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# A GENTLE INTRODUCTION TO FLOER HOMOLOGY THEORIES

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**Abstract**

This is a reading report on Greene's survey paper on Heegaard Floer homology. The aim of this thesis is to present the elements that center around this active research topic in an expository manner by following the development and evolution of Floer homology theories up to Heegaard Floer homology.

**Keywords:** Low-dimensional topology, Floer homology, history

# Contents

# 1 Introduction

Floer homology is an infinite-dimensional version of Morse homology first developed by Floer to attack the Arnold conjecture in symplectic topology. The development of this tool is a long and convoluted story with inspirations from symplectic topology to gauge theory, yielding numbers of variant theories that provide rich applications in many different fields.

In section 2, we introduce the Morse homology, the finite-dimensional case. Many ideas here, including the study of critical points, moduli spaces and compactification, will show up repeatedly in the future.

Section 3 is an introduction to the original idea of Floer in his work on Arnold conjecture. Here the Floer homology can be considered to be a Morse homology on the loop space of a symplectic manifold. We will also introduce an important variant called Lagrangian Floer homology that would eventually inspire Heegaar Floer homology.

Section 4 discusses another variant of Floer homology in gauge theory called instanton Floer homology. It is a topological invariant of 3-manifolds built from information of certain $SU(2)$-connections.

Section 5 introduces Heegaard Floer homology, which is an important invariant of 3-manifolds. Its inspiration came from low-dimensional topology, symplectic topology and gauge theory.

In the last section, we discuss a variant of Heegaard Floer homology in knot theory, and mention some of its properties.

# 2 A quick introduction to Morse homology

In this section, we briefly introduce the construction of Morse homology. Morse theory studies a particular type of function on manifolds that reflexes the topological information of the space itself. It has been the main inspiration for Floer homology theory, the latter can be seen as an infinite-dimensional version of Morse homology.

We assume all functions in this section be smooth, and manifold $M$ be compact. More detailed information on classic Morse theory can be found on [13], and that of Morse homology can be found on [1] and [2].

## 2.1 Morse functions

Let $M^n$ be a manifold and let $f : M \to \mathbb{R}$ be a smooth function on $M$, a point $p \in M$ is called a *critical point* if $df|_p = 0$. In a local coordinate $(x_1, \cdot, x_n)$, this would mean

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

**Theorem 2.1.1.** *The Hessian of $f$ at a critical point $p$, denoted by $(d^2f)_p$, is a symmetric bilinear functional on $TM_p$ given by formula $(d^2f)_p(V, W) = V \cdot (\widetilde{W} \cdot f)(p)$ ,where $\widetilde{W}$ denotes a vector field extending $W$ locally.*

$(d^2f)_p$ is symmetric since $V \cdot (\widetilde{W} \cdot f)(p) - W \cdot (\widetilde{V} \cdot f)(p) = ([\widetilde{V}, \widetilde{W}] \cdot f)(p)$, where the right hand side equals to 0, for $p$ is a critical point of $f$. It is well defined because it is independent of the extension of $W$, since $(d^2f)_p(V, W) = (d^2f)_p(W, V) = W \cdot (\widetilde{V} \cdot f)(p)$. Under a local coordinate system $(x_1, \cdot, x_n)$, this bilinear form can be represented by a symmetric matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{ij}$.

We denote the set of critical points of $f$ by $\text{Crit}(f)$. A critical point $p$ is called *non-degenerate* if $(d^2f)_p$ is a non-degenerate form. This would mean $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{ij}$ is non-singular in local coordinate. For those non-degenerate critical points, define their *index* $\text{ind}_f(p)$ to be the dimension of the largest negatively defined subspace of $(d^2f)_p$, and $\text{Crit}_i(f) = \{p \in \text{Crit}(f) | \, \text{ind}_f(p) = i\}$. The following is a classic result of Morse theory.

**Theorem 2.1.2** (Morse lemma). *Let $p$ be a non-degenerate critical point of the function $f : M^n \to \mathbb{R}$. Then there is a local coordinate $(U, x_1, \cdots, x_n)$ around $p$ such that*

$$f(x_1, \cdots, x_n) = f(p) - \sum_{i=1}^{\mathrm{ind}(p)} x_i^2 + \sum_{i=\mathrm{ind}(p)+1}^{n} x_i^2.$$

**Definition 2.1.3.** *A Morse function is a smooth function that has all its critical points non-degenerate.*

Notice that $\mathrm{Crit}(f)$ is therefore discrete for a Morse function $f$.

Morse functions turn out to be the object of interest in Morse theory. The chain complex in Morse homology is in fact $\mathrm{Crit}(f)$ together with a suitable differential. The good news is, Morse functions are dense in $C^\infty(M)$, and it is fairly easy to construct some simple ones.

**Proposition 2.1.4.** *Let $M^d$ be embedded in $\mathbb{R}^n$. For almost every point $p \in \mathbb{R}^n$, the function*

$$\begin{aligned} f_p : M &\longrightarrow \mathbb{R} \\ x &\longmapsto ||x-p||^2 \end{aligned}$$

*is a Morse function.*

*Proof.* The differential of $f_p$ at $x$ is $df_{p,x}(v) = 2\langle x - p, v\rangle$, so $x$ is a critical point if and only if $x - p$ is normal to $T_x M$. Choose a local parametrization $(u_1, \cdots, u_d) \mapsto x(u_1, \cdots, u_d)$ around $x$, we have

$$\begin{aligned} \frac{\partial f_p}{\partial u_i} &= 2(x-p) \cdot \frac{\partial x}{\partial u_i} \\ \frac{\partial^2 f_p}{\partial u_i \partial u_j} &= 2\left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + (x-p) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right). \end{aligned}$$

So $f_p$ is a Morse function if and only if all those $x \in M$ with $(x-p) \perp T_x M$ have the matrix above non-singular. Direct calculation shows that $p \in \mathbb{R}^n$ satisfying this are exactly the critical values of the map $E : NM \to \mathbb{R}^n$, $(x, v) \mapsto x + v$, and therefore the proposition is proved by Sard's theorem. □

Another simple example of Morse function is the height function that measure the "height" of the points in a manifold embedded in $\mathbb{R}^n$.

## 2.2 The moduli spaces of flow lines

We mentioned earlier that the chain complex in Morse homology called the Morse chain complex is $\mathrm{Crit}(f)$ together with a suitable differential. To be more precise, $C_i$ is exactly the free abelian group generated by $\mathrm{Crit}_i(f)$. For the differential, we need another ingredient, namely, the Riemannian metric. We want to count the flow lines flowing from one critical point in $C_i$ to another one in $C_{i-1}$.

Given a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and a function $f$ on $M$, the *gradient* vector field of $f$ is the vector field $\mathrm{grad}\, f$ defined by $\langle \mathrm{grad}\, f, \cdot \rangle = df$. The flow of gradient field is a family of diffeomorphism $\{\phi^s : M \to M\}_{s \in \mathbb{R}}$, with $\phi^0 = id_M$ and $\frac{d\phi^s(p)}{ds} = -\mathrm{grad}(f)_p$. Notice that $f \circ \phi^s(x)$ is increasing in $s$, and that $\phi^s(p)$ in constant for any $p \in \mathrm{Crit}(f)$.

**Definition 2.2.1.** *Let $f$ be a Morse function on $M$, $p \in \mathrm{Crit}(f)$, define its ascending and descending manifolds to be*

$$A(p) = \left\{ x \in M | \lim_{s \to +\infty} \phi^s(x) = p \right\}$$

*and*

$$D(p) = \left\{ x \in M | \lim_{s \to -\infty} \phi^s(x) = p \right\}.$$

The names for the two manifold comes from the example of height functions, where points in the ascending manifold flow to "higher" critical points, and points in the descending manifold do the opposite.

Notice that flow lines would flow towards $p$ in the directions of eigenvectors with negative eigenvalues, and flow away from $p$ in the directions of other eigenvectors. This inspired the following proposition.

**Proposition 2.2.2.** *$A(p)$ is diffeomorphic to an $(n - \operatorname{ind}(p))$-dimensional disk, and $D(p)$ is diffeomorphic to an $\operatorname{ind}(p)$-dimensional disk.*

The set of flow lines flowing away from $p$ and towards $q$ would be $A(q) \cup D(p)$, which is a manifold if the intersection is transverse.

**Definition 2.2.3.** *For a Morse function $f$ and a Riemannian metric $g$, the pair $(f, g)$ is called Morse-Smale if $A(q)$ intersects $D(p)$ transversely for every $p, q \in \operatorname{Crit}(f)$.*

We can perturb $f$ or $g$ to obtain a Morse-Smale pair.

Define the space of flow lines from $p$ to $q$ to be $\mathcal{M}(p, q) = A(q) \cup D(p)$, and the moduli space to be $\mathcal{L}(p, q) = \mathcal{M}(p, q)/\mathbb{R}$, where $\mathbb{R}$ acts on $A(q) \cup D(p)$ by the gradient flow. Under Morse-Samle condition, $\mathcal{L}(p, q)$ would be a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$. In this case, $M(p, q)$ would be 1-dimensional if $\operatorname{ind}(p) = \operatorname{ind}(q) - 1$, and we hope that it is a finite set so that we are able to count its elements. This lead to the study of compactification $\overline{\mathcal{L}(p, q)}$ of $\mathcal{L}(p, q)$.

**Theorem 2.2.4.** *Let $(f, g)$ be Morse-Smale, then for distinct $p, q \in \operatorname{Crit}(f)$, the compactification $\overline{\mathcal{L}(p, q)}$ is a manifold with corners, and the $k$-dimensional corners are*

$$\overline{\mathcal{L}(p, q)}_k = \bigcup_{\substack{r_1, \cdots, r_k \in \operatorname{Crit}(f) \\ \operatorname{ind}(q) < \operatorname{ind}(r_i) < \operatorname{ind}(p)}} \mathcal{L}(p, r_1) \times \mathcal{L}(r_1, r_2) \times \cdots \times \mathcal{L}(r_k, q)$$

Immediately we have $\overline{\mathcal{L}(p, q)} = \mathcal{L}(p, q)$ if $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$, so $\mathcal{L}(p, q)$ is a finite set of points. We also hope to orient $\mathcal{L}(p, q)$ in order to count its points with sign. To do this, we fix an orientation for every $D(p)$ and descend it to $M(p, q)$.

We are now ready to build the Morse complex.

## 2.3 Morse homology

Let $(f, g)$ be Morse-Smale, $C_i = \mathbb{Z}\operatorname{Crit}_i(f)$, and

$$\partial_i(p) = \sum_{\operatorname{ind}(q) = \operatorname{ind}(p) - 1} \#\mathcal{L}(p, q) \cdot q$$

where $\#\mathcal{L}(p, q)$ denote the signed count of its elements. Extend $\partial_i$ linearly on $C_i$.

**Proposition 2.3.1.** *The Morse complex $(C_*, \partial)$ is a chain complex.*

*Proof.* It suffices to verify that $\partial \circ \partial = 0$. For any $p \in C_i$,

$$\begin{aligned}
\partial \circ \partial(p) &= \sum_{q \in C_{i-2}} \sum_{r \in C_{i-1}} \#\mathcal{L}(p, r)\#\mathcal{L}(r, q) \cdot q \\
&= \sum_{q \in C_{i-2}} \# \left( \bigcup_{r \in C_{i-1}} \mathcal{L}(p, r) \times \mathcal{L}(r, q) \right) \cdot q \\
&= \sum_{q \in C_{i-2}} \#\partial\overline{\mathcal{L}(p, q)} \cdot q.
\end{aligned}$$

We have used theorem 1.2.4 here. Since $\overline{\mathcal{L}(p,q)}$ is a 1-dimensional manifold, the signed count of its boundary is zero. Thus $\partial \circ \partial(p) = 0$. □

The homology of this chain complex is called the Morse homology, denoted by $HM_*(M;\mathbb{Z})$. The modulo 2 homology $HM_*(M;\mathbb{Z}/2)$ can be similarly defined, with the only difference being that the orientation of moduli space becomes unnecessary. As an example, Figure 1 shows the flow lines of a height function on a torus, where $p \in \mathrm{Crit}_0$, $q, r \in \mathrm{Crit}_1$ and $s \in \mathrm{Crit}_2$. So its Morse homology is $HM_* = \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$, the same as the ordinary homology of a torus.

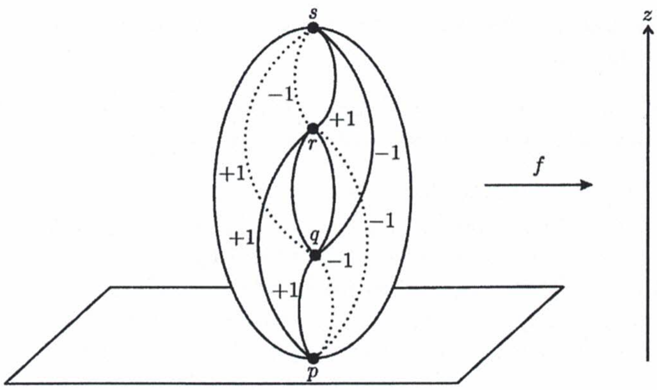


Figure 1: [2] Gradient flow lines with sign of a height function on a torus.

Morse homology turns out to be independent of the choice of $(f, g)$ and canonically isomorphic to the singular homology of the manifold. For application, we prove a classic result with Morse homology.

**Proposition 2.3.2** (Morse inequalities). *Let $f$ be a Morse function on $M$. Define $c_k(f)$ to be the number of its critical points of index $k$ and let $b_k$ be the $k$-th Betti number of $M$, then*

$$c_k(f) \geqslant b_k$$

*for all $k \geqslant 0$.*

*Proof.* Take the Morse complex $(C_*, \partial)$ of $M$. We have

$$\begin{aligned} c_k(f) &= \dim C_k \\ &= \dim \ker \partial_k + \dim \mathrm{im}\ \partial_k \\ &\geqslant \dim \ker \partial_k - \dim \mathrm{im}\ \partial_{k+1} \\ &= \dim HM_k(M) = b_k. \end{aligned}$$

□

# 3 Floer homology and the Arnold conjecture

In this section, we introduce the basic idea of Floer homology and Lagrangian Floer homology, the latter would become a vital inspiration for Heegaard Floer homology. We will focus mainly on the Floer homology. The two homologies were originally developed to attack the Arnold conjecture in symplectic topology.

Symplectic topology was originally motivated by Hamiltonian mechanics, which is a transformation of Lagrangian mechanics which turns physic problems into a matter of finding solutions of a Hamiltonian system with a time-dependent Hamiltonian. Since Floer's work on his homology theory, the study of Hamiltonian dynamics and $J$-holomorphic curves (see Appendix A) became a major driving force in the subject.

This section is based on [1], [11] and [15]. Manifolds throughout this section will be assumed to be smooth without boundary.

## 3.1 Symplectic systems and the Arnold conjecture

We first introduce some basic concepts in symplectic geometry.

**Definition 3.1.1.** *A Symplectic structure on a manifold $M$ is a closed 2-form $\omega \in \Omega^2(M)$ that is everywhere non-degenerate. A manifold with symplectic structure is called a symplectic manifold.*

Notice that non-degeneracy force the manifold to be of even dimension. The easiest example of symplectic manifold would be $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0(x,y) = \sum(x_i y_i - y_i x_i)$, that is, $\omega_0 = \sum dx_i \wedge dy_i$. It is also called a symplectic vector space. A symplectic vector bundle would be a vector bundle that has symplectic vector space as its fiber. All orientable surfaces are symplectic, since they have non-vanishing volume forms as symplectic forms.

**Definition 3.1.2.** *A symplectomorphism of a symplectic manifold $(M, \omega)$ is a diffeomorphism $\psi$ that preserve the symplectic form, i.e. $\psi^*\omega = \omega$.*

The set of all symplectomorphism is denoted by $\mathrm{Symp}(M, \omega)$ or simply $\mathrm{Symp}(M)$.

**Definition 3.1.3.** *$H : M \to \mathbb{R}$ is a smooth function on a symplectic manifold $M$. The vector field $X_H$ defined by*

$$-\iota(X_H)\omega = \omega_x(Y, X_H) = (dH)_x(Y) \text{ for every } Y \in T_x(M)$$

*is called the Hamiltonian vector field of the Hamiltonian $H$.*

We want to study the dynamical system, or Hamiltonian system, associated to this vector field. $X_H$ generates a 1-parameter group of diffeomorphism $\{\psi_H^t\}$, with $\psi_H^0 = \mathrm{id}_M$, called the Hamiltonian flow. We will later show that each diffeomorphism in a Hamiltonian flow is a symplectomorphism.

We now consider the situation when the Hamiltonian is "time-dependent". In this case, the Hamiltonian is $H : \mathbb{R} \times M \to \mathbb{R}$. For each $t$, the function $H_t := H(t, \cdot)$ would similarly generate a vector field $X_t := X_{H_t}$. As in the autonomous case, a time-dependent vector field also generates a group of diffeomorphism $\{\psi_H^t\}$, such that

$$\frac{d}{dt}\psi^t = X_t \circ \psi^t \text{ and } \psi^0 = \mathrm{id}_M.$$

We also call it a Hamiltonian flow.

**Definition 3.1.4.** *A diffeomorphism $\psi$ is called a Hamiltonian symplectomorphism if there exist a Hamiltonian flow $\psi^t$ such that $\psi^1 = \psi$. The set of all Hamiltonian symplectomorphism is denoted by* $\mathrm{Ham}(M, \omega)$ *or* $\mathrm{Ham}(M)$.

The following proposition justifies its name.

**Proposition 3.1.5.** *Every diffeomorphism in a Hamiltonian flow is a symplectomorphism*

*Proof.* Since $\psi^0 = \mathrm{id}_M$, it suffice to show that $(\psi^t)^*\omega$ is independent of $t$, so that $(\psi^t)^*\omega = (\psi^0)^*\omega = \omega$.

$$\begin{aligned}\frac{d}{dt}\left((\psi^t)^*\omega\right) &= (\psi^t)^*\mathcal{L}_{X_t}\omega \\ &= (\psi^t)^*(d \circ \iota(X_t)\omega) \\ &= (\psi^t)^*(-d \circ dH) = 0.\end{aligned}$$

Here we used Cartan formula

$$\mathcal{L}_X\omega = \iota(X)d\omega + d \circ \iota(X)\omega$$

and the fact that $\omega$ is a closed form. □

In Hamiltonian mechanics, periodic obits of Hamiltonian system are of great interest. In the 1960s, Arnold attempted to give the lower bound for the number of periodic solutions with his Arnold conjecture. Some notations are needed before we present the conjecture.

Let $M$ be a manifold, the minimum number of critical points of a function on $M$ is denoted by $\mathrm{Crit}(M)$, while the corresponding minimum for Morse functions is denoted by $\mathrm{Crit}_{\mathrm{Morse}}(M)$.

**Conjecture 3.1.6** (Arnold). *For any Hamiltonian symplectomorphism $\psi$ on a compact symplectic manifold $M$, there is*

$$\#\,\mathrm{Fix}(\psi) \geqslant \mathrm{Crit}(M).$$

*If the fixed points are all non-degenerate, then*

$$\#\,\mathrm{Fix}(\psi) \geqslant \mathrm{Crit}_{\mathrm{Morse}}(M).$$

A fixed point is called non-degenerate if no eigenvalue of $d_p\psi : T_pM \to T_pM$ is equal to one. Fixed points of a Hamiltonian symplectomorphism $\psi$ are exactly the points of $M$ that goes back to its original location after time 1 in the symplectic system that generates $\psi$. If the Hamiltonian $H_t$ happens to be 1-periodic, then these fixed points correspond to 1-periodic orbits of the system. We can in fact always assume $H_t$ to be 1-periodic. Take the Hamiltonian $K_t = \alpha'(t)H_{\alpha(t)}$ for $t \in [0,1]$, it generates the flow $\psi^{\alpha(t)}$. Let $\alpha(0) = 0$, $\alpha(1) = 1$ and $\alpha'(0) = \alpha'(1) = 0$, then $\psi^{\alpha(t)} = \psi^1$ and $K_t$ can be extended to a 1-periodic function. A periodic orbit is called non-degenerate if its corresponding fixed point is non-degenerate.

Floer's proved Arnold conjecture for monotone symplectic manifolds by developing a Morse homology on infinite-dimensional manifolds called Floer homology, which focuses on the periodic orbits of Hamiltonian systems. It deals with a weaker version of the conjecture.

**Theorem 3.1.7.** *For any 1-periodic Hamiltonian $H$ on a compact symplectic manifold $M$, suppose the 1-periodic orbits are all non-degenerate, then*

$$\#(1\text{-}\textit{periodic orbits}) \geqslant \sum_{k=0}^{\dim M} b_k(M).$$

It is indeed weaker than the original conjecture due to the Morse inequalities. To simplify the problem even more, we further assume manifold $M$ satisfies the following properties.

1. For every smooth map $w : S^2 \to M$, there is
$$\int_{S^2} w^*\omega = 0$$

2. For every smooth map $w : S^2 \to M$, the symplectic vector bundle $w^*TM$ is trivial.

To prove this theorem, we will construct the Floer homology. We start by considering the space of all loops on the manifold, which is a Banach manifold, and then give it a suitable functional. The critical points of it will be exactly the periodic orbit of a given Hamiltonian system. Similar to Morse homology, in order to construct the chain complex, we need to define indices for these critical points, and define the differential by counting some suitable gradient flows lines between critical points. The homology of this chain complex is called Floer homology, which coincides with ordinary homology of $M$. Theorem 3.1.7 will then be completely analogous to the Morse inequalities. Due to the complexity of the matter, our construction will be mostly expository.

## 3.2 The loop space and the action functional

Let $M$ be a compact symplectic manifold that satisfies the assumptions given before. Define the loop space $\mathcal{L}M$ to be the space of contractible free loops on $M$. It contains all smooth maps $x : S^1 \to M$ that are contractible. The loop space, together with its $C^\infty$ topology, is in fact a Banach manifold. We will not prove this statement but rather describe the space in a formal manner. To describe its tangent space, consider a curve $s \mapsto \tilde{x}(s)$ on $\mathcal{L}M$ passing the loop $x$ at $s = 0$. The curve $\tilde{x}$ can be seen as a map

$$\begin{aligned} \tilde{x} : \mathbb{R} \times S^1 &\longrightarrow M \\ (s, t) &\longmapsto \tilde{x}(s, t). \end{aligned}$$

Differentiate with respect to $s$, we get a vector field on the loop $x$. It is natural to see this vector field as the tangent vector of $\tilde{x}$ at $x$.

Similar to the construction of Morse homology, a Riemannian structure is needed on $\mathcal{L}M$. This is achieved by fixing an almost complex structure $J$ on $M$ compatible with the symplectic form $\omega$ (see Appendix A). It induces a Riemannian metric $g$ by

$$g(X, Y) = \omega(X, JY),$$

and therefore induces a metric on $\mathcal{L}M$ by

$$\langle X, Y \rangle = \int_0^1 g(X(t), Y(t))dt.$$

Given a 1-periodic Hamiltonian $H_t$ on $M$, define the action functional to be

$$\mathcal{A}_H(x) = -\int_D u^*\omega + \int_0^1 H_t \circ x(t)dt,$$

where $u$ is an extension of $x$ to a disk $D$. Such extension must exist since $x$ is contractible. $\mathcal{A}_H$ is well defined because of the assumption that $\int_{S^2} w^*\omega = 0$ for any $w : S^2 \to M$.

**Proposition 3.2.1.** *The critical points of $\mathcal{A}_H$ is exactly the 1-periodic solution of the Hamiltonian system of $H$.*

*Proof.* We compute the differential of $\mathcal{A}_H$ at $x$. Let $\tilde{x}_s$ be a path on $\mathcal{L}M$ which passes $x$ at $s = 0$ with tangent vector $Y(t)$, then

$$d_x\mathcal{A}_H(Y) = \frac{d}{ds}\mathcal{A}_H(\tilde{x}_s)|_{s=0}.$$

To compute the derivative in the right hand side, we need to find a smooth map $\tilde{u} : \mathbb{R} \times D \to M$ such that $\tilde{u}_s(z)$ extends $\tilde{x}_s(t)$ to a disk $D$ for every $s \in \mathbb{R}$. To achieve this, let $\tilde{x}$ converge to a constant path at $-\infty$, and define $\tilde{u}_s(e^{2\pi(l+it)}) = x_{s+l}(t)$ for $l < 0$. We also extend $Y$ to the disk, that is,

$$Y(z) = \frac{\partial \tilde{u}}{\partial s}(0, z).$$

Now

$$\mathcal{A}_H(\tilde{x}_s) = -\int_D \tilde{u}_s^*\omega + \int_0^1 H_t \circ \tilde{x}_s(t)dt.$$

Differentiate the first term,

$$\begin{aligned} -\int_D (\frac{d}{ds}\tilde{u}_s^*\omega)|_{s=0} &= -\int_D u^*(d \circ \iota(Y)\omega) \\ &= -\int_{S^1} x^*(\iota(Y)\omega) \\ &= \int_0^1 \omega(x'(t), Y(t))dt. \end{aligned}$$

The derivative of the second term would be $\int_0^1 \omega(Y(t), X_t(x(t)))dt$. So

$$d_x\mathcal{A}_H(Y) = \int_0^1 \omega(x'(t) - X_t(x), Y(t))dt,$$

So if $x$ is a critical point, then apparently $x'(t) = X_t(x)$, which is the desired result. □

With the explicit differential of $\mathcal{A}_H$ at hand, it is easy to calculate its gradient on $\mathcal{L}M$,

$$(\operatorname{grad}_x \mathcal{A}_H)(t) = Jx'(t) + \operatorname{grad}_{x(t)} H_t,$$

which is a vector field on $x$. We take the negative gradient just like when constructing Morse homology. A negative gradient flow $u$ on $\mathcal{L}M$ is a solution of the following PDE

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

This is called the Floer equation. We are only interested in all smooth contractible solutions of period 1.

## 3.3 The outline of the construction

There are still many problems to deal with in order to construct Floer homology. Here we state the outline the construction without proving. We will still assume $M$ to be compact and satisfies the two assumptions given before. We will also take coefficient $\mathbb{Z}/2$ to avoid orientation problem.

The first problem is that in infinite-dimensional case, a trajectory may not connect two critical points. So we focus ourselves to certain nice solutions with "finite energy". The energy of a solution is defined by

$$E(u) = \int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial s}\right|^2 dt\ ds.$$

**Proposition 3.3.1.** *Let $u : \mathbb{R} \times S^1 \to M$ be a solution of Floer equation. Then $u$ connects two critical points in $\mathcal{L}M$ if and only if $E(u) < \infty$.*

We will therefore consider the solution space

$$\mathcal{M} = \{u : \mathbb{R} \times S^1 \to M \mid u \text{ is a contractible solution of Floer equation with finite energy}\}.$$

We use $\mathcal{M}(x, y)$ to denote the space of all solution in $\mathcal{M}$ that flows from $x$ to $y$, and define the moduli space $\mathcal{L}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$.

Every non-degenerate critical point $x$ of $\mathcal{A}_H$ can be given an index $\mu(x)$ called Maslov index. We will assume all critical points to be non-degenerate from now. Just like in Morse theory, the study on the compactification of the moduli space gives the following result.

**Theorem 3.3.2.** *You can perturb the Hamiltonian $H$, without changing the critical points of $\mathcal{A}_H$, such that for any two critical points $x, y$ of $\mathcal{A}_H$, if $\mu(x) - \mu(y) = 1$, then $\mathcal{L}(x, y)$ is a compact 0-dimensional manifold. If $\mu(x) - \mu(y) = 2$, then the compactification $\overline{\mathcal{L}}(x, y)$ is a 1-dimensional manifold with its boundary being*

$$\bigcup_{\mu(z)=\mu(x)+1} \mathcal{L}(x, z) \times \mathcal{L}(z, y).$$

The Floer complex is constructed in exact same way as the Morse complex by letting $CF_i = \mathbb{Z}/2\,\mathrm{Crit}_i(\mathcal{A}_H)$ and

$$\partial_i(x) = \sum_{\mu(y)=\mu(x)-1} \#\mathcal{L}(x, y) \cdot y.$$

The proof of $\partial \circ \partial = 0$ is complete analogous to the one in the last section, and the homology of this complex is called the Floer homology, denoted by $HF_*(M, J)$, where $J$ is the almost complex structure we chose for $M$. It in fact coincides with ordinary homology of $M$. This is proved by using its independence of Hamiltonian. It is possible to choose a Hamiltonian $H$ that is independent of time, such that its 1-periodic orbits are all constant. In this case, the critical points of $\mathcal{A}_H$ will be exactly the critical points of $H$. This is because $\psi^t(x) = \mathit{const}$ if and only if $X_H = 0$, which happens if and only if $d_x H = 0$. So in this case, the Floer complex coincides with the Morse complex. Theorem 3.1.7 can then be proved in the same manner as the Morse inequality.

### 3.4 Arnold-Givental conjecture and Lagrangian Floer homology

In the 1980s, Arnold conjecture was further generalized by Arnold and Givental. We first introduce the concept of Lagrangian submanifold.

**Definition 3.4.1.** *A Lagrangian subspace $L$ of a symplectic vector space $V$ is a maximal isotopic subspace, that is, a maximal subspace where the symplectic form vanishes.*

A Lagrangian subspace $L$ is always of half the of the symplectic vector space it is in, since $L = L^{\perp}$ and $\dim L + \dim L^{\perp} = \dim V$.

**Definition 3.4.2.** *A submanifold $L$ of a symplectic manifold $(M, \omega)$ is called Lagrangian if $T_xL$ is a Lagrangian subspace of $T_xM$ for every $x \in L$.*

We then have $\dim L = \frac{1}{2}\dim M$ or $L$. As a result, if two Lagrangian submanifold of $M$ intersect transversely, then the intersection would be a 0-dimensional manifold. For the last definition, we call a diffeomorphism $\tau : M \to M$ on a symplectic manifold an anti-symplectic involution if $\tau \circ \tau = \mathrm{id}_M$ and $\tau^*\omega = -\omega$. $\mathrm{Fix}(\tau)$ is empty or a Lagrangian submanifold of $M$.

**Conjecture 3.4.3** (Arnold-Givental). *Let $M$ be a closed symplectic manifold. Let $L$ be the fixed point set of an anti-symplectic involution. Let $\psi \in \mathrm{Ham}(M)$, then*

$$\#(L \cap \psi(L)) \geqslant \mathrm{Crit}(L).$$

*If the intersection is transverse, then*

$$\#(L \cap \psi(L)) \geqslant \mathrm{Crit}_{\mathrm{Morse}}(L).$$

Since $L$ and $\psi(L)$ is of half the dimension of the compact manifold $M$, their intersection is a finite set. To see it indeed generalize Arnold conjecture, we look at the diagonal $\delta = \{(x, x) \mid x \in M\}$ in $(M \times M, (-\omega) \times \omega)$, which is the fixed point set of $\tau : (x, y) \mapsto (y, x)$. For any $\psi \in \mathrm{Ham}(M)$, the fixed points of $\psi$ is $\delta \cap \tilde{\psi}(\delta)$, where $\tilde{\psi} : (x, x) \mapsto (x, \psi(x))$ is also a Hamiltonian symplectomorphism.

Floer developed another homology in his work on Arnold-Givental conjecture, called Lagrangian Floer homology. Since $\psi(L)$ is also a Lagrange submanifold, which can be proved similar to proposition 3.1.5, Floer constructed the homology based on a pair of Lagrangian submanifold $(L_1, L_2)$, and studied the space of smooth paths from $L_1$ to $L_2$,

$$\mathcal{P}(L_1, L_2) = \{\gamma : \mathbb{R} \to M \mid \gamma \text{ is smooth and } \gamma(0) \in L_1, \gamma(1) \in L_2\}.$$

The constant paths in $\mathcal{P}(L_1, L_2)$ correspond to points in $L_1 \cap L_2$. Ignoring all the technical issues, Floer's general idea is to find a suitable action functional on this space so that the critical points are exactly those constant paths. Just like in Morse theory, $CF(L_1, L_2)$ is freely generated by the critical points, in other words, by $L_1 \cap L_2$. An almost complex structure is required to generate gradient flow. In this case, a flow from one critical point to another would be a disk between $L_1$ and $L_2$, called a $J$-holomorphic disk, and the differential is defined by counting $J$-holomorphic disks between the two Lagrangian submanifolds. This would later become a major inspiration for the construction of Heegaard Floer homology.

## 4 Floer homologies in gauge theory

Gauge theory studys the connections on principal bundles, which are often used to describe dynamics of elementary particles in physics. In 1950s, Yang and Mills described the strong nuclear force in terms of connections on principal $SU(2)$-bundle that satisfy Yang-Mills equations. In 4-dimensional space, there are some special solutions of these equations called instanton that reflex the topology of the space, which inspired the development in instanton Floer homology.

In this section, we provide some basic notion in gauge theory and give a very rough description of instanton Floer homology. We will see at last how the study of gauge theory and 4-manifold would eventually motivate Heegaard Floer homology. No proof will be given in this section. The notations in this section are based on [19] and [8]. More on instanton Floer homology can be found on [17] and [4].

## 4.1 Connections on $G$-bundles

Before we define connections, recall that a vector bundle $E$ of rank $k$ over $m$-manifold $X$ is a $(m+k)$-manifold together with a map $p: E \to X$, so that each fiber is (linearly) isomorphic to $\mathbb{R}^k$ and $p$ is locally a projection. So there is an atlas $\{U_\alpha\}$ of $X$ and a family of isomorphisms $\{\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k\}$, where the composition of projection and $\phi_\alpha$ is exactly $p$.

When $U_\alpha$ and $U_\beta$ overlap, $\phi_\alpha \circ \phi_\beta^{-1}$ defines a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k)$ by

$$\phi_\alpha \circ \phi_\beta^{-1}(x, w) = (x, g_{\alpha\beta}(x) \cdot w).$$

The collection $\{U_\alpha, g_{\alpha\beta}\}$ is called a cocycle, and in fact determines the vector bundle. For an arbitrary collection $\{U_\alpha, g_{\alpha\beta}\}$ to be a cocycle, that is, to determine a vector bundle, it need to satisfy the cocycle condition:

$$g_{\alpha\alpha} = \text{id}, \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \quad g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}.$$

The group $G$ where $g_{\alpha\beta}$ take values is called the structure group of $E$, and $E$ is called a $G$-bundle. What we have just defined is a $GL(k)$-bundle. It is sometimes possible to alter the cocycle for a vector bundle to make $g_{\alpha\beta}$ take values in a subgroup of $GL(k)$. For example, an orientable manifold $M^n$ have an oriented atlas, so the cocycle $g_{\alpha\beta}$ of $TM$ take values in $G = GL^+(n)$, and it is therefore a $GL^+(n)$-bundle. Further more, the tangent bundle of a (oriented) Riemannian manifold has a structure group $O(n)$ (or $SO(n)$).

We start by defining a connection on a $G$-bundle $E$ from the viewpoint of parallel transport. This is to set a "standard" way to transport a vector from one point to another, and define connections to detect how much a transportation differs from the standard one.

**Definition 4.1.1.** *Let $p: E \to X$ be a $G$-bundle. A parallel transport $\tau$ on $E$ is an association for each path $c: [0,1] \to X$ an isomorphism in $G$*

$$\tau_c : E_{c(0)} \to E_{c(1)},$$

*such that $\tau_{c' * c''} = \tau_{c''} \circ \tau_{c'}$, where $c' * c''$ is the join of the two paths.*

For example, for an $O(n)$-bundle, the parallel transport preserves metric , that is

$$\langle \tau_c(v), \tau_c(w) \rangle = \langle v, w \rangle.$$

For a path $c$ that starts at $x \in X$, given an element $e \in E_x$, $\tau$ lift the path to a section $\tau_{c|_{[0,t]}}(e)$ of $E$ defined on $c$. We call this section $\tau$-parallel. Now given a random section $\sigma : X \to E$ and a tangent vector $w \in T_x X$, we can measure the infinitesimal deviation of $\sigma$ from being parallel.

$$(\nabla_w \sigma)(x) = \frac{d}{dt} \tau_{c|_{[0,t]}}^{-1}(\sigma(t))$$

The is called a covariant derivative of $\sigma$ in the direction of $w$, which produces a vector in $E_x$. If $V$ is a vector field, $\nabla_V \sigma \in \Gamma(E)$ is defined in a natural way. The covariant derivative is therefore a map

$$\nabla : \Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E).$$

Since $\sigma$ is parallel on a path $c$ if and only if $\nabla_{\frac{d}{dt}c} \sigma = 0$, a covariant derivative determines the parallel transport on $E$.

**Definition 4.1.2.** *A covariant derivative is a map $\nabla : \Gamma(TX) \times \Gamma(E) \to \Gamma(E)$ such that*

1. *$\nabla$ is $\mathbb{R}$-bilinear*
2. *$\nabla_{fV} \sigma = f \nabla_V \sigma$*
3. *$\nabla_V(f\sigma) = df(V) \cdot \sigma + f \cdot \nabla_V \sigma$.*

A covariant derivative is equivalent to a map

$$d_\nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*X)$$

called a connection on $E$, satisfying the Leibnitz property

$$d_\nabla(f \cdot \sigma) = f \cdot d_\nabla(\sigma) + \sigma \otimes df.$$

This property implies that $d' - d''$ is $C^\infty(X)$ linear for any pair of connections $d'$ and $d''$, so $A = d' - d'' \in \Gamma(\mathrm{End}(E) \otimes T^*X)$. Notice that $A$ is in fact a 1-form whose values are endomorphisms of the fibers of $E$. When $E$ is a $G$-bundle, the endomorphisms acts by the Lie algebra, which means $A \in \Gamma(\mathfrak{g}(E) \otimes T^*X)$. In general, we denote the space of differential $k$-forms that take value in vector bundle $E$ by $\Omega^k(X, E)$. The concepts of parallel transport, covariant derivative, connection and connection 1-form are often interchangeable, and are usually just called connection.

To define the curvature of a connection $\nabla$, first extend $d_\nabla$ to

$$d_\nabla^k : \Omega^k(X, E) \longrightarrow \Omega^{k+1}(X, E)$$

by the Leibnitz property

$$d_\nabla(\sigma \otimes \alpha) = d_\nabla(\sigma) \wedge \alpha + \sigma \otimes d\alpha,$$

then it is easy to show $d_\nabla \circ d_\nabla$ is $C^\infty(X)$ linear, which implies

$$F_\nabla := d_\nabla^1 \circ d_\nabla^0 \in \Omega^2(X, \mathrm{End}(E)),$$

It is called the curvature 2-form of $\nabla$. Again, $F_\nabla \in \Omega^2(X, \mathfrak{g}(E))$ for a $G$-bundle.

**Definition 4.1.3.** *A flat connection is a connection with zero curvature form.*

For a trivial bundle $X \times \mathbb{R}^k$, we can define a flat connection by

$$d_{\text{flat}}(\sigma) = (d\sigma_1, \cdots, d\sigma_k).$$

## 4.2 Some basic concepts in gauge theory

We introduce some basic concepts we will need in the future.

**Definition 4.2.1.** *A fiber bundle $\pi : P \to X$ is called a principal $G$-bundle, where $G$ is a Lie group called the structure group of $P$, if its fibers are $G$ and*

1. *There is a smooth fiber-preserving action $r : P \times G \to P$ such that*

$$\begin{aligned} G &\longrightarrow P_x \\ g &\longmapsto r_g(p) = p \cdot g \end{aligned}$$

   *is a bijection for any $x \in X$, $p \in P$.*

2. *There is an atlas $\{U_\alpha, \phi_\alpha\}$ of $P$ such that*

$$\phi_\alpha(p \cdot g) = \phi_\alpha(p) \cdot g$$

   *for any $\alpha$, with $\phi_\alpha(p) \cdot g$ defined in the obvious way.*

To define a connection on principal bundle $P$, we can take the analogous approach by defining a parallel transport. The "differential form" version of connections is not obvious and will only be stated. We denote the set of all $k$-forms on $X$ with values in a vector space $W$ by $\Omega^k(X, W)$. These forms can be seen as taking value on the trivial bundle $W \times X$, that is $\Omega^k(X, W) = \Omega^k(X, W \times X)$.

**Definition 4.2.2.** *A connection 1-form or a gauge field on a principal $G$-bundle is an element $A$ in $\Omega^1(X, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of $G$, satisfying*

*1.* $r_g^* A = \mathrm{Ad}_{g^{-1}} \circ A$ *where* $r_g^*$ *is the pullback of differential form.*

*2. For every* $V \in \mathfrak{g}$ *and the associated fundamental vector field* $\tilde{V}$ *on* $P$*, there is* $A(\widetilde{V}) = V$*.*

The fundamental vector field associated to $V \in \mathfrak{g}$ describes infinitesimal behaviour of the $G$-action on $P$, defined by

$$\widetilde{V}_p = \left.\frac{d}{dt}\right|_{t=0} (p \cdot \exp(tV)).$$

We define the curvature 2-form by

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

where

$$[f \wedge g](V_1, \cdots, V_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\ [f(V_{\sigma(1)}, \cdots, V_{\sigma(k)}), g(V_{\sigma(k+1)}, \cdots, V_{\sigma(k+l)})].$$

A connection is called flat when if its curvature form vanish.

**Definition 4.2.3.** *A gauge transformation* $f$ *of principal bundle* $P$ *is a bundle automorphism, that is, it preserves the fibers and* $f(p \cdot g) = f(p) \cdot g$ *for all* $p \in P$ *and* $g \in G$*. The gauge group* $\mathcal{G}(P)$ *is the group of all gauge transformations of* $P$*.*

Gauge group acts on the space $\mathcal{A}$ of all connections by the rule $g^* A = g^{-1} dg + g^{-1} A g$. We are interested in the quotient space $\mathcal{B} = \mathcal{A} / \mathcal{G}$. To make it a smooth Banach manifold, take its subset of the orbits of all irreducible connections, which are the connections whose stabilisers are exactly the center of $\mathcal{G}$. This manifold is denoted by $\mathcal{B}^*$.

For another basic definition, we define the Hodge star operator on an oriented Riemannian manifold $X^m$. A Riemannian metric induces a metric for covectors which and can be generalize to $k$-form by Gram determinant

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle := \det(\langle \alpha_i, \beta_j \rangle)_{ij}.$$

The Hodge star operator $* : \Omega^k(X) \to \Omega^{m-k}(X)$ is defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}_X,$$

where $\mathrm{vol}_X$ is a fixed volume form on $X$. It can be proved that $*^2 = (-1)^{k(4-k)}$. This definition can actually be generalized to vector bundle valued forms $\Omega^k(X, W)$

When $X$ is 4-dimensional, Hodge operator is a map from $\Omega^2(X, W)$ to itself, and $*^2 = \mathrm{id}$. This splits $\Omega^2(X, W) = \Omega^2_+(X, W) \oplus \Omega^2_-(X, W)$ into its eigenspaces with eigenvalues $\pm 1$. These subspaces are respectively the spaces of self-dual (SD) and anti-self-dual (ASD) forms.

**Definition 4.2.4.** *Let* $A$ *be a connection on a principal bundle* $\pi : P \to X$*, where* $X$ *an oriented Riemannian 4-manifold.* $A$ *is called an instanton or an ASD connection if*

$$F_A + *F_A = 0.$$

Notice that all flat connections are instantons. The set of all instanton on $P$ is invariant under gauge group $\mathcal{G}$, we can thus talk about its moduli space $\mathcal{M}$ and $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$. Donaldson showed that the topology of the moduli space of instantons depends only on the topological information of $M$, which inspired the development of Instanton Floer homology.

## 4.3 Instanton Floer homology

Let $\Sigma$ be a homology 3-sphere with a suitable Riemannian metric, and $E = \Sigma \times SU(2)$ be a trivial $SU(2)$-bundle. Define the Chern-Simons functional $cs : \mathcal{B}^*(E) \to \mathbb{R} / \mathbb{Z}$

$$cs(\alpha) = \frac{1}{8\pi^2} \int_\Sigma \mathrm{tr}(\alpha \wedge d\alpha + \frac{2}{3} \alpha \wedge \alpha \wedge \alpha).$$

The critical points of this functional are those irreducible flat connections, which can be given $\mathbb{Z}_8$ valued indices $\mu$. The chain complex $IC_*$ is freely generated by these critical points and has a $\mathbb{Z}_8$-grading. Again, to define the differential, we want to count the flow lines (moduli space) from one critical to another. In order to do this, pull back $E$ to a cylinder $\mathbb{R} \times \Sigma$ and obtain another $SU(2)$-bundle $E'$. We are interested in instantons on $E'$ that limits to two different flat connections at minus and plus infinity respectively. In a sense, a moduli space of such instantons connects critical points of $cs$. The moduli space connecting $\alpha$ and $\beta$ is denoted by $\mathcal{M}(\alpha, \beta)$. It was proved that $\mathcal{M}(\alpha, \beta)$ is inside $\mathcal{B}^*$, implying that the construction is again a Morse homology on a Banach manifold.

Careful analysis on the moduli spaces shows that

$$\dim \mathcal{M}(\alpha, \beta) = \mu(\alpha) - \mu(\beta) \mod 8$$

Quotient out the $\mathbb{R}$-action on $\mathcal{M}(\alpha, \beta)$ defined by $A(t) \mapsto A(t+T)$, $T \in \mathbb{R}$, we get a compact manifold $\widehat{\mathcal{M}}$. The differential is then given by

$$\partial \alpha = \sum_{\mu(\beta)=\mu(\alpha)-1} \# \widehat{\mathcal{M}}(\alpha, \beta).$$

$(IC_*, \partial)$ is a (circular) chain complex. The homology of this complex is called instanton Floer homology. It depends only on the diffeomorphism type of $\Sigma$, and is denoted by $I_*(\Sigma)$

Given a pair of homology 3-sphere $(\Sigma_1, \Sigma_2)$ and a smooth compact oriented cobordism $W$ such that $\partial W = -\Sigma_1 \cup \Sigma_2$, view $W$ as an open manifold with two tubular neighbourhoods, there is a homomorphism $W_* : IC_*(\Sigma_1) \to IC_*(\Sigma_2)$ defined by

$$W_*(\alpha) = \sum_{\mu(\beta)=\mu(\alpha)+3(b_1-b_2^+)(W)} \# \mathcal{M}_W(\alpha, \beta),$$

with $\mathcal{M}_W(\alpha, \beta)$ the moduli space of finite Yan-Mills action connecting $\alpha$ and $\beta$. $W^*$ is a chain homomorphism whose induced map $W_* : I_*(\Sigma_1) \to I_*(\Sigma_2)$ only depends on the cobordism $W$. We in fact created a TQFT with instanton Floer homology.

## 4.4 Atiyah-Floer conjecture

There seem to be a connection instanton Floer homology and Lagrangian Floer homology which would became a key inspiration for Heegaard Floer homology. Before presenting the connection, we need to define an object called $SU(2)$-character variety. A group is called finitely presented if it is generated by finitely many elements and relations.

**Definition 4.4.1.** *The $SU(2)$-representation space of a finitely presented group $\pi$ is the space $R(\pi) :=$ $\mathrm{Hom}(\pi, SU(2))$ with compact open topology, where $\pi$ is endowed with discrete topology.*

*$SU(2)$ acts on $R(\pi)$ by conjugation, and the quotient $\mathcal{R}(\pi) := R(\pi)/SU(2)$ is a real compact algebraic variety called the $SU(2)$-character variety of $\pi$.*

$SU(2)$ can be viewed as a unit sphere in $\mathbb{R}^4$, and $R(\pi)$ can be viewed as an algebraic variety by representing each generator by a unit vector. The relations on $\pi$ will then define a polynomial equation for these vectors. We use $\mathcal{R}(X)$ to denote the $SU(2)$-character variety of $\pi_1(X)$, where $X$ is a compact manifold of dimension less than or equal to 3. $\pi_1(X)$ is always finitely presented since $X$ is homeomorphic to a finite simplicial complex. There is a canonical symplectic structure on $\mathcal{R}(X)$.

$SU(2)$-character variety played an important part in the development of instanton Floer homology. Atiyah-Floer conjecture asserts that, when splitting a homology sphere $Y$ "properly" into $U \cup V$, the Lagrangian Floer homology $HF(\mathcal{R}(U), \mathcal{R}(V))$ will coincide with the instanton Floer homology $I(Y)$. We present the simplified version of this conjecture.

**Conjecture 4.4.2.** *(Atiyah-Floer, simplified) Given a Heegaard splitting of a homology sphere $Y =$ $U \cup V$, the two Lagrangian submanifolds $\mathcal{R}(U)$ and $\mathcal{R}(V)$ in $\mathcal{R}(Y)$ induces a Lagrangian Floer homology isomorphic to the instanton homology of $Y$, that is*

$$HF_*(\mathcal{R}(U), \mathcal{R}(V)) \simeq I_*(Y).$$

We will define Heegaard splitting in the next section.

In 1990s, Seiberg–Witten monopole equations became a new popular object in studying 4-manifolds. The moduli spaces of monopole, the solutions of Seiberg–Witten equations, can also reflex topological information of manifolds. Feehan and Leness proved that monopole equations contain the same information as the instanton equations. A $(3+1)$-dimensional TQFT corresponding to instanton Floer homology was then constructed, called the monopole Floer homology. Inspired by the Atiyah-Floer conjecture, there should be an instance of Lagrangian Floer homology that coincides with monopole Floer homology. It turned out that there is indeed a homology theory very similar to Lagrangian Floer homology that coincides with it, called Heegaard Floer homology.

# 5 Heegaard Floer homology

In this section, we outline the construction of hat version of Heegaard Floer complex $\widehat{CF}(\mathcal{H})$ from a given pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and a complex structure on $\Sigma$. Ozsváth and Szabó [14] followed the framework of Lagrangian Floer homology by splitting a 3-manifold and produces a new space with two submanifolds within. $\widehat{CF}(\mathcal{H})$ will be freely generated by their intersection points and the differential is defined by directly counting holomorphic disks between the two manifolds.

More about Heegaard Floer homology can be found on [6]. Throughout this section, $Y$ denotes a closed and oriented 3-manifold.

## 5.1 Heegaard diagrams

We first define Heegaard diagrams for given 3 manifold. Detailed proofs can be found on [18].

**Definition 5.1.1.** *A Heegaard splitting of $Y$ is a decomposition*

$$Y = U \cup_f V,$$

*where $U$ and $V$ are handlebodies of genus $g$ and $f : \partial U \to \partial V$ is a diffeomorphism, which can also be seen as an automorphism of the surface $\Sigma_g$ of genus $g$.*

A Heegaard splitting used data of lower dimension to describe a 3-manifold. For example, $S^3$ can be split into two balls with $f = \text{id}$. For another splitting of $S^3$, consider two solid tori glued in the way shown below.

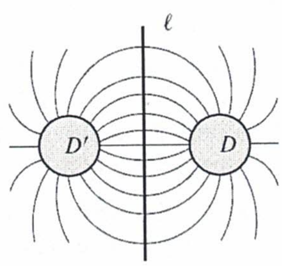


Figure 2: [18] Genus 1 Heegaard splitting of $S^3$.

Here we see $S^3$ as $\mathbb{R}^3 \cup \{\infty\}$. $D$ and $D'$ are section of one of the solid torus. Each arc that connect $D$ and $D'$ represents a section of the handle of another torus.

The good news is, every closed oriented 3-manifold admits a Heegaard splitting. To prove this, choose a triangulation for the manifold. The normal neighbourhood of its 1-dimensional skeleton and its complement will form a Heegaard splitting. We distinguish Heegaard splittings up to homeomorphism.

**Definition 5.1.2.** *Two Heegaard splitting $Y = U \cup_f V = U' \cup_{f'} V'$ are called homeomorphic if there exists a automorphism of $Y$ that takes $U$ to $U'$ and $V$ to $V'$*

Another way to understand Heegaard splitting is to imagine gluing two handlebodies $U, V$ to a surface $\Sigma$, one glued from "outside" and another one from "inside". We attach a set of circles to each handle of $U$ and $V$. To be precise, an attaching circle is a circle in $\Sigma$ that bound a disk in the handlebody. We then map these attaching circles to $\Sigma$ through the gluing, we now have a good way to represent a Heegaard splitting.

**Definition 5.1.3.** *A Heegaard diagram compatible with* $Y = U \cup_f V$ *is a triple* $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *where* $\Sigma = \partial U = \partial V$ *and* $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$ *and* $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$ *are the attaching circles for* $U$ *and* $V$ *that are mapped into* $\Sigma$.

Figure 3 shows the Heegaard diagram for the Heegaard splitting in Figure 2.

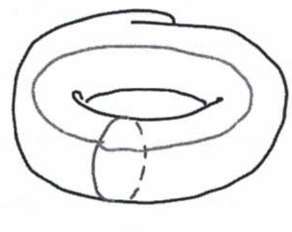

Figure 3: Heegaard diagram for $S^3$.

A Heegaard diagram completely determines $Y$, so we also say $\mathcal{H}$ is a Heegaard diagram for $Y$. However, $Y$ can have different Heegaard diagrams. We are interested in the operations on $\mathcal{H}$ that do not change the space it represents, which are called Heegaard moves. They consist of isotopies, handleslides, stabilizations and destabilizations. The first move, isotopic, is done by simply moving around the attaching circles in a smooth manner. Handlesliding is to connect one attaching circle to a parallel copy of another one, shown in Figure 4. Stabilizations add extra handles to both $U$ and $V$ and glue them in a trivial way, as shown in Figure 5, and destabilizations reverse the process.

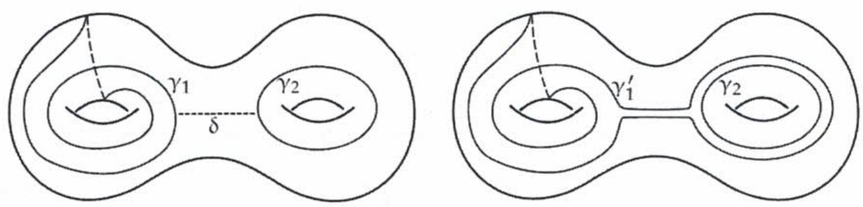


Figure 4: [9] Handlesliding.

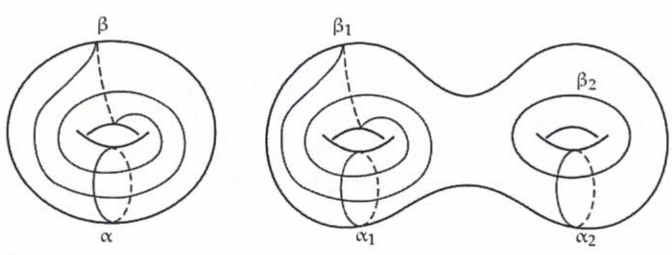


Figure 5: [9] Stabilization.

**Theorem 5.1.4.** *Every two Heegaard diagrams represent homeomorphic Heegaard splitting after applying finitely many Heegaard moves.*

A variation of Heegaard diagram is used to represent knots in 3-manifolds. We focus on knots in $S^3$ for simplicity.

**Definition 5.1.5.** *A doubly-pointed Heegaard diagram for a knot $K \subset S^3$ is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ such that $w, z \in \Sigma - \boldsymbol{\alpha} \cup \boldsymbol{\beta}$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for $S^3$, with which we are able to construct $K$ by the following procedure.*

*Find a arc in $\Sigma$ connecting $w$ and $z$ that does not intersect with $\boldsymbol{\alpha}$, and push it slightly into $U$. Then find another similar arc that does not intersect with $\boldsymbol{\beta}$, and push it slightly into $V$. The union of the two arcs will be a knot in $S^3$.*

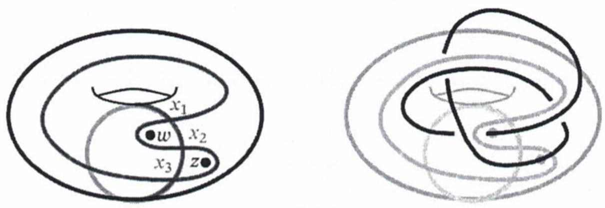


Figure 6: [7] A doubly-pointed Heegaard diagram of trefoil.

Every knot in $S^3$ admits a doubly-pointed Heegaard diagram. Figure 7 gives a way to generate a doubly-pointed Heegaard diagram from a given knot.

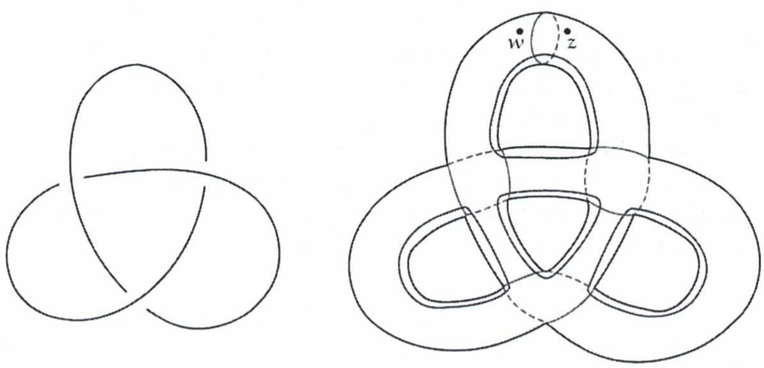


Figure 7: [9] Generating a doubly-pointed Heegaard diagram.

There are similarly pointed Heegaard diagrams with only one basepoint on $\Sigma$, which will be used when constructing Heegaard Floer homology. When applying Heegaard moves on pointed or double-pointed diagrams, we do not want $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to touch the basepoints. Theorem 5.1.4 still holds in either cases fortunately.

## 5.2 Symmetric products

We now want to define and study our ambient space in the construction of Heegaard Floer homology. We will start from a pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where $z \in \Sigma$ is a basepoint outside of $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, and consider the symmetric product $\mathrm{Sym}^g(\Sigma)$. The basepoint is essential. In fact, without the basepoint, the homology constructed will simply be the ordinary homology of $Y$.

**Definition 5.2.1.** *Let $M$ be a manifold, define its $k$-symmetric product to be*

$$\mathrm{Sym}^k(M) = \left( \prod^k M \right) / S_k,$$

*where $S_k$ acts on the space by permutations.*

$\mathrm{Sym}^k(M)$ can be seen as the space of all unordered $k$-tuples of points of $M$. It is not necessarily a manifold since $S_k$ does not act freely in general. However, the ambient space $\mathrm{Sym}^g(\Sigma_g)$ we are going to use is indeed a manifold.

**Proposition 5.2.2.** *Let $\Sigma_g$ be an oriented surface of genus $g$, then* $\mathrm{Sym}^k(\Sigma)$ *is a* $2k$*-dimensional manifold.*

*Proof.* Choose any point $z = (w_1, \cdots, w_1, w_2, \cdots, w_2, w_3, \cdots)$ in $\mathrm{Sym}^k(\Sigma)$, where all $w_i$ are distinct points in $\Sigma$ appear $l_i$ times in $z$. Take a sufficient small neighbourhood of $z$ so that the corresponding neighbourhoods for $w_i$ do not intersect with each other. This neighbourhood will then look like

$$\mathrm{Sym}^{l_1}(\mathbb{R}^2) \times \mathrm{Sym}^{l_2}(\mathbb{R}^2) \times \cdots .$$

Now it suffices to show $\mathrm{Sym}^l(\mathbb{R}^2) \approx \mathrm{Sym}^l(\mathbb{C})$ is a manifold. In fact, $\mathrm{Sym}^l(\mathbb{C}) \approx \mathbb{C}^l$. To construct the diffeomorphism $\Phi : \mathrm{Sym}^l(\mathbb{C}) \to \mathbb{C}^l$, see $\mathbb{C}^l$ as the space of monic polynomials with complex coefficients with degrees $\leqslant l$, and the desired diffeomorphism will be

$$\Phi : \{z_1, \cdots, z_l\} \longmapsto \prod(z - z_i)$$
$$\Phi^{-1} : f \longmapsto \text{ roots of } f \text{ with multiplicity.}$$

□

The fact that $\mathrm{Sym}^l(\mathbb{C}) \approx \mathbb{C}^l$ also implies that we can pass a complex structure from $\Sigma$ to $\mathrm{Sym}^g(\Sigma)$. However, the fact that $\Sigma$ is a symplectic manifold requires some works.

**Proposition 5.2.3.** $\mathrm{Sym}^k(\Sigma)$ *is a symplectic manifold.*

We will not prove this proposition. The idea is to consider the thick diagonal

$$\Delta = \{\{x_1, \cdots, x_k\} \mid x_i = x_j \text{ for some } i \neq j\}.$$

$\mathrm{Sym}^k(\Sigma) - \Delta$ is a part of the product $\prod \Sigma$, so the product form of a symplectic form on $\Sigma$, which must exist, would become a symplectic form on $\mathrm{Sym}^k(\Sigma) - \Delta$. It is proved that this form can be extended to $\Delta$.

Following the idea of Lagrangian Floer homology, we focus on two subspaces in the space $\mathrm{Sym}^g(\Sigma)$. Given a pointed Heegaar diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, define $\mathbb{T}_\alpha$ to be the image of $\boldsymbol{\alpha}$ in $\mathrm{Sym}^g(\Sigma)$, and $\mathbb{T}_\beta$ be that of $\boldsymbol{\beta}$. Since the attaching circles of the same handlebody do not intersect with each other, $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are in fact tori in $\prod \Sigma$, and are both $g$-dimensional manifolds. By making $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to intersect transversely in $\Sigma$, $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ will intersect transversely in $\mathrm{Sym}^g(\Sigma)$. In this case, since the two tori are of half the dimension of $\mathrm{Sym}^g(\Sigma)$, they intersect at finitely many points. The Heegaard Floer complex $\widehat{CF}(\mathcal{H})$ is then freely generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Since $\mathrm{Sym}^g(\Sigma)$ is symplectic, we might expect a construction completely analogous to Lagrangian Floer homology, but it turns out that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are not Lagrangian. In this case, a different procedure is required.

We need a lemma before heading on to the construction.

**Lemma 5.2.4.**

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \simeq \frac{H_1(\Sigma)}{[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]} \simeq H_1(Y).$$

*Proof.* The last isomorphism is a direct result of Mayer-Vietoris sequence, so the only problem is to prove $H_1(\mathrm{Sym}^g(\Sigma)) \simeq H_1(\Sigma)$. Fix a basepoint $z \in \Sigma$, set $i : \Sigma \to \mathrm{Sym}^g(\Sigma)$ defined by

$$i : x \longmapsto \{z, \cdots, z, x\}.$$

This induces a isomorphism $i_*$ on the first homology. For the inverse $j_*$, consider a path $\gamma$ in $\mathrm{Sym}^g(\Sigma)$ that represent an element $[\gamma] \in H_1(\mathrm{Sym}^g(\Sigma))$, move it by a homotopy so that it does not touch the thick diagonal $\Delta$ and still represent the same element. In this way, it becomes a path in $\prod \Sigma$ and therefore represents a collection of paths in $\Sigma$. This collection of paths will be the image of $[\gamma]$ under $j_*$. □

## 5.3 The outline of the construction

We will state without proving the construction of hat version of Heegaard Floer homology for $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where $g > 2$, as construction will be more difficult when $g = 2$ and impossible when $g < 2$. We will give $\Sigma$ a suitable complex structure and pass it to $\mathrm{Sym}^g(\Sigma)$, then count the moduli space of holomorphic representatives $\pi_2(x, y)$ defined as followed.

**Definition 5.3.1.** *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A Whitney disk connecting $x$ and $y$ is a continuous map*

$$\phi : D \longrightarrow \mathrm{Sym}^g(\Sigma)$$

*with $\phi(-i) = x$ and $\phi(i) = y$, and the two sides of the dick are mapped to $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ respectively, as shown in Figure 8. Denote the set of homotopy classes of Whitney disks connecting $x$ and $y$ by $\pi_2(x, y)$.*

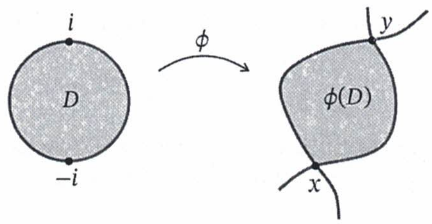


Figure 8: [7] A Whitney disk from $x$ to $y$.

Let $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of $\phi \in \pi_2(x, y)$. For each $\phi \in \pi_2(x, y)$ we can associate a Maslov index $\mu(\phi)$, which will be exactly the dimension of $\mathcal{M}(\phi)$ under suitable complex structures. $\mathcal{M}(\phi)$ also admits an $\mathbb{R}$-action given by the 1-parameter family of automorphisms of disk which fix $\pm i$. Let $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$. When $\mu(\phi) = 1$, it will be a compact oriented zero dimensional manifold under suitable complex structures.

Let $\widehat{CF}(\mathcal{H}) = \mathbb{Z}(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)$, to define a grading on $\widehat{CF}(\mathcal{H})$, we need two more functions.

**Definition 5.3.2.** *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Define $n_z : \pi_2(x, y) \to \mathbb{Z}$ by sending $\phi$ to its algebraic intersection number with $V_z := \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$.*

This is well-defined since $z$ is outside of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

**Definition 5.3.3.** *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Choose a pair of paths $a : [0, 1] \to \mathbb{T}_\alpha$, $b : [0, 1] \to \mathbb{T}_\beta$ from $x$ to $y$, then $a - b$ is a loop in $\mathrm{Sym}^g(\Sigma)$. Let $\epsilon(x, y)$ denote the image of $[a - b]$ in $H_1(Y)$ under the map in Lemma 5.2.4. It is a well defined map independent of the choice of $a$ and $b$. $\pi_2(x, y)$ is non-empty if and only if $\epsilon(x, y) = 0$.*

We then define a relative grading on $\widehat{CF}(\mathcal{H})$ by

$$\mathrm{gr}(x) - \mathrm{gr}(y) = \mu(\phi) - 2n_z(\phi),$$

where $\phi \in \pi_2(x, y)$. This however, is well defined only when $\epsilon(x, y) = 0$, so we need to partition $\widehat{CF}(\mathcal{H})$ into some classes where we are able to grade elements from the same class. A natural way to do this is to use the relation $\epsilon(x, y) = 0$. In reality, we use a more refined partition given by a map $s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$, where each class in $\widehat{CF}(\mathcal{H})$ is the preimage of a $\mathrm{Spin}^c$ structure $\mathfrak{s}$ of $Y$, denoted by $\widehat{CF}(\mathcal{H}, \mathfrak{s})$. We can now define the differential $\partial : \widehat{CF}(\mathcal{H}, \mathfrak{s}) \to \widehat{CF}(\mathcal{H}, \mathfrak{s})$ by

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \#\widehat{\mathcal{M}}(\phi) \cdot y.$$

Intuitively, it counts all the holomorphic disks representing those $\phi$ which do not intersect $V_z$ (algebraically) and only have finitely many holomorphic representatives. Since $\epsilon(x, y) = 0$ for every $y \notin \widehat{CF}(\mathcal{H}, \mathfrak{s})$, there is $\pi_2(x, y) = \varnothing$, and thus $\partial x \in \widehat{CF}(\mathcal{H}, \mathfrak{s})$. Analysis on the moduli space shows that $(\widehat{CF}(\mathcal{H}, \mathfrak{s}), \partial)$ is a chain complex.

**Theorem 5.3.4.** *Fix a* Spin$^c$ *structure on* $Y$. *Given two pointed Heegaard diagrams* $\mathcal{H}, \mathcal{H}'$, *there is*

$$\widehat{HF}(\mathcal{H}, \mathfrak{s}) \simeq \widehat{HF}(\mathcal{H}', \mathfrak{s})$$

This was proved by showing Heegaard moves produce chain homotopy equivalent complexes and then using Theorem 5.1.4. It make sense now to write $\widehat{HF}(Y, \mathfrak{s})$ for $\widehat{HF}(\mathcal{H}, \mathfrak{s})$. The Heegaard Floer homology of $Y$ is defined by

$$\widehat{HF}(Y) := \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}).$$

**Theorem 5.3.5** (Ozsváth, Szabó). $\widehat{HF}(Y)$ *is a invariant of* 3*-manifold* $Y$.

With great efforts, the analogue to the Atiyah-Floer conjecture was proved by Colin, Ghiggini, Honda, Kutluhan, Li, and Taubes.

**Theorem 5.3.6.** *The monopole Floer homology and the corresponding Heegaard Floer homology are isomorphic for every closed, oriented* 3*-manifold.*

# 6 Knot Floer homology

Knot Floer homology is an invariant of knots and links in three-manifolds and an important variants of Heegaard Floer homology, developed independently by Ozsváth-Szabó and Rasmussen. In this section, we will construct this homology and mention some of its properties.

## 6.1 Preliminaries in knot theory

We first have a quick review of some basic concepts in knot theory and introduce two classical invariants for knots and links: Jones polynomial and Alexander polynomial. All notations here are based on [10].

**Definition 6.1.1.** *A oriented link* $L$ *of* $m$ *components is a subset of* $S^3$ *consisting of* $m$ *piecewise linear, simple oriented closed curves. A link of* 1 *component is called a knot. Two links are equivalent if there is a orientation-preserving piecewise linear homeomorphism of* $S^3$ *that restricts to a homeomorphism between the two links.*

$S^3$ is often considered as $\mathbb{R}^3 \cup \{\infty\}$. A link can be projected to $\mathbb{R}^2$. By perturbing the link, we can make sure the intersections in image of the projection are all transverse intersections of 2 curves. The image together with the crossing information (see Figure 9) is called the link diagram of the given link. A link diagram determines the link itself, and we say two diagrams are equivalent if the links are equivalent. Equivalent diagrams are related by a finite sequence of Reidemeister moves shown in Figure 10.


Figure 9: [10] Two types of crossing.

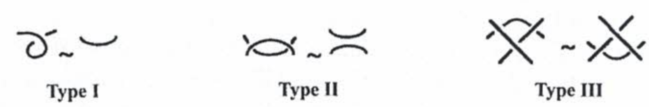


Figure 10: [10] Reidemeister moves.

Here is a useful tool in knot theory.

**Definition 6.1.2.** *A Seifert surface for an oriented link $L$ is a compact,connected and oriented surface in $S^3$ with $L$ being its oriented boundary.*

**Proposition 6.1.3.** *Every oriented link has a Seifert surface.*

*Proof.* Denote the diagram of $L$ by $D$ and resolve every crossing in the way shown in Figure ??, we get a diagram $\widehat{D}$ which consists of finitely many oriented circles and is thus the boundary of a collection of disks. To alter this collection of disks into a Seifert surface, Join them together at each crossing with half-twisted strips. Apply connected sum if the surface we get is not connected.□

Figure 11: [10] Resolving crossings.

**Definition 6.1.4.** *The genus of a knot $K$ is defined by*

$$g(K) = \min\{g(F) \mid F \text{ is a Seifert surface of } K\}.$$

We now head on to define Jones polynomial.

**Definition 6.1.5.** *The Kauffman bracket $\langle D \rangle$ of a given link diagram $D$ is a polynomial in $\mathbb{Z}[A^{-1}, A]$ characterised by*

1. $\langle \bigcirc \rangle = 1$
2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle D \rangle$
3. $\langle \times \rangle = A\langle )( \rangle + A^{-1}\langle \asymp \rangle$

Here $\bigcirc$ denotes an unknot, a curve that bounds a disk in $S^3$. Kauffman bracket is invariant under the second and third type of Reidemeister moves. However, when performing an first type move, an extra factor $-A^{\pm 3}$ will appears, where the sign depends on the orientation. To make it an invariant of links, denote $w(D)$ the sum of the signs (see Figure 9) of the crossings in $D$, then

**Proposition 6.1.6.** $(-A)^{-3w(D)}\langle D \rangle$ *is an invariant of oriented links.*

The proof is done by direct calculation.

**Definition 6.1.7.** *Given an oriented link $L$, its Jones polynomial is defined by*

$$V(L) = \left((-A)^{-3w(D)}\langle D \rangle\right)_{t^{\frac{1}{2}} = A^{-2}}$$

By induction on number of crossings in the diagram, it is easy to show $V(L) \in \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$.

Alexander polynomial on the other hands is much harder to construct. To sketch the construction, remove a regular neighbourhood of the link $L$ from $S^3$ to get a new manifold $X$, then remove a regular neighbourhood of the Seifert surface $F$ of $L$ from $X$, getting another manifold $Y$. There are two copies of $F$ in the boundary of $Y$, denoted by $F_+$ and $F_-$. Now take countable family of copies of $Y$, denoted by $\{Y_i\}$, each with a homeomorphism $h_i : Y_i \to Y$. Glue each $h_i F_-$ to $h_{i+1}F_+$, we form a space $X_\infty$. The process is illustrated in Figure 12.

There is a countable family of homeomorphism by "shifting", generated by $t : X_\infty \to X_\infty$, $t|_{Y_i} = h_{i+1}h_i^{-1}$. $\langle t \rangle$ acts on the homology group $H_1(X_\infty; \mathbb{Z})$, and thus making $H_1(X_\infty; \mathbb{Z})$ an $\mathbb{Z}[t^{-1}, t]$-module. The Alexander polynomial $\Delta_L(t)$ is the smallest principal ideal of $\mathbb{Z}[t^{-1}, t]$ that contains the first elementary ideal of this module. Here the $r$-th elementary ideal is the ideal generated by all the $(m - r + 1) \times (m - r + 1)$ minors of the presentation matrix of $H_1(X_\infty; \mathbb{Z})$, defined as followed.

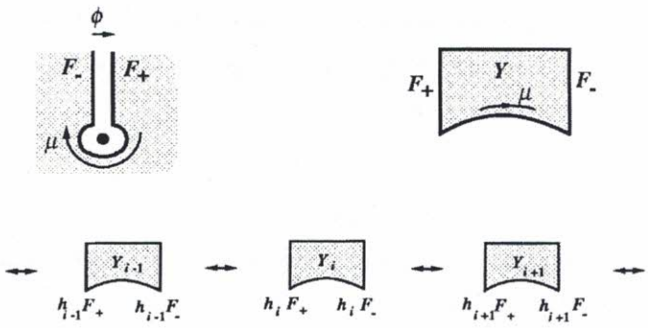

Figure 12: Gluing copies of $Y$.

**Definition 6.1.8.** *A finite representation of an $R$-module $M$ is an exact sequence*

$$F \longrightarrow E \longrightarrow M \longrightarrow 0$$

*where $E, F$ are free $R$-modules. The matrix of the first homomorphism is called the representation matrix of $M$.*

Alexander polynomial is much harder to compute, but it gives an important lower bound for genus of knots with the inequality $\deg \Delta(L) \leqslant g(L)$.

## 6.2 Construction and properties

We will construct the hat version of knot Floer homology. It is a bi-graded abelian group $HFK(K)$ associated to an oriented knot $K$. Start from a doubly-pointed Heegaard diagram $\mathcal{H}(\Sigma^g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ of the given knot, the construction is almost completely analogous to Heegaar Floer homology.

Define $\mathrm{Sym}^g(\Sigma)$, $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, $\pi_2(x, y)$ and $\mathcal{M}(\phi)$ the same way as in the last section. $\widehat{CFK}(\mathcal{H})$ is again freely generated by the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. In Heegaard Floer homology, a function $n_z$ counting algebraic intersections is was used in order to define the differential. Here we will need $n_z$ together with a similarly defined $n_w$. The differential in knot Floer homology is given by

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=n_w(\phi)=0}} \#\widehat{\mathcal{M}}(\phi) \cdot y.$$

Here we have $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ and $\mu$ the Maslov index. The homology of the chain complex $(\widehat{CFK}(\mathcal{H}), \partial)$ is the knot Floer homology, which is invariant of $K$ denoted by $\widehat{HFK}(\mathcal{H})$ or $\widehat{HFK}(K)$. Notice that we get $\widehat{HF}(S^3)$ if the second basepoint $w$ is dropped.

$\widehat{CFK}(\mathcal{H})$ is equipped with 2 grading, induced by

$$\begin{aligned} M(x) - M(y) &= \mu(\phi) - 2n_w(\phi) \\ A(x) - A(y) &= n_z(\phi) - n_w(\phi). \end{aligned}$$

The two gradings are called Maslov and Alexander grading respectively. We denote $M(x) = m$, $A(x) = a$. Notice that Maslov grading is exactly the same grading in $\widehat{CF}(\mathcal{H})$. The differential satisfies

$$\partial : \widehat{CFK}_m(\mathcal{H}, a) \longrightarrow \widehat{CFK}_{m-1}(\mathcal{H}, a),$$

so the two grading descends to homology

$$\widehat{HFK}(\mathcal{H}) = \bigoplus \widehat{HFK}_m(\mathcal{H}, a).$$

**Theorem 6.2.1.** *Let $K$ be an oriented knot.*

1. *Knot Floer homology categorifies Alexander polynomial by graded Euler characteristic, that is*

$$\Delta(K) = \sum_{m,a}(-1)^m \text{rank } HFK_m(K,a) \cdot t^a.$$

2. *Knot Floer homology detects the genus of the knot*

$$g(K) = \max\{a : HFK(K,a) \neq 0\}.$$

The Alexander grading also gives rise a spectral sequence. All terms in this spectral sequence are invariant of the knot $K$ and it converges to $\mathbb{Z}$.

**Definition 6.2.2.** *Define $\tau(K)$ be the Alexander grading of the surviving copy of $Z$ in the spectral sequence of $\widehat{HFK}(K)$*

Knot Floer homology is not the only homology theory that categorifies knot polynomial. Prior to this, Khovanov find a homology theory called Khovanov homology that categorifies Jones polynomial. The two homology theories share some common properties while being significantly different in construction. Some of the connections can be explained by spectral sequences. For example,

**Theorem 6.2.3.** *For any knot $K$ in $S^3$, there is a spectral sequence from reduced Khovanov homology of $K$ to $\delta$-graded knot Floer homology of the mirror image of $K$. The coefficient for both homology is $\mathbb{Q}$.*

The theorem accounts for the so called property FK found on many knots (see [16] and [5]).

# A Almost complex structures

We introduce some basic concepts related to almost complex structure here. More information can be found on [12], [11] and [3].

**Definition A.0.1.** *A (linear) complex structure on vector space $V$ over $\mathbb{R}$ is an automorphism $J : V \to V$ such that $J^2 = -\text{id}$.*

Computing determinant on both side of $J^2 = -\text{id}$, we know immediately that only even dimensional vector spaces can have a complex structure. A real vector space with a complex structure can be seen as a complex vector space, with the scalar multiplication given by

$$(a + bi, v) \longmapsto av + bJv.$$

**Definition A.0.2.** *An almost complex structure $J$ on a smooth manifold $M$ is an automorphism of its tangent bundle, such that $J^2 = -\text{id}$. In other words, $J_x$ is a linear complex structure on $T_xM$ for every $x \in M$. We call the pair $(M, J)$ an almost complex manifold.*

A bundle automorphism is a smooth map from bundle to itself that fixes the base space and preserves the structure of each fiber. Only even dimensional manifolds can be endowed with an almost complex structure.

Let $(M, \omega)$ be a symplectic manifold, an almost complex structure $J$ on $M$ is called $\omega$-tame if $\omega(v, Jv)$ is positive for every $v$, and is called $\omega$-compatible if at the same time $\omega(Jv, Jw) = \omega(v, w)$ for every $v$ and $w$, we also call the triple $(\omega, J, g)$ to be compatible in this case. It is easy to show that $g(v, w) := \omega(v, Jw)$ is a Riemannian metric on $M$ if and only if $J$ is compatible with $\omega$.

The simplest example of an (almost) complex structure would be $\mathbb{R}^{2n}$ with the standard basis $\{x_1, \cdots, x_n, y_1 \cdots, y_n\}$ and the symplectic form

$$\omega(x_i, y_j) = \delta_{ij},\ \omega(x_i, x_j) = \omega(y_i, y_j) = 0.$$

Here $J_0$ is given by the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

It is easy to verity $J_0$ is compatible with $\omega$. $J_0$ is called the standard complex structure. In fact, tangent spaces of an almost complex manifold are all isomorphic to $(\mathbb{R}^{2n}, J_0)$.

**Proposition A.0.3.** *Let $V$ be a $2n$-dimensional vector space with a linear complex structure $J$. There exists a linear isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $J = \Phi J_0 \Phi^{-1}$.*

*Proof.* Since there is a complex basis $(v_1, \cdots, v_n)$ for $V$, there is also a real basis

$$(v_1, Jv_1, \cdots, v_n, Jv_n).$$

Set

$$\Phi(p_1, \cdots, p_n, q_1, \cdots, q_n) = \sum (p_i v_i + q_i J v_i),$$

and we obtained the desired isomorphism. □

We now show that every symplectic manifold has at least one compatible almost complex structure.

**Proposition A.0.4.** *For any symplectic manifold $(M, \omega)$ with a Riemannian metric $g$, there is a canonical almost complex structure $J$ so that $\omega$ and $J$ are compatible.*

*Proof.* We start by proving the proposition for symplectic vector spaces $V$. Since $\omega$ and $g$ are non-degenerate, there is a unique isomorphism $A : V \to V$ such that $\omega(u, v) = g(Au, v)$, notice that $A$ is anti-self-adjoint. Perform polar decomposition to $A$ so that $A = PJ$, where $P$ is positive definite and self-adjoint and $J$ is orthogonal, that is, $JJ^* = \text{id}$. This is done by setting $P = (AA^*)^{\frac{1}{2}}$. Using the anti-self-adjointness of $A$, we can prove that it commutes with $P$, so it is easy to show $J$ commutes with $A$, and that

$$J^* = A^*(P^{-1})^* = -AP^{-1} = -P^{-1}A = -J.$$

So $J$ is an almost complex structure on $V$. It is compatible to $\omega$ since

$$\omega(Jv, Jw) = g(AJv, Jw) = g(Av, J^*Jw) = \omega(v, w),$$
$$\omega(v, Jv) = g(Av, Jv) = g(J^*Av, v) = g(Pv, v) \geqslant 0.$$

This is a canonical construction that depends smoothly on $g$ and $\omega$, so we can find a suitable $J_x$ for every $x$ in a $C^\infty$ way, which gives a canonical compatible almost complex structure $J$ on $(M, \omega)$. However, $(\omega, J, g)$ might not be compatible. □

Gromov's theory of $J$-holomorphic curve played a great part in Floer's original work on Arnold conjecture. Before we define it, recall that a function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We translate this condition to real vector spaces with complex structures. A smooth map $f : (\mathbb{R}^2, j) \to (\mathbb{R}^2, J)$, where $j = J = J_0$, is "holomorphic" if and only if

$$df + J \circ df \circ j = \begin{pmatrix} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \end{pmatrix} = 0,$$

or equivalently,

$$df \circ j = J \circ df.$$

This condition does not depend on local coordinate, thus can be easily generalized to almost complex manifolds.

**Definition A.0.5.** *A smooth map $\phi : M \to N$ between two almost complex manifolds $(M, j)$ and $(N, J)$ is called $(J, J')$-holomorphic if*

$$d_x\phi \circ j = J \circ d_x\phi$$

*for every $x \in M$.*

Another way to express the condition is to define the holomorphic and antiholomorphic derivatives by

$$\partial_J f = \frac{1}{2}(df - J \circ df \circ j)$$
$$\overline{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j),$$

then a map is holomorphic if and only if $\overline{\partial}_J f = 0$. Notice that $\partial_J + \overline{\partial}_J = d$. $J$-holomorphic curves would be defined in an obvious way.

**Definition A.0.6.** *A $J$-holomorphic curve on $M$ is a $(j, J)$-holomorphic map from a Riemann surface $(\Sigma, j)$ to $(M, J)$.*

We can similarly define $J$-holomorphics to be holomorphic maps from $D$ to $M$. The study of moduli space of $J$-holomorphic maps turned out to be crucial for dealing with technical problems presented in the construction of many versions of Floer homology.

# References


[1] Michele Audin, Mihai Damian, and Reinie Erné. *Morse theory and Floer homology*. Springer, 2014.

[2] Augustin Banyaga and David Hurtubise. *Lectures on Morse homology*. Vol. 29. Springer Science & Business Media, 2004.

[3] Ana Cannas Da Silva and A Cannas Da Salva. *Lectures on symplectic geometry*. Vol. 3575. Springer, 2008.

[4] Simon Kirwan Donaldson. *Floer homology groups in Yang-Mills theory*. Vol. 147. Cambridge University Press, 2002.

[5] Nathan Dowlin. "A spectral sequence from Khovanov homology to knot Floer homology". In: *arXiv preprint arXiv:1811.07848* (2018).

[6] David Ellwood et al. *Floer Homology, Gauge Theory, and Low-dimensional Topology: Proceedings of the Clay Mathematics Institute 2004 Summer School, Alfréd Rényi Institute of Mathematics, Budapest, Hungary, June 5-26, 2004*. Vol. 5. American Mathematical Soc., 2006.

[7] Joshua Evan Greene. "Heegaard Floer Homology". In: *Notices Amer. Math. Soc.* 68.1 (2021).

[8] Mark JD Hamilton. *Mathematical gauge theory*. Springer, 2017.

[9] Jennifer Hom. "Lectures notes on Heegaard Floer homology". In: *arXiv preprint arXiv:2008.01836* (2020).

[10] WB Raymond Lickorish. *An introduction to knot theory*. Vol. 175. Springer Science & Business Media, 2012.

[11] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Vol. 27. Oxford University Press, 2017.

[12] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*. Vol. 52. American Mathematical Soc., 2012.

[13] John Milnor. "Morse Theory.(AM-51), Volume 51". In: *Morse Theory.(AM-51), Volume 51*. Princeton university press, 2016.

[14] Peter Ozsváth and Zoltán Szabó. "Holomorphic disks and topological invariants for closed three-manifolds". In: *Annals of Mathematics* (2004), pp. 1027–1158.

[15] Andrés Pedroza. "A Quick View of Lagrangian Floer Homology". In: *Geometrical Themes Inspired by the N-body Problem*. Springer, 2018, pp. 91–125.

[16] Jacob Rasmussen et al. "Knot polynomials and knot homologies". In: *Geometry and topology of manifolds* 47 (2005), pp. 261–280.

[17] Nikolai Saveliev. *Invariants of Homology 3-spheres*. Vol. 140. Springer Science & Business Media, 2013.

[18] Nikolai Saveliev. "Lectures on the topology of 3-manifolds". In: *Lectures on the Topology of 3-Manifolds*. de Gruyter, 2011.

[19] Alexandru Scorpan. *The wild world of 4-manifolds*. American Mathematical Society, 2022.

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