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南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目： 非仿射连续函数的图像不是自相似的

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Undergraduate Thesis

Thesis Title: **Graphs of continuous but non-affine functions are never self-similar**

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非仿射连续函数的图像不是自相似的

奚敬华

（数学系　指导教师：刘博辰）

[摘要]：Bandt 和 Kravchenko [2] 证明了若一个自相似集生成 $\mathbb{R}^m$ 空间，则该集合在任何点处均不存在切超平面。特别地，这表明光滑平面曲线是自相似的当且仅当其为直线。当将曲线限制为连续函数图像时，我们可以证明：连续函数的图像是自相似的当且仅当该图像为直线，即对应的函数是仿射函数。

本文的证明可概括为以下三个关键步骤：

步骤 1: 证明对任意 $i \in [k]$，与相似变换 S_i 对应的等距映射 O_i 属于群

$$H_\theta := \left\{ I,\ -I,\ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},\ \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}。$$

步骤 2: 证明生成图像 G 的函数 f 满足 Lipschitz 连续性。

步骤 3: 推导出 f 必为仿射函数，即存在 $k, b \in \mathbb{R}$ 使得

$$f(x) = kx + b。$$

[关键词]：自相似性；连续函数图像；仿射函数

Graphs of continuous but non-affine functions are never self-similar

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[ABSTRACT]: Bandt and Kravchenko [2] proved that if a self-similar set spans $\mathbb{R}^m$, then there is no tangent hyperplane at any point of the set. In particular, this indicates that a smooth planar curve is self-similar if and only if it is a straight line. When restricting curves to graphs of continuous functions, we can show that the graph of a continuous function is self-similar if and only if the graph is a straight line, i.e., the underlying function is affine.

The proof can be summarized in the following three key steps:

Step 1: For any $i \in [k]$, the isometry O_i associated with similitude S_i belongs to the group

$$H_\theta := \left\{ I,\ -I,\ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},\ \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}.$$

Step 2: The underlying function f of generator G is proven to be Lipschitz continuous.

Step 3: f must be affine, i.e., there exist $k, b \in \mathbb{R}$ such that

$$f(x) = kx + b.$$

[Key words]: self-similar sets, graphs of continuous functions, affine functions.

Contents

1. Introduction

A map $S : \mathbb{R}^m \to \mathbb{R}^m$ is said to be a *(contracting) similitude* (e.g., [4]) if $S(x) = rOx + b$, where $r \in (0, 1)$, $b \in \mathbb{R}^m$, and O is an orthogonal matrix. A compact set $K \subset \mathbb{R}^m$ is *self-similar* if there are similitudes $\{S_i\}_{i=1}^k$, such that

$$K = \bigcup_{i=1}^{k} S_i(K). \tag{1.1}$$

The structure of a self-similar set becomes relatively well-understood when it satisfies the open set condition (**OSC**) introduced by Hutchinson in [5]. Here, the OSC is satisfied if there exists a nonempty open set $V \subset \mathbb{R}^m$ such that $\bigcup_{i=1}^k S_i(V) \subset V$ and $S_i(V) \cap S_j(V) = \emptyset$ for $i \neq j$. In this case, $\dim_{\mathrm{H}} K = s$, where dim_H is the Hausdorff dimension and s is the unique solution of $\sum_{i=1}^k r_i^s = 1$, with r_i being the ratio of S_i. Another characteristic of the OSC can be found in [1] by Bandt and Graf.

It is widely acknowledged that fractals are inherently non-smooth. Yet, there has been limited exploration into geometric objects that exhibit both self-similarity and smoothness. However, Bandt and colleagues have been pioneers in this field, achieving significant results. For instance, Bandt and Mubarak [3] established that any differentiable subcurve of the classical Sierpinski carpet must be a line segment. It is worth pointing out that Bandt and Kravchenko [2] demonstrated that a self-similar set spanning $\mathbb{R}^m$ cannot possess a tangent hyperplane at any point within the set, a finding with broad applications. For example, it suggests that a self-similar planar curve can only be a straight line if differentiable at some point.

In the present study, we are concerned with a special class of curves: the graphs of continuous functions, i.e.,

$$G := \{(x, y) \in \mathbb{R}^2 : y = f(x), x \in I\},$$

where f is a continuous function on a compact interval I. Many continuous but nowhere differentiable functions exhibit high "self-similarity" in their graphs. A notable example is Takagi's function (e.g., see [7]), as illustrated in Figure 1.1. However, because these functions are not smooth, it's hard to tell if they are self-similar by using existing results on smooth self-similar sets given by Bandt and Kravchenko [2]. This complexity has spurred our interest in exploring alternative approaches to address such questions.

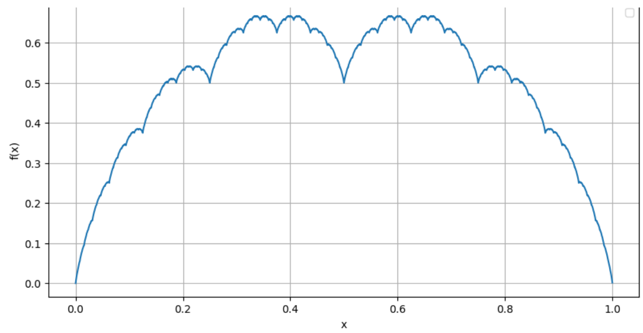

Figure 1.1: The graph of Takagi's function

In this work, we are concerned with which geometric shapes can be realized as a self-similar set. In particular, we propose the following problem for planar graphs associated with one real variable continuous functions:

When is the planar graph of a continuous function self-similar?

Bandt and Kravchenko's research reveals the absence of tangent spaces in self-similar curves (see [2, Theorem 1]), suggesting the C^1 regularity seems to be overly stringent for the graphs of such functions.

In this article, we are aiming to give an affirmative and comprehensive answer to the above question. To be more precise, our main result is stated as follows.

Theorem 1.1. *Let I be a compact interval, $f : I \to \mathbb{R}$ be a continuous function and $G = \{(x, f(x)) : x \in I\}$. Then the following two statements are equivalent:*

- *G is a self-similar set;*
- *The underlying function $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. In other words, f is an affine function on I.*

It is worth remarking that due to the existence of continuous self-similar planar curves, none of the hypotheses in Theorem 1.1 can be weakened. According to Theorem 1.1, graphs of continuous functions that are non-affine are not self-similar. In particular, the graphs of Weierstrass's function, Takagi's function, Cantor-Lebesgue's function, etc., are not self-similar.

The paper is organized as follows. In Section 2, we recall some notations and review some preliminaries. In Section 3, we prove our main Theorem 1.1. In subsection 3.1, we explain the strategy of the proof. The proof consists of three steps: occupy subsections 3.2, 3.3 and 3.4 respectively.

2. Notation and Preliminaries

For A is a subset of B, we denote by A^c the complement of A in B, assuming B is evident from the context. We use the convention that $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{Z}$ the set of integers, and $\mathbb{Q}$ (resp. $\mathbb{Q}^c$) the set of all rational (resp. irrational) numbers. For a compact interval I, the length of I is denoted by $|I|$. For each $k \in \mathbb{N}$, define $[k] := \{1, \dots, k\}$. For each integer $n \in \mathbb{N}$, define

$$[k]^n := \{(x_1, \dots, x_l) : x_i \in [k],\ i = 1, \dots, n\}.$$

For a continuous function f on a compact interval I, the *graph* of f, denoted as G, is defined by

$$G := \{(x, f(x)) \in \mathbb{R}^2 : x \in I\}.$$

Let $O(2)$ denote the orthogonal group of order 2, and $SO(2)$ the special orthogonal group of order 2.

Next, let us introduce some notations for self-similar sets on $\mathbb{R}^2$. We say a map $S : \mathbb{R}^2 \to \mathbb{R}^2$ is a *(contracting) similitude* if $S(x) = rOx + b$ with $r \in (0, 1), b \in \mathbb{R}^2$ and $O \in O(2)$. For a planar self-similar graph, (1.1) reduces to

$$G = \bigcup_{i=1}^{k} S_i(G),$$

where $S_i(v) := r_i O_i v + b_i$. Moreover, for $n \in \mathbb{N}$ and $\alpha = (i_1, \dots, i_n) \in [k]^n$, put

$$S_\alpha := S_{i_n} \circ \cdots \circ S_{i_1} \ \text{ and } \ r_\alpha := r_{i_n} \cdots r_{i_1}.$$

In addition, denote by $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ the unit circle centred at the origin, endowed with the circle metric. Denote by $\mathcal{N}$ and $\mathcal{S}$ the points $(0, 1)$ and $(0, -1)$ on S^1, respectively. We say each connected open subset in S^1 an *arc*.

By a well-known result on the minimality for irrational rotation on S^1 (see e.g., [6, Theorem 5.8]), we immediately have

Lemma 2.1. *Suppose J is an arc in S^1 and $\frac{\theta}{2\pi} \in \mathbb{Q}^c$, then*

$$\bigcup_{i=0}^{\infty} \rho_\theta^i(J) = S^1, \textit{where } \rho_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{2.1}$$

3. Proof of Theorem 1.1

3.1. Strategy of the Proof

Let's briefly outline the strategy of the proof of Theorem 1.1. The principal obstacle is addressing the function's lack of differentiability. Our argument steers clear of methods reliant on curvature or other differentiable mechanisms.

Our proof can be summarized in the following three key steps:

Step 1: We demonstrate that the isometry O_i associated with similitude S_i for any $i \in [k]$ is one of the elements in the group

$$H_\theta := \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \}$$

Step 2: We show that the underlying function f of G is Lipschitz continuous.

Step 3: We prove that f must be affine, i.e., $f(x) = kx + b$ for some $k, b \in \mathbb{R}$.

The proof of Theorem 1.1 is divided into three subsections (Subsections 3.2, 3.3, and 3.4), each subsection corresponds to the steps outlined above. The complete proof of Theorem 1.1 is provided at the end of Subsection 3.4. Without loss of generality, we reduce the hypothesis of Theorem 1.1 to the interval $[0, 1]$.

3.2. Possible Isometries for Similitudes

In this subsection, we systematically analyze the permissible isometries in the iterated function system generating a self-similar graph G.

First, consider similitudes with rotational isometries. By constructing an auxiliary function Φ that maps points on G to directional vectors relative to the unique fixed point p^* of the similitude S, we translate the geometric action of the similitude S into a dynamical system on S^1 under rotation. Invariance properties of ImΦ impose strict constraints: irrational rotations would densely cover S^1, contradicting the exclusion of polar points $\mathcal{N}, \mathcal{S}$; while rational rotations generate forbidden directions via backward invariance, the contradiction in the number of connected components forces ρ_θ to be either identity or inversion (see Proposition 3.1).

For similitudes involving reflections, observe that the compositions of two reflections yield rotations. Apply Proposition 3.1 to composite similitudes restricts the angle between any two reflection axes to 0 or $\frac{\pi}{2}$, thereby enforcing all reflections to share a common axis or to be mutually orthogonal. The synthesis of rotational and reflectional cases is unified in Corollary 3.3, fully characterizing the isometric group admissible for self-similar graphs of continuous functions.

Proposition 3.1. *Let G be the graph of a continuous function f on $[0,1]$, and $S(x) = r\rho_\theta(x)+b$ be a strict contracting similitude where $r \in (0,1), b \in \mathbb{R}^2$, and $\rho_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ with $\theta \in [0, 2\pi]$. If $S(G) \subset G$, then $\rho_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Since S is a (strict) contraction mapping from G to itself, it follows from the Banach contraction principle, similitude S has a unique fixed point in G, which we denote by p^*.

Next, define a function

$$\Phi : G \setminus \{p^*\} \to S^1, \quad p \mapsto \frac{p - p^*}{\|p - p^*\|}.$$

Given that G is the graph of a function, this ensures that the image of Φ lies in the unit circle S^1, excluding the points $\mathcal{N}$ and $\mathcal{S}$, i.e.,

$$\mathrm{Im}(\Phi) \subset S^1 \setminus \{\mathcal{N}, \mathcal{S}\}. \tag{3.1}$$

Recall that the points $\mathcal{N}$ and $\mathcal{S}$ were given in Section 2. Moreover, by the definitions of S and Φ and the hypothesis that $S(G) \subset G$, we then have

$$\Phi(S(p)) = \frac{S(p) - p^*}{\|S(p) - p^*\|} = \frac{S(p) - S(p^*)}{\|S(p) - S(p^*)\|} = \frac{\rho_\theta(p - p^*)}{\|p - p^*\|}, \quad \text{for every } p \in G\setminus\{p^*\}.$$

This means $\Phi(S(p)) = \rho_\theta(\Phi(p))$. It then yields that $\mathrm{Im}\Phi$ is forward invariant under the rotation ρ_θ, i,e.,

$$\rho_\theta(\mathrm{Im}\Phi) \subset \mathrm{Im}\Phi. \tag{3.2}$$

Since $G \setminus \{p^*\}$ has at most two connected components, it is worth mentioning that the continuity of Φ implies that ImΦ also has at most two connected components.

Lemma 3.2. *If G is not a straight line, then ImΦ contains an arc in S^1.*

Proof. We will prove Lemma 3.2 by contraposition. Suppose ImΦ doesn't include any arc in S^1. We aim to demonstrate that G is a straight line.

Recall that ImΦ has at most two connected components; under our premise, each must be a single point. We consider two cases:

Case 1: If ImΦ consists of a single point, then all directions from p^* to any point in G are constant, implying G is a straight line.

Case 2: If ImΦ consists of two points, these must be antipodal due to the invariance of ImΦ under ρ_θ. This implies all points in G are collinear, so G is again a straight line.

Consequently, in either case, if ImΦ does not include any arc in S^1, it follows that G must be a straight line, as we wanted.

□

With the aid of Lemma 3.2, we now proceed to the proof of Proposition 3.1.

Proof of Proposition 3.1. In fact, if G is already a straight line, then Proposition 3.1 trivially holds. If not, we can further assume that the graph G of a continuous function is not a straight line. Applying Lemma 3.2, then ImΦ must then contain an arc.

Choose such an arc $J \subset \mathrm{Im}\Phi$ on S^1. According to (3.2), we have $\rho_\theta^i(J) \subset \mathrm{Im}\Phi$ for any $i \in \mathbb{N}_0$. Consequently,

$$\bigcup_{i=0}^{n} \rho_\theta^i(J) \subset \mathrm{Im}\Phi \quad \text{for any } n \in \mathbb{N}_0, \tag{3.3}$$

and

$$\bigcup_{i=0}^{\infty} \rho_\theta^i(J) \subset \mathrm{Im}\Phi. \tag{3.4}$$

We now divide the remainder of the proof into two cases: when $\theta/2\pi$ is irrational and when $\theta/2\pi$ is rational.

Case 1: Suppose $\theta/2\pi \in \mathbb{Q}^c$. By Lemma 2.1, $\bigcup_{m=1}^{\infty} \rho_\theta^m(J)$ covers the entire circle S^1. Together with (3.4), this implies $\mathrm{Im}\Phi = S^1$. However, this contradicts to the fact that $\mathcal{N}$ and $\mathcal{S}$ are not in ImΦ. Hence, Case 1 is impossible.

Case 2: Now suppose $\theta/2\pi \in \mathbb{Q}$. We can write $\theta/2\pi = m/n$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$, with gcd$(m, n) = 1$.

For $n \geq 3$, denote by $\mathcal{N}_i = \rho_{-\theta}^i(\mathcal{N})$, $\mathcal{S}_i = \rho_{-\theta}^i(\mathcal{S})$, and $J_i = \rho_\theta^i(J)$ for $i = 0, \dots, n-1$. In this case, S^1 is partitioned into $2n$ segments by the $2n$ points $\mathcal{N}_0, \dots, \mathcal{N}_{n-1}, \mathcal{S}_0, \dots, \mathcal{S}_{n-1}$ if n is odd, and into n segments if n is even, with some points coinciding.

Regardless of n being odd or even, the intervals $J_0, \dots, J_{n-1}$ fall into n different segments, as illustrated in Figure 3.1.

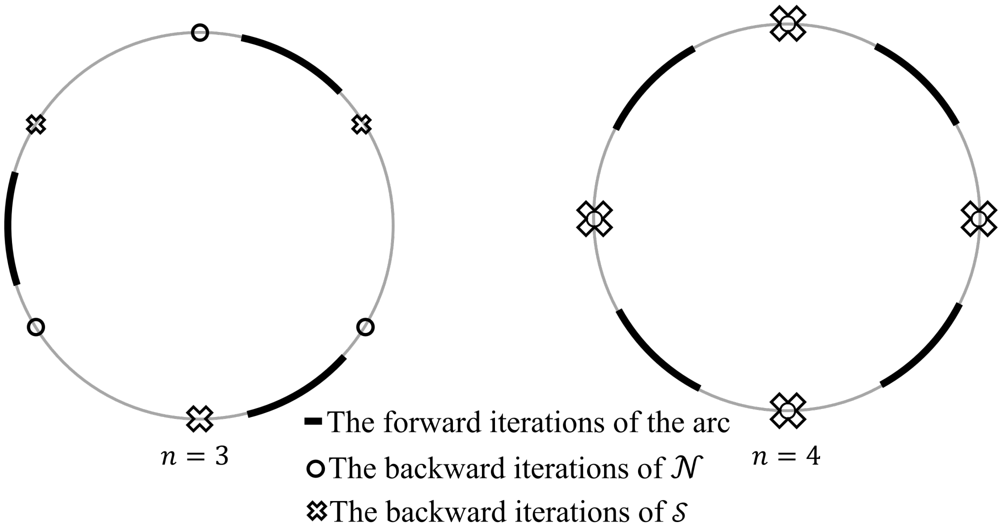

Figure 3.1: The iterations of $\mathcal{N}, \mathcal{S}$ and J on S^1

Observe that the complement of ImΦ in S^1 is backward invariant under the rotation ρ_θ, i.e.,

$$\rho_{-\theta}((\mathrm{Im}\Phi)^c) \subset (\mathrm{Im}\Phi)^c. \tag{3.5}$$

Combining (3.1) with forward and backward invariance (3.2) and (3.5) imply that the points $\mathcal{N}_i$ and $\mathcal{S}_i$, for $i = 0, \dots, n-1$, are not contained in ImΦ.

This fact together with (3.3) further implies that the number of connected components of ImΦ exceeds two, which is obviously a contradiction. Therefore, we conclude that $n \leq 2$.

When $n = 1$, we have $\theta = 2m\pi$, which means $\rho_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

When $n = 2$, we have $\theta = m\pi$, which means $\rho_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\square$

Corollary 3.3. *Let G be the graph of continuous function f on $[0,1]$. Suppose G is a self-similar set with IFS $\{S_i\}_{i=1}^k$. Then there exists a $\theta \in [0, 2\pi]$ such that for each similitude S_i*

in the IFS of self-similar graph G, the associated isometric part

$$O_i \in H_\theta := \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \}.$$

Proof. If $O_i \in SO(2)$ for all $i \in [k]$, then the result follows immediately from Proposition 3.1.

Now suppose, without loss of generality, $\det O_1 = -1$. For any other $O_i \in O(2) \setminus SO(2)$, note that $O_1 O_i, O_i O_1 \in SO(2)$. We can apply Proposition 3.1 to the strictly contractive similitudes $S_1 \circ S_i$ and $S_i \circ S_1$ to obtain

$$O_1 O_i, O_i O_1 \in \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}.$$

If we write

$$O_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, O_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ \sin\theta_i & -\cos\theta_i \end{pmatrix}$$

for some $\theta, \theta_i \in [0, 2\pi]$. Then

$$O_1 O_i = \begin{pmatrix} \cos(\theta - \theta_i) & \sin(\theta - \theta_i) \\ -\sin(\theta - \theta_i) & \cos(\theta - \theta_i) \end{pmatrix}, O_i O_1 = \begin{pmatrix} \cos(\theta_i - \theta) & \sin(\theta_i - \theta) \\ -\sin(\theta_i - \theta) & \cos(\theta_i - \theta) \end{pmatrix}.$$

Therefore, either $\theta_i = \theta$ or $|\theta_i - \theta| = \pi$. In both scenarios, we have $O_i \in H_\theta$ for all $i \in [k]$. □

3.3. Lipschitz Continuity

We prove the underlying function f is Lipschitz continuous when the isometric components of the IFS lie in a finite subgroup $H_\theta \subset O(2)$. Each self-similar copy of G is generated by first applying an element in H_θ to G, followed by some contraction and translation. This structure restricts the oscillation-to-length ratio $\frac{\omega_f(I_\alpha)}{|I_\alpha|}$ on the projected interval $\{I_\alpha\}$ of any copy $S_\alpha(G)$ to at most two values: one for rotational isometry h and another for reflectional one. If G is not a straight line, then these ratios are bounded. To globalize this local regularity, we can cover arbitrary intervals $[x, y]$ by projection intervals of self-similar copies with lengths $\leq |x - y|$. A minimal cover argument, combined with the uniform ratio bound, yields the Lipschitz condition.

Recall in the last section we denote by

$$H_\theta := \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \}$$

Proposition 3.4. *If there exists a $\theta \in [0, 2\pi]$ such that for each similitude S_i in the IFS of self-similar graph G, the associated isometric part $O_i \in H_\theta$ then the underlying function f is Lipschitz.*

Proof. The key observation is that H_θ forms a subgroup of $O(2)$. Whence, it follows that $O_\alpha \in H_\theta$ for all $\alpha \in [k]^n, n \in \mathbb{N}$.

Consider the height and width the image of G under O_α, $O_\alpha(G)$. There are two cases: If

$$O_\alpha \in \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \},$$

then the width and height of $O_\alpha(G)$ are precisely 1 and $\omega_f([0,1])$ respectively. If

$$O_\alpha \in \{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \},$$

then the width and height of $O_\alpha(G)$ are a and b respectively for some $a, b \geq 0$. Since $a = 0$ implies G is already a straight line, we may further assume $a > 0$ by excluding the trivial case where G is a straight line.

Let the interval I_α be the projection of S_α on x-axis. Note that $S_\alpha(G) \subset G$ is the graph of f restricted on I_α. That is the width and height of $S_\alpha(G)$ are $|I_\alpha|$ and $\omega_f(I_\alpha)$ respectively. Since $S_\alpha(G) = r_\alpha O_\alpha(G) + b_\alpha$, we have: If

$$O_\alpha \in \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \},$$

then $|I_\alpha|$ and $\omega_f(I_\alpha)$ are r_α and $r_\alpha \omega_f([0,1])$ respectively. If

$$O_\alpha \in \{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \},$$

then $|I_\alpha|$ and $\omega_f(I_\alpha)$ are $r_\alpha a$ and $r_\alpha b$ respectively.

Consequently, we have

$$\frac{\omega_f(I_\alpha)}{|I_\alpha|} \leq L := \max\{\omega_f([0,1]), \frac{b}{a}\} \tag{3.6}$$

for all $\alpha \in [k]^n, n \in \mathbb{N}$.

We aim to show that: For every $\delta > 0$,

$$|f(x) - f(y)| \leq 4L|x - y|,$$

for every $x, y \in [0,1]$ with $|x - y| = \delta$. Since δ, x, y are arbitrarily chosen, This directly yields that f is a $4L$-Lipschitz function, and thus will complete the proof.

Since $r_1, \ldots, r_k \in (0,1)$, we have $r_{\max} := \max\{r_1, \ldots, r_k\} \in (0,1)$. Thus, for each $\delta > 0$, we can choose a $n_0 := n_0(\delta) \in \mathbb{N}$ with $r_{\max}^{n_0} \cdot \max\{1, a\} \leq \delta$. This implies that for any $\alpha \in [k]^{n_0}$, we have

$$|I_\alpha| \leq \max\{r_\alpha, r_\alpha a\} \leq r_{\max}^{n_0} \cdot \max\{1, a\} \leq \delta. \tag{3.7}$$

The self-similarity of G indicates that

$$G = \bigcup_{i=1}^{k} S_i(G) = \bigcup_{\alpha \in [k]^{n_0}} S_\alpha(G).$$

Hence G is covered by $\{S_\alpha(R)\}_{\alpha \in [k]^{n_0}}$. Consequently, the interval $[0,1]$ is covered by a collection of compact intervals $\{I_\alpha\}_{\alpha \in [k]^{n_0}}$.

Due to the finiteness of the indices set $[k]^{n_0}$, we can select a finite indices subset $\Lambda \subset [k]^{n_0}$ that satisfies the following two conditions:

1. $\{I_\alpha\}_{\alpha \in \Lambda}$ forms a cover for $[x, y]$.
2. $\{I_\alpha\}_{\alpha \in \Lambda}$ is minimal, in the sense that if any $\alpha_0 \in \Lambda$ is removed, then $\{I_\alpha\}_{\alpha \in \Lambda \setminus \{\alpha_0\}}$ no longer forms a cover for $[x, y]$.

We arrange the set $\{I_\alpha\}_{\alpha \in \Lambda}$ into an ordered sequence $I_1, \ldots, I_m$ based on the left endpoints, from left to right. According to Condition (2) in our construction, for every indices $j \in [m-2]$, the intervals I_j and I_{j+2} are disjoint, i.e., $I_j \cap I_{j+2} = \emptyset$. To see this, assume for the sake of contradiction that there is an index j_0 in $[m-2]$ such that the intersection $I_{j_0} \cap I_{j_0+2}$ is non-empty. Given our ordering, this would suggest that I_{j_0+1} is entirely contained within the union of I_{j_0} and I_{j_0+2}. This implies that the set $I_1, \ldots, I_m$ excluding I_{j_0+1} would still provide a cover for the interval $[x, y]$, contradicting Condition (2) as defined in Λ.

Notice that the intervals I_j, where $j \in [m]\backslash\{1, m\}$ and j is odd, are mutually disjoint and their union is contained in $[x, y]$. Therefore,

$$\sum_{\substack{j\in[m]\backslash\{1,m\} \\ j \text{ odd}}} |I_j| \leq |[x, y]| = \delta.$$

Similarly, we also have

$$\sum_{\substack{j\in[m]\backslash\{1,m\} \\ j \text{ even}}} |I_j| \leq \delta.$$

On the other hand, due to (3.7), $|I_1|, |I_m| < \delta$. Hence the total length

$$\sum_{j=1}^{m} |I_j| \leq 4\delta. \tag{3.8}$$

Finally, we arbitrarily choose points $x_i \in I_i \cap I_{i+1}, i \in [m-1]$. Consider,

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(x_1)| + \sum_{i=1}^{m-2} |f(x_i) - f(x_{i+1})| + |f(x_m) - f(y)| \\
&\leq \omega_f(I_1) + \sum_{i=1}^{m-2} \omega_f(I_{i+1}) + \omega_f(I_m) \\
&= L|I_1| + \sum_{i=1}^{m-2} L|I_i| + L|I_m| && \text{(by (3.6))} \\
&\leq 4\delta L. && \text{(by (3.8))}
\end{aligned}$$

This proves that f is Lipschitz continuous. □

3.4. Proof of Theorem 1.1

The aim of this subsection is to show that the underlying function of a self-similar graph is affine.

To prove this, we demonstrate that any interval $[a, b]$ within $[0, 1]$ contains a subinterval $[s, t]$ of comparable length to $[a, b]$ satisfying the condition $\frac{f(t)-f(s)}{t-s} = f(1) - f(0)$ (see Proposition 3.5). By repeatedly applying Proposition 3.5, we can construct a Cantor-like subset within $[a, b]$. This construction, combined with the fact that f satisfies Lipschitz continuity (see Proposition 3.4), allows us to deduce that f is affine.

Proposition 3.5. *Let G be the graph of a continuous function f on $[0, 1]$. If G is self-similar, then there exists a constant $c \in (0, 1/2)$ such that for any closed interval $[a, b] \subset [0, 1]$, there exists a closed subinterval $[s, t] \subset [a, b]$ satisfying*

1. $c(b - a) \leq t - s \leq \frac{1}{2}(b - a)$;
2. $\frac{f(t)-f(s)}{t-s} = f(1) - f(0) := \lambda$.

Proof. Let $r_{\text{min}} := \min\{r_1, \ldots, r_k\} > 0$, and define $c := \frac{r_{\text{min}}^2}{2} > 0$. Fix an interval $[a, b] \subset [0, 1]$. By the self-similarity of G, for any integer $n \in \mathbb{N}$, there exists $\alpha \in [k]^n$ such that I_α contains the middle point $\frac{a+b}{2}$. Let $n^* \in \mathbb{N}$ be the smallest integer for which $|I_\alpha| \leq \frac{b-a}{2}$.

Case 1: $O_\alpha \in SO(2)$.

We claim the interval $I_\alpha := [s, t]$ satisfies both properties in the proposition.

Verification of Property (1):

Let $\alpha = (\alpha_1, \ldots, \alpha_{n^*}) \in [k]^{n^*}$ and define $\alpha' := (\alpha_1, \ldots, \alpha_{n^*-1})$. By the minimality of n^*, we have

$$|I_{\alpha'}| > \frac{b-a}{2}$$

. Since $|I_\alpha| = r_{\alpha_{n^*}} \cdot |I_{\alpha'}|$, it follows that

$$|I_\alpha| \geq r_{\text{min}} \cdot \frac{b-a}{2} \geq c(b - a).$$

which establishes the first inequality. The second inequity holds by the construction of I_α.

Verification of Property (2):

Let $\overrightarrow{x} := (x, f(x)) \in G$ for $x \in [0, 1]$. By Proposition 3.1, $O_\alpha \in \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\}$. Thus, S_α maps $\overrightarrow{x}$ to either $(r_\alpha x + b_\alpha, r_\alpha f(x) + b_\alpha)$ or $(-r_\alpha x + b_\alpha, -r_\alpha f(x) + b_\alpha)$. In both cases, the slope between $\{\overrightarrow{s}, \overrightarrow{t}\} = \{S_\alpha(\overrightarrow{0}), S_\alpha(\overrightarrow{1})\}$ equals the slope between $\{\overrightarrow{0}, \overrightarrow{1}\}$. This confirms Property (2).

Case 2: $O_\alpha \in O(2) \setminus SO(2)$.

If $O_\alpha \in O(2) \setminus SO(2)$, then there exists $i \in [k]$ such that $O_{\alpha''} \in SO(2)$ where $\alpha'' = (\alpha_1, \ldots, \alpha_{n^*}, i) \in [k]^{n^*+1}$. We show that $I_{\alpha''}$ satisfies the required properties.

Verification of Property (1):

Since $|I_{\alpha''}| = r_i \cdot |I_\alpha|$ and $r_{\min} \cdot \frac{(b-a)}{2} \leq |I_\alpha| \leq \frac{b-a}{2}$, we have

$$c(b-a) \leq t - s \leq \frac{1}{2}(b-a)$$

.

Verification of Property (2):

As $O_{\alpha''} \in SO(2)$, the argument from Case 1 applies directly to $I_{\alpha''}$.

This completes the proof of Proposition 3.5. □

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. " $\Rightarrow$ " : To show f is affine, it suffices to show that for any interval $[a, b] \subset [0, 1]$, we have $\frac{f(b)-f(a)}{b-a} = \lambda$.

Given an arbitrary interval $[a, b]$, we construct a Cantor-like subset of $[a, b]$ inductively in the following way.

For initial step, there exists a subinterval of $[a, b]$, say $[a_1^1, b_1^1]$, satisfying two properties in Proposition 3.5. We define $C_1 := [a, b] \backslash (a_1, b_1)$. This set is non-empty due to the first two properties in Proposition 3.5.

Suppose we have defined C_n. Notice there are 2^n intervals in C_n. For stage $n+1$, we apply Proposition 3.5 on each of these 2^n intervals, obtaining subintervals $[a_1^n, b_1^n], \ldots, [a_{2^n}^n, b_{2^n}^n] \subset C_n$. We then define $C_{n+1} := C_n \backslash \bigcup_{i=1}^{2^n} (a_i^n, b_i^n)$. This set is also non-empty by the first two properties in Proposition 3.5.

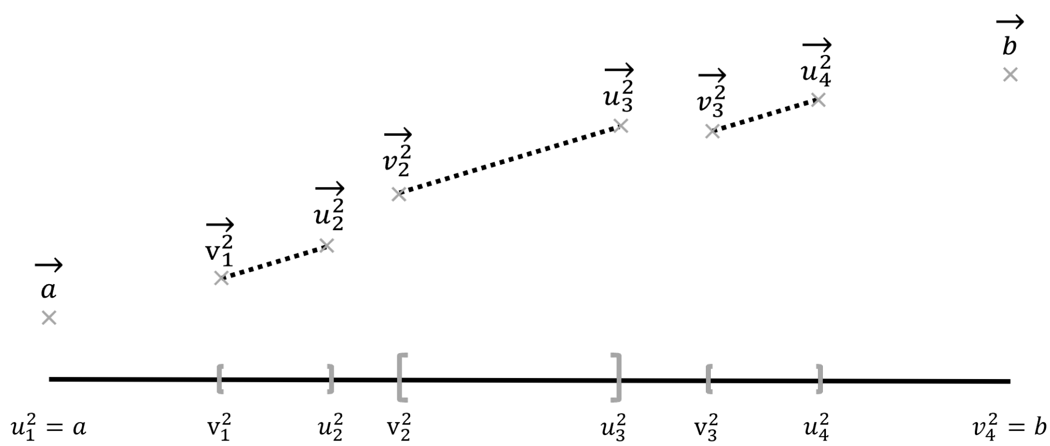

Figure 3.2: Illustration of the set C_2

Each C_n contains 2^n intervals. We denote the right and left endpoints of the i-th interval as u_i^n and v_i^n respectively. Thus, the intervals in C_n can be sequentially represented as $[u_1^n, v_1^n], [u_2^n, v_2^n], \ldots, [u_{2^n}^n, v_{2^n}^n]$, as depicted in Figure 3.2.

Next, we estimate the total length of C_n, $|C_n| = \sum\limits_{i=1}^{2^n}(v_i^n - u_i^n)$. By invoking Property (1) in Proposition 3.5, we deduce that $|C_{n+1}| \leq (1-c)|C_n|$. Combined this with the fact that $|C_1| \leq (1-c)(b-a)$, it follows that

$$|C_n| \leq (1-c)^n(b-a). \tag{3.9}$$

Meanwhile, by Proposition 3.4, there is a constant L, such that

$$|f(v_i^n) - f(u_i^n)| \leq L|v_i^n - u_i^n|, \text{ for all } n \in \mathbb{N}, i \in [2^n]. \tag{3.10}$$

On the other hand, by Property (2) in Proposition 3.5, it follows that

$$f(u_{i+1}^n) - f(v_i^n) = \lambda(u_{i+1}^n - v_i^n), \text{ for all } i \in [2^n - 1]. \tag{3.11}$$

Based on all the estimates, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|f(b) - f(a) - \lambda(b-a)| \\
=&|\sum_{i=1}^{2^n}(f(v_i^n) - f(u_i^n)) + \sum_{i=1}^{2^n-1}(f(u_{i+1}^n) - f(v_i^n)) - \lambda(b-a)| \\
=&|\sum_{i=1}^{2^n}(f(v_i^n) - f(u_i^n)) + \sum_{i=1}^{2^n-1}\lambda(u_{i+1}^n - v_i^n) - \lambda(b-a)| && \text{(by (3.11))} \\
\leq&\sum_{i=1}^{2^n}|f(v_i^n) - f(u_i^n)| + |\lambda(b-a-|C_n|) - \lambda(b-a)| \\
\leq&(L+|\lambda|)|C_n| && \text{(by (3.10))} \\
\leq&(L+|\lambda|)(1-c)^n(b-a). && \text{(by (3.9))}
\end{aligned}$$

Since $c = \frac{r_{\min}^2}{2} > 0$ and n is arbitrarily chosen, we conclude that $f(b)-f(a)-\lambda(b-a) = 0$ for any interval $[a,b] \subset [0,1]$. Therefore, f is an affine function.

" $\Leftarrow$ " : Conversely, suppose f is an affine function. Similitudes $S_1(v) = \frac{1}{2}v + (0, \frac{f(0)}{2})$ and $S_2(v) = \frac{1}{2}v + (\frac{1}{2}, \frac{f(1)}{2})$ satisfies $G = S_1(G) \cup S_2(G)$. Hence, G is self-similar. $\square$

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