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南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目：调和分析和 Hausdorff 维数专题

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专　　业：数学与应用数学

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2021 年 4 月 30 日

CLC ___________ Number___________

UDC___________ Available for reference ☐Yes ☐No

Southern University of Science and Technology

Undergraduate Thesis

Thesis Title:	**Harmonic Analysis and Hausdorff Dimension, a Brief Survey**
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Date: April 30, 2021

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调和分析和 Hausdorff 维数专题

林睿军

（数学系　指导教师：刘博辰）

[摘要]：调和分析是从 Fourier 分析出来的数学分支，主要研究将一个函数表示为若干基本的三角函数叠加，以及推广 Fourier 级数和 Fourier 变换的概念。Hausdorff 维数是一个比经典维数更好的描述集合尺度的指标，也是几何测度论中最重要的概念之一。几何测度论是分析学中用测度论方法解决几何问题的一个分支。

本篇毕业设计是一篇关于诸多不同形式的 Fourier 变换和 Hausdorff 维数相互作用的概述。并且讲述了一个应用，即证明了实数集上 Borel 环的一个特性：要么它的 Hausdorff 维数为 0，要么它是整个实数集。

[关键词]：Fourier 分析，Hausdorff 维数，能量积分，Borel 环.

[**ABSTRACT**]: Harmonic analysis is an area of analysis grown from Fourier analysis, which is concerned with the representation of functions as the superposition of basic trigonometric functions, and generalization of the notions of Fourier series and Fourier transforms. Hausdorff dimension is a finer index to measure the "mass" of sets than classical dimension. It is one of the most important concepts of geometric measure theory, an area of analysis concerned with solving geometric problems via measure-theoretic techniques.

This thesis is a survey on various types of Fourier analysis and the interplay with Hausdorff dimension, with an application in proving the behavior of a Borel ring on the real line: either has Hausdorff dimension zero or is the whole real line.

[**Keywords**]: Fourier Analysis, Hausdorff Dimension, Energy Integral, Borel Ring.

Contents

The content of this thesis is mainly based on Mattila[1].

1. Preliminaries

Definition 1.1 (Borel Set). *The Borel sets in a metric space X is the smallest σ-algebra of subsets of X containing all open subsets of X.*

Definition 1.2 (Borel Measure, Borel Regularity).

A Borel measure is a measure μ for which Borel sets are measurable.

A measure μ is called Borel regular if for any $A \subset X$ there is a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.

Definition 1.3 (Borel Measurable Function). *A function $f : X \to Y$ is said to be a Borel measurable function if for all Borel measurable sets $B \subset Y$, $f^{-1}(B)$ is Borel measurable in X.*

Definition 1.4. *The image or push-forward of a measure μ under a map $f : X \to Y$ is defined by*

$$f_{\#}(\mu(B)) = \mu(f^{-1}(B)) \text{ for } B \subset Y.$$

Lemma 1.5. *If μ is a Borel measure and f is a Borel measurable function, then*

$$\int g df_{\#}\mu = \int g \circ f d\mu$$

for all nonnegative Borel measurable functions g on X.

This can be proved by the monotone convergence theorem.

Definition 1.6 (Weak Convergence). *Let $C_0(\mathbb{R}^n)$ be the space of continuous functions with compact support on $\mathbb{R}^n$. The sequence $\{\mu_k\}_{k=1}^{\infty}$ of Borel measures on $\mathbb{R}^n$ is said to be converges weakly to a Borel measure μ if for all $\varphi \in C_0(\mathbb{R}^n)$,*

$$\int \varphi d\mu_k \to \int \varphi d\mu \text{ as } k \to \infty.$$

There is an important weak compactness theorem of Borel measures.

Theorem 1.7. *Any sequence of (finite) Borel measures $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k(\mathbb{R}^n)$ is bounded for all $k = 1, 2, \cdots$ has a weakly converging subsequence.*

The following proof of the theorem is mainly given by Mattila[2]. It relies on the Riesz representation theorem for measures (stated below), the so-called diagonal argument and the density argument, where the density comes from the following lemma without proof.

Lemma 1.8. *The space $C_0(\mathbb{R}^n)$ with L^∞-norm is separable.*

Proof. of Theorem 1.7. Define a sequence of operators $\{T_k\}_{k=1}^{\infty}$, where $T_k : C(\mathbb{R}^n) \to \mathbb{R}$, $T_k(\varphi) = \int \varphi d\mu_k$. T_k is well defined for each k and $\varphi \in C(\mathbb{R}^n)$ since the support of φ is compact and $\{\mu_k(\mathbb{R}^n)\}$ is bounded so that

$$|T_k(\varphi)| \leq \int |\varphi| d\mu_k \leq [\sup_{k\geq 1} \mu_k(\mathbb{R}^n)] \|\varphi\|_\infty < \infty$$

By Lemma 1.8, we take a countable dense subset D of $C_0(\mathbb{R}^n)$, written as $D = \{\varphi_1, \varphi_2, \varphi_3, \cdots\}$. Now comes to the diagonal argument.

First, for φ_1, $\{T_k(\varphi_1)\}$ is a bounded (numerical) sequence, so it has a converging subsequence denote by $T_{1,k}(\varphi_1)$ and we take out the corresponding operators $\{T_{1,k}\}$.

Second, for φ_2, $\{T_{1,k}(\varphi_2)\}$ is a bounded (numerical) sequence, so it has a converging subsequence denote by $T_{2,k}(\varphi_2)$ and we take out the corresponding operators $\{T_{2,k}\}$.

...

Proceeding in this fashion we obtain a countable array of operators $\{T_{m,k}\}$:

$$\begin{array}{cccc} T_{1,1} & T_{1,2} & T_{1,3} & \cdots \\ T_{2,1} & T_{2,2} & T_{2,3} & \cdots \\ T_{3,1} & T_{3,2} & T_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{array}$$

The diagonal sequence $\{T_{m,m}\}_{m=1}^{\infty}$ is the needed sequence that converges at each $\varphi \in D$. Since D is dense in $C_0(\mathbb{R}^n)$, $\{T_{m,m}\}_{m=1}^{\infty}$ converges at each $\varphi \in C_0(\mathbb{R}^n)$.

Finally, Theorem 1.7 gives the limit measure. ■

Theorem 1.7 will be used to prove the important Frostman's lemma linking the Hausdorff dimension and Borel measure in the next section.

We also list some definitions and theorems that will be used later.

The general definition of approximate identity is from Stein et al.[3].

Definition 1.9 (Approximate Identity)**.** *The family $\{\psi_t\}_{t>0}$ of continuous functions on $\mathbb{R}^n$ is said to be an approximate identity if it satisfies the following three conditions:*

(i)

$$\int_{\mathbb{R}^n} \psi_t(x)dx = 1$$

for every $t > 0$;

(ii) (Uniform Boundedness) There is a positive constant M such that

$$\int_{\mathbb{R}^n} |\psi_t(x)|dx < M$$

for every $t > 0$;

(iii) For every $\delta > 0$,

$$\int_{|x|\geq\delta} |\psi_t(x)|dx \to 0 \text{ as } t \to 0$$

There is a very important type of approximate identity called mollifiers, which is in addition nonnegative, infinitely differentiable and compactly supported. We give a description below.

Definition 1.10 (Standard Mollifier)**.** *The standard mollifier on $\mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$\psi(x) = \begin{cases} ce^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

where the constant c is chosen so that $\int \psi = 1$.

It is easy to check that ψ is compactly supported and infinitely differentiable. Using this stantard mollifier we can give a concrete approximate identity.

Proposition 1.11 (Standard Approximate Identity)**.** *Let ψ be defined as above. For $t > 0$ we define*

$$\psi_t(x) = t^{-n}\psi(t^{-1}x).$$

Then the family $\{\psi_t\}_{t>0}$ satisfies all the requirement of approximate identities. We call $\{\psi_t\}_{t>0}$ the standard approximate identity.

The proof of this proposition is just a simple verification of definitions.

Definition 1.12 (Convolutions)**.**

(1) The convolution of two functions f and g is defined by

$$(f * g)(x) = \int f(x-y)g(y)dy;$$

(2) The convolution of a function f and a Borel measure μ is defined by

$$(f * \mu)(x) = \int f(x-y)d\mu(y);$$

(3) The convolution of two Borel measures μ and ν on $\mathbb{R}^n$ is defined to satisfy the condition

$$\int \varphi d(\mu * \nu) = \iint \varphi(x+y)d\mu(x)d\nu(y) \text{ for all } \varphi \in C_0^+(\mathbb{R}^n).$$

The following two theorems (from for example, Duoandikoetxea[4]) explains where the term “approximate identity” comes from. It approximates functions through convolutions.

Theorem 1.13. *Let $\{\psi_t\}_{t>0}$ be an approximate identity. Then $\psi_t * f$ converges to f in $L^p(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$, and $\psi_t * f$ converges to f uniformly (i.e when $p = \infty$) if $f \in C_0(\mathbb{R}^n)$.*

To prove this theorem, we introduce the important Minkowski’s inequality for integrals.

Lemma 1.14 (Minkowski’s Inequality for Integrals)**.** *Suppose (X, μ) and (Y, ν) are two measure spaces with σ-finite measures. Then*

$$\left(\int_X \left|\int_Y f(x,y)d\nu(y)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y),$$

or in the form of norms,

$$\left\|\int_Y f(x,y)d\nu(y)\right\|_{L^p(X,\mu)} \leqslant \int_Y \|f(x,y)\|_{L^p(X,\mu)} d\nu(y).$$

Now comes the proof of Theorem 1.13.

Proof. of Theorem 1.13. Since by definition, $\{\psi_t\}_{t>0}$ has integral 1 (the first requirement in the definition),

$$(\psi_t * f)(x) - f(x) = \int_{\mathbb{R}^n} \psi_t(y)[f(x-y) - f(x)]dy.$$

If $1 \le p < \infty$, by the Minkowski's inequality for integrals,

$$\|(\psi_t * f) - f\|_p \le \int_{\mathbb{R}^n} |\psi_t(y)| \, \|f(\cdot - y) - f(\cdot)\|_p \, dy.$$

Recall that the translation is "continuous" in $L^p(\mathbb{R}^n)$ which means that for any given $\varepsilon > 0$, we can always choose a small $\delta > 0$ such that $\|f(\cdot + h) - f(\cdot)\|_p < \varepsilon/2M$ whenever $|h| < \delta$. By the third requirement in the definition of approximate identity, we choose t small enough such that $\int_{|y|\ge\delta} |\psi_t(y)|dy \le \varepsilon/4\|f\|_p$. Thus,

$$\begin{aligned}
\|(\psi_t * f) - f\|_p &\le \int_{\mathbb{R}^n} |\psi_t(y)| \, \|f(\cdot - y) - f(\cdot)\|_p \, dy \\
&= \int_{|y|<\delta} |\psi_t(y)| \, \|f(\cdot - y) - f(\cdot)\|_p \, dy + \int_{|y|\ge\delta} |\psi_t(y)| \, \|f(\cdot - y) - f(\cdot)\|_p \, dy \\
&\le \int_{|y|<\delta} M \, \|f(\cdot - y) - f(\cdot)\|_p dy + 2\|f\|_p \int_{|y|\ge\delta} |\psi_t(y)|dy \\
&\le M \cdot \frac{\varepsilon}{2M} + 2\|f\|_p \frac{\varepsilon}{4\|f\|_p}
\end{aligned}$$

where the first term of the last inequality above is because $\{\psi_t\}_{t>0}$ is uniformly bounded by the constant $M > 0$, and the second term is because of the properties of the norm, where $\|f(\cdot - y) - f(\cdot)\|_p \le \|f(\cdot - y)\|_p + \|f(\cdot)\|_p = 2\|f_p\| = \varepsilon$.

If $p = \infty$ instead, f has finite L^∞-norm. By continuity of f, for every $x \in \mathbb{R}^n$ we can choose a $\delta > 0$ such that $|f(x-y) - f(x)| < \varepsilon/2M$. The index t is chosen small enough so that $\int_{|y|\ge\delta} |\psi_t(y)|dy \le \varepsilon/(4\|f\|_\infty)$, analogous as above. Then,

$$\begin{aligned}
|(\psi_t * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} \psi_t(y)[f(x-y) - f(x)]dy \right| \\
&\le \int_{\mathbb{R}^n} |\psi_t(y)| \, |f(x-y) - f(x)|dy \\
&\le \int_{|y|<\delta} M \, |f(x-y) - f(x)|dy + 2\|f\|_\infty \int_{|y|\ge\delta} |\psi_t(y)|dy \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

The proof is complete. ■

Theorem 1.15. *Let $\{\psi_t\}_{t>0}$ be an approximate identity and μ a locally finite Borel measure on $\mathbb{R}^n$. Then $\psi_t * \mu$ converges weakly to μ as $t \to 0$, that is,*

$$\int \varphi(\psi_t * \mu)d\mathcal{L}^n \to \int \varphi d\mu \text{ as } t \to 0$$

for all $\varphi \in C_0(\mathbb{R}^n)$.

Proof. We perform direct computation.

$$
\begin{aligned}
\int \varphi(x)(\psi_t * \mu)(x)dx &= \int \varphi(x) \left(\int \psi_t(x-y)d\mu(y) \right) dx \\
&= \int \int \varphi(x)\psi_t(x-y)dx\, d\mu(y) \\
&= \int \int \varphi(x)\psi_t(y-x)dx\, d\mu(y) \\
&= \int (\varphi * \psi_t)(y)d\mu(y) \\
&\longrightarrow \int \varphi d\mu, \text{ as } t \to 0.
\end{aligned}
$$

The second equality is because of the Fubini's theorem, the third equality is by the reflection invariance of Lebesgue integration and the last line is by Lebesgue's dominated convergence theorem, and Theorem 1.13.

■

The Radon-Nikodym theorem and Riesz representation theorems builds a connection between measures and integrable functions. First we recall that a Radon measure on a metric space is an inner regular and locally finite measure. A Hilbert space is a complete inner product space.

Theorem 1.16 (Radon-Nikodym Theorem)**.** *Let $(X, \mathcal{X}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on the measurable space $(X, \mathcal{X})$ that is absolutely continuous with respect to ν. Then there is a nonnegative function f_ν on X that is measurable with respect to $\mathcal{X}$ for which*

$$\nu(E) = \int_E f_\nu d\mu, \quad \forall E \in \mathcal{X}$$

The function f_ν, is unique in the sense that if g is any nonnegative measurable function on X that also has this property, then $g = f_\nu$ μ-almost everywhere. Such a function f_ν, written formally as $f_\nu = \frac{d\nu}{d\mu}$, is called a Radon-Nikodym derivative. Moreover, if μ is a finite measure, i.e., $\mu(X) < \infty$, then such f_ν is integrable.

Theorem 1.17 (Riesz Representation Theorem for Measures)**.** *Let X be a locally compact metric space and $T : C_0(X) \to \mathbb{R}$ a positive linear functional. Then there is a unique Radon measure μ such that*

$$Tf = \int f d\mu \quad \text{for } f \in C_0(X)$$

Theorem 1.18 (Riesz Representation Theorem for Hilbert Spaces)**.** *Let H be a (Hilbert space). Given any $T \in H^*$ there exists a unique $u \in H$ such that*

$$Tv = \langle u, v \rangle \text{ for all } v \in H.$$

Moreover, the vector norm of u is the same as the operator norm of T:

$$\|u\| = \|T\|$$

(The above three theorems come from Royden[5], Mattila[2] and Muscat[6].)

The Baire category theorem states that complete metric space cannot be a countable union of nowhere dense sets. The Steinhaus's theorem states that the difference set of a Lebesgue measurable set of positive measure contains an open ball. We state them and prove the latter here which will be used later.

Theorem 1.19 (Baire Category Theorem)**.** *Let X be a nonempty complete metric space. Suppose there is a countable family of subsets $\{A_n\}_{n=1}^{\infty}$ such that*

$$X = \bigcup_{n=1}^{\infty} A_n,$$

then at least one member of this family $\{A_n\}_{n=1}^{\infty}$ has a nonempty interior.

Theorem 1.20 (Steinhaus's Theorem)**.** *Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ of positive measure. Define the difference set of E as*

$$E - E = \{x - y : x, y \in E\}.$$

Then there exists a $\delta > 0$ such that the open ball $B(0, \delta) \subset E - E$.

To prove Theorem 1.20 we need the following lemma.

Lemma 1.21. *Let $E \subset \mathbb{R}^n$ be a measurable subset with $\mathcal{L}^n(E) > 0$. For every λ satisfying $0 < \lambda < 1$ there is a cuboid I such that $\lambda|I| < \mathcal{L}^n(I \cap E)$, where $|I|$ stands for the volume of the cuboid I.*

Proof. The conclusion is trivial when $\mathcal{L}^n(E)$ is infinite, so we assume $\mathcal{L}^n(E) < \infty$. For $0 < \varepsilon < (\lambda^{-1} - 1)\mathcal{L}^n(E)$, we choose a cover of E by cuboids, say $\{I_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |I_k| < \mathcal{L}^n(E) + \varepsilon$. We claim there is a k_0 satisfying $\lambda|I_{k_0}| < \mathcal{L}^n(I_{k_0} \cap E)$. In fact, if for all $k \in \mathbb{N}^*$, $\lambda|I_k| \geq \mathcal{L}^n(I_k \cap E)$, then

$$\begin{aligned} \mathcal{L}^n(E) &\leq \sum_{k=1}^{\infty} \mathcal{L}^n(I_k \cap E) \\ &\leq \lambda \sum_{i=1}^{\infty} |I_k| \\ &\leq \lambda(\mathcal{L}^n(E) + \varepsilon) < \mathcal{L}^n(E) \end{aligned}$$

which leads to a contradiction. ■

Proof. of Theorem 1.20. Since $0 < \lambda < 1$, we can restrict λ satisfying $1 - 2^{-(n+1)} < \lambda < 1$. From the previous lemma, there is a cuboid I such that $\lambda|I| \leq \mathcal{L}^n(I \cap E)$. Now let δ be the smallest side length of I. Define a new open cube

$$J = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : |x_i| < \delta/2, i = 1, 2, \cdots n\}.$$

We now claim that $J \subset E - E$. This is equivalent to say that

$$\forall x_0 \in J, (E \cap I) \cap ((E \cap I) + \{x_0\}) \neq \varnothing$$

because this time we have $y, z \in E \cap I$, such that $y - z = x_0$.

Since J is a cube centered at the origin with side length δ, then $I + x_0$ still contains the origin. Thus we have in geometry that

$$\mathcal{L}^n(I \cap (I + \{x_0\})) > 2^{-n}|I|,$$

from which we have

$$\begin{aligned}\mathcal{L}^n\left(I \cup (I + \{x_0\})\right) &= |I| + \mathcal{L}^n\left(I + \{x_0\}\right) - \mathcal{L}^n\left(I \cap (I + \{x_0\})\right) \\ &< 2|I| - 2^{-n}|I|\end{aligned}$$

which means $\mathcal{L}^n\left(I \cup (I + \{x_0\})\right) < 2\lambda|I|$. By the translation invariance of Lebesgue measure, $(E \cap I)$ and $((E \cap I) + \{x_0\})$ is of the same measure larger than $\lambda|I|$, and they are both contained in $I \cup (I + \{x_0\})$. So they must have nonempty intersection. Otherwise

$$\mathcal{L}^n\left((E \cap I) \cup ((E \cap I) + \{x_0\})\right) > 2\lambda|I|$$

which leads to a contradiction. ∎

The Gamma functions and Bessel functions will be used later for in the discussion of some particular measures and functions, so we list the definitions here.

Definition 1.22 (Gamma Function)**.** *For $x > 0$, the integral*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

is well defined (which means converging absolutely). This integral is called the Gamma function.

Some properties of Gamma functions are listed below.

Properties 1.23.

(i) $\Gamma(x) > 0$ *for all* $x > 0$,

(ii) $\Gamma(1) = 1$,

(iii) $\Gamma(x + 1) = x\Gamma(x), \ x > 0$,

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition 1.24 (Bessel Function)**.** *The Bessel function $J_m : [0, \infty) \to \mathbb{R}$ of order $m > -1/2$ is defined by the formula*

$$J_m(u) = \frac{\left(\frac{u}{2}\right)^m}{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{iut}\left(1 - t^2\right)^{m-1/2} dt$$

We give the following recursion properties for Bessel functions (from Grafakos[7]) which will be used later.

Properties 1.25.

$$\begin{aligned}\frac{d}{dt}(t^{-m}J_m(t)) &= -t^m J_{m+1}(t), \\ \frac{d}{dt}(t^m J_m(t)) &= t^m J_{m-1}(t).\end{aligned}$$

Proof. For the first identity,

$$\begin{aligned}
\frac{d}{dt}\left(t^{-m}J_m(t)\right) &= \frac{i}{2^m\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} s e^{its}\left(1-s^2\right)^{m-\frac{1}{2}}ds \\
&= \frac{i}{2^m\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} \frac{it}{2}e^{its}\frac{\left(1-s^2\right)^{m+\frac{1}{2}}}{m+\frac{1}{2}}ds \\
&= -t^{-m}J_{m+1}(t)
\end{aligned}$$

For the second identity,

$$\begin{aligned}
&\frac{d}{dt}\left(t^m J_m(t)\right) \\
&= \frac{2mt^{2m-1}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}\left(1-s^2\right)^{m-\frac{1}{2}}ds + \frac{t^{2m}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} i\int_{-1}^{1} e^{its} is\left(1-s^2\right)^{m-\frac{1}{2}}ds \\
&= \frac{2mt^{2m-1}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}\left(1-s^2\right)^{m-\frac{1}{2}}ds + \frac{t^{2m}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} \left(\frac{e^{its}}{t}\right)'\left(1-s^2\right)^{m-\frac{1}{2}}sds \\
&= \frac{2mt^{2m-1}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}\left(1-s^2\right)^{m-\frac{1}{2}}ds - \frac{t^{2m}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} \frac{e^{its}}{t}\left(\left(1-s^2\right)^{m-\frac{1}{2}}s\right)'ds \\
&= \frac{t^{2m-1}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}\left[2m\left(1-s^2\right)^{m-\frac{1}{2}} - \left(\left(1-s^2\right)^{m-\frac{1}{2}}s\right)'\right]ds \\
&= \frac{t^{2m-1}2^{-m}}{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}(2m-1)\left(1-s^2\right)^{m-\frac{3}{2}}ds \\
&= \frac{t^{2m-1}2^{-(m-1)}}{\Gamma\left(m-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}\left(1-s^2\right)^{m-\frac{1}{2}}\frac{ds}{\sqrt{1-s^2}} \\
&= t^m J_{m-1}(t)
\end{aligned}$$

■

2. Hausdorff Dimension

Definition 2.1 (Hausdorff Content)**.** *Let (X,d) be a metric space and $A \subset X$ be a subset. Let $s \geq 0$ and $0 < \delta \leq \infty$ The s-dimensional δ-limited Hausdorff content of A is defined by*

$$\mathcal{H}_\delta^s(A) = \inf\left\{\sum_{j=1}^{\infty}\alpha(s)2^{-s}d\left(E_j\right)^s \,\middle|\, A \subset \bigcup_{j=1}^{\infty}E_j,\ \text{and } d\left(E_j\right) < \delta, j = 1,2,\cdots\right\}$$

where $d(E)$ is the diameter of the set E, $d(E) = \sup\{d(x,y) \mid x,y \in E\}$, and $\alpha(s)$ is a positive real number. For $s = n$ the integers, we let $\alpha(n)$ be the volume of the n-dimensional unit ball (especially, $\alpha(0) = 1$); for non-integer s we leave $\alpha(s)2^{-s} = 1$.

We say that $\{E_j\}_{j=1}^{\infty}$ is a covering of A if $A \subset \cup_{j=1}^{\infty}E_j$.

The definition n-dimensional δ-limited Hausdorff contents of a set is "essentially the same" as the case of s-dimensional contents. They just differ by a multiplication of the constant $\alpha(s)2^{-s}$. Actually, this constant is defined to make the Hausdorff measure consistent with Lebesgue measure when $s = n$ are integers.

The infimum in the definition of Hausdorff contents implies the following easy proposition:

Proposition 2.2. *For fixed $A \subset X$ and $s \geq 0$, $\mathcal{H}^s_\delta(A)$ is non-increasing about δ, that is, $\mathcal{H}^s_{\delta_1}(A) \geq \mathcal{H}^s_{\delta_2}(A)$ for $\delta_1 \leq \delta_2$.*

Because of this, fixing s, the limit of $\mathcal{H}^s_\delta(A)$ exists (or is infinity) as δ goes to 0.

Definition 2.3 (Hausdorff Measure)**.** *Let $0 \leq s \leq n$. The s-dimensional Hausdorff measure of a set $A \subset X$ is defined to be*

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

Proposition 2.4. *If for a set A, $\mathcal{H}^s_{\delta_0}(A)$ is zero for some $0 < \delta_0 \leq \infty$, then $\mathcal{H}^s_\delta(A) = 0$ for all $0 < \delta \leq \delta_0$ so that $\mathcal{H}^s(A) = 0$.*

Proof. We may just assume that $\alpha(s)2^{-s} = 1$. For every $\varepsilon > 0$, there exists a covering $\{E_j\}_{j=1}^\infty$ of A such that $d(E_j) < \delta_0$ and

$$\sum_{j=1}^{\infty} d\left(E_j\right)^s < \varepsilon$$

Thus every $d(E_j) < \varepsilon^{\frac{1}{s}}$ $j = 1, 2, \cdots$.

Now if $\delta \leq \delta_0$, take $\varepsilon > 0$ such that $\varepsilon < \delta^s \leq \delta_0^s$ and we obtain a covering $\{E_j\}_{j=1}^\infty$ with $d(E_j) < \varepsilon^{\frac{1}{s}} < \delta \leq \delta_0$. So we have a covering $\{E_j\}_{j=1}^\infty$ with $d(E_j) < \delta_0$ such that $\sum_{j=1}^\infty d\left(E_j\right)^s < \varepsilon$. Hence $\mathcal{H}^s_\delta(A) = 0$ for all $0 < \delta \leq \delta_0$, and $\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = 0$. ■

Combining Propositions 2.2 and 2.4, we have

Corollary 2.5. *$\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\delta(A) = 0$ for some $\delta > 0$.*

Lemma 2.6. *For $A \subset X$, there is a unique non-negative real number s_0, such that $H^s(A) = \infty$ if $s < s_0$ and $H^s(A) = 0$ if $s > s_0$.*

We omit the proof of this lemma. Because of the Lemma 2.6 we bave the concept of Hausdorff dimension.

Definition 2.7 (Hausdorff Dimension)**.** *The Hausdorff dimension of a set $A \subset X$ is defined as*

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$$

Corollary 2.5 tells us that we can express the definition of Hausdorff dimensions equivalently by

$$\dim A = \inf\left\{ s : \forall \varepsilon > 0, \exists E_1, E_2, \cdots \subset X \text{ s.t. } A \subset \bigcup_{j=1}^{\infty} E_j \text{ and } \sum_{j=1}^{\infty} d\left(E_j\right)^s < \varepsilon \right\}$$

3. Frostman's Lemma and Energy Integral

3.1 Frostman's lemma

It is in general not easy to evaluate the Hausdorff dimension of a subset of $\mathbb{R}^n$. The Frostman's lemma is a useful tool in determining such a quantity.

Theorem 3.1 (Frostman's Lemma). *Let $0 \le s \le n$. For a Borel set $A \subset \mathbb{R}^n, \mathcal{H}^s(A) > 0$ if and only if there is a $\mu \in \mathcal{M}(A)$ such that*

$$\mu(B(x,r)) \le r^s \text{ for all } x \in \mathbb{R}^n,\ r > 0. \tag{3.1}$$

In particular,

$$\dim A = \sup\{s : \text{ there is a } \mu \in \mathcal{M}(A) \text{ such that (3.1) holds.}\}.$$

Theorem 3.1 established the relation between Hausdorff measures and Borel measures. A measure satisfying (3.1) is often called a Frostman measure. The idea of the proof is to construct a sequence of Borel measures $\{\mu_k\}$ and the needed Frostman measure is its weak limit.

Proof. Suppose there is a $\mu \in \mathcal{M}(A)$ satisfies the condition: $\mu(B(x,r)) \le r^s$, for all $x \in \mathbb{R}^n,\ r > 0$. Let $\{B_j\}_{j=1}^{\infty}$ be a collection of balls covering A. We have

$$\sum_{j=1}^{\infty} d(B_j)^s \ge \sum_{j=1}^{\infty} \mu(B_j) \ge \mu(A) > 0,$$

Thus $\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A) > 0$.

Conversely, suppose $\mathcal{H}^s(A) > 0$. We may assume A is compact. By definition of Hausdorff measures there is a $c > 0$ such that

$$\sum_{j=1}^{\infty} d(E_j) \ge c \tag{3.2}$$

for all coverings $\{E_j\}_{j=1}^{\infty}$.

First we give a standard cubical partitioning of the whole space $\mathbb{R}^n$ with cubes of side-length 2^{-k}. Define a measure $\mu_{k,1}$ satisfying the condition on such cubes Q:

$$\mu_{k,1}(Q) = \begin{cases} d(Q)^s & \text{if } Q \cap A \neq \varnothing, \\ 0 & \text{if } Q \cap A = \varnothing. \end{cases}$$

This measure fits for cubes with side-length less than 2^{-k} but not necessarily for larger balls.

Second, we modify $\mu_{k,1}$ to a measure by giving a standard cubical partitioning of the whole space $\mathbb{R}^n$ with cubes of side-length doubled, i.e., 2^{1-k}. Define $\mu_{k,2}$ as follows:

$$\mu_{k,2}(Q) = \begin{cases} \mu_{k,1}(Q) & \text{if } \mu_{k,1}(Q) \le d(Q)^s, \\ d(Q)^s & \text{otherwise.} \end{cases}$$

Continue this process until we come to a single cube Q_0 containing the compact set A, and let μ_k be the final measure obtained in this way. In this process, we never increase the measure, so $\mu_k(Q) \leq d(Q)^s$ for all dyadic cubes with side-length at least 2^{-k}. Moreover the construction yields that every $x \in A$ is contained in some sub-cube $Q' \subset Q_0$ with side-length at least 2^{-k} such that $\mu_k(Q') = d(Q)^s$.

Choosing maximal, and hence disjoint such cubes $\{Q'_j\}$ covering A. Thus by (3.2), we have

$$\mu_k(\mathbb{R}^n) = \sum_{j=1}^{\infty} \mu_k(Q'_j) = \sum_{j=1}^{\infty} d(Q'_j)^s \geq c \tag{3.3}$$

Now take a weakly converging subsequence of $\{\mu_k\}$ using Theorem 1.7, still denoted by $\{\mu_k\}$ and denote its limit measure by μ. From the construction, the support of μ, suppμ is contained in A. Thus (taking $\varphi = \chi_Q$ in Theorem 1.7) for all cubes Q

$$\mu(Q) = \lim_{k\to\infty} \mu_k(Q) \leq d(Q)^s$$

for all cubes Q in $\mathbb{R}^n$. Since the definition of (normalized) Lebesgue measure of Borel sets using balls is equivalent with that using cubes, we conclude that $\mu(B) \lesssim_n d(B)^s$ for all balls B. Note that μ cannot be a zero measure because of (3.3). Finally a scaling of the obtained Borel measure μ by multiplying an appropriate number gives the needed new Borel measure. ■

A simple but useful application of the Frostman's lemma is the inequality for dimensions of product sets.

Theorem 3.2. *Let A and B be nonempty Borel sets in $\mathbb{R}^n$. Then*

$$\dim(A \times B) \geq \dim A + \dim B.$$

Proof. By Theorem 3.1, if $0 \leq s < \dim A$ and $0 \leq t < \dim B$, we can choose a $\mu \in \mathcal{M}(A)$ with $\mu(B(x,r)) \leq r^s$ and $\nu \in \mathcal{M}(B)$ with $\nu(B(x,r)) \leq r^t$. Then the product measure $\mu \times \nu \in \mathcal{M}(A \times B)$ with $(\mu \times \nu)(B((x,y),r)) \leq r^{s+t}$ from which the theorem follows. ■

3.2 Energy integral

In this section we will see an equivalent expression of Hausdorff dimensions by energy integrals.

Definition 3.3 (s-dimensional energy integral)**.** *The s-dimensional energy integral, or s-energy of a Borel measure μ is defined as the integral*

$$I_s(\mu) = \iint |x-y|^s d\mu(x) d\mu(y)$$

This integral can be written as the form of a convolution

$$I_s(\mu) = \int (k_s * \mu)(x) d\mu(x)$$

where $k_s(x) = |x|^{-s},\ x \in \mathbb{R}^n$ is called the *Riesz kernel*.

A direct observation is that if μ has compact support we have trivially $I_s(\mu) < \infty$ implies $I_t(\mu) < \infty$ for $0 < t < s$, by considering the integral on $|x - y| \leq 1$ and $|x - y| > 1$, respectively.

The Hausdorff dimension of a set can be expressed by s-energy, as stated in the following theorem.

Theorem 3.4. *Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ if and only if there is a $\mu \in \mathcal{M}(A)$ such that the s-energy integral $I_s(\mu) < \infty$. Equivalently,*

$$\dim A = \sup\{s : \textit{there is } \mu \in \mathcal{M}(A) \textit{ such that } I_s(\mu) < \infty\}.$$

The proof is to use the above Frostman's lemma (Theorem 3.1) and some arguments. First we need the following lemma about integrals.

Lemma 3.5. *Let $\mu \in \mathcal{M}(A)$ be a Borel measure on $\mathbb{R}^n$ and let f be a non-negative μ-integrable function. For every $r > 0$, define the strict hypograph E_r to be $E_r = \{y : f(y) > r\}$. Then we have*

$$\int_{\mathbb{R}^n} f(y)d\mu(y) = \int_0^\infty \mu(E_r)dr.$$

Now it is time to prove the Theorem 3.4.

Proof. of Theorem 3.4. Let μ such that (3.1) in the Frostman's lemma holds.

Using Lemma 3.5 by taking $f(y) = |x - y|^{-s}$, then $E_r = \{y : |x - y|^{-s} > r\} = \{y : |y - x| < r^{-1/s}\}$. This is an open ball centered at x with radius $r^{-1/s}$. Thus $\mu(E_r) = \mu(B(x, r^{-1/s}))$. For any $\varepsilon > 0$,

$$I_{s-\varepsilon}(\mu) = \int \int_0^\infty \mu(B(x, r^{-1/(s-\varepsilon)}))dr \; d\mu(x) = \mu(\mathbb{R}^n) \int_0^\infty \mu(B(x, r^{-1/(s-\varepsilon)}))dr$$

We can restrict the lower limit of the integral to $d(\text{supp})^{-(s-\varepsilon)}$, because when

$$r^{-\frac{1}{s-\varepsilon}} > d(\text{supp}\mu)$$

the value of the integral no longer increases.

Applying Theorem 3.1,

$$I_{s-\varepsilon}(\mu) \leq \int_{d(\text{supp})^{-(s-\varepsilon)}}^\infty r^{-1-\frac{\varepsilon}{s-\varepsilon}} dr < \infty$$

Thus $\dim A \leq \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}$.

On the other hand, suppose $\mu \in \mathcal{M}(A)$ satisfies $I_s(\mu) < \infty$. Then $\int |x - y|^s d\mu(x) < \infty$ for almost all $y \in \mathbb{R}^n$. We can find a $0 < M < \infty$ such that the set $C = \{y : \int |x - y|^s d\mu(x) < M\}$ has positive μ measure. Then $\mu|_C(B(x, r)) \leq 2^s M r^s$ for all $x \in \mathbb{R}^n$. Thus we have

$$\dim A \geq \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}$$

Combining the results above, we have

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$

■

4. Fourier Transform

4.1 Fourier transform in $L^1(\mathbb{R}^n)$

Definition 4.1 (Fourier Transform in $L^1(\mathbb{R}^n)$)**.** *For Lebesgue integrable functions $f \in L^1(\mathbb{R}^n)$, the Fourier transform is defined by*

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx,\ \xi \in \mathbb{R}^n$$

where the $\cdot$ is the dot product in $\mathbb{R}^n$.

Some easy properties are listed below.

Properties 4.2.
(i) $\widehat{f}$ is well defined, bounded and continuous.
(ii) Product formula

$$\int \widehat{f}g = \int f\widehat{g},\ f,g \in L^1(\mathbb{R}^n)$$

(iii) Convolution formula

$$\widehat{f * g} = \widehat{f}\widehat{g},\ f,g \in L^1(\mathbb{R}^n)$$

(iv) Define the translation $\tau_a(x) = x + a$ for $a \in \mathbb{R}^n$ and the dilation $\delta_r(x) = rx$ for $r \in \mathbb{R}$. Then

$$\begin{aligned}\widehat{f \circ \tau_a}(\xi) &= e^{2\pi i a\cdot\xi}\widehat{f}(\xi)\\ \widehat{e^{2\pi i a\cdot x}f}(\xi) &= \widehat{f}(\xi - a) = \widehat{f} \circ \tau_a(\xi)\\ \widehat{f \circ \delta_r}(\xi) &= r^{-n}\widehat{f}(r^{-1}\xi)\end{aligned}$$

Lemma 4.3 (Riemann-Lebesgue Lemma)**.** *For $f \in L^1(\mathbb{R}^n)$,*

$$\widehat{f}(\xi) \to 0 \textit{ as } |\xi| \to \infty.$$

Proof. Note that

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx\\ &= -\int_{\mathbb{R}^n} f(x)e^{-2\pi i\left(x\cdot\xi + \frac{1}{2}\right)}\\ &= -\int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot\left(x + \frac{\xi}{2|\xi|^2}\right)}\\ &= -\int_{\mathbb{R}^n} f\left(x - \frac{\xi}{2|\xi|^2}\right)e^{-2\pi i x\cdot\xi}dx\end{aligned}$$

Thus

$$\widehat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}\left[f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right]e^{-2\pi i x\cdot\xi}dx \to 0$$

as $|\xi| \to \infty$ by the Lebesgue's dominated convergence theorem. ■

One of the most important conclusions in the Fourier transform is the Fourier inversion formula.

Theorem 4.4 (Fourier Inversion Formula). *If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then we have the Fourier inversion formula*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

for almost every $x \in \mathbb{R}^n$.

The Fourier inversion of such a function f is often denoted by $\mathcal{F}^{-1}(f)$ or $\check{f}$. To prove the Fourier inversion formula, we first introduce some notations.

Notations. Define $f_\varepsilon(x) = f(\epsilon x) = f \circ \delta_\varepsilon(x)$, and $f^\varepsilon(x) = \varepsilon^{-n} f(\varepsilon^{-1} x)$. Then we have the following practical identities:

$$\widehat{f_\varepsilon} = (\widehat{f})^\varepsilon$$
$$\widehat{f^\varepsilon} = (\widehat{f})_\varepsilon$$

Definition 4.5 (Gauss Funcion, Gauss Kernel). *For $x \in \mathbb{R}^n$, the Gauss kernel $\Psi(x)$ is given by*

$$\Psi(x) = e^{-\pi |x|^2}$$

and the Gauss kernels is defined to be the family $\{\Psi^\varepsilon\}_{\varepsilon > 0}$.

Lemma 4.6. *The Fourier transform of the Gauss function Ψ is itself, that is*

$$\widehat{\Psi} = \Psi.$$

The proof is based on complex analysis method and we omit it here.

Lemma 4.7. *The Gauss kernel $\{\Psi^\varepsilon\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}^n)$,is an approximate identity.*

This can be checked directly by definition of approximate identities.

Proof. of Theorem 4.4 Define

$$I_\varepsilon(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-\pi \varepsilon^2 |\xi|^2} e^{2\pi i \xi \cdot x} d\xi.$$

On the one hand, by the Lebesgue's dominated convergence theorem,

$$I_\varepsilon(x) \longrightarrow \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi, \quad \varepsilon \to 0;$$

On the other hand, consider $g(x, y) = e^{-\pi \varepsilon^2 |y|^2} e^{2\pi i y \cdot x}$. Fixing x, let $g_x(y) = g(x, y)$, we have that $\widehat{g_x}(y) = \widehat{\Psi_\varepsilon}(y - x) = \Psi^\varepsilon(x - y)$. Using the multiplication formula and Theorem 1.15 and Lemma 4.7,

$$I_\varepsilon(x) = \int \widehat{f} g_x = \int f \widehat{g_x} = (\Psi^\varepsilon * f)(x) \to f(x) \text{ in } L^1.$$

Hence we can take a subsequence $\{I_{\varepsilon_n}\}_{n=1}^\infty$ of $\{I_\varepsilon\}$ converging to f pointwise almost everywhere. Also, $\{I_{\varepsilon_n}\}$ converges to $\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$ pointwise by the previous argument. This yields the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

for almost every $x \in \mathbb{R}^n$. ■

There are several equivalent expressions of Fourier inversion formula. If we define the reflection $\widetilde{f}(x) = f(-x)$ and the conjugation $\overline{f}(x) = \overline{f(x)}$, then the Fourier inversion formula can be expressed by

$$\check{f} = \widetilde{\widehat{f}} = \widehat{\widetilde{f}}, \ \widehat{\widehat{f}} = \widetilde{f}, \ \overline{f} = \widehat{\overline{\widehat{f}}}. \tag{4.1}$$

Of course these equations hold almost everywhere.

Corollary 4.8. *If f and $\widehat{f}$ are both belong to $L^1(\mathbb{R}^n)$, then f is continuous.*

This is because of the Fourier inversion and the continuity of the Fourier transform.

We can also deduce the so-called "reversed convolution formula" from Fourier inversion.

Corollary 4.9 (Reversed Convolution)**.** *If $f, g, fg, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$, then*

$$\widehat{fg} = \widehat{f} * \widehat{g} \quad \textit{almost everywhere.}$$

4.2 Fourier transform in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$

Notations. Let $\alpha \in \mathbb{N}^n$, that is, $\alpha = (\alpha_1, \cdots, \alpha_n), \alpha_i \in \mathbb{N}, i = 1, \cdots n$ and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $x \in \mathbb{R}^n$, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Define the partial differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

This is to take α_i-th order partial derivatives on x_i respectively.

Definition 4.10 (Rapidly Decreasing Function; Schwartz Space)**.** *A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be rapidly decreasing if it is infinitely differentiable and for all $\alpha, \beta \in \mathbb{N}^n$,*

$$\sup |x^\alpha D^\beta f(x)| < \infty.$$

The space $\mathcal{S}(\mathbb{R}^n)$ of all such functions is called the Schwartz space on $\mathbb{R}^n$.

Some observations come as follows. Firstly, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is closed under addition and scalar multiplication, which makes $\mathcal{S}(\mathbb{R}^n)$ into a vector space. Secondly, if f is a rapidly decreasing function, then so is $x^\alpha D^\beta f$ for every given α and β. Thirdly, if f and g are rapidly decreasing functions, then the multiplication fg is also a rapidly decreasing function.

We can directly check that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ by definition, so we can perform the Fourier transform directly on the Schwartz space. Since rapidly decreasing functions are smooth, hence continuous, then for $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier inversion of $\widehat{f}$ is precisely f.

The less direct conclusion is that the convolution of two rapidly decreasing functions is also rapidly decreasing. We just give this proposition and do not prove it here, but we will give some explanations later.

Proposition 4.11. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$.*

We list two properties of Fourier transforms of derivatives.

Properties 4.12. *For $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$$
$$(-2\pi i x_j f)\widehat{\ }(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi)$$

Proposition 4.13 (Parseval's Identity)**.** *For $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have the identity*

$$\int f\overline{g} = \int \widehat{f}\widehat{\overline{g}}$$

Proof.

$$\begin{aligned}
\int f\overline{g} &= \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx \\
&= \int_{\mathbb{R}^n} \widehat{\widehat{f}}(-x)\overline{g}(x)dx \quad \text{(Fourier Inversion)} \\
&= \int_{\mathbb{R}^n} \widehat{\widehat{f}}(x)\overline{g}(-x)dx \quad \text{(Reflection invariance of Lebesgue integral(} \\
&= \int_{\mathbb{R}^n} \widehat{f}(x)\widetilde{\overline{g}}(x)dx \quad \text{(Multiplication formula)} \\
&= \int_{\mathbb{R}^n} \widehat{f}(x)\widehat{\overline{g}}(x)dx
\end{aligned}$$

The last equation is because

$$\begin{aligned}
\widehat{\overline{g}}(x) = \int_{\mathbb{R}^n} \overline{g}(-x) &= \int_{\mathbb{R}^n} \overline{g(y)}e^{-2\pi i y\cdot(-x)}dx \\
&= \int_{\mathbb{R}^n} g(y)e^{-2\pi i y\cdot x}dx \\
&= \widehat{g}(x)
\end{aligned}$$

∎

In particular, when taking $g = f$ we obtain a conclusion that the Fourier transform of rapidly decreasing functions preserves L^2-norms.

Corollary 4.14 (Plancherel's Identity for $\mathcal{S}(\mathbb{R}^n)$)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$, $\|\widehat{f}\|_2 = \|f\|_2$.*

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is somewhat a nice space because the Fourier transform performed on $\mathcal{S}(\mathbb{R}^n)$ is a linear bijection. We now discuss on this by the following several propositions.

Proposition 4.15. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in L^1(\mathbb{R}^n)$, so f is bounded as in Properties 4.2. Thus $\widehat{D^\alpha x^\beta f}$ is bounded because $D^\alpha x^\beta f \in \mathcal{S}(\mathbb{R}^n)$ for any given α and β. Using Properties 4.12 repeatedly,

$$\widehat{D^\alpha x^\beta} f(\xi) = (2\pi i)^{|\alpha|}(-2\pi i)^{-|\beta|}\xi^\alpha D^\beta \widehat{f}(\xi).$$

If we let $1/C = (2\pi i)^{|\alpha|}(-2\pi i)^{-|\beta|}$ then

$$\xi^\alpha D^\beta \widehat{f}(\xi) = C\widehat{D^\alpha x^\beta} f(\xi)$$

which is bounded as argued above. This means $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$. ∎

Lemma 4.16. *The Fourier transform is a surjection from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $\widetilde{f}(x) = f(-x)$. Then $\widehat{\widetilde{f}} \in \mathcal{S}(\mathbb{R}^n)$ by Proposition 4.15. The equivalent expressions of Fourier inversion implies that

$$\widehat{\widehat{\widetilde{f}}}(x) = \widetilde{f}(-x) = f(x).$$

Hence we find that $\widehat{\widetilde{f}}$ is a pre-image of f under the Fourier transform. Since f is any rapidly decreasing function, we conclude that the Fourier transformation is a surjection from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. ∎

The Fourier inversion formula implies directly that the Fourier transform is an injection from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, since if $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{f} = \widehat{g}$, then the Fourier inversion implies $f = g$. Combining this with Lemma 4.16, we can infer that the Fourier transform is bijective and hence an isomorphism on $\mathcal{S}(\mathbb{R}^n)$.

Corollary 4.17. *The Fourier transform $\mathcal{F}$ is a linear isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.*

Back to Proposition 4.11, the quickest way to see this is using the convolution formula mentioned in Properties 4.2. The functions $\widehat{f}$ and $\widehat{g}$ are rapidly decreasing and so is $\widehat{f}\widehat{g} = \widehat{f * g}$. Apply Fourier inversion and Corollary 4.17 we can conclude that $f * g \in \mathcal{S}(\mathbb{R}^n)$.

4.3 Fourier transform in $L^2(\mathbb{R}^n)$

Corollary 4.14 inspires us to think whether we can perform Fourier transforms in the space $L^2(\mathbb{R}^n)$.

Let's recall some concepts from functional analysis (for example, from Muscat[6]).

Strictly speaking, $L^2(\mathbb{R}^n)$ is a normed vector space consists of square integrable functions, modulo an equivalence relation defined as follows: two functions are equivalent if the L^2-norm of their difference is zero. That is to say,

$$L^2(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty \right\} \Big/ \sim .$$

Define L^2-norm to be $\|f\|_2 = \int_{\mathbb{R}^n} |f(x)|^2 dx$. The equivalence relation $\sim$ is defined by

$$f \sim g \iff \|f - g\|_2 = 0.$$

This equivalence relation is equivalent to the statement that $f \sim g$ if and only if $f = g$ almost everywhere.

A Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a (real or complex) inner product space which is complete with respect to the norm induced by the inner product $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

$L^2(\mathbb{R}^n)$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. The norm defined from the inner product coincide with the L^2-norm.

Lemma 4.18. *$\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$ with L^p-norm for $1 \leq p < \infty$.*

This is because $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ using the standard approximate identity.

Because of this we can give a unique isometric linear extension from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Theorem 4.19. *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometric isomorphism on $\mathcal{S}(\mathbb{R}^n)$, and can be uniquely extended from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, and the Fourier transform in $L^2(\mathbb{R}^n)$ is isometric.*

The unique extension means that the extension is unique in $L^2(\mathbb{R}^n)$. In other words, it means that if two functions in $L^2(\mathbb{R}^n)$ are the Fourier transform of the same function, then they coincide almost everywhere.

Proof. Isometric extension: Let $f \in L^2(\mathbb{R}^n)$. By Lemma 4.18 we can choose a sequence $\{g_k\}_{k=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$g_k \to f \text{ as } k \to \infty.$$

Apply $\mathcal{F}$ to g_k and denoted by $\mathcal{F}(f)$ the obtained limit of $\mathcal{F}(g_k)$. The limit exists because we can check that $\{\mathcal{F}(g_k)\}_{k=1}^\infty$ is a Cauchy sequence. Then

$$\begin{aligned}\|\mathcal{F}(f)\|_2 &= \|\mathcal{F}(f) - \mathcal{F}(g_k)\|_2 + \|\mathcal{F}(g_k)\|_2 \\ &= \|\mathcal{F}(f) - \mathcal{F}(g_k)\|_2 + \|g_k\|_2\end{aligned}$$

Let $k \to \infty$, and note that $|\|g_k\|_2 - \|f\|_2| \leq \|g_k - f\|_2 \to 0$ as $k \to \infty$. We have $\|g_k\|_2 \to \|f\|_2$ as $k \to \infty$. Hence $\|\mathcal{F}(f)\|_2 = \|f\|_2$, equivalently, the operator norm of $\mathcal{F}$, $\|\mathcal{F}\| = 1$.

Uniqueness: If $\mathcal{F}(g_k) \to h$ in L^2 for another $h \in L^2(\mathbb{R}^n)$, then

$$\|\mathcal{F}(f) - h\|_2 \leq \|\mathcal{F}(f) - \mathcal{F}(g_k)\|_2 + \|\mathcal{F}(g_k) - \mathcal{F}(h)\|_2 \to 0 \text{ as } k \to \infty,$$

proving that $\mathcal{F}(f) = h$ almost everywhere. ■

Because of Theorem 4.19, the translation and dilation formulas in Properties 4.2 continue to hold for L^2 functions. The identity $\|\mathcal{F}(f)\|_2 = \|f\|_2$ for $L^2(\mathbb{R}^n)$ is known as the Plancherel's identity for L^2 space, analogous to the Proposition 4.14 for rapidly decreasing functions. With a further argument can show that the Parseval's identity continue to hold in L^2 space.

Theorem 4.20 (Parseval's Identity for $L^2(\mathbb{R}^n)$)**.** *For $f, g \in \mathbb{R}^n$, we have the identity*

$$\langle \mathcal{F}(f), \mathcal{F}(g) \rangle = \langle f, g \rangle.$$

Proof. The key is to express the inner product of two elements using the norm.

For $L^2(\mathbb{R}^n)$ as a real inner product space we have

$$\langle f, g \rangle = \frac{\|f+g\|^2 - \|f-g\|^2}{4} \tag{4.2}$$

for all $f, g \in L^2(\mathbb{R}^n)$.

For $L^2(\mathbb{R}^n)$ as a complex inner product space we have

$$\langle f, g \rangle = \frac{\|f+g\|^2 - \|f-g\|^2 + \|f+ig\|^2 i - \|f-ig\|^2 i}{4} \tag{4.3}$$

for all $f, g \in L^2(\mathbb{R}^n)$.

With the identities (4.2) and (4.3) the claim in this theorem can be immediately verfied by Theorem 4.19. ■

Corollary 4.17 and Theorem 4.19 motivate us to continue thinking whether the Fourier transform extended from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ is also an isometric isomorphism on $L^2(\mathbb{R}^n)$. Fortunately, this is true as we expect.

Recall that the adjoint operator of a linear operator $T : V \to W$ is the operator $T^* : W \to V$ such that $\langle Tv, w \rangle = \langle v, T^* w \rangle$ for every $v \in V$ and $w \in W$. The Riesz representation theorem for Hilbert spaces guarantees that such a vector $T^* w$ is unique so that T^* is well defined. T^* is indeed linear which can be checked directly by definition.

A unitary operator U on a Hilbert space H is a bounded linear operator $U : H \to H$ satisfying $U^*U = UU^* = I$, where I is the identity operator.

We present a lemma in functional analysis without proof which states the equivalent definition of unitary operators.

Lemma 4.21. *A bounded linear operator on a Hilbert space $U : H \to H$ is a unitary operator if and only if*

•*the range of U is dense in H and*

•*U preserves the inner product of H, that is, for all vectors x and y in H we have:*

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

With these preparations, we can now conclude that the Fourier transform on $L^2(\mathbb{R}^n)$ is indeed a unitary operator, and hence is an isometric isomorphism.

Theorem 4.22. *The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator.*

Proof. We have seen that $L^2(\mathbb{R}^n)$ is a Hilbert space with its inner product defined before. By Lemma 4.21 we need to verify that the range of $\mathcal{F}$ is dense in $L^2(\mathbb{R}^n)$ and $\mathcal{F}$ preserves the inner product. But this already hold because of Lemma 4.16, Lemma 4.18, and the Parseval's identity for $L^2(\mathbb{R}^n)$ (Theorem 4.20). ■

4.4 Fourier transform of (finite) measures

Definition 4.23 (Fourier Transform of Finite Borel measure). *Given a finite Borel measure μ on $\mathbb{R}^n$, the Fourier transform of μ on $\mathbb{R}^n$ is defined by*

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x),\ \xi \in \mathbb{R}^n.$$

When $\mu \in \mathbb{R}^n$, μ is bounded Lipschitz function, that is, $\|\widehat{\mu}\|_\infty \leq \mu(\mathbb{R}^n)$ and there is an $R > 0$ such that $|\widehat{\mu}(x) - \widehat{\mu}(y)| \leq R \cdot \mu(\mathbb{R}^n)|x - y|$ for all $x, y \in \mathbb{R}^n$, if supp$\mu \subset B(0, R)$. However $\widehat{\mu}$ need not be in $L^p(\mathbb{R}^n)$ for any $p < \infty$.

The Fourier transform of finite measures generalizes the Fourier transform of functions in $L^1(\mathbb{R}^n)$, where we can identify $d\mu = f_\mu d\mathcal{L}^n$ according to the Radon-Nikodym theorem.

The following product and convolution properties of Fourier transform of measures are analogous to those of functions, and can be checked using Fubini's theorem.

Properties 4.24. *For $f \in L^1(\mathbb{R}^n), \mu, \nu \in \mathcal{M}(\mathbb{R}^n)$,*

(i) Multiplication formulas

$$\int \widehat{\mu} f = \int \widehat{f} d\mu,$$
$$\int \widehat{\mu} d\nu = \int \widehat{\nu} d\mu.$$

(ii) Convolution formulas

$$\widehat{f * \mu} = \widehat{f}\widehat{\mu},$$
$$\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}.$$

We can appproximate measures with smooth compactly supported functions using convolutions. Let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity in $C_0^\infty(\mathbb{R}^n)$ such that $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon), \psi \geq 0$, supp$\psi \subset B(0, 1)$ and $\int \psi = 1$. Then $\widehat{\psi_\varepsilon}(\xi) = \widehat{\psi}(\varepsilon\xi) \to \psi(0) = \int \psi = 1$ as $\varepsilon \to 0$. For a finite Borel measure μ, setting $\mu_\varepsilon = \psi_\varepsilon * \mu$, by Theorem 1.15 we have μ_ε converges to μ weakly as $\varepsilon \to 0$ and $\widehat{\mu_\varepsilon} = \widehat{\psi_\varepsilon}\widehat{\mu} \to \widehat{\mu}$ uniformly.

Since $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, using Fourier inversion formula we can draw a conclusion that two compactly supported measure coincide if and only if their Fourier transformation are equal.

Proposition 4.25. *For $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$, $\widehat{\mu} = \widehat{\nu}$ if and only if $\mu = \nu$.*

Proof. If $\mu = \nu$, then $\widehat{\mu} = \widehat{\nu}$ immediately by taking Fourier transforms on both sides.

Conversely if $\widehat{\mu} = \widehat{\nu}$, then by the above discussion, $\widehat{\mu_\varepsilon} \to \widehat{\mu}$ and $\widehat{\nu_\varepsilon} \to \widehat{\nu}$ uniformly and the right-hand sides are equal. Since by definition $\mu_\varepsilon, \nu_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$, the Fourier inversions performed on their Fourier transforms bring themselves back, so

$$\mu_\varepsilon(x) = \int_{\mathbb{R}^n} \widehat{\mu_\varepsilon}(\xi) e^{2\pi i x\cdot\xi} d\xi \longrightarrow \int_{\mathbb{R}^n} \widehat{\mu}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$
$$\text{and } \nu_\varepsilon(x) = \int_{\mathbb{R}^n} \widehat{\nu_\varepsilon}(\xi) e^{2\pi i x\cdot\xi} d\xi \longrightarrow \int_{\mathbb{R}^n} \widehat{\nu}(\xi) e^{2\pi i x\cdot\xi} d\xi,\ \text{as } \varepsilon \to 0.$$

Here we use the uniform convergence of $\{\mu_\varepsilon\}$ and $\{\nu_\varepsilon\}$. Thus the two limits are equal.

Meanwhile, $\int \varphi\mu_\varepsilon d\mathcal{L}^n \to \int \varphi d\mu$ and $\int \varphi\nu_\varepsilon d\mathcal{L}^n \to \int \varphi d\nu$ as $\varepsilon \to 0$ for every $\varphi \in C_0(\mathbb{R}^n)$, by Theorem 1.15. Also,

$$\int \varphi(x)\mu_\varepsilon(x)dx \longrightarrow \int \varphi(x)\left(\int \widehat{\mu}(\xi)e^{2\pi i x\xi}d\xi\right)dx$$

$$\text{and} \int \varphi(x)\nu_\varepsilon(x)dx \longrightarrow \int \varphi(x)\left(\int \widehat{\nu}(\xi)e^{2\pi i x\xi}d\xi\right)dx \text{ as } \varepsilon \to 0.$$

The above two limits are obtained by the Lebesgue's dominated convergence theorem, with dominating functions $|\mu_\varepsilon|\|\varphi\|_\infty$ and $|\nu_\varepsilon|\|\varphi\|_\infty$, respectively. We can see that right-hand sides in the above formulas are equal. By the uniqueness of limits,

$$\int \varphi d\mu = \int \varphi(x)\left(\int \widehat{\mu}(\xi)e^{2\pi i x\xi}d\xi\right)dx = \int \varphi(x)\left(\int \widehat{\nu}(\xi)e^{2\pi i x\xi}d\xi\right)dx = \int \varphi d\nu$$

for every $\varphi \in C_0(\mathbb{R}^n)$. Thus $\mu = \nu$ because we can approximate the characteristic function on any set pointwise by continuous functions.

■

Properties 4.26. *For $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$,*

$$\text{(Reversed convolution) } \widehat{f\mu} = \widehat{f} * \widehat{\mu}$$
$$\int \overline{f}d\mu = \int \overline{\widehat{f}}\widehat{\mu}$$
$$\int \widehat{f}\overline{\widehat{g}}d\mu = \int f(\widehat{\mu} * \overline{g}).$$

If we have known the behavior of the Fourier transform of finite Borel measures, we can also infer back the property of the measure themselves. Recall the Radon-Nikodym theorem. If μ is a finite, absolutely continuous measure (with respect to the Lebesgue measures $\mathcal{L}^n$), then there is a function $f_\mu \in L^1(\mathbb{R}^n)$ unique up to a set of measure zero, called a Radon-Nikodym derivative, such that $d\mu = f_\mu d\mathcal{L}^n$ and we may identify μ with f_μ in the following contexts.

Theorem 4.27. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $\widehat{\mu} \in L^2(\mathbb{R}^n)$, then $f_\mu \in L^2(\mathbb{R}^n)$.*

Proof. By Theorem 4.22, when we identify $\widehat{\mu}$ with a function in $L^2(\mathbb{R}^n)$, there is indeed a function $f \in L^2(\mathbb{R}^n)$ such that $\widehat{\mu} = \widehat{f}$. Define $\mu_\varepsilon = \psi_\varepsilon * \mu$ and $f_\varepsilon = \psi_\varepsilon * f$. Then by the convolution formula we have

$$\widehat{\mu_\varepsilon} = \widehat{\psi_\varepsilon}\widehat{\mu} = \widehat{\psi_\varepsilon}\widehat{f} = \widehat{f_\varepsilon},$$

so $\mu_\varepsilon = f_\varepsilon$ as rapidly decreasing functions. As $\mu_\varepsilon \to \mu$ weakly and $f_\varepsilon \to f$ in L^2, we have $f_\mu = f$ almost everywhere. ■

Theorem 4.28. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $\widehat{\mu} \in L^1(\mathbb{R}^n)$, then f_μ is almost a continuous function, which means that f_μ is equal to a continuous function almost everywhere.*

Proof. Let μ_ε be as in the previous proof. Then $\mu_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$. By the Fourier inversion formula and the Lebesgue's dominated convergence theorem,

$$\mu_\varepsilon(x) = \int \widehat{\mu}_\varepsilon(\xi) e^{2\pi i \xi \cdot x} d\xi = \int \widehat{\psi}(\varepsilon\xi)\widehat{\mu}(\xi) e^{2\pi i \xi \cdot x} d\xi$$
$$\longrightarrow \int \widehat{\mu}(\xi) e^{2\pi i \xi \cdot x} d\xi, \text{ as } \varepsilon \to 0.$$

Denote by $g(x)$ the limit integral. g is continuous since $\widehat{\mu} \in L^1$. On the other hand, μ_ε converges weakly to μ, so $f_\mu = g$ almost everywhere according to the Radon-Nikodym theorem. ■

4.5 Fourier transform of tempered distributions

Definition 4.29 (Tempered Distribution)**.** *A tempered distribution is a continuous linear functional on the Schwartz space:* $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$.

Definition 4.30 (Fourier Transform of a Tempered Distribution)**.** *Let* $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ *be a tempered distribution. The Fourier transform of* T *is the tempered distribution* $\widehat{T}$ *satisfying*

$$\widehat{T}(\varphi) = T(\widehat{\varphi}) \quad \textit{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Generally, every locally integrable function f with an additional condition: $|f(x)| \lesssim |x|^m$ when $|x| > 1$ for some fixed m, can be regarded as a tempered distribution and we can define its distributional Fourier transform. Such a function is called a tempered function (in the sense of Wolff[8]).

Definition 4.31 (Tempered Function)**.** *A tempered function* $f : \mathbb{R}^n \to \mathbb{C}$ *is a locally integrable function such that*

$$\int_{\mathbb{R}^n} (1 + |x|)^{-m} |f(x)| dx < \infty$$

for some constant $m \geq 0$.

Definition 4.32 (Fourier Transform of a Tempered Function)**.** *Let* f *be a tempered function. The Fourier transform of this tempered function* f *is another tempered function* $\widehat{f}$*, such that the following holds:*

$$\textit{(multiplication formula)} \quad \int \widehat{f}\varphi = \int f\widehat{\varphi}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For a tempered function f, consider the operator $T_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$T_f(\varphi) = \int f\varphi.$$

Then T_f is indeed a tempered distribution and we call T_f the tempered distribution induced by f.

Lemma 4.33. *If* f_1 *and* f_2 *are two tempered functions on* $\mathbb{R}^n$ *inducing the same tempered distributions, i.e.,* $T_{f_1} = T_{f_2}$*, then* $f_1 = f_2$ *almost everywhere.*

We just sketch the idea of the proof. First consider the case when $\varphi \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R})$. The Riesz representation theorem for measures and the Radon-Nikodym theorem imply that there is a unique f up to a set of measure zero, such that

$$T_f(\varphi) = \int f\varphi.$$

Then we conclude the theorem using the density of $C_0^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R})$ with the Lebesgue's dominated convergence theorem.

The Fourier transform of a tempered function is the Fourier transform of a tempered distribution if we identify $\widehat{T_f}$ with $T_{\widehat{f}}$:

$$T_{\widehat{f}}(\varphi) = \int \widehat{f}\varphi = \widehat{T_f}(\varphi) = \int f\widehat{\varphi} = T_f(\widehat{\varphi}).$$

From the above definitions, we see that functions in L^1, $\mathcal{S}$ and L^2 are tempered functions. An observation gives us that $L^1 + L^2$ defined by

$$L^1 + L^2 = \{f_1 + f_2 : f_1 \in L^1, f_2 \in L^2\},$$

is also a tempered functions, so that their original Fourier transforms are compatiable with their distributional Fourier transforms.

4.6 Fourier transform of radial functions

Definition 4.34 (Radial Function). *A radial function f on $\mathbb{R}^n$ is a function whose value at each point depends only on the length of the variable, that is, $f(x) = \psi(|x|), x \in \mathbb{R}^n$ for some $\psi : [0, \infty) \to \mathbb{C}$.*

A simple observation yields the following proposition.

Proposition 4.35. *A function f on $\mathbb{R}^n$ is a radial function if and only if it preserves rotations. That is to say, $f \circ \rho = f$ for all $\rho \in \mathrm{SO}(n)$, the special orthogonal group on $\mathbb{R}^n$.*

An useful conclusion is that the Fourier transform of radial functions can be expressed expressed explicitly by Bessel functions.

Theorem 4.36 (Fourier Transform for Radial Functions). *Let $f \in L^1(\mathbb{R}^n)$ be a radial function. The Fourier transform of f is given by*

$$\widehat{f}(x) = c(n)|x|^{-(n-2)/2} \int_0^\infty \psi(s) J_{(n-2)/2}(2\pi|x|s) s^{n/2} ds,$$

where $c(n)$ is some constant depending on n which need not be determined, ψ is the function of a one-dimensional variable defined in the above definition and $J_{(n-2)/2}$ is the Bessel function of order $(n-2)/2$.

From Theorem 4.36 we can see that the Fourier transform of a radial function is also a radial function.

To derive this formula we should first recall the change of variable formula of Lebesgue integration in polar coordinates. If $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f d\mathcal{L}^n = \int_{S^{n-1}} \left(\int_0^\infty f(rx) r^{n-1} dr \right) d\sigma^{n-1}(x) \tag{4.4}$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and σ^{n-1} is the stantard spherical measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

For the second, if we fix a direction $e \in S^{n-1}$ and let $S_\theta = \{x \in S^{n-1} | e \cdot x = \cos\theta\}$ for $0 \leq \theta \leq \pi$, then S_θ is an $(n-2)$-dimensional sphere of radius $\sin\theta$ (which is a circle when $n = 3$). If we denote by $\sigma^{n-2}_{\sin\theta}$ the surface measure (area) of S_θ we have

$$\sigma^{n-2}_{\sin\theta} = \sigma^{n-2}(S^{n-2})(\sin\theta)^{n-2}. \tag{4.5}$$

Then for $g \in L^1(S^{n-1})$,

$$\int_{S^{n-1}} g d\sigma^{n-1} = \int_0^\pi \left(\int_{S_\theta} g(x) d\sigma^{n-2}_{\sin\theta}(x) \right) d\theta. \tag{4.6}$$

After these preparations, we begin our proof of this formula.

Proof. of Theorem 4.36. By the change of variable formula and Fubini's theorem,

$$\widehat{f}(re) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i r e \cdot y} dy = \int_0^\infty \psi(s) s^{n-1} \left(\int_{S^{n-1}} e^{-2\pi i r s e \cdot x} d\sigma^{n-1}(x) \right) ds. \tag{4.7}$$

Apply the equation (4.6) to the inside integral of the right-hand side,

$$\begin{aligned} \int_{S^{n-1}} e^{-2\pi i r s e \cdot x} d\sigma^{n-1}(x) &= \int_0^\pi e^{-2\pi i r s \cos\theta} \sigma^{n-2}_{\sin\theta} (S_\theta) \, d\theta \\ &= \sigma^{n-2}(S^{n-2}) \int_0^\pi e^{-2\pi i r s \cos\theta} (\sin\theta)^{n-2} d\theta \end{aligned} \tag{4.8}$$

where the first equality comes from (4.5)

Changing variable by letting $\cos\theta = -t$ and introducing Bessel functions

$$J_m(u) = \frac{\left(\frac{u}{2}\right)^m}{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{iut} \left(1 - t^2\right)^{m - 1/2} dt,$$

we obtain

$$\begin{aligned} \int_{S^{n-1}} e^{-2\pi i r s e \cdot x} d\sigma^{n-1}(x) &= \sigma^{n-2}(S^{n-2}) \int_{-1}^1 e^{2\pi i r s t} (1 - t^2)^{(n-3)/2} dt \\ &= c(n) (rs)^{-(n-2)/2} J_{(n-2)/2}(2\pi r s) \end{aligned}$$

Substitute the above formula in (4.8) and we obtain the needed result. ■

The following property of radial functions in $L^2(\mathbb{R}^n)$ will be used in the next section.

Proposition 4.37. *Let $f \in L^2(\mathbb{R}^n)$ be a radial function. Then its Fourier transform $\mathcal{F}(f)$ is also a radial function, in the sense that $\mathcal{F}(f)$ is a radial outside a null set.*

Proof. The idea is to construct a sequence of radial an rapidly decreasing functions approximating f for $f \in L^2(\mathbb{R}^n)$.

Let $\{\psi_t\}_{t>0}$ be the standard approximate identity, which, as we observe, is radial. Then $f_t = f * \psi_t \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is radial for any $t > 0$, because it is easy to check that the convolution of two radial functions is again a radial function. Thus $\widehat{f_t}$ is a radial function by the formula given in Theorem 4.36. By Theorem 1.13 f_t converges to f in L^2 as t goes to 0. We know from Theorem 4.19 that $\mathcal{F}(f_t)$ converges to $\mathcal{F}(f)$ in $L^2(\mathbb{R}^n)$. Then there is a subsequence $\{\mathcal{F}(f_{t_k})\}_{k=1}^{\infty}$ of $\mathcal{F}(f_t)$ converging to $\mathcal{F}(f)$ almost everywhere as $k \to \infty$.

Since $\{\mathcal{F}(f_{t_k})\}_{k=1}^{\infty}$ is a family of radial functions and converges to $\mathcal{F}(f)$ outside a null set, then $\mathcal{F}(f)$ is radial outside a null set. ■

4.7 Fourier transform of Riesz kernels

Recall that Theorem 3.4 tells us that the s-dimensional Hausdorff measure of $A \subset \mathbb{R}^n$ is nonzero if and only if there is a finite Borel measure $\mu \in \mathcal{M}(A)$ such that its s-dimensional energy integral is finite. The s-dimensional energy integral of a measure μ can be written as $I_s(\mu) = \int (k_s * \mu)(x) d\mu(x)$, where k_s is the Riesz kernel $k_s(x) = |x|^{-s}$. It is easy to veryfy that the Riesz kernel k_s is a tempered function. We now compute the distributional Fourier transform of the Riesz kernel.

Theorem 4.38 (Fourier transform of Riesz kernel)**.** *Define the Riesz kernel by $k_s(x) = |x|^{-s}$, $s > 0$. For s such that $0 < s < n$, then there is a positive and finite constant $\gamma(n, s)$ such that*

$$\int k_s \widehat{\varphi} = \gamma(n, s) \int k_{n-s} \varphi \quad \text{for } \varphi \in \mathcal{S}\left(\mathbb{R}^n\right). \tag{4.9}$$

That is to say, the Fourier transform of the Riesz kernel k_s (as a tempered function), is $\widehat{k_s} = \gamma(n, s) k_{n-s}$.

Moreover, the constant $\gamma(n, s)$ can be computed explicitly:

$$\gamma(n, s) = \pi^{s-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \tag{4.10}$$

We list a lemma ahead of the proof of Theorem 4.38 which will be used in the proof.

Lemma 4.39. *Suppose that g is a tempered even function on $\mathbb{R}^n$ such that its distributional Fourier transform f is a tempered function. Then*

$$\widehat{f} = g.$$

Proof. Using the product formula and equivalent expressions of Fourier inversion, we have for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} T_{\widehat{f}}(\varphi) = T_f(\widehat{\varphi}) &= \int f \widehat{\varphi} = \int \widehat{g} \widehat{\varphi} = \int g \widehat{\widehat{\varphi}} \\ &= \int g(x) \varphi(-x) dx = \int g(-x) \varphi(x) dx = \int g \varphi, \end{aligned}$$

from which the lemma follows. ■

Proof. of Theorem 4.38 We split the proof in three cases according to s.

Case I. $\frac{n}{2} < s < n$. We observe that $k_s \in L^1 + L^2$ because we can decompose k_s into $k_s = f_1 + f_2$ with $f_1 \in L^1$ and f_2 in L^2, where $f_1 = k_s \chi_{B(0,1)}$ and $f_2 = k_s \chi_{\mathbb{R}^n \setminus B(0,1)}$. Since k_s is radial and $k_s(rx) = r^{-s} k_s(x)$ for $r > 0$, then $\widehat{k_s}$ is also radial by Theorem 4.36, with

$\widehat{k_s}(rx) = r^{s-n}\widehat{k_s}(x)$ by the dilation formula in Properties 4.2. We see that $\widehat{k_s}(x)$ is of the form $\gamma(n,s)k_{n-s}$. To be rigorous, since $\widehat{k_s}(rx) = r^{s-n}\widehat{k_s}(x)$ and $\widehat{k_s}(x) = h(|x|)$ for some $h : [0,\infty) \to \mathbb{C}$, then for every $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$\begin{aligned} \widehat{k_s}\left(\frac{x}{|x|}\right) &= h(1), \text{ and} \\ \widehat{k_s}\left(\frac{x}{|x|}\right) &= \widehat{k_s}(x)\frac{1}{|x|^{s-n}} \end{aligned}.$$

Thus $\widehat{k_s}(x)/|x|^{s-n} = h(1)$ and $k_s(x) = h(1)|x|^{s-n}$. The case $x = 0$ automatically suits the equation. The constant $h(1)$ is denoted as $\gamma(n,s)$ depending on $n \in \mathbb{N}^*$ and $s > 0$, and $|x|^{s-n} = k_{n-s}(x)$. Combining all obtained above, we have

$$\widehat{k_s}(x) = \gamma(n,s)k_{n-s}(x).$$

The $\widehat{k_s}$ is indeed the distributional Fourier transform of k_s because $L^1 + L^2$ is clearly a subset of the collection of tempered functions.

Case II. $0 < s < \frac{n}{2}$. We should show that $\widehat{k_s}(x) = \gamma(n,s)k_{n-s}(x)$ is also the distributional Fourier transform of k_s. By Lemma 4.39 and the discussion in Case I, the Fourier form of $k_s = \gamma(n,n-s)^{-1}\widehat{k_{n-s}}$ (as $n/2 < n-s < n$) is

$$\widehat{k_s} = \gamma(n,n-s)^{-1}k_{n-s}$$

.

Case III. $s = \frac{n}{2}$. This requires harder and more technical work work. We use a limiting argument.

If $\widehat{k_{n/2}}$ is the distributional Fourier transform of $k_{n/2}$, then

$$\int \widehat{k_{n/2}}\varphi = \int k_{n/2}\widehat{\varphi} = \int \lim_{s\to n/2} k_s\varphi = \lim_{s\to n/2}\int k_s\widehat{\varphi} = \lim_{s\to n/2}\gamma(n,s)\int k_{n-s}\varphi = \int k_{n/2}\varphi.$$

The interchange of the limit in the third equation is because of the Lebesgue's domimated convergence theorem. Now we should explain why $\lim_{s\to n/2}\gamma(n,s) = 1$.

Take $\Psi(x) = e^{-\pi|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ to be the Gauss function, so that $\widehat{\Psi} = \Psi$. For $n/2 < s < n$, we have by the validity of (4.9),

$$\int k_s\Psi = \int k_s\widehat{\Psi} = \gamma(n,s)\int k_{n-s}\Psi,$$

that is,

$$\int |x|^{-s}e^{-\pi|x|^2}dx = \gamma(n,s)\int |x|^{s-n}e^{-\pi|x|^2}dx. \tag{4.11}$$

For $0 < s < n/2$, we replace $\gamma(n,s)$ by $\gamma(n,n-s)^{-1}$ so that (4.11) becomes

$$\int |x|^{-s}e^{-\pi|x|^2}dx = \gamma(n,n-s)^{-1}\int |x|^{s-n}e^{-\pi|x|^2}dx.$$

Taking the limit as $s \to n/2$ on both sides with Lebesgue's domimated convergence theorem gives that $\lim_{s\to n/2^+}\gamma(n,s) = \lim_{s\to n/2^-}\gamma(n,n-s) = 1$. Recall the definition of the Gamma function. Computing the integrals on both sides of (4.11) we find that the formula for $\gamma(n,s)$ is precisely (4.10). Moreover by (4.10), $\gamma(n,s)=\gamma(n,n-s)^{-1}$ so that the constants for Case I, II, and III can be unified. ■

4.8 Expression of energy integrals by Fourier transforms

The following theorem shows that energy integrals of μ can be precisely expressed by the corresponding Fourier transformations. This builds a bridge between the Hausdorff dimension and Fourier transform.

Theorem 4.40. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0<s<n$. Then*

$$I_s(\mu) = \gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}dx \tag{4.12}$$

Proof. If we still have the validity of Parseval's identity, convolution formulas, then by Theorem 4.38, we have

$$\begin{aligned} I_s(\mu) &= \int k_s * \mu \, d\mu \\ \text{(Parseval's identity)} \quad &= \int \widehat{k_s * \mu}\overline{\widehat{\mu}} \\ \text{(Convolution formulas)} \quad &= \int \widehat{k_s}|\widehat{\mu}|^2 \\ \text{(Theorem 4.38)} \quad &= \gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}dx. \end{aligned}$$

Therefore we should check the validity.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a real-valued function. Changing the variable $z=y-x$ and denoting $\widetilde{\varphi}(x)=\varphi(-x)$ we have

$$\begin{aligned} I_s(\varphi) &= \iint k_s(y-x)\varphi(x)\varphi(y)dxdy \\ &= \iint k_s(z)\varphi(y-z)\varphi(y)dzdy = \int k_s(\tilde{\varphi} * \varphi). \end{aligned}$$

By Corollary 4.9 and equivalent forms of Fourier inversion formula, with Theorem 4.38, we have

$$I_s(\varphi) = \gamma(n,s)\int k_{n-s}|\widehat{\varphi}|^2 = \gamma(n,s)\int |x|^{s-n}|\widehat{\varphi}(x)|^2dx.$$

So we finished proving the theorem for "smooth measures" φ.

For general $\mu \in \mathcal{M}(\mathbb{R}^n)$, we approximate μ_ε with $\mu_\varepsilon = \psi_\varepsilon * \mu$, where $\{\psi_\varepsilon\}_{\varepsilon>0}$ is the standard approximate identity defined before. With $\varphi = \mu_\varepsilon$ applied in the above smooth case, we have

$$
\begin{aligned}
&\iint \left(\iint |x-y|^{-s}\psi_\varepsilon(x-z)\psi_\varepsilon(y-w)dxdy\right) d\mu(z)d\mu(w)\\
=&\iint \left(|x-y|^{-s}\int \psi_\varepsilon(x-z)d\mu(z)\int \psi_\varepsilon(y-w)d\mu(w)\right) dxdy\\
=&I_s(\mu_\varepsilon)=\gamma(n,s)\int |\widehat{\mu}(x)|^2|\widehat{\psi}(\varepsilon x)|^2|x|^{s-n}dx \longrightarrow \gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}dx, \text{ as } \varepsilon\to 0.
\end{aligned}
$$

We perform another change of variables by taking $u=(x-z)/\varepsilon$, and $v=(y-w)/\varepsilon$. Then looking at the inner integral of the first term,

$$
\begin{aligned}
&\iint |x-y|^{-s}\psi_\varepsilon(x-z)\psi_\varepsilon(y-w)dxdy\\
=&\iint |\varepsilon(u-v)+z-w|^{-s}\psi(u)\psi(v)dudv \longrightarrow |z-w|^{-s} \text{ as } \varepsilon\to 0 \text{ and } z\neq w.
\end{aligned}
$$

With the above identity we have the estimate:

$$
\iint |x-y|^{-s}\psi_\varepsilon(x-z)\psi_\varepsilon(y-w)dxdy \lesssim |z-w|^{-s}.
$$

Then when $I_s(\mu)<\infty$, we can conclude the proof by the Lebesgue's dominated convergence theorem.

When $I_s(\mu)=\infty$, we get by Fatou's lemma,

$$
\begin{aligned}
\infty=I_s(\mu)\leq &\liminf_{\varepsilon\to 0}\iint \left(\iint |x-y|^{-s}\psi_\varepsilon(x-z)\psi_\varepsilon(y-w)dxdy\right) d\mu(z)d\mu(w)\\
=&\gamma(n,s)\liminf_{\varepsilon\to 0}\int |\widehat{\mu}(x)|^2|\widehat{\psi}(\varepsilon x)|^2|x|^{s-n}dx=\gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}dx.
\end{aligned}
$$

The proof is complete. ■

5. Projection of Sets

Definition 5.1. *Given a direction $e\in S^{n-1}, n\geq 2$, the projection of a point in $\mathbb{R}^n$ onto this direction $P_e:\mathbb{R}^n\to\mathbb{R}$ is defined by*

$$
P_e(x)=x\cdot e,
$$

where $\cdot$ means the standard dot product in $\mathbb{R}^n$.

In other words, this is the orthogonal projection onto the line $\{te|t\in\mathbb{R}\}$.

A simple observation is that $\dim P_e(A)\leq \dim A$ because the projection map is a Lipschitz map which does not increase dimensions.

Theorem 5.2. *Let $A\subset\mathbb{R}^n$ be a Borel set and $s=\dim A$.*
(i) If $s\leq 1$, then $dim P_e(A)=s$ for σ^{n-1}-almost all $e\in S^{n-1}$;

(ii) If $s > 1$, then $\mathcal{L}^1(P_e(A)) > 0$ for σ^{n-1}-almost all $e \in S^{n-1}$.

Proof. If $\mu \in \mathcal{M}(A)$ and $e \in S^{n-1}$, define the image $\mu_e = (P_e)_\# \mu$ under the projection $P_e : \mathbb{R}^n \to \mathbb{R}$, that is

$$\mu_e(B) = \mu(P_e^{-1}(B)), \quad B \subset \mathbb{R}.$$

Then $\mu_e \in \mathcal{M}(P_e(A))$ and

$$\widehat{\mu_e}(r) = \int_{-\infty}^{\infty} e^{-2\pi i r x} d\mu_e(x) = \int_{\mathbb{R}^n} e^{-2\pi i r (y \cdot e)} d\mu(y) = \widehat{\mu}(re) \text{ for all } r \in \mathbb{R} \tag{5.1}$$

To prove (i), suppose $0 < s = \dim A \leq 1$. For every $0 < t < s$, pick a $\mu \in \mathcal{M}(A)$ such that $I_t(\mu) < \infty$ by Theorem 3.4. Using Theorem 4.40 and (5.1) together with the change of variable formula in polar coordinates (4.4),

$$\begin{aligned}
&\int_{S^{n-1}} I_t(\mu_e) d\sigma^{n-1}(e) \\
\text{((4.12) in Theorem 4.40)} &= \gamma(1,t) \int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)|^2 \, |r|^{t-1} dr \right) d\sigma^{n-1}(e) \\
\text{((5.1) and } \widehat{\mu_e}(r) \text{ is even)} &= 2\gamma(1,t) \int_{S^{n-1}} \left(\int_0^{\infty} |\hat{\mu}(re)|^2 r^{t-1} dr \right) d\sigma^{n-1}(e) \\
\text{((4.4))} &= 2\gamma(1,t) \int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{t-n} dx \\
\text{((4.12))} &= 2\gamma(1,t)\gamma(n,t)^{-1} I_t(\mu) < \infty
\end{aligned}$$

In particular $I_t(\mu_e) < \infty$ for σ^{n-1}-almost all $e \in S^{n-1}$ and $\dim(P_e(A)) \geq t$ for such e by Theorem 3.4. Since $t < s$ is chosen arbitrarily, we conclude that $\dim P_e(A) \geq s$ for σ-almost all $e \in S^{n-1}$.

To prove (ii), suppose $s > 1$. By Theorem 3.4 there is a $\mu \in \mathcal{M}(A)$ such that $I_1(\mu) < \infty$ (because $I_s(\mu) < \infty$ implies $I_t(\mu) > \infty$ for $0 < t < s$).

Using the similar argument as above with $t = 1$,

$$\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)|^2 dr \right) d\sigma^{n-1}(e) = 2\gamma(n,1)^{-1} I_1(\mu) < \infty$$

whence $\mu_e \in L^2(\mathbb{R})$ for σ^{n-1}-almost all $e \in S^{n-1}$.

By Theorem 4.40, μ_e, if regarded as a nonnegative integrable function, is in $L^2(\mathbb{R})$, which means that there exists a nonnegative function $f_{\mu_e} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $d\mu_e = f_{\mu_e} d\mathcal{L}^1$. In particular, μ_e(identified with f_{μ_e}) is absolutely continuous with respect to $\mathcal{L}^1$ for σ^{n-1}-almost all $e \in S^{n-1}$. As $\mu_e \in \mathcal{M}(P_e(A))$, we conclude that $\mathcal{L}^1(P_e(A)) > 0$ for such e.

■

Theorem 5.3. *Let $A \subset \mathbb{R}^n$ be a Borel set and $\dim A > 2$. Then the projection $P_e(A)$ has nonempty interior for σ^{n-1} almost all $e \in S^{n-1}$.*

Proof. Let $2 < s < \dim A$. We can choose a measure $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$, by Theorem 3.4. Define $\mu_e(B) = \mu(P_e^{-1}(B)), B \subset \mathbb{R}$ as in the previous theorem.

Now, we consider the integral

$$\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e)$$

where $\widehat{\mu_e}$ is the Fourier transform of the measure μ_e.

$$\begin{aligned}
&\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e) \\
\leq &2 \int_{S^{n-1}} \left(\int_{1}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e) + 2\mu(\mathbb{R}^n)\sigma^{n-1}(S^{n-1}) \\
\leq &2 \left[\int_{S^{n-1}} \int_{1}^{\infty} |\widehat{\mu}(re)|^2 r^{s-1} dr d\sigma^{n-1}(e) \right]^{1/2} \cdot \left[\int_{S^{n-1}} \int_{1}^{\infty} r^{1-s} dr d\sigma^{n-1}(e) \right]^{1/2} + C(\mu)
\end{aligned} \tag{5.2}$$

The first inequality is because the Fourier transform of a Borel measure is always an even function, and

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)|\, dr &= 2 \int_{1}^{\infty} |\widehat{\mu_e}(r)|\, dr + 2 \int_{0}^{1} |\widehat{\mu_e}(r)|\, dr \\
&\leq 2 \int_{1}^{\infty} |\widehat{\mu_e}(r)|\, dr + 2 \int_{0}^{1} |\widehat{\mu_e}(0)|\, dr \\
&= 2 \int_{1}^{\infty} |\widehat{\mu_e}(r)|\, dr + 2 \int_{0}^{1} \mu\left(\mathbb{R}^n\right) dr \\
&= 2 \int_{1}^{\infty} |\widehat{\mu_e}(r)|\, dr + 2\mu\left(\mathbb{R}^n\right)
\end{aligned}$$

The second equality is because of the Schwartz's inequality:

$$\|\, |\widehat{\mu}(re)|\, \|_{L^1([1,\infty)\times S^{n-1})} \leq \|\, |\widehat{\mu}(re)| r^{s-1}\, \|_{L^2([1,\infty)\times S^{n-1})} \cdot \|r^{1-s}\|_{L^2([1,\infty)\times S^{n-1})}.$$

We continue to estimate the integral:

$$\begin{aligned}
&\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\widehat{\mu_e}(r)| dr \right) d\sigma^{n-1}(e) \\
&\leq 2 \left(\frac{\sigma^{n-1}(S^{n-1})}{s-2} \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{s-n} dx \right)^{\frac{1}{2}} + C(\mu) \\
&= C(n,s) I_s(\mu)^{1/2} + C(\mu) < \infty
\end{aligned}$$

The inequality above is obtained by direct computation of the second term in (5.2) and $r^{s-1} \leq r^{s-n}$ when $n \geq 1$ and $r \geq 1$.

Hence $\widehat{\mu_e} \in L^1(\mathbb{R})$ for σ^{n-1}-almost all $e \in S^{n-1}$ and by Theorem 4.28, μ_e is a continuous function, in the sense that there is a continuous function $g_{\mu_e} \in L^1(\mathbb{R})$ such that $d\mu_e = g_{\mu_e} d\mathcal{L}^1$. As $\mu_e \in \mathcal{M}(P_e(A))$, we conclude that the interior of $P_e(A)$ is nonempty for σ^{n-1}-almost all $e \in S^{n-1}$. ■

6. Dimension of Borel rings

The projection theorem of sets can be applied to the discussion on the dimension of the so-called Borel rings. The goal of this section is to show that if the dimension of a Borel ring on the real line $\mathbb{R}$ is stricly grater than zero, then this Borel ring must be the $\mathbb{R}$ itself.

Recall that an algebraic subring of a ring (with multiplicative identity) $(R, +, \cdot, 0, 1)$ is a subset S of R that preserves the structure of a ring under $+$ and $\cdot$. A Borel ring is a Borel set equipped with a ring structure.

The main theorem is stated as follows.

Theorem 6.1. *Let $E \subset \mathbb{R}$ be a Borel set which is also an algebraic subring of $\mathbb{R}$. Then either E has Hausdorff dimension zero or E is the whole real line $\mathbb{R}$.*

Proof. To prove Theroem 6.1, we just show that if such $E \subset \mathbb{R}$ has dimension strictly larger than zero, then $\dim E = 1$.

We first do some observations. Suppose $\dim E > 0$. From Theorem 3.2 we have $\dim E^k \geq k \dim E$ for any $k \in \mathbb{N}^*$, where E^k is the k-fold Cartesian product of E. Because of this we can choose a sufficiently large k so that $\dim E^k > 2$. Consider the linear functional $\varphi = P_e : \mathbb{R}^k \to \mathbb{R}$ by just choosing the projection operator. Theorem 5.3 shows that $\varphi(E^k)$ has nonempty interior, and since the image of the linear functional $\varphi(E^k)$ is a subspace of $\mathbb{R}$, $\varphi(E^k) = \mathbb{R}$.

The following two lemmas imply that continuing the above discussion, k can only be 1 and $E = \mathbb{R}$, which conclude the proof of the theorem.

■

The first lemma is a purely linear algebraic proposition.

Lemma 6.2. *Let $E \subset \mathbb{R}$ be a subring. Assume that there is a $k \in \mathbb{N}^*$ and a linear functional $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that $\varphi(E^k) = \mathbb{R}$, then such a k can be chosen so that φ maps E^k bijectively onto $\mathbb{R}$.*

Proof. The above discussion guarantees the existence of integers k satisfying $\varphi(E^k) = \mathbb{R}$. Then we can find a smallest k such that $\varphi(E^k) = \mathbb{R}$ holds. We still denote the smallest integer by k. Now φ is already surjective on E^k. We claim that the restriction of φ to E^k, $\varphi|_{E^k} : E^k \to \mathbb{R}$ is injective. Let $\{e_1, \cdots, e_k\}$ be the standard basis of $\mathbb{R}^k$. Denote by r_j the image of each basis vector, $r_j = \varphi(e_j)$. Now $\varphi(E^k) = \mathbb{R}$ implies that the span of the vectors $\{r_j; 1 \leq j \leq k\}$ with coefficients in E is the real line $\mathbb{R}$, that is,

$$\left\{ \sum_{j=1}^{k} a_j r_j : a_j \in E, j = 1, \cdots k \right\} = \mathbb{R}. \tag{6.1}$$

Suppose on the contrary that $\varphi|_{E^k}$ is not injective. Then there are $b_1, \cdots b_k \in E$ not all zero, such that $\sum_{j=1}^{k} b_j r_j = 0$. We may assume that $b_k \neq 0$ so we can express r_k by

$$r_k = \sum_{j=1}^{k-1} \frac{-b_j}{b_k} r_j.$$

Let $s \in \mathbb{R}$ and then $s/b_k \in \mathbb{R}$. By (6.1), there exists $a_1, \cdots a_k \in E$ such that $s/b_k =$

$\sum_{j=1}^{k} a_j r_j$. Therefore

$$s = \sum_{j=1}^{k-1} b_k a_j r_j + b_k a_k \left(\sum_{j=1}^{k-1} \frac{b_j}{b_k} r_j \right) = \sum_{j=1}^{k-1} (b_k - a_j - a_k - b_j) r_j.$$

This implies that

$$\left\{ \sum_{j=1}^{k} a_j r_j : a_j \in E, j = 1, \cdots k-1 \right\} = \mathbb{R}$$

and we obtain the restriction of φ to E^{k-1}, $\varphi|_{E^{k-1}} : E^{k-1} \to \mathbb{R}$ mapping E^{k-1} onto $\mathbb{R}$. This contradicts the minimality of k, and thus prove the claim that φ is injective on E^k. ■

The second lemma forces the Borel subring of $\mathbb{R}$ with dimension strictly larger than zero to be $\mathbb{R}$.

Lemma 6.3. *Let $E \subset \mathbb{R}$ be a Borel subring. Let k be a positive integer and $\varphi : \mathbb{R}^k \to \mathbb{R}$ a linear functional mapping E^k bijectively onto $\mathbb{R}$. Then $k = 1$ and $E = \mathbb{R}$.*

Proof. Let $\psi : \mathbb{R} \to E^k$ be the inverse map of $\varphi|_{E^k}$, the restriction of φ to E^k. Recall that φ, as the projection, is continuous and one-to-one when restricted to E^k. So, ψ maps Borel subsets of E_k onto Borel sets (by a standard result on Borel sets). Thus ψ is measurable linear homomorphism. Using the same notation, let $\{e_i; i = 1, \cdots k\}$ be the standard basis of $\mathbb{R}^k$ and let $r_j = \varphi(e_j)$. Let $\pi_1 : \mathbb{R}^k \to \mathbb{R}$ be the map taking out the first coordinate. Then $\tau = \pi_1 \circ \psi$ is a map from $\mathbb{R}$ to $\mathbb{R}$,satisfying $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathbb{R}$ and τ is Borel measurable because it is the composition of a Borel measurable morphism with a continuous map. We claim that there is a constant c such that $\tau(x) = cx$ for all $x \in \mathbb{R}$.

To verify the claim, we first observed that $\tau(x+y) = \tau(x)+\tau(y)$ implies that τ is linear on the rational numbers, i.e., $\tau(q) = \tau(1)q$ for all $q \in \mathbb{Q}$. This is because if we let $q = a/b$, assuming $a, b \in \mathbb{N},\ b \neq 0$, then $a = bq = q + q + \cdots q$ where there are b copies of q so that $\tau(bq) = b\tau(q)$. Similarly, $\tau(a) = a\tau(1)$. Thus, $a\tau(1) = \tau(a) = b\tau(q)$ and

$$\tau(q) = \tau(1)q,\ q \in \mathbb{Q}.$$

Second, if we show that τ is continuous, then it follows that $\tau(x) = cx$ for all $x \in \mathbb{R}$. Since the linearity condition holds on $\mathbb{Q}$, it suffices to show that τ is continuous at the origin 0. We do this with the help of the Steinhaus's theorem. By the denseness of $\mathbb{Q}$ in $\mathbb{R}$, for every $\varepsilon > 0$, the balls centered at rational points (with radius $\varepsilon/2$) cover the real line $\mathbb{R}$, that is

$$\bigcup_{q \in \mathbb{Q}} B(q, \varepsilon/2) = \mathbb{R}.$$

The inverse image of τ preserves unions.

$$\bigcup_{q \in \mathbb{Q}} \tau^{-1} \left(B(q, \varepsilon/2) \right) = \tau^{-1} \left(\bigcup_{q \in \mathbb{Q}} B(q, \varepsilon/2) \right) = \mathbb{R}.$$

We know from the Baire category theorems applied here that there is a rational index $q_0 \in \mathbb{Q}$ for which $\tau^{-1}(B(q_0, \varepsilon/2)) \neq \varnothing$, so that $\mathcal{L}^1(\tau^{-1}(B(q_0, \varepsilon/2))) > 0$. Then by the Steinhaus's theorem, there exists a $\delta > 0$ such that

$$\begin{aligned} B(0,\delta) &\subset \tau^{-1}(B(q_0,\varepsilon/2)) - \tau^{-1}(B(q_0,\varepsilon/2)) \\ &\subset \tau^{-1}(B(q_0,\varepsilon/2) - B(q_0,\varepsilon/2)) \\ &= \tau^{-1}(B(0,\varepsilon)), \end{aligned}$$

which shows that the inverse image of any neighborhood of $\tau(0) = 0$ is a neighborhood of 0. This is equivalent to say that τ is continuous at the 0.

Finally, $\tau(r_1) = \pi_1(e_1) = 1$, so c is nonzero. But if $k > 1$, there would be an $r_2 \neq 0$ with $\tau(r_2)\pi_1(e_2) = 0$ which is a contradiction. Therefore k can only be 1 so that $\varphi : \mathbb{R} \to \mathbb{R}$ has the form $\varphi(x) = ax$ for some constant $a \in \mathbb{R}$. Since φ maps E to $\mathbb{R}$, we have $E = \mathbb{R}$. ■

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Acknowledgements

I would like to appreciate my thesis advisor Bochen Liu first, who has been selflessly imparting the knowledge of harmonic analysis to me and bringing me to the vast world of analysis. I also want to express my sincere gratitude to my academic advisor Yifei Zhu. He gives me support in all respects for learning mathematics, which is equally important. I want to thank Professor Anyue Chen especially for he has been caring about me all the way through my undergraduate life. Professors who have taught me about mathematics all deserve my respect. Moreover, I want to thank one of my schoolmates Yilin Li who motivated me to strengthen my mind to get out of another major to pursue my degree in mathematics.