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南方科技大学

SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目： 表示论视角下模空间的同调稳定性

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Undergraduate Thesis

Thesis Title: **Homological stability of moduli spaces from the viewpoint of representation theory**

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表示论视角下模空间的同调稳定性

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（数学系　指导教师：朱一飞）

[摘要]：在本读书报告中，我们系统地介绍模空间的同调稳定性问题，并揭示其与表示论之间的深刻联系。首先，我们阐述模空间的基本理论框架及其与常见数学概念的关联。进一步的，基于群 (上) 同调理论及谱序列的计算方法，我们通过跟随 Quillen 的经典证明，对辫群情形给出完整的同调稳定性证明。进一步的，当考虑具有紧性质的更一般的模空间 (如流形 M 的无序构型空间 $C_n(M)$) 时，Quillen 提出的拓扑稳定性理论出现局限。为此，我们引入表示稳定性，借助 **FI**-模范畴 (主要突出‘单射’和‘有限’的代数结构的范畴) 这一强大的工具，证明有序构型空间同调群 $\mathrm{H}_*(F_n(M);\mathbb{Q})$ 和上同调群 $\mathrm{H}^*(F_n(M);\mathbb{Q})$ 具有 **FI**-模结构。在此基础之上，我们通过推导特征多项式存在性及维数多项式增长性等关键性质，最终给出构型空间同调的表示稳定性定理。

[关键词]：同调稳定性；　表示论

[**ABSTRACT**]: In this reading report, we systematically introduce the homological stability problem of moduli spaces revealing a profound connection with representation theory. Firstly, we outline the basic theoretical framework of moduli spaces and their relationship with common mathematical concepts. Furthermore, based on the theory of group (co)homology and the computational method of spectral sequences, we follow Quillen's classical proof to provide a complete proof of homological stability for the case of braid groups. Furthermore, when considering more general moduli spaces with compact property (such as the unorder configuration space $C_n(M)$ of fixed n points on a manifold M), Quillen's topological stability theory encounters limitations. To address this, we introduce representation stability and, with the powerful tool of the **FI**-module category (a category focus on 'injective' and 'finite'), prove that the homology groups $\mathrm{H}_*(F_n(M);\mathbb{Q})$ and cohomology groups $\mathrm{H}^*(F_n(M);\mathbb{Q})$ of ordered configuration spaces possess **FI**-module structures. On this basis, by deriving key properties such as the existence of characteristic polynomials and the growth of dimension polynomials, we ultimately characterize the representation stability theorem for the homology of configuration spaces.

[**Key words**]: homological stability; representation theory

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1. Introduction

Moduli spaces are an important type of research object in mathematics such as algebraic geometry, topology, and mathematical physics. What they contain is the parameterization of geometric objects, that is, how to consider a family of geometric objects with the same structure as a whole according to some primary rules. The homological stability of moduli Spaces, that is, the asymptotic behavior of the homology groups of some specific moduli space families with the change of indicators, is a key issue for studying their structures, classifications, and connections with other branches of mathematics.

1.1 Background and motivation

At the end of the 20th century, mathematicians began to notice that the configuration spaces of certain geometric objects exhibited the phenomenon of homological stability. In the 1970s, Quillen took out the concept of "homological stability"[1]. Sooner, McDuff[2] proved the homological stability of $\{C_n(M)\}_n$ and Segal gave explicit stable ranges[3].

Theorem 1 *Let M be the interior of a compact connected manifold with nonempty boundary. For each $k \geq 0$ the maps $(s_n)_* : H_k(C_n(M);\mathbb{Z}) \to H_k(C_{n+1}(M);\mathbb{Z})$ are isomorphisms for $n \geq 2k$.*

And, Quillen provided a proof of homology stability in the manifold configuration space. We called it "Quillen's argument", as Quillen 's argument.

In 2012, Thomas Church published the research on "homological stability of manifold configuration spaces"[4]. Specifically, for a closed manifold M, when the number of configuration points tends to infinity, the homology group of its configuration space exhibits a stable pattern.

In 2013, Church and Benson Farb formally proposed the concept of stability[5].

In 2014, Church, Ellenberg and Farb further studied the representation stability of algebraic varieties over finite fields[6].

In 2018, Galatius, Kupers and Randal-Williams proposed a new stability on the homology of linear groups and mapping groups, which was called "secondary homological stability”[7].

In 2024, Sierra and Wahl utilized the arc complex to reveal the homological stability of the symplectic group[8].

1.2 Outline

In this reading report[9], I will present it in the following order:

1. Basic Notions and Tools: In this part, we will induce some tools, such as spectral sequence, to compute the cohomology and category to describe the patten of homological stability and reprersentation stability.

2. Topological Homological Stability: In this part, we will describe the Quillen's argument and give out a brief proof.

3. Representation Stability: In this part, we find out the limitations of the homological stability. Then, we will using the tools in representation theory to describe the representation stability.

2. Basic notions and tools

2.1 Moduli space

In mathematics, a very important question is how to classify. Or, how to use it to express the equivalence and differences between mathematical objects.

Moduli spaces serve as geometric solutions to classification problems in mathematics, providing a structured way to parameterize equivalence classes of objects such as Riemann surfaces, vector bundles, and algebraic varieties.

These spaces arise naturally in algebraic geometry, differential geometry, and topology, offering deep insights into both the objects being classified and the geometric structure of the moduli space itself. By transforming abstract equivalences into geometric structures, they offer deep insights into the objects being classified and their variations.

2.1.1 Some Examples of Moduli Spaces

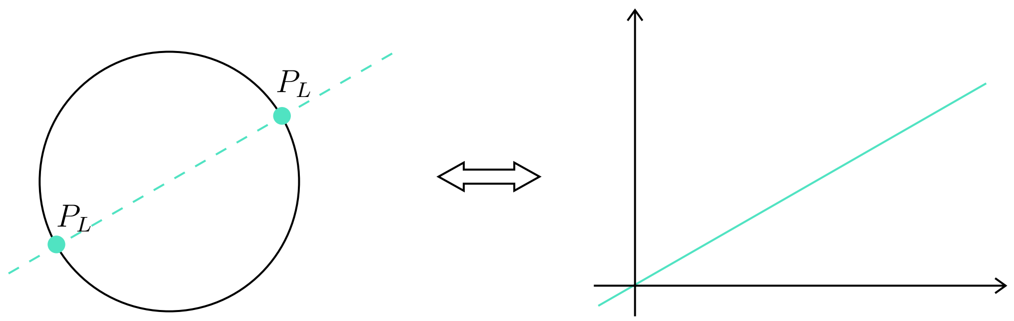
Lines in the Plane without paralleling to the y-axis

These kinds of lines can be written as $y = kx+b$. Each line is one-to-one corresponding to a unique binary array (k, b). Thus, the moduli space of line with finite slope on $\mathbb{R}$ is

$$\{y = kx + b | k, b \in \mathbb{R}\}$$

Lines in the Plane

The moduli space of lines passing through the origin in $\mathbb{R}^2$ is the real projective line $\mathbb{RP}^1$, which is topologically a circle. Each line is represented by an angle $\theta \in [0, \pi)$, with the endpoints 0 and π identified to reflect the continuity of lines. This space captures the idea of continuous families of lines through maps from a parameter space X to $\mathbb{RP}^1$.



Elliptic Curves

An elliptic curve is a genus-1 Riemann surface with a marked point, often described as the quotient $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$. The **Teichmüller space** $\mathcal{T}_{1,1}$, modeled by the upper half-plane $\mathbb{H}$, parametrizes marked elliptic curves via the modular parameter τ. The **coarse moduli space** $\mathfrak{M}_{1,1} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ is isomorphic to $\mathbb{C}$.

2.1.2 Applications and Connections

Number Theory

Modular forms, which are functions on moduli spaces like $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, play a central role in the Langlands program. This program connects number theory to harmonic analysis and has led to significant breakthroughs, such as Wiles' proof of Fermat's Last Theorem through the modularity of elliptic curves.

Characteristic Classes

Families of vector bundles over a space X induce cohomology classes (e.g., Euler class) via pullback to moduli spaces. These invariants measure topological twisting in parameterized families and provide tools for understanding the global structure of moduli spaces.

2.2 Braid group

The Braid group is a concept in mathematics. Compared with permutation groups, it pays more attention to the process of permutation. It extends the concept of discrete symmetric operations to continuous "paths" and characterizes the path of permutation between points.

More information on the relationship between braid group and moduli space will be used in the rest of this article.

2.2.1 As the trace of permutation

To figure out the first definition of braid group, imagine hanging several strings vertically, allowing them to crisscross through space without being cut or overlapping.

Definition 1 *(First definition of (pure) braid group)*

For fixed n points, let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{C}$. Let $(f_1, \ldots, f_n)$ be a n-tuple of continuous functions,

$$f_i : [0,1] \to \mathbb{C}$$

such that

$$f_i(0) = p_i, \quad f_i(1) = p_j \text{ for some } j$$

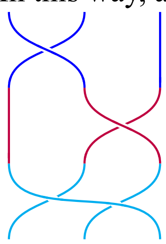
and

$$\{t | f_i(t) = f_j(t)\} = \emptyset \text{ for any } i \neq j$$

under compounding action, these n-tuples form a group, called ***braid group****, denoted as* $\boldsymbol{B}_n$*.*

Furthermore, if we require $f_i(1) = p_i$, then we get another group called ***pure braid group****, denoted as* $\boldsymbol{P}_n$*, which is the action permutate n points and required n points fixed after permutation.*

In this way, an element of the braid group can be represented as the figure below.



2.2.2 As the fundamental group of a manifold

Because the story we want to make will be closely related to module space, we need another definition, one that comes from module space.

Definition 2 *(ordered Configuration Spaces)*

Let M be a topological space. The configuration space is

$$F_n(M) := \{(p_1, \dots, p_n) | p_i \neq p_j \textit{ if } i \neq j\} \subset M^n$$

For example,

- $F_0(M)$ is a singular point,
- $F_1(M)$ is the topological space M itself,
- $F_1(\mathbb{C})$ consists of two distinguish triangle.

Definition 3 *(unordered Configuration Spaces)*

Let M be a topological space. The configuration space is

$$C_n(M) := \{\{p_1, \dots, p_n\} | p_i \neq p_j \textit{ if } i \neq j\} \subset M^n / S^n$$

Or, more simply, $C_n(M) = F_n(M)/S^n$.

Definition 4 *(Second definition of (pure) braid group)*

$\pi_1(C_n(\mathbb{C}))$ *called braid group.* $\pi_1(F_n(\mathbb{C}))$ *called pure braid group.*

Actually, the two different definitions of (pure) braid group are equivalent, since the loop on $C_n(\mathbb{C})$ is one-to-one corresponding to one method to permutate n points.

2.3 Group homology and group cohomology

Imagine a topological space where the fundamental group captures information about "loops" within the space, while the homology groups describe the dimensions and the quantity of "holes" present in the space. As the group representation theory uses the characteristics of a group mapping the properties of the group, group homology adopts the concept of treating abstract groups G as an "algebraic shadow of a space", constructed in such a way that G reflects a virtual "shape" which gives out its homology groups.

Definition 5 *(acts freely)*

Say that G acts freely on a space X if the map $G \times X \to X \times X$, $(g, x) \mapsto (x, gx)$, is a homeomorphism from $G \times X$ onto its image.

If we get such a topological space, we can consider the quotient space X/G. Through covering space theory in algebraic topology, the fundamental group of the quotient satisfies $\pi_1(X/G) \cong G$. In this way, we can consider the group G as an "algebraic shadow of a space" as the foundamantal group of a algebraic topological space.

Definition 6 *(the classifying space)*

For a group G,

- *The group homology of G with coefficients in an abelian group A is defined as:*

$$H_*(G; A) := H_*(BG; A)$$

- *The group cohomology of G with coefficients in an abelian group A is defined as:*

$$H^*(G; A) := H^*(BG; A)$$

Here, EG is a contractible space on which G acts freely, and $BG = EG/G$ is the classifying space of G.

By this definition, **B**G is unique up to (weak) homotopy equivalence.
Furthermore, if G is a discrete group, then **B**G is precisely an Eilenberg-MacLane space $K(G, 1)$.

For example,

Consider the additive action of $\mathbb{Z}$ on $\mathbb{R}$; the classifying space **B**$\mathbb{Z}$ is a circle.

Consider the action of $\mathbb{Z}_2$ on $\mathbb{S}^\infty$ sending a point to its opposition; the classifying space **B**$\mathbb{Z}_2$ is $\mathbb{R}\mathrm{P}^\infty$

2.4 Spectral sequence

Spectral sequence is a tool to calculate some homology groups which is difficult to calculate directly. The core idea is to transform high-dimensional or complex homology calculations into a series of low-dimensional approximation problems by layer by layer decomposition and recursion.

For a fix H^* where H^* is a graded R-module or a graded k-vector space or a graded k-algebra or $\cdots$

Consider the filtered:

$$H^* \supset \cdots \supset F^n H^* \supset F^{n+1} H^* \supset \cdots \supset \cdots \supset \{0\}$$

In actual calculations, it is always difficult for us to calculate the homology group. The spectral sequence mainly separates the homology group differentials how the operator kills things at each page. As we turn the pages from 1 to infity, we approach the relationship between homology groups bit by bit.

A filtration of H^*, say F^*, can be collapsed into another graded vector apace called the **associated graded vector space** and defined by $E_0^p(H^*) = F^p H^* / F^{p+1} H^*$.

If H^* is locally finite graded vector space, then we have $H^* \cong \bigoplus_{p=0}^{\infty} E_0^p(H^*)$

$$E_0^{p,q} = F^p H^{p+q} / F^{p+1} H^{p+q}$$

The index q is called the **complementary degree**

The index p is called the **filtration**.

Definition 7 *(a sketch definition of spectral sequence)*[10]

A (first quadrant, cohomological) spectral sequence is a sequence of differential bi-graded vector spaces, that is, for $r = 1, 2, 3, \ldots$,and for p and $q \geq 0$,we have a vector space $E_r^{p,q}$. Furthermore, each ***bi-graded*** *vector space,$E_r^{\cdot,\cdot}$,is equipped with a linear mapping*

$$d_r : E_r^{\cdot} \longrightarrow E_r^{\cdot}, \textit{which is a differential}, d_r \circ d_r = 0, \textit{of bidegree}(r, 1-r),$$

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

Finally, for all $r \geq 1$, $E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r)$, *that is,*

$$E_{r+1}^{p,q} = \frac{\ker d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}}{d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}}$$

2.4.1 Serre spectral sequence

There is a useful theorem called 'Serre spectral sequence theorem'

Theorem 2 *Let* $F \to X \to B$ *be a fibration with* B *path-connected. If* $\pi_1(B)$ *acts trivially on* $H_*(F; G)$, *then there is a spectral sequence* $\{E_r^{p,q}, d_r\}$ *with:*

1. $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ *and* $E_{r+1}^{p,q} = \ker d_r / \operatorname{Im} d_r$ *at* $E_r^{p,q}$.

2. *Stable terms* $E_\infty^{p,n-p}$ *isomorphic to the successive quotients* F_n^p / F_n^{p-1} *in a filtration* $0 \subset F_0^n \subset \cdots \subset F_n^n = H_n(X; G)$ *of* $H_n(X; G)$.

3. $E_2^{p,q} \cong H_p(B; H_q(F; G))$.

2.5 Category

A category is a kind of algebra structure which only focus on the object itself and homomorphism between objects.

Definition 8 *(category)*

Category *is a langurage to describe the mathmatics and their relations. A category* $\mathcal{C}$ *consists of the following parts:*

- ***Objects*** $\mathrm{Ob}(\mathcal{C})$ *Some mathematic objects, such as sets, groups,* R*-modules or topology space.*

- ***homomorphism*** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *the mapping between objects which preserve some structure.*

- ***composition*** $\circ$: *satisfying* $(f \circ g) \circ h = f \circ (g \circ h)$;

- ***Identity*** $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$: *for any* $f : X \to Y$, *we have* $f \circ \mathrm{id}_X = f$ *and* $\mathrm{id}_Y \circ f = f$。

For example, we give out some familiar category

- **Sets**: objects are sets, homomorphisms are mapping

- **Gp**: objects are groups, homomorphisms are group homomorphism

- **Top**: objects are topological space, homomorphisms are continuous functions.

In the following statements, we will use categories to represent some special patterns, such as **FI**-module.

2.6 Simplex

In geometry and algebraic topology, the **simplex** serves as the fundamental building block for constructing more complex geometric structures. It is intrinsically a **convex set** generated by convex combinations of affinely independent points. Formally, given $n+1$ affinely independent points $v_0, v_1, \ldots, v_n$ in $\mathbb{R}^m$-space ($m \geq n$), the corresponding **n-dimensional simplex** is defined as:

$$\Delta^n = \left\{ \sum_{i=0}^{n} \lambda_i v_i \,\middle|\, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\},$$

.

For example:

- A 0-simplex is a point

- A 1-simplex forms a line segment

- A 2-simplex corresponds to a triangle
- A 3-simplex represents a tetrahedron

The combinatorial structure of a simplex is characterized through its **faces** - every k-dimensional face ($k \leq n$) is generated by any subset of $k + 1$ vertices from the original set. This property endows simplices with a hierarchical recursive framework, establishing their foundational role in **simplicial complexes**.

3. Topological homological stability

3.1 Calculation of the group cohomology of braid groups

Consider small n, we have

$$F_1(\mathbb{C}) \cong \mathbb{C}, F_2(\mathbb{C}) \cong \mathbb{S}^1, F_3(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0, 1\}$$

.

When n is small, we can know what the manifolds corresponding to the braid group are, so we can use our algebraic topological methods to calculate the cohomology groups.

The following are some calculation results on the homology group of braid groups in small n.[11]

$n\backslash k$	0	1	2	3	4	5
0	$\mathbb{Z}$					
1	$\mathbb{Z}$					
2	$\mathbb{Z}$	$\mathbb{Z}$				
3	$\mathbb{Z}$	$\mathbb{Z}$				
4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$			
5	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$			
6	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
7	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
8	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$
9	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$

Therefore, around 1970, Arnol'd proposed the representational stability of braid groups.

Theorem 3 *[11]*

Let M be the interior of a compact connected manifold with nonempty boundary. For each $k \geq 0$, the induced map

$$(s_n)_* : H_k(\boldsymbol{B}_n; \mathbb{Z}) \to H_k(\boldsymbol{B}_{n+1}; \mathbb{Z})$$

is an isomorphism for $n \geq 2k$.

3.2 Quillen's argument

Theorem 4 *[12]* ***Thm. Quillen's argument for homological stability***

Let $0 \to G_1 \to G_2 \to G_3 \to \cdots \to G_n \to \cdots$ be a sequence of discrete groups.

For each n, let W_n be a simplicial complex with a simplicial action of G_n with

1. *W_n is $\left(\frac{n-2}{2}\right)$-connected.*

2. *$\forall p > 0$, G_n act transitively on the set of p-simplices.*

3. *$\forall \sigma_p$ in W_n, we have $\{g \in G_n : g|_{\sigma_p} = Id_{\sigma_p}\} := \mathrm{stab}(\sigma_p) := \{g \in G_n : g\sigma_p = \sigma_p\}$.*

4. *$\exists h \in G_n$ s.t. $h^{-1}\,\mathrm{stab}(\sigma_p)h = G_{n-p-1}$.*

5. *$\forall$ edge $[v_0, v_1]$ in W_n, there exist $g \in G$ s.t. $gv_0 = v_1$ and for all $h \in G$ if $h|_{[v_0,v_1]} = Id_{[v_0,v_1]}$ then $gh = hg$.*

Then, the sequence $\{G_n\}_n$ is ***homologically stable****.*

Specifically, $H_k(G_n) \to H_k(G_{n+1})$ is an isomorphism for $n \leq 2k+1$ and a surjection for $n = 2k+1$.

Proof.

To connect that $\mathrm{B}G_{n-p}$ to $\mathrm{B}G_n$ for each n we obtain a homology spectral sequence by using $W_n \times_{G_n} EG_n$ to build an approximation to $\mathrm{B}G_n$ from the spaces $\mathrm{B}G_{n-p}$ for $p > 0$.

Since W_n is $(\frac{n-2}{2})$-connected, we have $H_n(W_n \times_{G_n} EG_n) = 0$ for $n \leq \frac{n-1}{2}$.

By Shapiro's lemma[13], we get a spectral sequence:

$$E^1_{p,q} = \bigoplus_{\text{orbits}} H_q(\mathrm{stab}(\sigma_p), \mathbb{Z}) = H_q(G_{n-p-1}, \mathbb{Z}) \Rightarrow E^\infty_{p,q} = H_{p+q}(W_n \times_{G_n} EG_n)$$

Thus, we have $E^\infty_{p,q} = 0$ for $p + q \leq \frac{n-1}{2}$.

Now, to finish the proof, we want to show that

$$d^1 : E^1_{0,i} = H_i(G_n) \to E^{-1}_{-1,i} = H_i(G_{n+1})$$

is sur when $n \geq 2i$ and inj when $n \geq 2i+1$.

Then, we prove by induction on i, case $i = 0$ is trivial.

For the **surjection** of $d^1 : E^1_{0,i} = H_i(G_n) \to E^{-1}_{-1,i} = H_i(G_{n+1})$, we only need to check.

(1) $E^{\infty}_{-1,i} = 0$;

(2) $E^2_{p,q} = 0$ for $p + q = i$ with $q < i$.

As $E^{\infty}_{p,q} = 0$ when $p + q \le \frac{n-1}{2}$ and $i - 1 \le \frac{n-1}{2}$ when $2i \le n$

when $q < i$, we claim that

$$E^1_{p,q} = \oplus_{\text{orbits}} H_q(\mathrm{St}(\sigma_p), \mathbb{Z}) \xrightarrow{\cong} \oplus_{\text{orbits}} H_q(G_{n+1}, \mathbb{Z})$$

is an iso when $p + q \le i$ and is a sur when $p + q = i + 1$.

For a p-simplex σ_p, st(σ_p) is conjugate to G_{n-p-1}.

denote d_i as the boundary operator of σ_p and c_h as the induced map by conjugate action.

$$\begin{tikzcd}
H_q(\mathrm{St}(\sigma_p), \mathbb{Z}) \arrow[r, "d_i"] \arrow[d] & H_q(\mathrm{St}(d_i\sigma_p), \mathbb{Z}) \arrow[r, "c_h"] \arrow[dl] & H_q(\mathrm{St}(\sigma_{p-1}), \mathbb{Z}) \arrow[d] \\
H_q(G_{n+1}, \mathbb{Z}) \arrow[rr, "\mathrm{id}"] & & H_q(G_{n+1}, \mathbb{Z})
\end{tikzcd}$$

commute because c_h acts as identity on $H_q(G_{n+1}, \mathbb{Z})$. Thus, we get a map from the q-line of E^1-page to the chain complex of W_{n+1}/G_{n+1}. And this map is iso when $p + q \le i$ and sur when $p + q = i + 1$.

Because of $H_*(W_{n+1}/G_{n+1})$ is trivial when $* < n - 1$ by condition 2 and condition 5, we have proved $E^2_{p,q} = 0$ for $p + q = i$ with $q < i$ for i since $i < n - 1$ when $2i \le n$ and $i \ge 1$.

For the **injection** of $d^1 : E^1_{0,i} = H_i(G_n) \to E^{-1}_{-1,i} = H_i(G_{n+1})$, we can translate the injection of $d^1 : E^1_{0,i} = H_i(G_n) \to E^{-1}_{-1,i} = H_i(G_{n+1})$ to three conditions on spectral sequence when $n > 2i + 1$.

(1) $E^{\infty}_{0,i} = 0$;

(2) $E^2_{p,q} = 0$ for $p + q = i + 1$ with $q < i$;

(3) $d^1 : E^1_{1,i} \to E_{0,i}$ is the 0-map.

For (1), we need $i \le \frac{n-1}{2}$, which is equivalent to $n \ge 2i + 1$.

For (2), it is similar to (2) for surjection.

For (3), the boundary map is $d^1 = d^1_1 - d^1_0$ on each orbit σ_1 with $d^1_i = c_{h_i} \circ d_i$ for some

$h_0, h_1 \in G_{n+1}$. Never loss of generality, we can assume that $h_0 = \mathrm{id}$.

Furthermore, by condition 5, we can assume that h_1 taking one vertex of σ_1 to the other. And, we have h_1 commuting with every element in $\mathrm{st}(\sigma_1)$ On the group level, we have

$$\begin{tikzcd}
\mathrm{st}(\sigma_1) \arrow[r, "d_1"] \arrow[d, "c_{h_0}=\mathrm{id}"] & h_1 \mathrm{st}(\sigma_0) h_1^{-1} \arrow[d, "c_{h_1}"] \\
\mathrm{st}(\sigma_1) \arrow[r, "d_0"] & \mathrm{st}(\sigma_0)
\end{tikzcd}$$

Hence, we have $d_0^1 = d_1^1$ implies $d^1 = 0$.

We finish the proof.

□

3.3 Examples

Now, we use Quillen's argument to prove the homology stability of the braid group.

To use the Quillen's argument, we need to construct a sequence of complex W_n satisfying the condition of Quillen's argument.

Consider the construction given by Hatcher and Waul called **arc complex**.

Let $\mathbb{D}^2$ be the closed disk. Fix n points $\{v_1, \dots, v_n\}$ in its interior and a distinguished point $*$ on its boundary. Define the vertices of W_n as the isotopy class of $\mathbb{D}^2 \setminus \{v_1, \dots, v_n\}$ joining $*$ with one of the marked points v_i. The vertices of W_n form a p-simplex if and only if the corresponding isotopy classes can be described as arcs that do not intersect.

Hetcher and Waul prove the high connectedness of W_n.

Lemma 1 *If S has at least one pure boundary, then $\mathcal{F}(S; \Delta_0, \Lambda_n)$ is contractible for all $n \geq 1$.*

Proof. The general idea of the proof is to consider an arc with one end on $\partial_0 S$ (this has a point in the simplex).

Consider $< I_0, \dots, I_p >$ as one of the simplices, $P = \sum_j t_j I_j$ as one of the points, and consider the representation with the least intersection. Let $\theta = \sum_j a_j t_j$ where $a_j = |I_j \cap I|$ (thickness). Then prove that the entire simplicial complex can be deformed into $< I >$, which is a star-shaped domain with I as the center point.

□

Actually, B_n givess out the action on S, that is said, B_n gives out a simplicial group action on S.

Consider a fixed p-simplex and fixed a representation by describing the corresponding isotopy classes as some arcs with points $\{p_{i_1}, \ldots, p_{i_r}\}$. denote the arcs as $f_{i_j} : I \to \mathbb{D}^2$ with $f_{i_j}(0) = *$, $f_{i_j}(x) \notin \partial\mathbb{D}^2$ for $i \in (0, 1]$ and $f_{i_j}(1) = p_{i_j}$.

Choose another point a_0 different from $*$ on the boundary of $\mathbb{D}^2$.

Lemma 2 *for condiction* 2*, B_n act transitively on p-simplices.*

Proof.

First, consider the point p_{i_1}. We observe that after removing all arcs, the remaining part of the disk $\mathbb{D}^2$ is a simply connected component. Consequently, there must exist a path connecting a_0 to p_{i_1} that is disjoint from all other arcs and the points $\{p_{i_j}\}$. Furthermore, by the Jordan curve theorem, this path will divide the disk D into two simply connected components. And, the point $*$ and a_0 must be contained in the boundary of these components.

Next, we consider each point p_{i_j} individually, which must be in some connected branch that was previously divided. So think of this connected branch as a $\mathbb{D}^2$ with fewer points. We can again construct a path connecting a_0 to p_{i_j} that is disjoint from all other arcs and another point $\{p_i\}$. This operation still divides the connected branch into two new simply connected branches.

Then, we get a partition that gives a coincidence trace variation of $g \in G$ such that $g\Delta_p$ can be represented as connected $*$ to each point using straight arcs.

Consider the conjunctive transformation $h \in G$ to transform all connected points to the left and all unconnected points to the right, and we get the following figure1.

That is, for every Δ_p, there exists $h \in G$ such that $g\Delta_p$ can be represented as above, so B_n acts transitively on p-simplices.

□

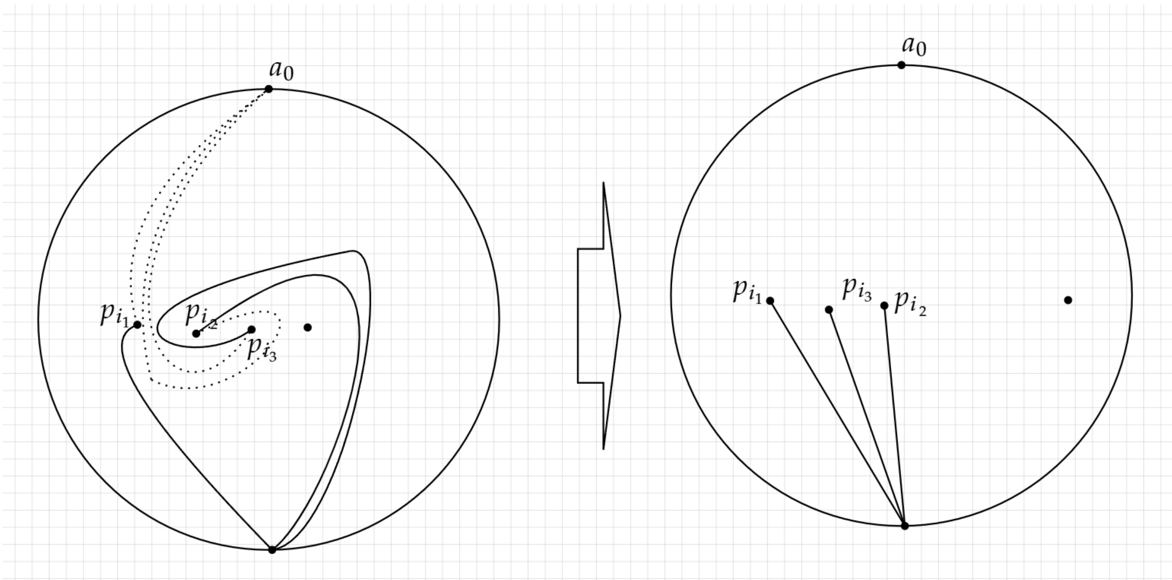


Figure 1 normal form

Following the step of 2, we called “normalform” the $\Delta_p \subset W_n$ which can be represented as connected $*$ to each point using straight arcs.

For the condition 3 and 4, every p-simplex in W_n can be transformed into a normal form. This implies that the stabilizer subgroup of a p-simplex in B_n consists precisely of those elements that act trivially on individual points. Formally, for any $\sigma_p \in W_n$, we have:

$$\left\{g \in \mathrm{B}_n \,\middle|\, g|_{\sigma_p} = \mathrm{id}_{\sigma_p}\right\} = \mathrm{stab}(\sigma_p) := \left\{g \in \mathrm{B}_n \,\middle|\, g \cdot \sigma_p = \sigma_p\right\}.$$

and

$$\exists h \in G_n \text{ s.t. } h^{-1}\mathrm{stab}(\sigma_p)h = G_{n-p-1}$$

For 5, we consider the normal form of 2-simplex. Let $g \in \mathrm{B}_n$ be the action that transforms the two leftmost points counterclockwise. If $h|_{[v_0,v_1]} = Id_{[v_0,v_1]}$, then h will act on p_0, p_1 trivially. By the figure2, $gh = hg$ is clear.

To summarize, we have checked every condition of Quillen's argument. Then, the sequence $\{\mathrm{B}_n\}_n$ is homologically stable.

Corollary 1 *[11]*

Let M be the interior of a compact connected manifold with nonempty boundary. For

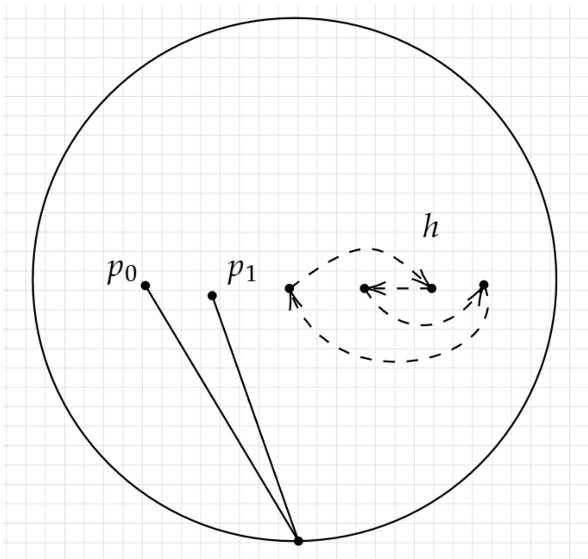


Figure 2 $gh = hg$

each $k \geq 0$, the induced map

$$(s_n)_* : H_k(\boldsymbol{B}_n(M);\mathbb{Z}) \to H_k(\boldsymbol{B}_{n+1}(M);\mathbb{Z})$$

is an isomorphism for $n \geq 2k$.

Similarly, we have

Corollary 2 *Let M be the interior of a compact connected manifold with nonempty boundary. For each $k \geq 0$ the maps*

$$(s_n)* : H_k(C_n(M);Z) \to H_k(Cn+1(M);Z)$$

are isomorphisms for $n \geq 2k$.

4. Representation stability

4.1 Caculation of the group cohomology of pure braid group

However, the condition of Quillen's arguement seems to be a little too strong, if it is also for some very characteristic modular Spaces, such as $F_n(M)$, the theorem will not be hold.

Consider $\mathrm{P}_n^{\mathrm{ab}} \cong \mathbb{Z}^{\binom{n}{2}}$ as an abelianization group of $\mathbf{P}_n$, we get an abelian group consisting of the image α_{ij} of the generators T_{ij} as figure 3.

Thus, we have $H_1(F_n(M);\mathbb{Z}) \sim n^2$ as $n \to \infty$.

Clearly, **Homological stability fails.**

But, from the view point of representation theory, we found more stability called representation stability.

Church and Farb, proposed a new paradigm for stability in spaces like the ordered configuration spaces $F_n(M)$of a manifold M. Because (co)homology is functorial, the S_n-action on $F_n(M)$ induces an action of S_n on the (co)homology groups. Even though the (co)homology does not stabilize as a sequence of abelian groups, they proposed, it does stabilize as a sequence of S_n-representations.

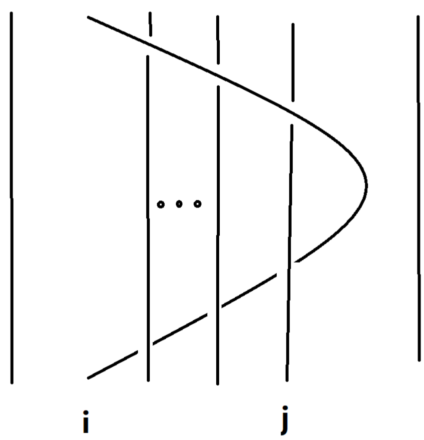


Figure 3 Artin's generator T_{ij}

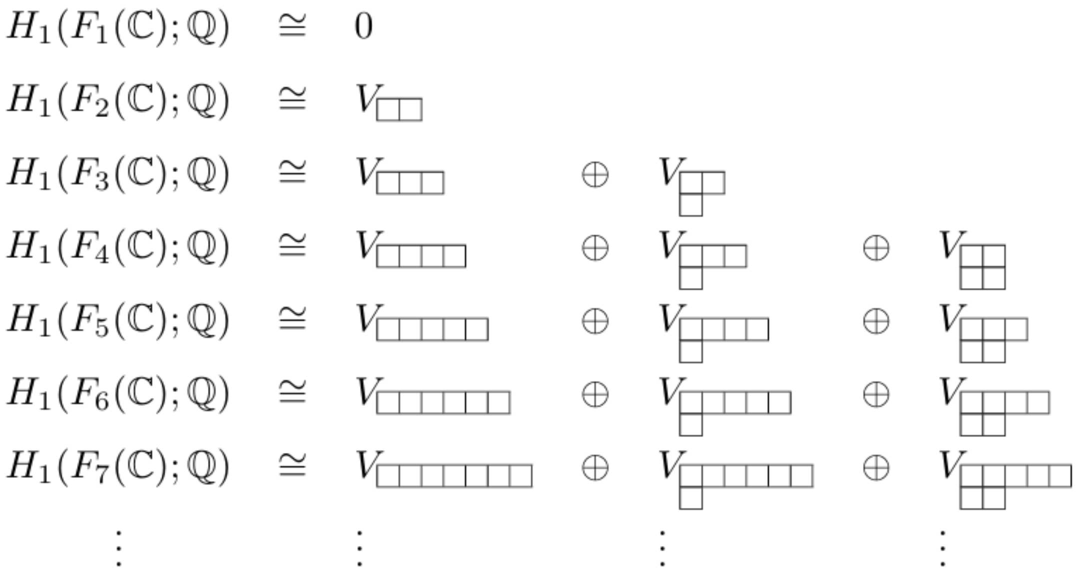


Figure 4 $F_n(M)$**representation**

4.2 S_n-representations

In order to better characterize this representation stability, we only need to consider the irreducible S_n-representations, since S_n is finite group implies V is semi-simple.

4.2.1 Young graph

The representation theory give out a way to corresponding every irreducible S_n-representation to a Young graph one-to-one.

Theorem 5 (Young's Correspondence Theorem) *The representation theory of S_n establishes a canonical bijection:*

$$\{\textit{Irreducible } S_n\textit{-representations}\} \leftrightarrow \left\{\lambda \vdash n \,\middle|\, \textit{Young diagram } \lambda\right\}$$

where $\lambda \vdash n$ denotes a partition of n. The irreducible representation V_λ corresponding to λ is uniquely determined by the Specht module construction.

Consider the action of S_n on $F_n(\mathbb{C})$, we can induce the action of S_n on $H_k(F_n(\mathbb{C});\mathbb{Q})$.[9]

When $n \geq 4k$, the H_{n-1} to H_n no longer increases with new parts, but continues to add a square to the right side of the first row of the Young graph of each part. Subsequently, Church proved that $H^k(F_n(M);\mathbb{Q})$ also has a similar property, and it is not necessarily limited to $F_n(M)$; a series of other spaces also exhibit similar properties and stability.

Thus, there is a impotent question: **Are there any hidden patterns between these changes in representation?**

Church, Ellenberg, Farb, Nagpal, Putman answer this question and proposed representation stability.

4.3 **FI**-module

In order to accurately and reasonably characterize the mathematical properties of homotopy stability, describe how other parts change at the same time when n approaches infinity. We firstly describe the process of n increasing as a category.

4.3.1 A general introduction of **FI**-module

FI-module V over R

we want to use **FI**-module to describe the "embedding" and the "automorphism" induced by S_n.

Definition 9 *(**FI**-module V over R)*

*Let **FI** be the category whose objects are finite sets (including$\emptyset$), and whose morphisms are all injective maps. Given a commutative ring R (typically $\mathbb{Z}$ or $\mathbb{Q}$), an **FI**-module V over R is a functor from **FI** to the category of R-modules.*

For example,

- $V_n = \mathbb{Q}$ the trivial S_n -representations, ι_n the identity maps.
- $V_n = \mathbb{Q}^n$, S_n permutes the standard basis, $\iota_n : \mathbb{Q}^n \cong (\mathbb{Q}^n \times 0) \hookrightarrow \mathbb{Q}^{n+1}$.
- $V_n = \mathbb{Q}[x_1, \ldots, x_n]$ the polynomial algebra with S_n permuting the variables, ι_n the inclusion.

are **FI**-modules.

Although this is essentially a concept on category, because of its special structure, we can also define its generated set like a "module" and then consider a "finitely-generated" **FI**-module

Definition 10 *(finitely generated FI-module)*

Let V be an FI-module. A subset $S \subseteq \bigcup_{n\geq 0} V_n$ is said to generate *V if either of the following equivalent conditions holds:*

1. *The images of S under all FI-morphisms span V_n for every $n \geq 0$;*

2. *The smallest FI-submodule of V containing S is V itself.*

We say V is finitely generated in degree $\leq d$ *if there exists a finite set of elements $S \subseteq \bigcup_{n\leq d} V_n$ that generates V.*

For example, consider the **FI**-module V over a commucative ring R such that $V_n = R[x_1, \ldots, x_n]_{(2)}$ is the submodule containing all degree-2 homogeneous polynomial in $R[x_1, \ldots, x_n]$, and $l_n : V_{n-1} \to V_n$ is the inclusion map. Then, let $S = \{x_1^2, x_1x_2\}$ which $x_1^2 \in V_1, x_1x_2 \in V_2$

1. $V_1 =< x_1 >$

2. $V_2 =< x_1^2, x_1x_2, x_2^2 >$

3. $V_3 =< x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3 >$

4. $V_n =< x_1^2 \ldots, x_n^2, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n >$

Clearly, S span every V_n. Hence, V is a finite generated **FI**-module.

4.4 the (co)homology group of $F_n(M)$ is **FI**-module

So, what is the relationship between **FI**-module and representation stability? Then a very important point worthy of our attention is that the cohomology and homology of $F_n(M)$ are a **FI**-module.

Consider a point in $F_n(M)$ as an index embedding $\rho : [n] \to M$

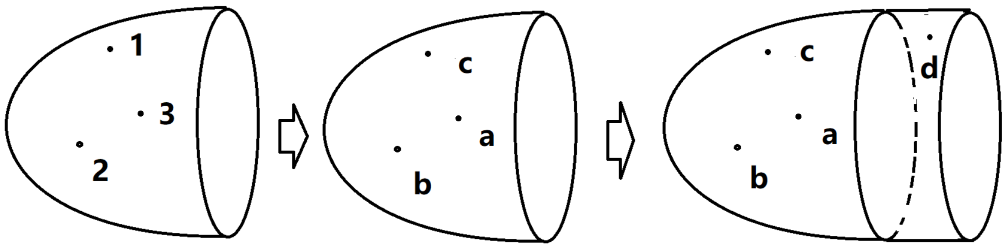


Figure 5 $f_\circ$ **defined using bourdary**

For any **FI**-morphism $f : [n] \to [m]$, we can define that

$$f' : F_m(M) \to F_n(M); \rho \to \rho \circ f$$

the we get a covariant functor on cohomology groups given by f'.

But, in this way, we can only get a contravariant functor on the homology groups. That goes against the direction we need to go in.

So we need to try to induce a covariant functor on F_n using S_n. That is why we need to further assume that dim $M \geq 2$ also has at least one non-empty boundary.

Then, we can define $f_\circ$ as follows: For any $x = (x_1, \ldots, x_n) \in F_n(M)$, $f_1(x) = (x_{f(1)}, \ldots, x_{f(n)}, y_{n+1}, \ldots, y_m)$, where we relabel the indices of x_i using f and add some points $y_{n+1}, \ldots, y_m$ at infinity using the non-empty boundary.

For example, if $f : [3] \to [4] = \{a, b, c, d\}$, then we can figure it out as figure5.

That is a covariant functor from **FI**-category to the category of homology groups.

4.5 Representation stability

Church–Ellenberg–Farb and(independently) Snowden proved that **FI**-modules over $\mathbb{Q}$ satisfy a **Noetherian** property:

Lemma 3 *submodules of finitely generated modules are themselves always finitely generated.*

then, we can claim the representation stability as below,

Definition 11 (Representation stability[14]) *Let V be an FI-module over $\mathbb{Q}$, finitely generated in degree $\leq d$. The following hold.*

- ***Finite generation.*** *For $n \geq d$,*

$$S_{n+1} \cdot i_n(V_n) \text{ spans } V_{n+1}.$$

- ***Polynomial growth.*** *There is a polynomial in n of degree $\leq d$ that agrees with the dimension $\dim_{\mathbb{Q}}(V_n)$ for all n sufficiently large.*

- ***Multiplicity stability.*** *For all $n \geq 2d$ the decomposition of V_n into irreducible constituents stabilizes.*

- ***Character polynomials.*** *The character of V_n is independent of n for all $n \geq 2d$.*

The characters of V are in fact eventually equal to a character polynomial, independent of n.

For the first item, it is relatively obvious. The third rule is a clear definition of the rule we observed before.

So, we will now focus on explaining second rule and forth rule.

4.5.1 Characteristic polynomial

To explain representation stability, we have to define the 'character polynomial' as follow,

Definition 12 *(character polynomial)*

For all $i > 0$, consider $X_i : S_n \to \mathbb{N}$ defined by

$$X_i(\sigma) := \text{number of } i\text{-cycles in } \sigma.$$

Polynomials in $\mathbb{Q}[X_1, X_2, \dots]$ are called caracter polynomial. Any character polynomial $P \in \mathbb{Q}[X_1, X_2, \dots]$ also define a mapping from S_n to $\mathbb{Q}$ for all $n \geq 1$.

The degree of a character polynomial is defined by setting $\deg(X_i) = i$.

For example, $\sigma = (123) \in S_4$, then $X_1(\sigma) = 1, X_2(\sigma) = 0, x_3(\sigma) = 1$.

If $P = X_1^2 + X_2^2$, then $P(\sigma) = 1^2 + 0^2 = 1$ and $\deg(P) = \max\{1 \cdot 2, 2 \cdot 2\} = 4$

Based on this concept, we can redefine rule 2 and 4 more mathematically. we denote that weight(V) as the minimal number of element that can generate the **FI**-module V.

we using stab-deg(V)[14] to describe where the **FI**-module stable. Ifstab-deg$(V) = s$, then when$n \geq s$, FI-module$V$ no longer loses information when splicing new elements (by mapping$T$), and the old and new structures are completely compatible.

Theorem 6 *[14]*

let V *a finite generated* ***FI****-module over a field* R *with characteristic* 0.

There exist a polynomial $P_V \in \mathbb{Q}[X_1, X_2, \dots]$ *with* $\deg P_V \leq$ weight(V) *such that for all* $n \geq$ stab-deg(V) + weigh(V) *and all* $\sigma \in S_n$*,*

$$\mathcal{X}_{V_n}(\sigma) = P_V(\sigma).$$

For example, If we define a **FI**-module V with $V_n = \mathbb{Q}[x_1, \dots, x_n]$ and natural inclusion mapping $l_n : V_{n-1} \to V_n$. Then, we have $P_V(\sigma) = X_1 + X_2$. (For every $\sigma \in S_n$, we only need to find the number of fix points on the basis $< x_1^2, \dots, x_n^2, x_1x_2, \dots, x_{n-1}x_n >$.)

4.5.2 Why does the polynomial stability appear after $2d$?

The proof is a bit long, we will sketch the proof[14].

Firstly, we consider a special **FI**-module,

Definition 13 *The FI-module* $M(W)$

We define the functor $M(-)$: *FB-Mod* $\to$ *FI-Mod as the left adjoint of* π: *FI-Mod* $\to$ *FB-Mod. Explicitly, if* W *is an FB-module, then the FI-module* $M(W)$ *satisfies*

$$M(W)_S = \text{colim}_{(T \in FB, f:T \hookrightarrow S)} W_T \bigoplus_{T \subseteq S} W_T,$$

with the map $f\colon M(W)_S \to M(W)_{S'}$ induced by $f\colon S \hookrightarrow S'$ being the sum of $(f_T)\colon W_T \to W_{f(T)}$.

Definition 14 *The irreducible representation $V(\lambda)_n$*

Given a partition λ, for any $n \geq \lambda_1$ we define the padded partition

$$\lambda[n] := (n - \lambda, \lambda_1, \ldots, \lambda_\ell).$$

For $n \geq \lambda_1$, we define $V(\lambda)_n$ to be the irreducible S_n-representation

$$V(\lambda)_n := V_{\lambda[n]}.$$

Definition 15 *The FI-module $M(\lambda)$*

When k is a field of characteristic 0, given a partition λ we write $M(\lambda)$ for the FI-module $M(\lambda) : M(V_\lambda)$.

Lemma 4 *For any partition λ , the FI-module $M(\lambda)$ over a field of characteristic 0 has stab-deg$((M(\lambda)) = \lambda_1)$.*

This is very important because soon we will know that in fact most of the **FI**-module we care about can be decomposed into these basis parts.

And, for each finite-generated **FI**-module, we have a classification theorem.

Theorem 7 *Every finite generated FI-module is form of the*

$$V = \bigoplus_\lambda M(\lambda)^{\oplus c_\lambda}.$$

Then, we have

$$\text{stab-deg}(M(\lambda)) = \lambda_1 \leq |\lambda| = \text{weight}(M(\lambda)).$$

To sum up,

$$\text{stab-deg}(V) \leq \text{weight}(V).$$

That is why the representation stability appeared when $i \geq 2d = d + d$.

5. Conclusion

In this thesis, we study the homology stability of moduli spaces from the perspective of representation theory and conducts an in-depth analysis taking braid groups as an example. By introducing tools such as the **FI**-module and characteristic of polynomials, we revealed the essential characteristics represention stability and proved the representation stability of the braid groups. Compared with the traditional homological stability, representation stability provides richer information and a deeper understanding, offering a new perspective for us to study the properties and behaviors of moduli spaces.

This topic is still very active. Now, there are many new developments in this topic, such as secondary homological stability, which was introduced by Galatius–Kupers–Randal-Williams in 2018.

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