v/\mathcal{I}_Z$ where $\mathcal{O}_v$ is the sheaf of holomorphic functions on V and $\mathcal{I}_Z$ is the ideal sheaf generated by f_1,..., f_m.*

Definition 2.2 (complex analytic space). *A complex analytic space X is a locally ringed space $(X, \mathcal{O}_x)$ such that for each $x \in X$, there is an open neighborhood U of x which is isomorphic to a local model space as locally ringed $\mathbf{C}$-space.*

Definition 2.3. *A morphism between analytic varieties is a morphism between them as locally $\mathbf{C}$ ringed space.*

Complex analytic spaces, also called complex analytic varieties in some texts, are generalizations to complex manifolds allowing the existence of singularities.

Definition 2.4 (quasi-coherent sheaf). *A quasi-coherent sheaf $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is an $\mathcal{O}_X$-module such that for each $x \in X$, there is an open neighborhood U around X such that $\mathcal{F}|_U$ is isomorphic to the co-kernel of map of free $\mathcal{O}_X$-modules.*

Definition 2.5. *For a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module of finite type is an $\mathcal{O}_X$-module such that for each $x \in X$, there is an open neighborhood U around X such that $\mathcal{F}|_U$ can be generated by finitely many global sections.*

Definition 2.6 (coherent sheaf). *A coherent sheaf $\mathcal{F}$ on a ringed space is an $\mathcal{O}_X$-module of finite type satisfying the additional condition that for any open set $U \in X$ and a morphism $\mathcal{O}_X^n \xrightarrow{f} \mathcal{F}$, the kernel is of finite type.*

Remark 2.1. *A coherent sheaf $\mathcal{F}$ on a separable analytic space, is endowed with a natural Fréchet topology depending on the uniformly converging of analytic functions on a compact subset.*

Definition 2.7. *For a ringed space $(X, \mathcal{O}_X)$, we say $\mathcal{O}_X$ is a coherent sheaf of rings if it is a coherent sheaf as a module over itself.*

The following lemmas are useful in Deligne's proof of the relative version of the classical Riemann–Hilbert correspondence.

Lemma 2.2 (Oka's lemma)**.** *The sheaf of analytic functions on* $\mathbf{C}^n$ *is coherent.*

Lemma 2.3 (syzygy)**.** *For a coherent sheaf* $\mathcal{F}$ *on a complex manifold* M*,* $\mathcal{F}$ *locally admits a free resolution.*

Proof. Since this is local on M, WLOG M is an open subset of $\mathbf{C}^n$ for some n. For each $x \in X$, there is an open neighborhood U such that there are morphism off sheaves

$$\mathcal{O}^{\oplus m_1} \xrightarrow{i_1} \mathcal{O}^{\oplus m_0} \xrightarrow{i_0} \mathcal{F}$$

such that i_0 is surjective and the sequence is exact in the middle.
By Oka's lemma, $\mathcal{O}^{\oplus m_1}$ and $\mathcal{O}^{\oplus m_0}$ are also coherent. So one can continue this process by shrinking U and adding i_2,...,i_k,... It is also known that the local ring of germs at x is regular. By Hilbert's syzygy, the stalk of $\ker i_k$ at X will finally become free. It is known that a coherent analytic sheaf with a free stalk at a point x is free in an open neighborhood of x. Thus $\mathcal{F}$ admits a free resolution in a neighborhood of x. □

Lemma 2.4 (generalized Oka's lemma)**.** *The structure sheaf* $\mathcal{O}_X$ *of any complex analytic space* $(X, \mathcal{O}_X)$ *is coherent.*

For morphisms of analytic spaces, there are operations or formally named functors for sheaves and coherent sheaves on the spaces.

Definition 2.8 (direct image sheaf)**.** *For a continuous map* $X \xrightarrow{f} Y$ *of topology spaces. The direct image sheaf* $f_*\mathcal{F}$ *of a sheaf* $\mathcal{F}$ *on* X *is a sheaf on* Y *defined by* $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ *for any open set* U *on* Y*.*

Definition 2.9 (sheaf theoretic inverse image)**.** *For a continuous map* $X \xrightarrow{f} Y$ *of topology spaces. The sheaf theoretic inverse image of a sheaf* $\mathcal{F}$ *on* Y *is a sheaf on* X *associated to a presheaf* $\mathcal{G}$ *on* X *defined by* $\mathcal{G}(U) = \varinjlim_{f(U)\subset V} \mathcal{F}(V)$

Definition 2.10 (pullback)**.** *For a morphism* $X \xrightarrow{f} Y$ *of ringed spaces, and a* $\mathcal{O}_Y$*-module* $\mathcal{F}$ *on* Y*. Note that* $f^{-1}\mathcal{F}$ *is a* $f^{-1}\mathcal{O}_Y$*-module and there exists a morphism* $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$*. So we define the pullback of* $\mathcal{F}$ *to be* $f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$

There is another functor that is less common but very important in the study of perverse sheaves.

Definition 2.11 (direct image with proper support)**.** *For a continuous map $X \xrightarrow{f} Y$ of locally compact Hausdorff topology spaces which is either an open inclusion or a closed inclusion. For a sheaf $\mathcal{F}$ on X, define the direct image sheaf of $\mathcal{F}$ with proper support to be*

$$f_!\mathcal{F}(U) = \{s \in \mathcal{F}(f^{-1}(U)) |\ f|_{Supp_s}\, is\, proper\}$$

.

The so-called GAGA[3] is an important work of J.P.Serre. It compares algebraic varieties over **C** with analytic varieties and allows us to use techniques in complex analysis to study algebraic geometry.

Definition 2.12. *For a* **C***-scheme X of finite type, we define X_h to be the associated analytic space of X in the following way: Assume $X = \cup_{i\in I} SpecY_i$ be an open covering of X . Each Y_i is a finite type* **C***-algebra. Assume $Y_i = [x_1, ..., x_n]/(f_1, ..., f_m)$ for some n and f_1,..., f_m. Therefore one has the local model space $(SpecY_i)_h$ associated with Y_i. Then one glues the $(SpecY_i)_h s$, in the same way, gluing $SpecY_i s$(It is clear how we translate the morphism between quasi-affine complex varieties into holomorphic maps(thus morphism between analytic varieties) between analytic varieties) to a space X_h. It follows that X_h does not depend on the choice of the covering.*

The associated analytic space of a **C**-scheme of finite type can be understood as a functor $(-)^{an}$ from the category of **C**-schemes of finite type to the category of complex analytic spaces. Many important properties are preserved by this functor.

For example, regularity, normality, being irreducible, being connected, and dimension are all preserved. All finite limits are preserved. Many common properties of morphisms are also preserved.

2.2 Connections

Definition 2.13. *For a locally ringed space X, we called a locally free $\mathcal{O}_X$-module of finite rank a vector bundle on X.*

Definition 2.14. *For a morphism of locally ringed space* $X \xrightarrow{f} S$*, a relative local system on* X *is a sheaf of* $f^{-1}\mathcal{O}_S$*-modules which is locally isomorphic to the sheaf-theoretic inverse image of a coherent sheaf on* S*.*

Remark 2.5. *A complex analytic space* X *is considered as a morphism* $X \xrightarrow{f} Spec\mathbf{C}$*. Therefore a local system on* X *always refers to a locally constant sheaf of finite-dimensional* $\mathbf{C}$*-vector spaces on* X*.*

Remark 2.6. *Note that for a locally constant sheave of* $\mathbf{C}$*-vector spaces on an analytic space* X*, by tensoring with* $\mathcal{O}_X$ *one gets a vector bundle on* X*.*

Also, note that for a vector bundle $\mathcal{E}$ *on an analytic space* X *and a point* $x \in X$*, we can consider the fibre of* $\mathcal{E}$ *at* x *to be* $\mathcal{E}_x = \mathcal{E}_{(x)} \otimes \mathcal{O}_{(x)}/m_x$*, where* $\mathcal{E}_{(x)}$ *is the localization of* $\mathcal{E}$ *at* X*,* $\mathcal{O}_{(x)}$ *is the local ring at* X*, and* m_x *is the maximal ideal of* $\mathcal{O}_{(x)}$*.*

Definition 2.15. *Let* X *be a complex analytic space. Consider the diagonal* Δ *in the product* $X \times X$*. Then* Δ *is a locally closed subspace of* $X \times X$*. Let* $\mathcal{J}$ *be the sheaf of ideal of* Δ *in* $X \times X$*. By abuse of notations, let* Δ *be the diagonal morphism for* X *to* $X \times X$*. The sheaf of differentials, also called the sheaf of differential forms of first-order* Ω_X *on* X *is defined to be* $\Delta^*(\mathcal{J}/\mathcal{J}^2)$*.*

For complex manifold X of dimension n, Ω_X is locally free of rank n and is isomorphic to the sheaf of holomorphic 1-from on X.

The definition of the sheaf of differentials can also be given in a relative context. For morphism $X \xrightarrow{f} Y$ of analytic spaces, one can define the sheave of relative differentials $\Omega_{X/Y}$ on X.

Definition 2.16 (derivation)**.** *Let* $X \xrightarrow{f} Y$ *be a morphism of complex analytic spaces. A* $f^{-1}\mathcal{O}_Y$*-derivation of* $\mathcal{O}_X$ *is a* $\mathcal{O}_Y$*-linear map:*

$$\mathcal{O}_X \xrightarrow{d} \mathcal{M}$$

that satisfies the Leibniz rule where $\mathcal{M}$ *is a coherent sheaf on* X*.*

Lemma 2.7. *There is a derivation* d *from* $\mathcal{O}_{X/Y}$ *to* Ω_X *which is given by* $f \in \mathcal{O}_X(U) \mapsto 1 \otimes f - f \otimes 1 \in \mathcal{J}/\mathcal{J}^2(U \times U)$*. And* d *is initial among all derivations.*

Definition 2.17. *For a morphism $X \xrightarrow{f} Y$, the tangent bundle $\mathcal{T}_{X/Y}$ is defined to be the sheaf $\mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X)$.*

Lemma 2.8. *For each open set U on X, $\mathcal{T}_{X/Y}$ is canonical isomorphic to $Der_{f^{-1}\mathcal{O}_Y|_U}(\mathcal{O}_U, \mathcal{O}_U)$.*

A morphism $X \xrightarrow{f} Y$ of complex analytic spaces gives a homomorphism $f^*\Omega_Y \to \Omega_X$.

Also, for a commutative diagram of morphisms of analytic spaces:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\ Y & \xrightarrow[f]{} & X \end{array}$$

There is a canonical $\mathcal{O}_{Y'}$-homomorphism

$$g^*\Omega_{Y/X} \to \Omega_{X'/Y'}$$

induced by

$$1 \otimes g^{-1}(d_{X/Y}(s)) \mapsto d_{X'/Y'}(1 \otimes g^{-1}(s))$$

Moreover, if this is a pull-back diagram, then $g * \Omega_{Y/X} \to \Omega_{Y'/X'}$ is an isomorphism. By the adjoint property of g^* and g_*, one also knows that for a pull-back diagram, there is a canonical isomorphism $\Omega_{Y/X} \to g_*\Omega_{Y'/X'}$ when the diagram is a pull-back diagram.

Definition 2.18. *A smooth morphism of relative dimension n of analytic spaces is a morphism $X \xrightarrow{f} Y$ such that:*
For each $x \in X$,there is a open neighborhood U such that the restriction of f on U is isomorphic to a projection $D^n \times Y \to Y$.

Lemma 2.9. *For a smooth morphism $X \xrightarrow{f} S$ of relative dimension n, $\Omega_{X/S}$ is locally free of dimension n.*

Definition 2.19. *For a complex analytic space X, and a vector bundle $\mathcal{E}$ on it, a connection on $\mathcal{E}$ is a $\mathbf{C}$-linear map:*

$$\nabla : \mathcal{E} \longrightarrow \Omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

that satisfies the Leibniz rule:

$$\nabla(fs) = df \otimes S + f \cdot \nabla(s)$$

for each section f of $\mathcal{O}_X$ and each section s of $\mathcal{E}$ on a open set.

One should note that the definition of a connection only requires $\mathcal{E}$ to be a coherent sheaf. However, it is a nontrivial fact that a coherent sheaf on a smooth analytic space equipped with an integrable must be locally free. For the definition in the relative context, i.e., for a morphism $X \xrightarrow{f} S$, we only require $\mathcal{E}$ to be a coherent sheaf on X.

There are operations on connections. For connections $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$, we can define$(\mathcal{E}_1 \oplus \mathcal{E}_2, \nabla_1 \oplus \nabla_2)$ by

$$\mathcal{E}_1 \oplus \mathcal{E}_2 \longrightarrow (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_1) \oplus (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_2) \xrightarrow{isomorphism} \Omega_X \otimes_{\mathcal{O}_X} (\mathcal{E}_1 \oplus \mathcal{E}_2)$$

and define $(\mathcal{E}_1 \otimes \mathcal{E}_2, \nabla_1 \otimes \nabla_2)$ by

$$s_1 \otimes s_2 \mapsto \nabla_1(s_1) \otimes s_2 + \nabla_2(s_2) \otimes s_1$$

We can also define dual connections. For a connection $(\mathcal{E}, \nabla)$, we define $(\mathcal{E}^\vee, \nabla^\vee)$ by

$$\nabla^\vee(l) = d \circ l - 1 \otimes l \circ \nabla$$

For an $\mathcal{O}_X$-homomorphism $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ of vector bundles on X with connections ∇ and ∇'. We say f is horizontal if

$$\nabla' \circ f = f \circ \nabla$$

Definition 2.20 (homomorphism between connections). *A homomorphism between connections $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ is a horizontal $\mathcal{O}_X$-homomorphism between them.*

By some computations, one can show that the kernels and cokernels of a homomorphism between connections still form connections.

The connections on an analytic space X form an abelian category, denoted by $MC(X)$. The relative connections on $X \xrightarrow{f} Y forms$ an abelian category denoted by $MC(X/Y)$.

There are also pull-backs for connections. For a commutative diagram of analytic spaces:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[f]{} & X \end{array}$$

And a relative connection $(\mathscr{E}, \nabla)$ on Y, $g^* \mathscr{E}$ is equipped with a connection $g^*\nabla$ given by the composition of maps:

$$g^* \mathscr{E} \xrightarrow{\nabla} g^*(\Omega_{x/Y} \otimes_{oo_Y} \mathscr{E}) \xrightarrow{\tau} \Omega_{X'/Y'} \otimes_{\mathscr{O}_{Y'}} g^* \mathscr{E}$$

Where τ is induced by the $\mathscr{O}_Y$-homomorphisms $g^*\Omega_{x/Y} \to \Omega_{X'/Y'}$.

Definition 2.21 (De Rham complex)**.** *For a complex analytic space X, define $\Omega_X^P = \wedge^p \Omega_X$, the p^{th} exterior power of Ω_X. We can extend the derivation $d : \mathscr{O}_X \xrightarrow{d} \Omega_X$ to $\mathbf{C}$-linear maps from Ω_X^P to Ω_X^{P+1} for each non-negative integer p by a unique way such that*

$$d(f\omega) = df \wedge \omega + (-1)^p f \cdot d\omega$$

where f is a local section of $\mathscr{O}_X$ and ω is a local section of Ω_X^P. Then all Ω_X^Ps together with d is called the de Rham complex of X.

As for the cotangent bundle, pull-back also works for De-Rham complexes.

Definition 2.22. *Let $\mathscr{E}$ be a vector bundle on a complex analytic manifold X. Define $\Omega_X^p(\mathscr{E}) = \wedge^p \Omega_X \otimes_{\mathscr{O}_x} \mathscr{E}$. We can define a $\mathbf{C}$-linear morphisms*

$$\nabla : \Omega_X^p(\mathscr{E}) \mapsto \Omega_X^{p+1}(\mathscr{E})$$

by letting

$$\nabla(\omega, s) = d\omega \otimes s + (-1)^p \omega \wedge s$$

where ω is a local section of Ω_X^P and s is a local section of $\mathscr{E}$.

Definition 2.23. *The curvature R of the connection $(\mathscr{E}, \nabla)$ is defined to be the composition*

$$\nabla^2 : \mathscr{E} \longrightarrow \Omega_X^2(\mathscr{E})$$

Definition 2.24. *A connection is said to be integrable if its curvature is 0.*

Lemma 2.10. *The pull-back of an integrable connection is integrable.*

If X is smooth, then an equivalence condition for a connection $(\mathscr{E}, \nabla)$ to be integrable is that ∇ is a Lie-algebra homomorphism from $\mathscr{T}_X$ to $\mathscr{E}nd\, \mathscr{E}$.

Note that if a connection is integrable, $\nabla^2 = 0$ so that $\Omega_X^p(\mathcal{E})$ forms a differential complex.

We see in dimension 1, that a holomorphic connection is automatically integrable, since $\wedge^2\Omega_X = 0$.

Definition 2.25. *Let $(\mathcal{E}, \nabla)$ be a connection on X, $Ker(\nabla)$ is said to be the sub-sheaf of horizontal sections of $(\mathcal{E}, \nabla)$.*

One should take care that $Ker(\nabla)$ is only a sheaf of $\mathbf{C}$-vector space but not a coherent sheaf on X since ∇ is not a homomorphism of $\mathcal{O}_X$-modules.

2.3 D-modules

Definition 2.26. *Let X be a complex manifold of dimension n. The sheaf $\mathcal{D}_X$ of the algebra of differential operators is defined by*

$$\mathcal{D}_X(U) = \{ \sum_{\lambda \in \mathbf{Z}_{\geq 0}{}^n} f_\lambda \prod_{i=1}^{n} \partial_{x_i}^{\lambda_i} | \text{ each } f_\lambda \text{ is holomorphic in } U\} \tag{2}$$

The multiplication is given by $PQ(h) = P(Q(h))$ for P, $Q \in \mathcal{D}_X(U)$ and $h \in \mathcal{O}_U$

Remark 2.11. *For an algebraic version of D-modules, one can still use a local coordinate system instead of a holomorphic chart in the definition.*

Another definition that does not rely on the coordinate is as follows.

Definition 2.27. *The sheaf of holomorphic vector fields on a complex manifold is defined as:*

$$\Theta_X = \{\theta \in \mathcal{E}nd_{\mathbf{C}_X}(\mathcal{O}_X) \mid \theta(fg) = \theta(f)g + f\theta(g)\} \tag{3}$$

Here $\mathbf{C}_X$ just stands for the constant sheaf of $\mathbf{C}$ on X.

And $\mathcal{D}_X$ is defined as the subring of $\mathcal{E}nd_{\mathbf{C}_X}$ generated by $\mathcal{O}_X$ and Θ_X.

Since generally the $\mathcal{D}_X$ does not commute, it is necessary to distinguish left and right $\mathcal{D}_X$-modules.

Lemma 2.12. *For a vector bundle $\mathcal{E}$ on X, giving a $\mathcal{D}$-module structure on $\mathcal{E}$ is the same as giving a $\mathcal{O}_X$-linear homomorphism:*

$$\nabla : \Theta_X \to \mathcal{E}nd_{\mathbf{C}_X}\,\mathcal{E}$$

such that

$$\nabla_\theta(fv) = \theta(f)\nabla_\theta(v)$$

for each sections θ, f, andv, and

$$\nabla_{[\theta_1,\theta_2]}(v) = [\nabla_{\theta_1}, \nabla_{\theta_2}](v)$$

for each sections θ_1,θ_2 and v.

Lemma 2.13. *When $\mathcal{E}$ is a free(or locally free) $\mathcal{D}_X$-module of finite rank, the map ∇ in the previous lemma is equivalent to a connection defined in previous sections for a complex manifold, and it is integrable.*

Lemma 2.14. *For a vector bundle $\mathcal{E}$ on X with integrable connection ∇, consider the left action of $\mathcal{D}_X$ on it by*

$$\xi \cdot v =< \nabla_\xi, v >$$

for sections ξ and v of θ and $\mathcal{E}$, then $\mathcal{E}$ is given a $\mathcal{D}_X$-module structure.

Definition 2.28. *A coherent $\mathcal{D}_X$-module is a $\mathcal{D}_X$-module locally isomorphic to the cokernel of free $\mathcal{D}_X$-modules.*

The sheaf $\mathcal{D}_X$ is equipped with a natural filtralization F from the order of differential operators. Consider $gr^F\mathcal{D}_X := \oplus_{i\geq 0}F_i\mathcal{D}_X/F_{i+1}dd_X$ which is a sheaf of commutative rings generated by differential operators.

Let X be a smooth complex algebraic variety. Then the cotangent sheaf Ω_X is locally free, thus giving rise to a complex algebraic vector bundle $T^*X \xrightarrow{\pi} X$. And $\pi_*\mathcal{O}_{T^*X}$ is a sheaf of $\mathcal{O}_X$ algebra generated by Ω_X, which is naturally isomorphic to $gr^F\mathcal{D}_X$. Let M be a coherent $\mathcal{D}_X$-module with a good filtration F. Then $gr^F M$ is a coherent $\pi_*\mathcal{O}_{T^*X}$ module. So $\mathcal{O}_{T^*X}\otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}}\pi^{-1}gr^F M$ is a coherent sheaf on T^*X.

Definition 2.29. *The characteristic variety of M is defined to be*

$$Ch(M) = \operatorname{supp} \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_* \mathcal{O}_{T^*X}} \pi^{-1} gr^F M$$

which is a closed subvariety of T^*X.

It turns out that $Ch(M)$ does not rely on the filtration.

Definition 2.30 (holonomic D-module)**.** *A nonzero coherent $\mathcal{D}_X$-module is said to be holonomic if* $\dim Ch(M) = dim X$*.*

2.4 Intersection homology and perverse sheaves

2.4.1 Introduction

Perverse sheaves are generalizations of locally constant sheaves. It comes from the study of the homology of spaces with singularities. For manifolds, there are good results for its homology groups, such as the Poincare duality. However, these fail for varieties with singularities. In the intersection homology and cohomology theory, a good homology for varieties with singularities is constructed. And later this is described with the language of derived categories and perverse sheaves. This section aims to provide some basic concepts and results of this theory organized according to the historical order, mainly following [4].

In this section, we only consider reasonable topological spaces and sheaves in $Sh(X, \mathbf{K})$ (sheaves of $\mathbf{K}$-modules on a reasonable topological space X) where $\mathbf{K}$ is a commutative Noetherian ring.

2.4.2 Stratifications and intersection chain complexes

Definition 2.31 (topological pseudo manifolds)**.** *A topological manifold is a topological stratified space*

$$X \supset X_{n-2} \supseteq X_{n-3} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$$

Such that $X \setminus X_{n-2}$ is dense in X.

To discuss intersection homology, we restrict to the case X is a PL pseudo manifold. In this case, the intersection chains are all PL chains and their intersection with strata is easy to discuss.

Definition 2.32 (perversity)**.** *A perversity $\bar{p}$ is a sequence $p(2),\ldots,p(n)$ such that $p(2) = 0$ and $p(k) \leq p(k+1) \leq p(k)+1$ for all k.*

Definition 2.33 (p-allowable chains)**.** *A PL chain ξ of dimension i is said to be $\bar{p}$-allowable if for all $k \geq 2$*

$$\begin{cases} \dim(\xi \cap X_k) \leq p(k) + i - k \\ \dim(\partial\xi \cap X_k) \leq p(k) + i - k - 1 \end{cases}$$

Now we define the intersection homology groups $\mathrm{IH}^{\bar{p}}_{\bullet}(X)$ of a PL pseudo manifold w.r.t a perversity p to be the homology groups given by the chain complex consisting of p-allowable chains.

2.4.3 Deligne's construction of intersection cohomology

Let $D^b(X,\mathbf{K})$ be the bounded full subcategory of derived category of $Sh(X,\mathbf{K})$ derived category of the category of complexes of sheaves on X. It turns out that $D^b(X,\mathbf{K})$ is no longer an abelian category, instead it is a triangulated category.

Definition 2.34 (constructible sheaves)**.** *Let S be a stratification of X, with strata $\{X_i \xrightarrow{h_i} X\}$.*
We say $\mathcal{F} \in Sh(X,\mathbf{K})$ is constructible w.r.t. S if each $h_i^{-1}\mathcal{F}$ is a locally constant sheaf of finitely generated $\mathbf{K}$-modules.
We say $\mathcal{F}^\bullet \in D^b(X,\mathbf{K})$ is constructible if each $\mathcal{H}^j(\mathcal{F}^\bullet)$ is constructible w.r.t. S.
For $\mathcal{F}^\bullet \in D^b(X,\mathbf{K})$, we say in is constructible on X if it is constructible w.r.t. some stratification S.
Denote the above 3 categories by $Sh_S(X,\mathbf{K})$, $D^b_S(X,\mathbf{K})$, and $D^b_c(X,\mathbf{K})$.

Definition 2.35 (Shriek pull-back of sheaves)**.** *Let $X \xrightarrow{h} Y$ be a locally closed inclusion of topological spaces. For a sheaf $\mathcal{F}$ on Y, we define the Shriek pull-back of $\mathcal{F}$, $h^!\mathcal{F}$ to be the sheafification of the presheaf*
$\mathcal{G}(U) = \varinjlim_{V\cap\overline{X}=U}\{s \in \mathcal{F}(V) \mid supp(s) \in U\}$.

Remark 2.15. *If h is open, the $h^! = h^{-1}$.*

Lemma 2.16. *Let* $X \xrightarrow{h} Y$ *be a locally closed inclusion of topological spaces. Then* $h^!$ *is a right adjoint to* h_i.

Remark 2.17. *For arbitrary continuous maps between topological spaces, it is not able to define a right adjoint of the pushforward-with-proper-support functor for the categories of sheaves. One has instead to work in derived categories.*

Deligne constructed an intersection cohomology complex in ${}^p\mathrm{D}^{\mathrm{b}}_{\mathrm{S}}(\mathrm{X}, \mathbf{K})$, whose hypercohomology computes the intersection homology of X.

Definition 2.36. *Deligne's truncation functor For* $k \in \mathbf{Z}$, *the truncation of* $\mathcal{F}^\bullet \in D^b_c(X, \mathrm{K})$ *is given by:*

$$(\tau_{\leq k}\mathcal{F}^\bullet)^i = \begin{cases} \mathcal{F}^i & i \leq k-1 \\ \ker d^i & i = k \\ 0 & i > k \end{cases}$$

Let X be a topological pseudo manifold with filtration

$$X \supset X_{n-2} \supseteq X_{n-3} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$$

Let $U_k = X \setminus X_{n-k}$.

Let $\mathcal{F}$ be a locally constant sheaf on the regular part U_2. Let p be a perversity. Then one can define a sheaf on X inductively by $\mathcal{F}^\bullet_0$ be the complex containing only $\mathcal{F}$ in degree 0 on U, let

$$\mathcal{F}^\bullet_2 = \mathcal{F}^\bullet_0[n]$$

and for each $k \geq 2$, set

$$\mathcal{F}^\bullet_{k+1} = \tau_{\leq p(k)-n} i_* \mathcal{F}^\bullet_k$$

.

Definition 2.37. *Deligne complex Following the above construction, we define*

$$IC^\bullet_{\bar{p}}(\mathcal{F}) = \mathcal{F}^\bullet_{n+1}$$

Especially, we define $IC^\bullet_{\bar{p}}$ *to be* $IC^\bullet_{\bar{p}}(\mathcal{K})$, *where* $\mathcal{K}$ *is the constant sheaf with coefficient* K.

Remark 2.18. *The construction of the Deligne complex is due to Deligne, while Goresky and MacPherson developed a set of Axioms for the intersection cohomology complex to prove that its hypercohomology computes the intersection homology groups. To be explicit,* $\mathrm{H}^{-i}(IC^{\bullet}_{\bar{p}}) = I^{BM}\mathrm{H}^{\bar{p}}_{i}(X)$

Remark 2.19. *It is further shown that the intersection cohomology groups do not depend on the choice of stratification.*

2.4.4 Perverse sheaves

From now on we on consider complex algebraic or analytic varieties with strata being constructible subsets. As the middle perversity behaves well under duality, from now on we restrict our discussion to the middle perversity.

Perverse sheaves are not sheaves, instead, they form an abelian subcategory of the the derived category of complexes of sheaves on a complex algebraic or analytic variety.

Definition 2.38 (Perverse t-structure). *The Perverse t-structure on $D^b_c(X, \mathrm{K})$ of a complex algebraic or analytic variety x is given by:*

$$D^b_c(X)^{\leq 0} := \{\ \mathcal{F}^{\bullet} \in D^b_c(X, \mathrm{K}) \mid dim_{\mathbf{C}} supp^{-j}(\mathcal{F}^{\bullet}) \leq j,\ \ for\, any\, j \in \mathbf{Z}\}$$

$$D^b_c(X, \mathrm{K})^{\geq 0} := \{\ \mathcal{F}^{\bullet} \in D^b_c(X, \mathrm{K}) \mid dim_{\mathbf{C}} cosupp^{-j}(\mathcal{F}^{\bullet}) \leq j,\ \ for\, any\, j \in \mathbf{Z}\}$$

where

$$supp^{-j} := \overline{\{x \in X \mid i^{*}_{x}\mathcal{F}^{\bullet} \neq 0\}}$$

$$cosupp^{-j} := \overline{\{x \in X \mid i^{!}_{x}\mathcal{F}^{\bullet} \neq 0\}}$$

Definition 2.39 (Perverse sheaves). *With the above definition, we set*

$$Per(X, \mathbf{K}) = {}^{p}\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{K})^{\leq 0} \cap {}^{p}\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{K})^{\geq 0}$$

Definition 2.40 (The topological Deligne complex). *Following the previous section, we set the topological Deligne complex IC^{top}_{X} of a complex algebraic or analytic variety of pure dimension n to be $IC^{\bullet}_{\bar{m}}$ of X. And we define IC_X to be $IC^{top}_{X}[-n]$.*

Lemma 2.20. *With the condition in the above definition, IC_X is a perverse sheaf.*

Remark 2.21. *Perverse sheaves satisfy good gluing properties like sheaves. This is a reason why they are called "perverse sheaves".*

Let $X_1 \xrightarrow{i} X_0$ be an inclusion of an open constructible subset of analytic spaces with $X_0 \setminus X_1$ closed, and a perverse sheaf $\mathscr{F}^\bullet$ on X_1, one can use an intermediate extension to extend $\mathscr{F}^\bullet$ to a perverse sheaf on X_0.

Definition 2.41. *Let U be an open constructible subset of a complex algebraic or analytic variety X. Let i be the inclusion map. A sheaf complex $\mathscr{G}^\bullet$ in $D_c^b(X)$ is said to be a extension of $\mathscr{F}^\bullet$ in $D_c^b(U)$, if*

$$i^{-1}\mathscr{G}^\bullet \cong \mathscr{F}^\bullet$$

Definition 2.42 (intermediate extension)**.** *The intermediate extension $i_{i*}\mathscr{F}^\bullet$ of $\mathscr{F}^\bullet \in Per(U)$ is the image of the natural morphism $i_i\mathscr{F}^\bullet \to i_!\mathscr{F}^\bullet$ in $Per(X)$*

Remark 2.22. *The intermediate extension is between the minimal extension $i_!$ and the "largest" extension i_*.*

Remark 2.23. *It can be shown that for a complex algebraic(or analytic) variety X with an abritrary open subvariety $U \xrightarrow{i} X$, the intermediate extension of the IC sheaf on U is isomorphic to that of X. $i_{!*}IC_U \cong IC_X$.*

3 Classical Riemann–Hilbert correspondences

A classical version of Riemann-Hilbert correspondence states that the integrable holomorphic connections on a complex manifold and locally constant sheaves of finite dimensional complex vector spaces can be identified. The proof requires the Cauchy-Kowalevski theorem from PDE theory.

4 The relative Riemann–Hilbert correspondence for analytic manifolds by Deligne

Deligne also gave a beautiful but brief proof in[2] for Riemann—Hilbert correspondence in the relative context. In this section, I will state the relative version of the Riemann–Hilbert correspondence and expand some of the details of Deligne's proof in[2].

First, we give the formal statement of this version of Riemann–Hilbert correspondence:

Theorem 4.1. *Let* $X \xrightarrow{f} S$ *be a smooth morphism of complex analytic spaces. There exist functors* F *from the category of relative systems on* X *to the category of integrable relative connections on* X *and* G *from the category of relative integrable connections on* X *to the category of local relative systems on* X *such that the following holds:*
(a)For every local relative system V *on* X*,* $F(V) = (V \otimes_{f^{-1}(\mathcal{O}_s)} \mathcal{O}_X, \nabla)$ *such that* ∇ *is a relative integrable connection with* $\ker(\nabla) = V$
(b)For every relative integrable connection $(\mathcal{E}, \nabla)$ *on* X*,* $G(\nabla) = \ker(\nabla)$
(c)The functors F *and* G *are quasi-inverse to each other. So give an equivalence of categories.*
(d)The complex

$$\Omega_{X/S}(\mathcal{E}) : 0 \longrightarrow V \longrightarrow (F(V)) \longrightarrow \Omega_{X/S}(F(V)) \longrightarrow ... \longrightarrow \Omega^p_{X/S}(F(V)) \longrightarrow ...$$

is a resolution of V*.*

This is rephrased from [2], page 15, theorem 2.23, while still confusing and to be modified in the final submission.

The proof of this consists of several parts. First, prove that F gives integrable connections and F is fully faithful. Next, prove (d) which shows that GF is naturally isomorphic to id. Then it remains to prove that F is dense.

Proof: First, construct F. For a local relative system V, define

$$\nabla : V \otimes_{f^{-1}(\mathcal{O}_s)} \mathcal{O}_X \longrightarrow \Omega_{X/S}(V \otimes_{f^{-1}(\mathcal{O}_S)} \mathcal{O}_X)$$

by $\nabla(fs) = df \otimes s$. Then it follows that ∇ is an integrable relative connection. Next, prove part(d). Since f is smooth, it is locally isomorphic to a projection $pr_2 : D^n \times S \to S$.

First, consider the special case where $S = D^n$, $X = D^n \times D^m$, f is the projection pr_2 and V is simply $f^{-1}(\mathcal{O}_S)$. Then one can show the complex

$$0 \to \Gamma(f^{-1}(\mathcal{O}_S)) \to \Gamma(\mathcal{O}_X) \to \Gamma(\Omega_{X/S}) \to ...$$

is acyclic by constructing the following homotopy functor H:

(1) $H : \Gamma(\mathcal{O}_X) \to \Gamma(f^{-1}\mathcal{O}_S) = \Gamma(\mathcal{O}_S, S)$ is given by the zero section of f.

(2) Each element ω of $\Gamma(\Omega^p_{X/S})$ can be expressed into a unique converging series:

$$\omega = \sum_{\substack{I\subset[1,m] \\ |I|=p}} \sum_{\underline{n}\in\mathbf{N}^{m+n}} a^I_{\underline{n}} \left(\wedge_{i\in I} x_i^{n_i} dx_i\right) \left(\prod_{i\in[1,m+n]\setminus I} x_i^{n_i}\right)$$

and set

$$H(\omega) = \sum_{I\subset[1,m]} \sum_{j\in I} \sum_{\underline{n}\in\mathbf{N}^{m+n}} sign_I(j) a^I_{\underline{n}} \left(\wedge_{\substack{i\in I \\ i\neq j}} x_i^{n_i} dx_i \frac{x_j^{n_j+1}}{n_j+1}\right) \left(\prod_{i\in[1,m+n]\setminus I} x_i^{n_i}\right)$$

, where $sign_I(j)$ refers to the signature of j in I, to be explicit, if j is the r^th element of I (the same order with that appears in the wedge product), then $sign(j) = (-1)^{r-1}$. In the following equations, if $j \notin I$, then $sign(j)$ refers to the signature of j in $I \cup \{j\}$.

There is a grading of the complex. For a relative monomial p-form $\omega = \prod_{k\in[1,m]} x_k^{n_k} \wedge_{i\in I} dx_i$, we define its degree to be $p + |\{k \in [1, m] \setminus I | n_k \neq 0\}|$. It follows that both d and H preserve this degree. To avoid misreads with the degree of polynomials or dimension of chain complexes, I will call this degree the skew degree in the following paragraphs.

First, consider the subcomplex of skew degree 0, which only intersects
$\Gamma(f^{-1}(\mathcal{O}_S), X) = \Gamma(\mathcal{O}_S, S)$ and $\Gamma(\mathcal{O}_s)$. And it is direct to check this sub complex is exact.

Next, our computation will show that for the subcomplex of skew degree $q \geq 1$. The $\Gamma(\mathcal{O}_S)$-linear map $H \circ d + d \circ H$ equals the multiplication by q map. Then for each positive integer q the subcomplex of skew degree q is exact. If w is a form in $\ker d$, then consider $w = \sum_{k=0}^{\infty} w_k$. It follows that $w_k \in \ker d$ for each k. Then it follows that $w = d\left(\sum_{k=1}^{\infty} \frac{H(w_k)}{k} + w_0\right)$. Note that the coefficients of $\sum_{k=1}^{\infty} \frac{H(w_k)}{k}$ is well controlled by the coefficients of w, so $\sum_{k=0}^{\infty} \frac{H(w_k)}{k}$ is also converging. Therefore this will prove the exactness of the complex.

Now we begin our calculation. We only need to show the result for monomials.

Assume $f = \prod_{i=1}^{q} x_i^{n_i} \in \Gamma(\mathcal{O}_X)$. Then $H(f) = 0$, and

$$d(f) = \sum_{i=0}^{q} n_i x_i^{n_i - 1} \prod_{\substack{j\in[1,q]\\ j\neq i}} x_j^{n_j}$$

So

$$\begin{aligned}(H \circ d + d \circ H)(f) &= H(d(f))\\ &= H(\sum_{i=1}^{q} n_i x_i^{n_i-1} \prod_{\substack{j\in[1,q]\\ j\neq i}} x_j^{n_j})\\ &= \sum_{i=1}^{q} \prod_{i=1}^{q} x_i^{n_i} = qf\end{aligned}$$

Assume

$$\omega = (\wedge_{i\in I} x_i^{n_i} dx_i) \left(\prod_{i\in[1,m+n]\setminus I} x_i^{n_i} \right)$$

is a p-form of skew degree q, where $p \geq 1$.

Note that

$$\begin{aligned} d\omega &= d\left(\prod_{i\in[1,m+n]} x_i^{n_i} \right) \wedge (\wedge_{i\in I} dx_i)\\ &= \sum_{j\in[1,m]\setminus I} \left(n_j x_j^{n_j-1} \prod_{\substack{i\in[1,m+n]\\ i\neq j}} x_i^{n_i} \right) dx_j \wedge (\wedge_{i\in I} dx_i) \end{aligned} \tag{4}$$

And

$$(H\circ d)\omega = \sum_{\substack{j\in[1,m]\setminus I\\ n_j\neq 0}} \left(x_j^{n_j} \prod_{\substack{i\in[1,m+n]\\ i\neq j}} x_i^{n_i} \right) \wedge_{i\in I} dx_i \tag{5}$$

$$+ \sum_{j\in[1,m]\setminus I} \left(n_j x_j^{n_j-1} \prod_{\substack{i\in[1,m+n]\\ i\neq j}} x_i^{n_i} \right) dx_j \wedge \left(\sum_{k\in I} \frac{-sign_I(k) x_k}{n_{k+1}} \wedge_{\substack{i\in I\\ i\neq k}} dx_i \right) \tag{6}$$

$$= \sum_{\substack{j\in[1,m]\setminus I\\ n_j\neq 0}} \omega + \sum_{j\in[1,m]\setminus I} \left(n_j x_j^{n_j-1} \prod_{\substack{i\in[1,m+n]\\ i\neq j}} x_i^{n_i} \right) dx_j \wedge (\sum_{k\in I} \frac{-sign_I(k) x_k}{n_{k+1}} \wedge_{\substack{i\in I\\ i\neq k}} dx_i) \tag{7}$$

While

$$(d \circ H)\omega = d\left(\sum_{k\in I} \frac{sign_I(k)x_k}{n_k+1} \wedge_{\substack{i\in I\\ i\neq k}} dx_i \prod_{i\in[1,m+n]} x_i^{n_i}\right)$$
$$= \sum_{k\in I}\left(\omega + \frac{sign_I(k)x_k}{n_k+1} \sum_{j\in[1,m]} n_j x_j^{n_j-1} dx_j \wedge (\wedge_{\substack{i\in I\\ i\neq k}} dx_i) \prod_{\substack{i\in[1,m+n]\\ i\neq j}} x_i^{n_i}\right) \tag{8}$$

Then equation (7)+(8) shows that

$$(H \circ d + d \circ H)(w) = qw$$

Now we have proved the result we want in this step.

The same construction works on smaller poly-cylinders of D^{m+n}, the complex of sections on which is also exact. Since smaller polydiscs form a base of the polydisc, the complex of sheaf is exact on stalks. Thus the complex of sheaves in (d) is exact.

Remark 4.2. *This is a relative version of the holomorphic Poincare lemma.*

For an exact sequence of coherent sheaves on S:

$$0 \to V \to V' \to V'' \to 0$$

, since f is smooth and $\Omega^P_{X/S}$ is flat over $\mathcal{O}_S$ for each p, the sequence of differential complexes

$$0 \to \Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_s} f^{-1}V \to \Omega^{\cdot}_{X/s} \otimes_{\mathcal{O}_s} f^{-1}V' \to \Omega^{\cdot}_{X/s} \otimes_{\mathcal{O}_s} f^{-1}V'' \to 0$$

is exact in each dimension. So the snake lemma shows that if (d) holds for two of V, V', and V'', it holds for the third.

By the syzygy lemma, any coherent sheaf on D^n locally has a finite free resolution. Therefore (d) holds for the case where $S = D^n$.

For the general case, it still suffices only to consider the situation where $X = S \times D^m$, f is the projection pr_2, and the local system is the sheaf-theoretic inverse image of a coherent sheaf $\mathcal{F}$ on S.

Since (d) is local on S, one can assume S is a closed analytic subset of D^n with inclusion i. For a coherent sheaf $\mathcal{F}$ on S, consider the direct image sheaf $i_*\mathcal{F}$, then $i_*\mathcal{F}$ is also a coherent sheaf on D^n. Also, note that the push-forward of the complex

$$0 \to f^{-1}\mathcal{F} \to f^*\mathcal{F} \to f^*\mathcal{F} \otimes \Omega_{X/S} \to ...$$

onto $D^m \times D^n$ is just the complex

$$0 \to pr^{-1}i_*\mathcal{F} \to pr^*i_*\mathcal{F} \to pr^*i_*\mathcal{F} \otimes \Omega_{D^m \times D^n/D^n} \to ...$$

where pr is the projection from $D^m \times D^n$ to D^n. It follows from the special case that this complex is exact.

Now we have proved (d). This directly implies (a), which means $G \circ F = Id$. It remains to show that for every integrable relative connection $(\mathcal{E}, \Delta)$, $(\mathcal{E}, \Delta) = F(V)$ for some local relative system V. This is again local on both X and S.

The last part is proved by induction on relative dimensions of f and reduction to the case where $X = D^n \times D^1$, $S = D^n$, $f = pr_1$, and the bundle is free.

If $X = D^n \times D^1$, $S = D^n$, $f = pr_1$, and $\mathcal{E} = \mathcal{O}_X^m$. Then finding a horizontal section s of $\mathcal{O}_X^m$ is such that $s|_{0 \times S} = s_0$ for a given $s_0 \in \mathcal{O}_S^m$ is solving a series differential equation on D^1 of order n with initial value parameterized holomorphically by D^n. Therefore each equation in the holomorphic series has a unique global solution. Moreover, the solutions form a global section of $f^{-1}\mathcal{O}_S$. Therefore the horizontal sections $\in \mathcal{E}$ forms a free $\Gamma(\mathcal{O}_S, S)$ module of rank m.

If $X = D^n \times D^1$, $S = D^n$, $f = pr_1$, and $\mathcal{E}$ is an arbitrary coherent sheaf on X. By shrinking S and X, we can assume that $\mathcal{E}$ has a finite representation

$$\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \xrightarrow{\beta} \mathcal{E}$$

We shall show the existence of an open neighborhood of 0 where $\mathcal{E}_1$ and $\mathcal{E}_0$ admit integrable connections ∇_1 and ∇_0 compatible with α and β.

Let (e_i) be a basis of global sections of $\mathcal{E}_0$. First, consider $(g_i) = (\beta(e_i))$ which generates $\mathcal{E}$. We can let $g_i \neq g_j$ when we construct $\mathcal{E}_0$. To be accurate, choose an open neighborhood of 0 where $\mathscr{E}$ can be generated by global sections and let $\mathscr{E}$ be a free sheaf generated by these global sections. Consider $\nabla(\beta(e_i)) = f_i$. There exists an open neighborhood U of 0 such that each $f_i|_U$ is expressed in the form $\sum_j \omega_j \otimes g_j$ where $\omega_j \in \Omega_{X/S}(U)$ and $g_j \in \{g_i\}$. Then let $\nabla_0(f_i) = \sum_j \omega_j \otimes f_j$. This defines a connection on $\mathscr{E}_0$ in U. Since $X \xrightarrow{f} S$ is smooth of dimension 1, ∇_0 is automatically integrable. And we can do the same procedure to $\mathscr{E}_1$.

Then shrink X and S and by the previous conclusion we may assume $(\mathscr{E}_0, \nabla_0) = F(V_0)$ and $(\mathscr{E}_1, \nabla_1) = F(V_1)$ for relative local systems (in fact free $\Gamma(\mathscr{O}_S)$ modules) V_0 and V_1. Then $\mathscr{E} = \beta V_1 \otimes_{\mathscr{O}_s} \mathscr{O}_X$, and βV_1 consists of only horizontal sections of ∇. By (d) βV_1 consists of all horizontal sections of ∇. Therefore $(\mathscr{E}, \nabla) = F(\beta V_1)$. The conclusion holds.

Now consider the case that $X \xrightarrow{f} S$ is of relative dimension 1. Again it suffices to consider S is a closed analytic subset of D^n, $X = D^1 \times S$, and $f = pr_2$.

Again we consider the diagram:

$$\begin{array}{ccc} D \times S & \xrightarrow{i'} & D \times D^n \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} \\ S & \xrightarrow[i]{} & D^n \end{array}$$

It follows that local S-relative systems(resp. coherent sheaves with integrable S-relative connections) on $D \times S$ are identified with local D^n-relative systems(resp. coherent sheaves with D^n-relative connections) on $D \times D^n$ that are annihilated by the inverse image of the ideal sheaf of S. So the conclusion follows from the previous case.

For the general case, induct on the relative dimension n of $X \xrightarrow{f} S$. The case $n = 0$ is trivial since f is locally isomorphism, $\Omega_{X/S} = 0$, and each coherent sheaf on X is a relative local system. And the case $n = 1$ is proved. Assume the conclusion holds for n, for $n + 1$, again WLOG $X = S \times D^n \times D^1$ with $f = pr_1$. Consider the section i at 0:

$$X_0 = S \times D^n \xrightarrow{i} S \times D^n \times D^1$$

. Note that $i^*(\mathscr{E}, \nabla)$ is a connection on X_0. By inductive hypothesis $(i^*\mathscr{E}, i^*\nabla) = F_{X_0/S}(V_0)$ for some local S-relative system V_0 on X_0. Consider the projection

$$X \xrightarrow{pr} X_0$$

The relative connection ∇ also induces a S-relative connection $\widetilde{\nabla}$ for pr. Then by the case n = 1, $(\mathcal{E}, \nabla) = F_{X/X_0}(V')$ for some X_0-relative system V on X. Also, note that

$\mathcal{E}|_{X_0} = V'|_{X_0} = i^* \mathcal{E}$. And $\mathcal{E}|_{S\times 0\times 0} = V'|_{S\times 0\times 0} = i^* \mathcal{E}|_{S\times 0} = V_0|_{S\times 0}$ is a coherent sheaf $\mathcal{F}$ on S. And $\mathcal{E} = pr_1^{-1}\mathcal{F} \otimes_{pr^{-1}\mathcal{O}_S} \mathcal{O}_X$.

It remains to show sections of $pr_1^{-1}\mathcal{F}$ are horizontal with respect to ∇.

We can consider the tangent sheaf $T_{X/S}$ with the basis ∂_{x_1}, ∂_{x_2},..., ∂_{x_n}. For a section s of $pr_1^{-1}\mathcal{F}$, $\nabla_{\partial_{x_n}}(s) = 0$, and $\nabla|_{x_0 \partial_{x_i}}(s|_{X_0}) = 0$.

Since ∇ is integrable $[\nabla_{\partial_{x_i}}, \nabla_{\partial_{x_n}}] = \nabla_{[\partial_{x_i}, \partial_{x_n}]} = 0$. Therefore $\nabla_{\partial_{x_n}} \nabla \partial_{x_i}(s) = \nabla_{\partial x_i} \nabla \partial_{x_n}(s) = 0$. Consider then section $\nabla_{\partial_{x_i}}(s)$, it is zero at X_0 and horizontal along x_n. Therefore $\nabla_{\partial_{x_i}}(s) = 0$, which shows s is horizontal. Q.E.D.

5 Degline's Riemann–Hilbert correspondence for regular singularities

Discussing the relationship between regular singularities and integrable connections is very complicated. Deligne used too many techniques from analysis and algebraic geometry in [2] to be included in this short thesis. Therefore I will only give a brief introduction to this topic.

5.1 Dimension one

According to the former sections, we know that in dimension 1 any holomorphic connection is integrable. Meanwhile, a connection may give an irregular differential equation. However, after a suitable coordinate change on the bundle, it could still give rise to a regular differential equation. First, we look at this easy example:

Consider $U = \mathbf{C}^1 \setminus 0$. Consider a connection ∇ on $\mathcal{O}_U$ given by

$$f \mapsto df - z^{-2} f dz$$

The ∇ gives a irregular(recall introduction) differential equation:

$$\frac{df}{dz} = \frac{f}{z^2}$$

The global solution of the equation exists and is of the form $f(z) = c \cdot e^{-\frac{1}{z}}$. We can apply a base change to $\mathcal{O}_U$ by $f \mapsto \frac{f}{z}$.
Then after the base change, the connection becomes the trivial connection on $\mathcal{O}_U$. This explains why the equation would have global solutions.

Assume X is a Riemann surface and Σ is a finite set of points on X. Let $X^* = X \setminus \sigma$ and let $j : X^* \to X$ be the open immersion. Now we define a connection with regular singularities along Σ:

Definition 5.1 (holomorphic connections with regular singularities)**.** *For a connection $(\tilde{\mathcal{E}}, \nabla)$ on X^*, we say this connection has regular singularity along Σ if, for any $\sigma \in \Sigma$, there exists a punctured neighborhood of σ such that ∇ has meromorphic coefficients with at most only simple poles at σ for some base.*

Theorem 5.1. *Each holomorphic connection on X^* has regular singularity along Σ.*

Proof. One can construct a vector bundle on X with a meromorphic connection whose restriction on X^* is regular and has the same monodromy as the original connection. Then we are done with the Riemann–Hilbert correspondence. □

Remark 5.2. *Consider the corresponding algebraic varieties for X and X^*, denoted by X_{ag} and X^*_{ag}. Let $(\mathcal{E}, \nabla)$ be an algebraic variety on X^*. Then after analytification, $(\mathcal{E}_{an}, \nabla_{an})$ on X^* should be regular. However $(\mathcal{E}, \nabla)$ itself may not be 'regular' since the isomorphism between analytic vector bundles with connections might not be algebraic. This leads us to find a definition of regular algebraic connections.*

5.2 Higher dimensions

Let's introduce the definition of an algebraic connection with regular singularities in higher dimensions.

First, define an algebraic version of regular connections for curves, the following definitions are taken from [5]

Definition 5.2. *Let* $(C,\ K)$ *be an abstract nonsingular curve over* $\mathbf{C}$*. Let* p *be a closed point on* C*. Denote the quotient field of* $\mathcal{O}_{C,p}$ *by* $K_{C,p}$*. A meromorphic connection* (M, ∇) *is a finite-dimensional* $K_{C,p}$ *vector space* M *with a* $\mathbf{C}$*-linear map:* $M \xrightarrow{\nabla} \Omega_{C,p} \otimes \mathcal{O}_{C,p} M$

Definition 5.3. *An algebraic connection* (M, ∇) *is called regular if there is a finitely generated* $\mathcal{O}_{C,p}$*-submodule of* M *such that* $M = K_{C,p}L$ *and* $x\nabla(L) \subset \Omega_{C,p} \otimes_{\mathcal{O}_{C,p}} L$ *for some local parameter* x *at* p*.*

Definition 5.4. *Let* X *be a smooth algebraic variety. An integrable connection* $(\mathcal{E}, \nabla)$ *on* X *is said to be regular if for any morphism* $C \xrightarrow{i_C} X$ *from a smooth algebraic curve* C $i_C^* \mathcal{E}$ *is regular.*

There is another definition :

Definition 5.5 (log sheaves)**.** *Let* X *be a proper, smooth* $\mathbf{C}$*-scheme. Let* $D = \cup D_i$ *be the union of finitely many connected smooth divisors in* X *with normal crossings. Let* $Der_D(X/Spec\mathbf{C})$ *be the sheaf on* X *of derivations which preserves the ideal sheaf of each branch of* D*. The sheaf of differentials on* X *with logarithmic singularities along* D *is defined by*

$$\Omega_X(log\, D) := \mathcal{H}om_{\mathcal{O}_X}(Der(S/Spec\mathbf{C}),\, \mathcal{O}_X)$$

Definition 5.6 (Regular connections)**.** *Let* $U = X \setminus D$ *as above with* $U \xrightarrow{i} X$*. For an algebraic connection* $(\mathcal{E}, \nabla)$ *on* U*, if* $\mathcal{E}$ *is a union of* Der_D*-stable coherent submodules, the* $\mathcal{E}$ *is said to be regular along* D*.*

Deligne showed the equivalence of regular algebraic integrable connections and analytic integrable connections in [2], theorem 5.9.

6 Kashiwara's Riemann–Hilbert correspondence

Now we consider derived categories $D_c^b(X)$ and $D_rh^b(\mathscr{D}_X)$

There are two functors defined as follows

$$DR : D_{rh}^b(\mathscr{D}_X)^{op} \to D_c^b(X)$$

:

$$\mathscr{M}^\bullet \mapsto \mathscr{R}hom_{\mathscr{D}_X}(\mathscr{M}^\bullet, \mathscr{O}_X)$$

and

$$SOL : D_{rh}^b(\mathscr{D}_X) \to D_c^b(X)$$

:

$$\mathscr{F}^\bullet \mapsto \Omega_X \otimes_{\mathscr{D}_X} \mathscr{F}^\bullet$$

Kashiwara's Riemann–Hilbert correspondence says that the above two functors map $D_{rh}^b(\mathscr{D}_X)^{op}$ and $D_{rh}^b(\mathscr{D}_X)$ into $Perv(X)$ an induces equivalences of categories.

7 Concluding remarks

In this thesis, we expand on some details of Deligne's proof of a relative version of Riemann-Hilbert correspondence for smooth morphisms of analytic varieties and give a little introduction to the theory of connections with regular singularities and Kashiwara's work for regular holonomic D-modules. The Riemann–Hilbert correspondence is also an important foundation of the nonabelian Hodge theory. It might be considered a non-abelian analog of the Derham-constant comparison. This topic is still very active. Now there are many new processes in this topic. For example Kashiwara's work for irregular holonomic D-modules and generalizations for varieties over fields of positive characteristics. Meanwhile the Riemann—Hilbert correspondence also has applications in integrable systems and other fields. To make a conclusion, the Riemann–Hilbert correspondence is a very interesting and active topic that is worth further investigation.

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致谢

首先感谢朱一飞老师向我对毕设的主题提供灵感，虽然过程十分不顺利。具体来说，一开始我和朱一飞老师商量的是利用相交上同调来研究与一个具体物理问题相关的几何对象的性质，这个课题非常有价值而且还有可以上手的简单例子。但是由于这个问题其实非常开放而且我缺乏对于其背后的物理意义足够的了解以及物理上的直觉，一直不知道如何进行有实际意义的研究，加上缺乏足够的交流以及我确实因为种种主观和客观的原因在这上面投入的时间不够，以至于临近初稿截止日期还缺乏值得写入毕业论文的进展，只好匆忙换题，也因此不得不推迟一点提交初稿（在这里我表示诚挚的道歉并感谢数学系的宽容）。原来的课题在朱老师和其他同学手上目前有一些初步的进展。现在这篇毕设的来源，其实和原来的课题还是有一些联系。朱老师找到了一篇和原来的课题存在联系的长篇论文，其中涉及到 Nonabelian Hodge 相关的一些东西，我看到之后产生了一些兴趣。由于我了解到黎曼-希尔伯特对应是 Nonabelian Hodge 理论中的一个基础，遂最终选择它作为我毕设的主题。刚刚换题的时候我想结合之前对于相交上同调的一些了解，写一份以介绍反常层和 regular holonomic D 模的对应为主的报告。但是由于时间仓促进展不尽人意，最终只好写了一份介绍性质为主加上对于较为简单的版本的证明的读书报告。尽管如此，这个过程还是让我学到一些知识。在这里非常感谢 Hosgood 教授无私地将自己对 Deligne 的论文的几乎完全还原的英语翻译公开在 Github 上，使得我免于阅读法语原文的困难。同时我也感谢我的电脑在这段时间连续工作却没有出现任何严重的故障，避免了我初稿完成的时间进一步往后拖。我在这篇毕设中完成的实际的内容尚且相当有限，所以在最终定稿之前我还有许多工作可以做，比如对 Deligne 或者 Kashiwara 的工作给出更详细的介绍以及优化背景知识部分定义和结论的铺陈。再次感谢朱一飞老师在这样危急的情况下仍然对我的毕设给予指导和支持。