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GROUP ACTIONS ON MANIFOLDS AND PRODUCT IDENTITIES OF MODULAR FORMS

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Abstract

This thesis is a reading report on recent work of Bringmann, Castro, Sabatini, and Schwagenscheidt, which derived topological and number-theoretical consequences from the rigidity of elliptic genera. The goal of this thesis is to state the rigidity theorem and to deduce some apparently new product identities of modular forms by applying this theorem, following their work. For these purposes, we study the S^1 actions on almost complex manifolds and symplectic manifolds, as well as basic theory of modular forms.

Key words: elliptic genus, rigidity, group action, modular form.

Contents

1 Introduction

A genus of a manifold is a kind of cobordant invariant, and an elliptic genus is a special type of genus developed to deal with problems about quantum field theory. A elliptic genus of level N associated to a compact almost complex manifold (M^{2n}, J), denoted by $\phi_N(M)$, is a modular form with respect to the group $\Gamma_1(N)$ of weight n. The elliptic genus of level N should have relied on a parameter t. However, it is independent of t in fact. This result is so-called the rigidity theorem.

The proof of the rigidity theorem has a long history and it is involved with Landweber, Ochanine, Taubes, Bott, Atiyah, Hirzebruch, Hattori and so on. We will give the statement of the rigidity theorem. Before that, we are going to study the prerequisites first.

In Section 2, we discuss the almost complex manifolds and symplectic manifolds with a S^1-action, which will be main geometric objects in the later sections.

Roughly speaking, the relations among the next Sections are as follows:

$$\text{Section } 3 \Rightarrow \text{Section } 4 \Rightarrow \text{Section } 5 \Rightarrow \text{Section } 6$$

Section 6 can give another description of elliptic genus of level N i.e. an index of certain virtual bundle, which is important in the statement of the rigidity theorem.

Section 7 is independent of the previous sections. In this section, we will introduce some basic knowledge of modular forms and define a special Eisenstein series, which is just the coefficients of some formal power series in defining the elliptic genus of level N.

In Section 8, after introducing some conceptions of genus theory, we can define the elliptic genus of level N in Subsection 8.3, and state the rigidity theorem in Subsection 8.4 by using the knowledge in the previous sections.

Section 9 is an application of the rigidity theorem. When applying this theorem on a certain almost complex manifold, we can obtain some non-trivial relations of modular forms. Specially, we will compute some such relations when the manifold is $\mathbb{CP}^2$.

2 Almost complex and symplectic circular manifolds

In this section, the properties of action of S^1 on almost complex and symplectic manifolds will be introduced. The genera (which will appear in section 8) is a map sending a manifold to an element in a certain ring. When restricting some genera on these manifolds, we will get some interesting consequences of geometry and number theory.

2.1 Almost complex structure and symplectic structure

Definition 2.1.1. *If a real manifold M admits a tensor field J of type $(1,1)$ s.t. $\forall\ x \in M$,*

$$J_x^2 = -Id$$

then M is called an almost complex manifold.

In a word, an almost complex manifold is equipped with an endomorphism $J_x : T_xM \to T_xM$ with $J_x^2 = -Id$ and it depends on x smoothly.

Proposition 2.1.2. *An almost manifold is of even dimension.*

Proof. Assume $dim\ M = N$, $\forall\ x \in M$,

$$det(J_x)^2 = det(J_x^2) = det(Id_n) = (-1)^n$$

Since $det(J_x)$ is real, it must be 1. Hence n is even. □

Proposition 2.1.3. *An almost manifold of dimension $2n$ is orientable.*

Proof. Choose a metric g compatible with J i.e. $\forall X, Y \in \Gamma(M, TM)$, $g(X,Y) = g(JX, JY)$. Such metric always exists (for example, start with an arbitary metric h, then let $g(X,Y) := h(X,Y) + h(JX, JY)$). Define $\omega(X,Y) := g(X, JY)$. Then ω is skew-symmetric since

$$\omega(X,Y) = g(X,JY) = g(JY,X) = g(J^2Y, JX) = -g(Y, JX) = -\omega(Y,X)$$

Thus ω is a 2-form. Let $\omega^n := \bigwedge^n \omega$. Then ω^n is nowhere vanishing, thus it is an orientable form. □

Let M be an almost complex manifold. Then $T^{\mathbb{C}}M$ denotes the complexification of TM i.e. $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$.

Since $J_x^2 = -Id$, J has eigenvalues $\sqrt{-1}$ *and* $-\sqrt{-1}$. Thus we have the following decomposition

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ and $T^{0,1}M$ are eigenspaces with respect to $\sqrt{-1}$ *and* $-\sqrt{-1}$. They are called holomorphic and antiholomorphic tangent bundle respectively.

Definition 2.1.4. *For an alomost complex manifold M, define the following bundles:*

$$\bigwedge^k M := \bigwedge^k (T^{\mathbb{C}}M)^*$$

$$\bigwedge^{p,q} M := \bigwedge^p (T^{1,0}M)^* \wedge \bigwedge^q (T^{0,1}M)^*$$

Their sheaves of sections are denoted by $\mathcal{A}_M^k$ and $\mathcal{A}_M^{p,q}$ respectively. The elements in $\Gamma(M, \mathcal{A}_M^{p,q}) =: \mathcal{A}^{p,q}(M)$ are called forms of type (p,q).

There is a natural direct sum decomposition

$$\bigwedge^k M = \bigoplus_{p+q=k} \bigwedge^{p,q} M$$

$$\mathcal{A}_M^k = \bigoplus_{p+q=k} \mathcal{A}_M^{p,q}$$

Definition 2.1.5. *Let M be an almost complex manifold. d is the $\mathbb{C}$-linear extension of the exterior differential. We define two operators:*

$$\partial := \Pi^{p+1,q} \circ d : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q}$$

$$\bar{\partial} := \Pi^{p,q+1} \circ d : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p,q+1}$$

where $\Pi^{m,n}$ is the natural projection form $\mathcal{A}_M^{m+n}$ to $\mathcal{A}_M^{m,n}$.

Definition 2.1.6. *A symplectic manifold (M, ω) is a manifold M equipped with a non-degenerate closed 2-form ω. Such a form is called a symplectic form.*

In the local coordinates, the symplectic form can be written as

$$\omega_{ij} dx^i \wedge dx^j, \ d\omega = 0$$

Being non-degenerate means the matrix $(\omega_{i,j})$ is invertible. Since $(\omega_{i,j})$ is also skew-symmetric, it must have an even rank. Hence a symplectic manifold is of an even dimension.

Also, the non-degeneracy implies $\bigwedge^n \omega$ is nowhere vanishing if ω is a symplectic form on a $2n$-dimensional manifold. Thus every syplectic manifold is orientable.

Definition 2.1.7. *Let $(M_1, \omega_1), (M_2, \omega_2)$ be two symplectic manifolds. A symplectomrphism is a diffeomorphism $f : M_1 \to M_2$ s.t.*

$$f^* \omega_2 = \omega_1$$

Theorem 2.1.8 (Darboux). *Let (M, ω) be a symplectic manifold of dimsion $2n$. Then $\exists$ a local coordinate $\{x_i, y_j\}_{i,j=1}^n$ of each point of M, s.t. the symplectic form in this local coordinate is of the form:*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

Proof. See [3]. □

This means that all symplectic manifolds are "the same" locally. Thus the global properties of a symplectic is much more interesting.

Theorem 2.1.9. *Let (M, ω) be a symplectic manifold. Then every differential function H on M determines a vector field X_H which generates a symplectomorphism in the sense of*

$$\mathcal{L}_{X_H}\omega = 0$$

Proof. In a local coordinate, define

$$X_H^i := \omega^{ij}\partial_j H,\ X_H := X_H^i \partial_i$$

This is equivalent to

$$\iota_{X_H}\omega = dH$$

By Cartan's magic formula, we have

$$\mathcal{L}_{X_H}\omega = \iota_{X_H} d\omega + d(\iota_{X_H}\omega) = d(dH)) = 0$$

□

Remark 2.1.10. *The function H is called a Hamitonian and X_H is called a Hamiltonian vector field. The converse is true. Namely, any generator of a symplectomorphism comes from a Hamiltonian vector field.*

Theorem 2.1.11. *A symplectic manifold is almost complex.*

Proof. Choose a Riemannian metric g. Define an linear operator $A : \Gamma(M, TM) \to \Gamma(M, TM)$ as follows:

$$\omega(X, Y) = g(AX, Y)$$

In a local coordinate

$$A_j^i = \omega_{jk} g^{ki}$$

A is an anti-hermitian operator:

$$g(AX, Y) = \omega(X, Y) = -\omega(Y, X) = -g(AY, X) = -g(X, AY)$$

Therefore $AA^H = -A^2$ is hermitian and positive define. Thus we can take the square root of AA^H and define

$$J := (\sqrt{AA^H})^{-1} A$$

Since

$$J^2 = (AA^H)^{-1} A^2 = -Id$$

J is an almost complex structure. □

2.2 Circle actions on almost complex manifolds and symplectic manifolds

Lemma 2.2.1. *The complex irreducible representations of S^1 are of the form $z \to z^n,\ n \in \mathbb{Z}$.*

Proof. Since S^1 is abelian, its irreducible representation is one-dimensional. Thus an irreducible representation is an element in $Hom(S^1, S^1)$. Thus it is of the form $z \to z^n,\ n \in \mathbb{Z}$. □

Let the almost complex manifold (M, J, S^1) of dimension $2n$ admit a circle action $\mu : M \times S^1 \to M$ and the action preserves J i.e. $\forall s \in S^1,\ \mu_s \circ J = J \circ \mu_s$. We always assume that this action has fixed points and denote the set of the fixed point as M^{S^1}.

Recall that if $\mu : G \times M \to M$ is a Lie group action on a manifold M with a fixed point p, then it induces a representation of G as follows:

$$G \to \mathrm{Aut}(T_pM),\ g \mapsto d\mu_g$$

By the Lemma 2.2.1, if $p \in M^{S^1}$, then $\exists$ a complex coordinates $\{z_1, \cdots, z_n\}$ on $T_pM \cong \mathbb{C}^n$ s.t. the induced S^1 action on T_pM is of the form

$$d\mu_s \cdot (z_1, \cdots, z_n) = (s^{w_1(p)} z_1, \cdots, s^{w_n(p)} z_n)$$

for some integers $w_1(p), \cdots, w_n(p)$.

Definition 2.2.2. *If S^1 acts on a manifold with a fixed point p, then the integers $w_1(p), \cdots, w_n(p)$ defined above are called weights of this action at p.*

Proposition 2.2.3. *$p \in M^{S^1}$ is isolated fixed point iff every weight at p is non-zero.*

Proof. Choose an S^1-invariant metric g on M, that is, $\forall s \in S^1, X, Y \in \Gamma(M, TM)$, $g(X,Y) = g(d\mu_s X, d\mu_s Y)$. Then μ_s is an isometry. Thus, $exp \circ d\mu_s = \mu_s \circ exp$.

Suppose $w_i = 0$. Then

$$\mu_s \circ exp(0, \cdots z_i \cdots, 0) = exp \circ d\mu_s(0, \cdots z_i \cdots, 0) = exp(0, \cdots z_i \cdots, 0)$$

i.e. $exp(0, \cdots z_i \cdots, 0)$ is a fixed point. Similarly, $\forall \lambda \in \mathbb{R}$, we have $\lambda(0, \cdots z_i \cdots, 0)$ is a fixed point. Thus $(0, \cdots z_i \cdots, 0)$ is not isolated.

Conversely, if p is not isolated, then for the neighborhood U on which exp is a diffeomorphism, there is a fixed point $q \in U$.

$$exp \circ d\mu_s(exp^{-1}(q)) = \mu_s \circ exp(exp^{-1}(q) = \mu_s(q) = q$$

implies $\mu_s(exp^{-1}(q))$ is fixed$\Rightarrow$ some weights are 0's. □

Given a G-manifold M for some Lie group G, $\forall \xi \in Lie(G)$, there is an associated vector field $\xi^{\#}$ on M given by

$$\xi^{\#}_x = \frac{d}{dt}|_{t=0} exp(t\xi) \cdot x$$

let a symplectic manifold (M, ω) admit a S^1-action. Then we say the action preserves the symplectic structure if

$$\mathcal{L}_{\xi^{\#}}\omega = d(\iota_{\xi^{\#}}\omega) = 0$$

If the closed form $\iota_{\xi^{\#}}\omega$ is exact, then this action is called Hamiltonian. More specifically, we have the following definition:

Definition 2.2.4. *Let (M, ω, S^1) be a S^1-symplectic manifold and the action of S^1 preserves the symplectic structure. The S^1 action is called Hamiltonian if $\exists$ a map $\phi : M \to Lie(S^1)^*$ s.t.*

(1) ϕ is S^1 invariant. That is, $\forall x \in M$, $s \in S^1$ we have

$$s \cdot \phi(x) = Ad^*_s(\phi(x))$$

(2) $\forall \xi \in Lie(S^1)$, we have

$$\iota_{\xi^{\#}}\omega = d\phi^{\xi}$$

where $\phi^{\xi}(x) := \langle \phi(x), \xi \rangle$.

The map $\phi : M \to Lie(S^1)^*$ is called moment map, which has a physical background.

Proposition 2.2.5. *Let symplectic manifold (M, ω) admit a Hamiltonian S^1-action with the moment map $\phi : M \to Lie(S^1)^*$. $\forall \xi \in Lie(S^1)$, we have* $\mathrm{Crit}(\phi^{\xi}) = M^{S^1}$

Proof.

$$\begin{aligned} p \in M^{S^1} &\Leftrightarrow \xi^{\#}_p = 0 \\ &\Leftrightarrow \iota_{\xi^{\#}}\omega|_p = 0 \\ &\Leftrightarrow d\phi^{\xi}_p = 0 \\ &\Leftrightarrow p \in \mathrm{Crit}(\phi^{\xi}) \end{aligned}$$

□

Corollary 2.2.6. *If (M, ω, S^1) is compact, then a Hamiltonian action always has fixed points.*

Proof. Since the integral of every exact form on a compact manifold is 0, every exact form on a compact manifold has vanishing points. Thus $\exists p \in M$ s.t. $d\phi^{\xi}|_p = \iota_{\xi^{\#}}\omega|_p \neq 0$. By the Proposition 2.2.5, we obtain the result. □

Theorem 2.2.7. *Let (M, ω) be a compact symplectic manifold of dimension $2n$ and suppose S^1 acts as a Hamiltonian action on M with the moment map $\phi : M \to Lie(S^1)^*$. Then, $\forall \xi \in Lie(S^1)$,the function ϕ^ξ is a perfect Morse-Bott function. Furthermore, the critical submanifolds of ϕ^ξ are symplectic with all its indeces even. (See the appendix)*

Proof. See [11] □

Moreoer, If M^{S^1} is discrete, then the moment map is a Morse function with only even indices. Since it is perfect, we have

$$b_{2i}(M) = N_i, \ \forall i \in \{0, \cdots, n\}$$

where N_i is the number of the critical point of the moment map with index i; thus N_i is also the number of fixed points with i negative weights.

Since ω is non-degenerated, $b_{2i}(M) \neq 0 \, \forall i \in \{0, \cdots, n\}$. Thus, if there is a Hamiltonian S^1-action on a symplectic manifold of dimension $2n$, the there are at least $n+1$ fixed points.

3 Universal bundles, classifying spaces, and equivariant cohomology

Every fiber bundle can be pulled back by a continuous maps. After being pulled back, the bundle carries less information than before. So here is a question: Does there exist "the most complicated bundle" s.t. every bundle is a pull-back of this bundle? Under some circumstances, the answer is "yes".

3.1 Universal bundles and classifying spaces

Definition 3.1.1 (Principal bundles). *Let G be a topological group. If the continuous map $p : E \to B$ from a G-space E to a topological space B satisfies the following conditions, then (E, B, p) (sometimes denoted as E only) is called a principal G-bundle.*

$\exists$ a countable open covering $\{U_i\}_{i \in I}$ of B and homeomorphisms $\phi_i : U_i \times G \to p^{-1}(U_i)$ satisfying that $\forall \ b \in U_i$ and $g, h \in G$

(1) $p \circ \phi_i(b, g) = b$

(2) $\phi_i(b, gh) = g \cdot \phi_i(b, h)$

Such (U_i, ϕ_i) is called a trivialization and E is called the total space.

Definition 3.1.2. *A principal bundle is called universal if the total space is weakly contractible. i.e. every homotopy group of the total space is 0.*

The universal bundle with respect to the topological group G is always denoted as $E_G \to B_G$. B_G is called the classifying space with respect to G.

Remark 3.1.3. (1) *This is not the origin definition but it is convenient. Later we will see why it is called "universal".*

(2) *The Whitehead theorem says that if f is a continuous map between CW-complexes X, Y inducing isomorphisms on all homotopy groups, then f is a homotopy equivalence. Thus if a CW-complex is weakly contractible, then it is contractible.*

For an arbitrary topological group G, dose there exist a universal principal G-bundle?

Theorem 3.1.4 (Milnor). *Every topological group has a universal bundle.*

Proof. Milnor constructed E_G directly as follows: Define the join of two spaces X,Y as $X \times Y \times [0,1]/\sim$ where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ (See the figure 2), and it is denoted as $X * Y$. Just let $E_G := G * G * \cdots * G * \cdots$, $B_G := E_G/G$. G has a natural action on E_G and E_G is weakly contractible. Hence $E_G \to B_G$ is a universal bundle with respect to G. □

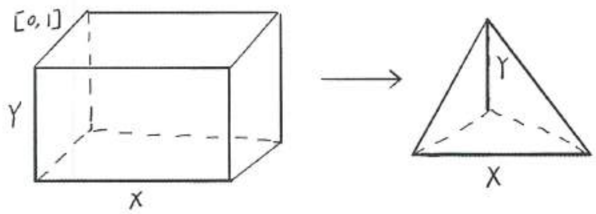

Figure 1: $X \times Y \times [0,1] \to X * Y$

Example 3.1.5. (1) *Let $G = \{e\}$, then $pt \to pt$ is the associated universal bundle.*

(2) *Let $G = \mathbb{C}^*$. G acts on $\mathbb{C}^n \backslash \{0\}$ by*

$$c \cdot (x_1, \cdots, x_n) = (cx_1, \cdots, cx_n)$$

Then $\mathbb{C}^n \backslash \{0\} \to \mathbb{CP}^{n-1}$ is a principal G-bundle.

However, this is not a universal bundle, since $\pi_{2n-1}(\mathbb{C}^n \backslash \{0\}) \neq 0$, although the previous homotopy goups are all 0. How to fix this problem? Note $\mathbb{C}^n \backslash \{0\} \hookrightarrow \mathbb{C}^{n+1} \backslash \{0\}$ $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0)$ is G-equivariant. Therefore we get inclusions of G-bundles:

$$\begin{array}{ccc} \cdots \hookrightarrow \mathbb{C}^n \backslash \{0\} & \hookrightarrow & \mathbb{C}^{n+1} \backslash \{0\} \hookrightarrow \cdots \\ \downarrow & & \downarrow \\ \cdots \hookrightarrow \mathbb{CP}^{n-1} & \hookrightarrow & \mathbb{CP}^n \hookrightarrow \cdots \end{array}$$

This forms a direct system. Take the direct limit. Then we get

$$\mathbb{C}^\infty \backslash \{0\} \to \mathbb{CP}^\infty$$

is a universal principal G-bundle, since $\mathbb{C}^\infty \backslash \{0\}$ *is weakly contractible.*

(3) *Let* $G = S^1$. G *acts on* S^{2n+1} *in the similar way as that in* (2). *Then* $S^{2n+1} \to \mathbb{CP}^n$ *is a principal G-bundle.* S^{2n+1} *is k-connected whenever* $k < 2n+1$ *but it is not* $2n+1$*-connected, so this is not a universal bundle. Using the similar method to fix it, we get*

$$S^\infty \to \mathbb{CP}^\infty$$

is the universal bundle with respect to S^1.

(4) *Let* $G = GL_n(\mathbb{C})$, $M^{full}_{m,n}$ *denote the space of* $m \times n$ *matrices of full rank and* $Gr_n(\mathbb{C}^m)$ *denote the Grassmannian. Then G has a natural action on* $M^{full}_{m,n}$.

$$M^{full}_{m,n} \to Gr_n(\mathbb{C}^m), \ [v_1, \cdots, v_n] \to [span\{v_1, \cdots, v_n\}]$$

is a principal G-bundle. It can be shown that $M^{full}_{m,n}$ *is k-connected whenever* $k \leq m - n$. *Hence*

$$M^{full}_{\infty,n} \to Gr_n(\mathbb{C}^\infty)$$

is the universal bundle with respect to $GL_n(\mathbb{C})$.

(5) *Let* $G = U_n$. *Then*

$$\begin{aligned} E_G &= \{(e_1, \cdots, e_n) | \langle e_i, e_j \rangle = \delta_{ij}, \ e_i \in \mathbb{C}^\infty\} \\ B_G &= E_G/G = \{V \subset \mathbb{C}^\infty | dimV = n\} = Gr_n(\mathbb{C}^\infty). \end{aligned}$$

(6) *Let G be a Lie group and H is a closed Lie subgroup of G. If* $E_G \to B_G$ *is a universal bundle of G, then* $E_G \to E_G/H$ *is a universal bundle for H.*

Before explaining the properties of the universal bundles, we need to make some preparations.

Definition 3.1.6. *If X is a right G-space and Y is a left G-space, the balanced product* $X \times_G Y$ *is the quotient space* $X \times Y/\sim$, *where* $(xg, y) \sim (x, gy)$. *Equivalently, we can regard* $X \times Y$ *as a right G-space:* $(x, y)g = (xg, g^{-1}y)$. *Then* $X \times_G Y = (X \times Y)/G$.

Lemma 3.1.7. *Every fiber bundle with weakly contractible fibers admits a section.*

Proof. See [14], Lemma 4.0.1. □

Lemma 3.1.8. *Given two principal G-bundles* $P \to B$ *and* $P' \to B'$. *There is a bijective correspondance between* $Mor_G(P, P')$ *and* $\Gamma(B, P \times_G P')$.

Proof. See [14], Corollary 4.0.1. ☐

Now, it is ready to explain what makes the universal bundle universal.

Theorem 3.1.9. *Suppose $E_G \to B_G$ is a universal G-bundle. Then $\forall$ CW-complex X, the map*

$$[X, B_G] \to \mathcal{P}_G X, \ [f] \mapsto [f^* E_G]$$

is bijective, where $[X, B_G]$ denotes the homotopic classes of continuous map from X to B_G and $\mathcal{P}_G X$ denotes the principal bundles over X up to isomorphisms.

Proof. For surjectivity: Suppse $P \to X$ is a principal G-bundle. It is equivalent to finding a G-equivariant map $\phi : P \to E_G$ and putting $f : X \to B_G$ the induced map.

By the Lemma 3.1.7, it suffices to find a section of the bundle $P \times_G E_G \to X$.

By the Lemma 3.1.8, it suffices to show $P \times_G E_G$ is weakly contractible.

Since E_G is a universal bundle, it is weakly contractible. P is a principal G-bundle, which implies G acts on P transitively. Hence, $P \times_G E_G$ is weakly contractible.

The injectivity part requires a lot of preparations so it is omitted. One can find the proof in [14], Theorem 4.0.1. ☐

From this theorem, we know that every pincipal G-bundle is a pull-back of the universal bundle. B_G is called the classifying space for G. If $P \to X$ is a principal G-bundle, then any map $f : X \to B_G$ s.t. $P = f^*(E_G)$ is called a classifying map for P.

Lemma 3.1.10. *Let $F \to E \to B$ be a fiber bundle. Then there is a long exact sequence of homotopy groups:*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Proof. See [5, Theorem 4.41, page 376]. ☐

Proposition 3.1.11. *Let $E_G \to B_G$ be a universal bundle. Then*

(1) B_G can be taken to be a CW-complex. Henceforth, B_G's that appear below default to CW-complexes.

(2) E_G is unique up to homotopy equivalence.

Proof. (1) Let $\phi : B'_G \to B_G$ be a CW-approximation. Then we have the following commutative diagram:

$$\begin{array}{ccc} \phi^*(E_G) & \xrightarrow{p_1} & E_G \\ p_2 \downarrow & & \downarrow \\ B'_G & \longrightarrow & B_G \end{array}$$

Only need to show ϕ^* is weakly contractible. We have the following exact sequences and commutative diagram by the Lemma 2.1.10:

$$\cdots \to \pi_n(G) \to \pi_n(E_G) \to \pi_n(B_G) \to \pi_{n-1}(G) \to \cdots$$

$$\uparrow \cong \qquad \uparrow p_{2*} \qquad \uparrow \phi_* \qquad \uparrow \cong$$

$$\cdots \to \pi_n(G) \to \pi_n(\phi^*(E_G)) \to \pi_n(B'_G) \to \pi_{n-1}(G) \to \cdots$$

Since ϕ is a CW-approximation, ϕ_* is an isomorphism. Thus p_{2*} is also an isomorphism by Five Lemma. Hence, $\forall\ n$,

$$\pi_n(\phi^*(E_G)) = \pi_n(E_G) = 0$$

i.e. $\phi^*(E_G)$ is weakly contractible.

(2) Choose two classifying maps

$$f : B'_G \to B_G,\ g : B_G \to B'_G$$

s.t.

$$E'_G \cong f^*(E_G),\ E_G \cong g^*(E'_G)$$

Then

$$f \circ g : B_G \to B_G$$

is a classifying map of E_G to itselfe since

$$(f \circ g)^*(E_G) \cong g^*(f^*(E_G)) \cong g^*(E'_G) \cong E_G$$

By the injectivity of the map $[X, B_G] \to \mathcal{P}_G X$ defined in Theorem 3.1.9, we have

$$f \circ g \simeq Id_{B_G}$$

Similarly,

$$g \circ f \simeq Id_{B'_G}$$

Thus, $B_G \simeq B'_G$ □

3.2 Equivariant cohomology

Cohomology rings of a topological space can tell us some topological information about this space. But if we compute the Cohomology rings of a G-space directly, this does not reflect any information about the group action (namely, the certain symmetry of this space). A naive idea is to compute the cohomology rings of the orbit space instead, but it does not work because the orbit space may have some "singularities" if the action is not "good" enough. For example, if the action is not free, then the quotient of a manifold will not be a manifold again in general. Next, we will introduce so-called equivariant cohomology to fix this problem.

Definition 3.2.1. *Let G be a topological group and X a G-space. Choose a universal bundle $E_G \to B_G$. The equivariant cohomology ring with coefficient in R is defined to be*

$$H_G^*(X, R) := H^*(X \times_G E_G, R)$$

Proposition 3.2.2. *The definition above is independent of the choice of the universal bundle.*

Proof. Suppose $E'_G \to B'_G$ is another universal bundle. Consider the projections $X \times_G E_G \times_G E'_G \to X \times_G E_G$ and $X \times_G E_G \times_G E'_G \to X \times_G E'_G$. These are fiberations with firbers E'_G and E_G respectively. Then we have the exact sequence:

$$\cdots \to \pi_n(E'_G) = 0 \to \pi_n(X \times_G E_G \times_G E'_G) \to \pi_n(X \times_G E_G) \to \pi_{n-1}(E'_G) = 0 \to \cdots$$

By the exactness, we have

$$\forall\ n,\ \pi_n(X \times_G E_G \times_G E'_G) \cong \pi_n(X \times_G E_G)$$

Similarly,

$$\pi_n(X \times_G E_G \times_G E'_G) \cong \pi_n(X \times_G E'_G)$$

Thus, $\forall\ n$,

$$\pi_n(X \times_G E_G) \cong \pi_n(X \times_G E'_G)$$

Hence

$$H^*(X \times_G E_G, R) = H^*(X \times_G E'_G, R)$$

□

Sometimes, we don't need to compute the all equivariant cohomology groups of a G-space. To deal with the classifying space is not very easy. If we only need the finite certain equivariant cohomology groups, the following proposition will be a convenient way.

Proposition 3.2.3. *Let $n \in \mathbb{N}^+$ and $E^n \to B^n$ a principal G-bundle which is n-connected. Then $H_G^m(X, R) = H^m(X \times_G E^n, R)$, $\forall$ compact topological group G, manifold X of dimension at most n, $m \leqslant n$.*

Proof. See [8], Theorem 13.1. □

Example 3.2.4. (1) *If $X = pt$, then $H_G^*(X, R) = H^*(E_G/G, R) = H^*(B_G, R)$. Moreover, if $G = \mathbb{T}^m = (S^1)^m$, Then*

$$\begin{aligned} H_G^*(X, R) &= H^*((\mathbb{CP}^\infty)^m, R) \\ &\cong \varprojlim H^*((\mathbb{CP}^n)^m, R) \\ &\cong \varprojlim R[x_1, \cdots, x_m]/(x_1^{n+1}, \cdots, x_m^{n+1}) \\ &\cong R[x_1, \cdots, x_m] \end{aligned}$$

where x_i *is of degree* 2.

(2) *If* G *is trivial, then* $pt \to pt$ *is a universal bundle. Thus*

$$H_G^*(X, R) \cong H^*(X, R).$$

(3) *If* G *acts freely on* X, *then* $\exists$ *a fiberation*

$$E_G \to X \times_G E_G \to X/G.$$

Thus

$$H_G^*(X) = H^*(X \times_G E_G) = H^*(X/G).$$

The last "=" is due to the argument in Proposition 3.2.2. This example says that if the action is free, then the equivariant cohomology coincides with the "naive idea" mentioned in the beginning of this subsection.

(4) *Let* H *be a closed group of the Lie group* G. $E_G \to E_G/H$ *is a universal bundle for* H, *which implies that*

$$\begin{aligned} H_G^*(G/H, R) =& H^*(G/H \times_G E_G, R) \\ \cong& H^*(pt \times_H G \times_G E_G, R) \\ \cong& H^*(pt \times_H E_G, R) \cong H_H^*(pt, R) \\ =& H^*(B_H, R). \end{aligned}$$

The equivairant cohomology has following properties:

(1) Pull back (functorial). Let $X \to Y$ be a G-equivariant continuous maps of G-spaces. Then

$$f \times Id_{E_G} : X \times E_G \to Y \times E_G$$

is a morphism of G-spaces, which induces a continuous map

$$f' : X \times_G E_G \to Y \times_G E_G.$$

Further, f' can induce the pull-back of the cohomology groups of $Y \times_G E_G$ to $X \times_G E_G$ and hence the pull-back from the equivariant cohomology groups of Y to X.

(2) $H_G^*(pt)-mod$ structure. The G-equivariant map $f : X \to pt$ gives a monomorphism $H_G^*(pt) \to H_G^*(X)$.

The equivariant cohomology ring can often tell us more information than the non-equivariant one, since the inclusion $\{e\} \hookrightarrow G$ induces a pull-back

$$H_G^*(M, R) \to H_{\{e\}}^*(M, R) \cong H^*(M, R)$$

Lemma 3.2.5 (Localization formula). *Let (M, J, S^1) be a compact almost complex manifold of dimension $2n$ acted by S^1 with discrete fixed point set M^{S^1}. Let α be an equivariant cohomology class. Then*

$$\int_M \alpha = \sum_{p \in M^{S^1}} \frac{1}{x^n} \frac{\alpha(p)}{w_1(p) \cdots w_n(p)}$$

Where x is as in $H^*_{s^1}(pt, R) \cong R[x]$.

4 Characteristic classes and equivariant Chern classes

"A characteristic class is a way of associating to each principal bundle of X a cohomology class of X. The cohomology class measures the extent the bundle is 'twisted' and whether it possesses sections. Characteristic classes are global invariants that measure the deviation of a local product structure from a global product structure. They are one of the unifying geometric concepts in algebraic topology, differential geometry, and algebraic geometry." –Wikipedia

In this section, the Chern classes, a kind of important characterastic classes, will be introduced. There are mainly 4 approches to define Chern classes: axiomatic approach, Obstruction theory, Chern-Weil theory and moduli space theory. We only introduce the axiomatic approach and moduli space approach, which will be enough in the later sections. Obstruction theory and Chern-Weil theory are geometric and they are harder to be constructed, so they will be omitted.

4.1 Two approaches to Chern classes

Definition 4.1.1. *A characteristic class c of a vector bundle or a principal G-bundle is an assignment to each bundle a class in the cohomology ring of the base space that is natural: if $f : N \to M$ is a map, then $c(f^*E) = f^*(c(E)) \in H^*(N)$, where E is a bundle on M.*

Definition 4.1.2 (axiomatic approach). *The Chern classes are characteristic classes for a complex vector bundle $E \to M$. For each $i \geqslant 0$ the i^{th} Chern class $c_i(E) \in H^{2i}(M, \mathbb{Z})$. The total Chern class $c(E) := c_0(E) + c_1(E) + \cdots$. The Chern class of M, $c_i(M)$, is defined to be $c_i(TM)$. Chern classes satisfy the following condition:*

(1) $c_0(E) = 1$.

(2) *The Whitney sum formula: $c(E \oplus F) = c(E)c(F)$. Hence,*

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E)c_j(E)$$

(3) *Let x be the generator of $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$. Then $c(H) = 1 - x$ where H is the tautological bundle of $\mathbb{CP}^2$.*

Theorem 4.1.3 (Grothendieck) *The Chern classes exist and are unique.*

Example 4.1.4. (1) *Let $\epsilon_n \to M$ be the trivial bundle of rank n. Then $c(\epsilon_n) = 1$ since ϵ_n is the pull-back of the trivial bundle over a point.*

(2) *$c(E \oplus \epsilon_n) = c(E)c(\epsilon_n) = c(E)$. Thus if E is stably trivial, then $c(E) = 1$. Hence the total Chern class can tell us a necessary condition for a bundle to be stably trivial.*

(3) *$T\mathbb{CP}^n \oplus \mathbb{C} \cong H^{*\oplus n+1}$ implies $c(\mathbb{CP}^n) = (1+x)^{n+1}$.*

There is another way to think about the characteristic class. Given a complex bundle $E \to M$ of rank n, we get a principal U_n-bundle, hence a classifying map $f_E : M \to B_{U_n}$. If $c \in H^*(B_{U_n})$, then let $c(E) := f_E^*(c)$. Then $c(E)$ is a characteristic class. On the other hand, all characteristic classes for E arise this way since all principal U_n-bundle are pull-backs of the universal bundle $E_{U_n} \to B_{U_n}$ by Theorem 3.1.9. Namely, a characteristic class is a cohomology class of the classifying space.

Now, let's define Chern classes by classifying space. First, the map

$$U_n \hookrightarrow U_{n+1}, \ A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

induces a map

$$B_{U_n} \to B_{U_{n+1}}$$

This is a direct system so we can take the direct limit

$$B_U := \varinjlim B_{U_n}$$

In fact B_{U_n} has a concrete expression: $Gr_n(\mathbb{C}^\infty)$ (see Example 3.1.5 (5)).

Theorem 4.1.5. *$H^*(B_U) \cong \mathbb{Z}[c_1, c_2, \cdots]$, with $|c_k| = 2k$.*

Proof. See [6], Theorem 3.9. □

Then the k^{th} Chern class $c_k(E)$ can be defined to be $f_E^*(c_k)$ where f_E is the classifying map of E.

Remark 4.1.6. *This approach allows us to define the Chern classes for a principale G-bundle, not just vector bundles.*

Definition 4.1.7. *Given a real vector bundle E over M, the k^{th} Pontryjagin class $p_k(E)$ is defined to be*

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M, \mathbb{Z})$$

4.2 Chern roots, Chern numbers, and equivariant Chern classes

Definition 4.2.1. *Let E be a complex bundle, the Chern polynomial c_t of E is given by:*

$$c_t(E) := 1 + c_1(E)t + \cdots + c_n(E)t^n$$

If we use the axiomatic approach to define the Chern class, then it is easy to see thath the Chern polynomial satisfies the Whitney sum formula:

$$c_t(E \oplus F) = c_t(E)c_t(F)$$

If $E = L_1 \oplus \cdots \oplus L_n$ is a direct sum of complex line bundles, then it follows from the Whitney sum formula that

$$c_t(E) = (1 + \gamma_1(E)t) \cdots (1 + \gamma_n(E)t)$$

where $\gamma_i(E) = c_1(L_i)$. $\gamma_i(E)$ is called the Chern root of E, which determine the coefficients of the Chern polynomial:

$$c_k(E) = e_k(\gamma_1(E), \cdots, \gamma_n(E))$$

where e_k is the k^{th} elementary symmetric polynomial.

Definition 4.2.2. *The Chern character of a complex bundle $E \to X$ is defined to be*

$$\mathrm{ch}(E) := e^{\gamma} + \cdots + e^{\gamma} \in H^*(X, \mathbb{Q})$$

where a_i are Chern roots.

Equivalently, by Chern-Weil theory,

$$\mathrm{ch}(E) = tr\left(exp\left(\frac{i\Omega}{2\pi}\right)\right)$$

where Ω is the curvature matrix.

The Chern character satisfies

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F)$$

Definition 4.2.3. *Let (M, J) be a almost complex manifold , $\lambda = [\lambda_1, \cdots, \lambda_k]$ be a partition of n. The Chern number of (M, J) with respect to λ is defined to be*

$$C_\lambda(M) := \int_M c_{\lambda_1} \cdots c_{\lambda_k} \in \mathbb{Z}.$$

where the integral of a characteristic class is the integral of its associated differential form class by de Rham theorem

$$H^n(M, \mathbb{R}) \cong H^n_{dR}(M)$$

Simliar to the reasons why the equivariant cohomology is introduced, we will introduce so-called equivariant Chern class.

Definition 4.2.4. *If a complex vector bundle $E \to M$ is equivariant with respect to G, then the equivariant Chern class $c_k^G(E) := c_k(E \times_G E_G \to M \times_G E_G)$.*

Proposition 4.2.5. *Let (M^n, J, S^1) be an almost complex manifold with an action of S^1. If p is a fixed point with weights $w_1, \cdots, w_n$, then*

$$c_k^{S^1}(p) = e_k(w_1, \cdots, w_n)x^k$$

where e_k is the k^{th} elementary symmetric polynomial.

Proof.

$$c^{S^1}(p) = c(T_pM \times_{S^1} S^\infty \to \{p\} \times_{S^1} S^\infty) = c(\bigoplus_i \mathcal{O}(w_i) \to \mathbb{CP}^n) = \prod_i (1 + w_i x).$$

Thus,

$$c_k^{S^1}(p) = e_k(w_1, \cdots, w_n)x^k.$$

□

Corollary 4.2.6. *Assume the same conditions in Lemma 3.2.5. Using Lemma 3.2.5 and Proposition 4.2.5, we get*

$$C_\lambda(M) = \sum_{p \in M^G} \frac{e_{\lambda_1}(w_1(p), \cdots, w_n(p)) \cdots e_{\lambda_k}(w_1(p), \cdots, w_n(p))}{w_1(p) \cdots w_n(p)}.$$

This gives another way to compute the Chern number.

4.3 Todd classes

Todd class is a kind of characteristic class, which plays a fundamental role in generalising the Riemann-Roch theorem. The relation between the Chern class and the Todd class is somehow like the relation between the tangent bundle and normal bundle.

Definition 4.3.1. *Consider the formal power series*

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

where B_i is the i^{th} Bernouli number. Then the Todd class $\mathrm{td}(E)$ is defined to be

$$\mathrm{td}(E) := \prod_i Q(\gamma_i)$$

where γ_i are Chern roots.

The Todd class has a explicit expression in Chern classes:

$$\mathrm{td}(E) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \cdots$$

Although this seems a little scary, the Todd class will be a polynomial if the dimension of X is finite.

The Todd class is multiplicative:

$$\mathrm{td}(E \oplus F) = \mathrm{td}(E)\mathrm{td}(F)$$

Example 4.3.2. (1) *Let L be a line bundle. Then $c_k(L) = 0$ whenever $k > 1$. Thus*

$$\mathrm{td}(L) = 1 + \frac{c_1}{2} + \frac{c_1^2}{12} - \frac{c_1^4}{720} + \cdots = Q(c_1)$$

(2) *There is a famous exact sequence, called Euler sequence (see [9], Proposition 2.4.4)*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to T\mathbb{CP}^n \to 0$$

Then $\mathrm{td}(\mathbb{CP}^n) = \mathrm{td}(\mathbb{CP}^n)\mathrm{td}(\mathcal{O}) = \mathrm{td}(\mathcal{O}(1)^{\oplus n+1}) = (\mathrm{td}(\mathcal{O}(1))^{n+1}) = \left(\frac{-x}{1-e^x}\right)^{n+1}$

Theorem 4.3.3 (Hirzebruch-Riemann-Roch). *Let E be a holomorphic vector bundle on a compact complex manifold X. Then its Euler characteristic is given by*

$$\chi(X, E) = \int_X \mathrm{ch}(E)\mathrm{td}(X)$$

5 K-theory and equivariant K-theory

Roughly speaking, the K-theory is the study of a ring generated by vector bundles over a topological space, and it is a generalized cohomology theory. The early work on topological K theory is due to Michael Atiyah and Friedrich Hirzebruch.

5.1 The Grothendieck completion

In order to study the vector bundles over a manifold M, we want to bring all the vector bundles together. The bundles form a set, say $Vect(M)$. However, this set does not have a good algebraic structure. To fix it, we need to introduce so-called Grothendieck completion.

Definition 5.1.1. *An abelian monoid is a set A with a binary operation $+$, satisfying*

(1) *$+$ is commutative and associative.*

(2) *There is an element 0 s.t. $\forall\ a \in A$, $0 + a = a$.*

Example 5.1.2. (1) *Every abelian group is an abelian monoid.*

(2) $\mathbb{N}$ *is an abelian monoid but not an abelian group.*

(3) *Let* $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$; $\forall\ a \in \mathbb{N}_\infty^+$, $a + \infty = \infty$. *Then* $\mathbb{N}_\infty^+$ *is an abelian monoid.*

Proposition 5.1.3. *Let* A *be an abelian monoid. Define* $(a_1, b_1) \sim (a_2, b_2)$ *if* $\exists\ c \in A$ *s.t.* $a_1 + b_2 + c = b_1 + a_2 + c$. *It is clear that* $\sim$ *is an equivalence relation on* $A \times A$. *Then* $A \times A/\sim$ *is an abelian group with the addition*

$$[(a_1, b_1)] + [(a_2, b_2)] = [(a_1 + a_2, b_1 + b_2)]$$

Proof. It is easy to see that $[(0,0)]$ is the identity and the inverse of $[(a,b)]$ is $[(b,a)]$. $\square$

Definition 5.1.4. *For every abelian monoid* A, *the group* $A\times A/\sim$ *is called the Grothendieck complition of* A.

Just like the way to define the fractional field of a ring, we denote $[(a,b)]$ as $a - b$ instead. This will be more intuitionistic.

"$+c$" in the relation is necessary because an abelian monoid dose not have cancellation in general, such as Example 5.1.2(3).

Example 5.1.5. (1) *If* A *is an abelian group, then the completion of* A *is* A *itself.*

(2) *The completion of* $\mathbb{N}$ *is* $\mathbb{Z}$.

(3) *The completion of* $\mathbb{N}_\infty$ *is* $\{0\}$ *since* $\forall\ a,\ b \in \mathbb{N}_\infty$, $a + 0 + \infty = b + 0 + \infty = \infty$ *implies* $a - b = 0 - 0$.

5.2 Complex K-theory

Let $Vect(X)$ be the set of all complex bundles over a topological space X. It is an abelian monoid with the addition $\oplus$. Denote the trivial bundle of rank n as ϵ_n.

Definition 5.2.1. *Let* X *be a compact Hausdorff space.The Grothendieck completion of* $Vect(X)$ *is denoted as* $K(X)$.

Remark 5.2.2. (1) $K(X)$ *has a ring srtucture:*

Addition:

$$(E_1 - F_1) + (E_2 - F_2) := (E_1 \oplus E_2) - (F_1 \oplus F_2)$$

Product:

$$(E_1 - F_1)(E_2 - F_2) = (E_1 \otimes E_2 \oplus F_1 \otimes F_2) - (E_1 \otimes F_2 \oplus F_1 \otimes E_2)$$

Zero: ϵ_0 *the bundle with each fiber*$= \{0\}$

Identity: ϵ_1 *the trivial bundle of rank 1.*

(2) *If* $f : X \to Y$ *is continuous. Then*

$$f^* : K(Y) \to K(X),\ E - F \mapsto f^*(E) - f^*(F)$$

is a ring homeomorphism. That is, $K(-)$ *is a contravariant functor.*

(3) *Every element of* $K(X)$ *has a representation of the form* $E - \epsilon_n$ *for some* n*, since if we start with* $E - F$ *we can add both* E *and* F *a bundle* G *s.t.* $F \oplus G \cong \epsilon_n$ *for some* n.

Example 5.2.3. (1) $Vect(pt) \cong \mathbb{N}$. *Thus* $K(pt) \cong$ *the Grothendieck completion of* $\mathbb{N}$*, i.e.* $\mathbb{Z}$.

(2) *If* X *is contractible and* $f : X \to pt$ *is a homotopy equivalence, then* f^* *is an isomorphism between* $K(X)$ *and* $K(pt)$*. Thus*

$$K(X) \cong \mathbb{Z}$$

(3) *The quotient map*

$$[0,1] \to S^1 = [0,1]/0 \sim 1$$

determines a pull-back of the vector bundle over S^1 *to* $[0,1]$*. Since every vector bundle over* $[0,1]$ *is trivial, the complex vector bundle of rank* n *over* S^1 *is determined by an isomorphism between the fiber at* 0 *and the fiber at* 1*, i.e. an element in* $GL_n(\mathbb{C})$*. Since* $GL_n(\mathbb{C})$ *is path connected, every complex vector bundle over* S^1 *is trivial. Hence,*

$$K(S^1) \cong \mathbb{Z}$$

(4) *Let* D^n_+ *and* D^n_- *be the upper and the nether (closed) semi-sphere of* S^n *respectively.* S^n *can be written as*

$$S^n = D^n_+ \coprod D^n_- / \partial D^n_+ \sim \partial D^n_-$$

Since every vector bundle on D^n_+ *or* D^n_- *is trivial, every vector bundle of rank* k *on* S^n *is of the form*

$$E_g := (D^n_+ \times \mathbb{C}^n) \coprod (D^n_- \times \mathbb{C}^n)/(x,v) \sim (x, g(x)v)$$

for some $g : S^{n-1} \to GL_k(\mathbb{C})$*. Since* $E_g \cong E_f$ *iff* $g \simeq f$*,* $Vect^k(S^n) = [S^{n-1}, GL_k(\mathbb{C})] = \pi_{n-1}(GL_k(\mathbb{C}))$*. Thus* $Vect(S^n) = \pi_{n-1}(GL_\infty(\mathbb{C}))$.

Lemma 5.2.4. *Let* $f, g : S^{n-1} \to GL_k(\mathbb{C})$*. Then*

$$E_{fg} \oplus \epsilon_k \cong E_f \oplus E_g$$

where E_f, E_g *were defined in Example 5.2.3* (4).

Proof. Since $GL_i(\mathbb{C})$ is connected, there is a path $\gamma_t \in GL_{2k}(\mathbb{C})$ from the identity matrix to the matrix of the transformation which interchanges the two factor of $\mathbb{C}^k \times \mathbb{C}^k$. Then $(f \oplus I)\gamma_t(I \oplus g)\gamma_t$ gives a homotopy form $f \oplus g$ to $fg \oplus I$. The later one is the associated with the bundle $E_{fg} \oplus \epsilon_k$. □

Corollary 5.2.5. *Let H be the tautological line bundle over $\mathbb{CP}^1 = S^2$. Then*

$$H \otimes H \oplus \epsilon_1 \cong H \oplus H$$

Proof. Let $f = g : S^1 \to GL_2(\mathbb{C})$ be

$$z \mapsto (z)$$

Then use the lemma above. □

In the ring $K(S^2)$ this conclusion turns to be $H^2 + 1 = 2H$, i.e. $(H-1)^2 = 0$. So there is is a natural homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$.

Theorem 5.2.6. $\mathbb{Z}[H]/(H-1)^2 \cong K(S^2)$

Proof. See [6], Corollary 2.3. □

Definition 5.2.7. *There is another relation equivalence $\approx$ in $Vect(X)$, where X is a connected compact Hausdorff space: $E \approx F$ if $E \oplus \epsilon_n \cong F \oplus \epsilon_m$ for some n and m. Then $Vect(X)/\approx$ is an abelian group with respect to $\oplus$, denoted as $\tilde{K}(X)$, called the reduced K-theory group.*

There is a natural homomorphism $\phi : K(X) \to \tilde{K}(X)$ sending $E - \epsilon_n$ to the $\approx$ class of E. This is clearly well-defined and surjective. $Ker\phi = \{\epsilon_m - \epsilon_n | m, n \in \mathbb{N}\} \cong \mathbb{Z}$. Thus we have a short exact sequence

$$0 \to \mathbb{Z} \to K(X) \to \tilde{K}(X) \to 0.$$

The sequence splits since $rank : K(X) \to \mathbb{Z}$ is a left inverse of $\mathbb{Z} \to K(X)$. Thus we have $K(X) \cong \tilde{K}(x) \oplus \mathbb{Z}$.

Definition 5.2.8. *Let Y be a closed subspace of X. The space obtained by contracting Y to a base point y is denoted by X/Y, and $\iota : \{y\} \to X/Y$ is the inclusion map. The relative K group is defined to be*

$$K(X,Y) := \tilde{K}(X/Y)$$

If $Y = \varnothing$, we define X/Y as $X \coprod \{pt\}$. Then

$$K(X,Y) = \tilde{K}(X \coprod \{pt\}) = Ker(\iota^* : K(X \coprod \{pt\}) \to K(pt)) = K(X)$$

Equivalently, the relative K group can be defined to be

$$K(X,Y) := \{E \overset{\alpha}{\to} F | \alpha \textit{ is a bundle morphism and } \alpha|_Y \textit{ is an isomrophism}\}$$

Definition 5.2.9. *The K theory with compact support on a compact base X is defined to be*

$$K_c(X) := \{E \overset{\alpha}{\to} F | \alpha \textit{ is a bundle morphism and is an isomrophism outside a compact set}\}$$

Definition 5.2.10. *Let X be a locally compact space. We define $K(X) = \tilde{K}(\bar{X})$ where $\bar{X} = X \coprod \{pt\}$ is the one-point compactification of X and*

$$\tilde{K}(\bar{X}) := Ker(\iota^* : K(\bar{X}) \to K(pt))$$

In fact, $K(X)$ here is always denoted as $K^0(X)$ in other materials. More generally, we can define $K^n(X)\ \forall\ n \in \mathbb{Z}$. K^* satisfies the Eilenberg-Steenrod axioms for cohomology theory. That is, K-theory is a kind of generized cohomology.

5.3 Equivariant K-theory

Definition 5.3.1. *A G-vector bundle over a G-space is a complex vector bundle $p : E \to X$, together with a G-space structure on E, s.t.*

(1) p is G-equivariant.

(2) $\forall\ g \in G,\ g \cdot (-) : p^{-1}(x) \to p^{-1}(gx)$ is a linear map.

The set of isomorphism classes of G-vector bundles on X forms an abelian monoid under $\oplus$. This set is denoted as $Vect_G(X)$.

Definition 5.3.2. *The Grothendieck completion of $Vect_G(X)$ is called the equivariant K group, and it is denoted as $K_G(X)$*

The tensor product of G-vector bundles induces a structure of commutative ring in $K_G(X)$.

If $f : X \to Y$ is a G-equivariant map between compact G-spaces. Then there is a pull-back $f^* : K_G(Y) \to K_G(X)$, which is a ring homomorphism. Thus K_G is a contravariant functor from compact G-spaces to commutative rings.

Definition 5.3.3. *Given a group G, the set of isomorphism classes of finite dimensional linear $\mathbb{C}$-representation of G is an abelian monoid. The Grothendieck completion of this set is called the representation ring with respect to G, and it is denoted as $R(G)$.*

$R(G)$ has a ring structure: the addition is the direct sum of the representations and the product is tensor product.

Example 5.3.4. (1) *It is clear that*

$$K_{\{e\}}(X) = K(X)$$

(2) *Because a G-vector bundle over a point is just a complex vector space with a linear action of G which is just the representation of G, we have*

$$K_G(pt) \cong R(G)$$

Specially, if $G = S^1$, *then*

$$K_G(pt) = \mathbb{Z}[t, t^{-1}]$$

(3) *If G acts freely on X, then*

$$K_G(X) \cong K(X/G)$$

Consider an extreme case. When G acts trivially on X. Then there is a ring homomorphism

$$K(X) \to K_G(X)$$

Combining this with the natural map

$$R(G) \to K_G(X)$$

we have a morphism of rings

$$R(G) \otimes K(X) \to K_G(X)$$

Proposition 5.3.5. *If X is a trivial G-space, then*

$$R(G) \otimes K(X) \cong K_G(X)$$

6 Index theory and equivariant index

In differential geometry, the Atiyah–Singer index theorem, proved by Michael Atiyah and Isadore Singer (1963), states that for an elliptic differential operator on a compact manifold, the analytical index (related to the dimension of the space of solutions) is equal to the topological index (defined in terms of some topological data). It includes many other theorems, such as the Chern–Gauss–Bonnet theorem and Riemann–Roch theorem, as special cases, and has applications to theoretical physics. Equivariant index theory is to study the index of a G-spaces.

6.1 Analytic index and topological index

Definition 6.1.1 (Differential operator). (1) *The linear map*

$$D : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is a differential operator of order k *if*

$$Df(x) = \sum_{|\alpha| \leqslant k} a_\alpha \partial_1^{\alpha_1} \cdots \partial_1^{\alpha_n} f$$

and $a_\alpha \neq 0$ *for some* α *with* $|\alpha| = k$.

(2) *More generally, let* E, F *be two vector spaces. The linear map*

$$D : C^\infty(\mathbb{R}^n, E) \to C^\infty(\mathbb{R}^n, F)$$

is a differential operator of order k *if*

$$Df(x) = \sum_{|\alpha| \leqslant k} a_\alpha \partial_1^{\alpha_1} \cdots \partial_1^{\alpha_n} f$$

and $a_\alpha \neq 0$ *for some* α *with* $|\alpha| = k$, *where* $a_\alpha \in C^\infty(\mathbb{R}^n, Hom(E, F))$

(3) *More generally, let* E, F *be two vector bundles over a manifold* X. *The linear map*

$$D : \Gamma(X, E) \to \Gamma(X, F)$$

is a differential operator of order k *if locally*

$$Df(x) = \sum_{|\alpha| \leqslant k} a_\alpha \partial_1^{\alpha_1} \cdots \partial_1^{\alpha_n} f$$

and $a_\alpha \neq 0$ *for some* α *with* $|\alpha| = k$, *where* $a_\alpha \in \Gamma(X, Hom(E, F))$

Example 6.1.2. (1) *A vector field on a manifold is a differential operator.*

(2) *The exterior differential* d *is a differential operator.*

Definition 6.1.3. *Let* $D : \Gamma(X, E) \to \Gamma(X, F)$ *be a differential operator of order* k. *The symbol of* D *is a vector bundle morphism*

$$\sigma(D) : \pi^*(E) \to \pi^*(F)$$

where $\pi : T^*X \to X$ *is the projection from the cotangent space to* X. *On each fiber*

$$\sigma(D)_\xi : \pi^*(E)_{(x,\xi)} = E_x \to \pi^*(F)_{(x,\xi)} = F_x$$

is defined to be

$$\sigma(D)_\xi = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha$$

where $T^*X \ni \xi = \sum_i \xi^i dx_i$ *locally.*

Definition 6.1.4. *A differential operator of order k is called elliptic if*

$$\sigma(D)_\xi : \pi^*(E)_{(x,\xi)} \to \pi^*(F)_{(x,\xi)}$$

is invertible $\forall\ \xi \neq 0$.

Remark 6.1.5. (1) *If $D : \Gamma(X, E) \to \Gamma(X, F)$ is elliptic, then $rank(E) = rank(F)$. Thus the Example 6.1.2* (2) *is not an elliptic operator.*

The vector field is clearly non-invertible; thus it is not an elliptic operator either.

(2) *The symbol of the elliptic operator*

$$\sigma(D) : \pi^*(E) \to \pi^*(F)$$

represents a class in $K(T^*(X), T^*(X)_0)$, where V_0 denotes the zero section of a bundle V, and we denote this class as $[\sigma_D]$.

Example 6.1.6. *On a Riemannian manifold, the dual operator of the exterior differential d^* can be defined (similarly, $\bar{\partial}^*$ is defined).*

(1) *$d + d^*$ is an elliptic operator.*

(2) *The Hodge laplacian $\Delta := (d + d^*)^2 = dd^* + d^*d$ is an elliptic operator.*

(3) *$\bar{\partial} + \bar{\partial}^*$ is an elliptic operator.*

Theorem 6.1.7. *If D is an elliptic operator on a compact manifold, then D is a Fredholm operator.*

Definition 6.1.8. *The analytic index of an elliptic operator D on a compact manifold is defined to be*

$$\mathrm{a} - \mathrm{Ind}(D) := \dim\ \mathrm{Ker}D - \dim\ \mathrm{Coker}D$$

Since D is Fredholm, this is well-defined.

Remark 6.1.9. dim Ker$D$ *may jump discontinuously and so does* dim Coker$D$. *However, they jump simultaneously, thus the analytic index is continuous. So there is a question, is the analytic index a pure topological data?*

Theorem 6.1.10 (Thom isomorphism). *Let $p : E \to X$ be a real,oriented, k-dimensional vector bundle over a compact base X. Then there is an isomorphism, called Thom isomorphism:*

$$\phi : H^i(X, \mathbb{Q}) \to H^i(E, E_0, \mathbb{Q}),\ a \mapsto p^*(a) \cdot u$$

where u is called the Thom class.

There is another version of Thom isomorphism. Assume the same condition. Then there is an isomorphism:

$$\phi' : K_c(X) \to K_c(E)$$

Note that the Chern character defined in 4.2.2 can be natural extended to $K(X)$ by defining $\mathrm{ch}(E - F) := \mathrm{ch}(E) - \mathrm{ch}(F)$. This is actually a ring homomorphism.

Definition 6.1.11. *Let D be an elliptic operator on a compact manifold X of dimension $2n$. Then the topological index of D is defined to be*

$$\mathrm{t} - \mathrm{Ind}(D) := \mathrm{ch}(D)\mathrm{td}(X)[X] = \int_X \mathrm{ch}(D)\mathrm{td}(X)$$

where $\mathrm{ch}(D) := (-1)^n \phi^{-1}\mathrm{ch}([\sigma_D])$.

Remark 6.1.12. *There is another way to define the topological index. Assume the same conditions. Let $f : X \hookrightarrow \mathbb{R}^N$ be a smooth embedding for N large enough (guaranteed by Sard's theorem). This induces an embedding*

$$f_! : K(T^*X, (T^*X)_0) \to K(T^*\mathbb{R}^N, (T^*\mathbb{R}^N)_0)$$

(see [10]). Now, consider $T^*\mathbb{R}^N \cong \mathbb{R}^N \oplus \mathbb{R}^N \cong \mathbb{C}^N$ and think $\mathbb{C}^N$ as a complex vector bundle $q : \mathbb{C}^N \to pt$. Let $q_! : K_c(\mathbb{C}^N) \to K_c(pt) = K(pt) \cong \mathbb{Z}$ be the inverse of the Thom isomorphism ϕ'. Then the topological index can be defined to be

$$\mathrm{t} - \mathrm{Ind}(D) := q_! f_!([\sigma_D])$$

Actually, the two definitions are equivalent.

6.2 The Atiyah-Singer index theorem

Theorem 6.2.1 (Atiyah-Singer). *Let D be an elliptic operator on a compact manifold. Then*

$$\mathrm{a} - \mathrm{Ind}(D) = \mathrm{t} - \mathrm{Ind}(D)$$

This theorem says that the analytic index is a purely topological data. Many classical theorems can be obtained by Atiyah-Singer theorem.

Example 6.2.2. (1) *Take X to be a complex manifold with a holomorphic vector bundle V. Let E, F be the sums of the bundles of differential forms with coefficients in V of type $(0, i)$ with i even or odd. Let $D = (\bar{\partial} + \bar{\partial}^*)|_E$. The analytical index is*

$$\mathrm{a} - \mathrm{Ind}(D) = \sum_k (-1)^k dim H^k(X, V) = \chi(X)$$

the topological index is

$$\mathrm{t-Ind}(D) = \int_X ch(V)td(X)$$

Thus, we get Hirzebruch-Riemann-Roch theorem by Atiyah-Singer index theorem:

$$\chi(X) = \int_X ch(V)td(X)$$

Particularly, if C *is a connected compact curve and* L *is a holomorphic line bunde on it, then*

$$\chi(C) = \int_C c_1(L) + \frac{c_1(C)}{2} = deg(L) + \frac{K_C^*}{2}$$

which is the ordinary Riemann-Roch theorem.

(2) *Let* $D = \Delta = dd^* + d^*d$. *Then the analytical index is*

$$\mathrm{a-Ind}(D) = Sgn(X)$$

where $Sgn(X)$ *is the Hirzebruch's signature of a* $4k$*-dimensional manifold (See [15, Chapter 7]). the topological index is*

$$\mathrm{t-Ind}(D) = \int_X \mathrm{ch}(\bigoplus \Omega_X^p)\mathrm{td}(X) = \int_X L(X)$$

Where $L(X)$ *is the L-genus which is defined by Chern roots:*

$$L(X) = \prod_i \frac{a_i}{tanh(\frac{a_i}{2})}$$

Then we get the Hirzebruch's signature theorem:

$$Sgn(X) = \int_X L(X)$$

(3) *Let* $D = (d + d^*)|_{\Omega^{even} \to \Omega^{odd}}$. *Then the analytical index of* D *is*

$$\mathrm{a-Ind}(D) = \chi(X)$$

the topological index of D *is*

$$\mathrm{t-Ind}(D) = \int_X e(X)$$

where $e(X) = c_n(X) = \prod_i \gamma_i$ *is called the Euler class. Then we get Chern-Gauß-Bonnet formula:*

$$\chi(X) = \int_X e(X)$$

6.3 Equivariant index

Let M be a compact Riemannian manifold with a smooth action of a compact Lie group G compatible with the metric. Consider the closed subset $T^*_G M$ of T^*M, which is defined to be

$$(T^*_G M)_x := \{\xi \in (T^*M)_x | \xi(V^{\#}_{\pi(\xi)}) = 0, \ \forall\ V \in Lie(G),\ x \in M\}$$

where $\pi : T^*M \to M$ is the projection, is then a closed G-invariant subspace. There is a linear structure on $(T^*_G M)\ \forall\ x \in M$, but the fiber dimension may not be locally constant.

Definition 6.3.1. *Let E, F be smooth G-equivariant vector bundles over M, $D : \Gamma(M, E) \to \Gamma(M.F)$ be a differential operator on M. D is said to be G-transversally elliptic if it is G-invariant and its symbol is invertible on $(T^*_G M)_0$.*

If D is a G-transversally elliptic operator, then its kernel is a G-invariant vector space. Moreover,

$$g|_{KerD} : G \to GL(KerD),\ g \mapsto g \cdot (-)|_{KerD}$$

is a representation of G.

Similarly, the adjoint operator D^* is also G-transversally elliptic with symbol

$$\sigma(D^*)(x, \xi) = \sigma(D)(x, \xi)^*$$

and $KerD^*$ is also associated to a representation of G.

Definition 6.3.2. *Let D be a G-transversally elliptic operator on M. Then the G-equivariant index of D is defined to be*

$$\mathrm{Ind}_G(D) : G \to \mathbb{C},\ g \mapsto tr(g|_{KerD}) - tr(g|_{KerD^*})$$

The ordinary index can be viewed as a map $K(T^*M, (T^*M)_0) \to K(pt) \cong \mathbb{Z}$. Meanwhile, the equivariant index can ve viewed as a map $K_G(T^*M, (T^*_G M)_0) \to K_G(pt) \cong R(G)$. Particularly, Ind_{S^1} can be viewed as a map $K_G(T^*M, (T^*_G M)_0) \to R(S^1) \cong \mathbb{Z}[t, t^{-1}]$.

Lemma 6.3.3 (Localization formula). *Let (M, J, S^1) be a compact almost complex manifold acted by a circle with isolated fixed points. Given an equivariant bundle $E \in K_{S^1}$, its equivariant index is given by*

$$\mathrm{Ind}_{S^1}(E) = \sum_{p \in M^{S^1}} \frac{E(p)}{\prod_i (1 - t^{-w_i(p)})} \in \mathbb{Z}[t, t^{-1}]$$

(See [2, Lemma 2.2])

The ordinary index of E mentioned above can be computed by its equivariant index, namely

$$\mathrm{Ind}(E) = \lim_{t \to 1} \sum_{p \in M^{S^1}} \frac{E(p)}{\prod_i (1 - t^{-w_i(p)})} \in \mathbb{Z}$$

7 Modular forms

Modular forms are bridges connecting geometry and number theory. Later we will construct a genus using modular forms so that we obtain a correspondence between a certain manifold (geometric object) and a certain modular form (object of number theory).

7.1 The modular group, congruence subgroups, and modular forms

Definition 7.1.1. (1) $SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$. $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}/\{\pm I\}$ *is called modular group.*

(2) *Let* $N \in \mathbb{N}$, *the principal congruence subgroup of level* N, $\Gamma(N)$ *is defined to be* $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1\ (mod\ N),\ b \equiv c \equiv 0\ (mod\ N) \right\}$.

(3) Γ *is a congruence subgroup if* $\Gamma(N) < \Gamma < SL_2(\mathbb{Z})$ *for some* $N \in \mathbb{N}$.

Example 7.1.2. (1) $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0\ (mod\ N) \right\}$

(2) $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1\ (mod\ N),\ c \equiv 0\ (mod\ N) \right\}$

$\Gamma_0(N)$ *and* $\Gamma_1(N)$ *are both congruence subgroups. Moreover, there is a relation* $\Gamma(N) < \Gamma_1(N) < \Gamma_0(N) < SL_2(\mathbb{Z})$.

$SL_2(\mathbb{Z})$ has a natural action on the upper half complex plane $\mathbb{H}$ (so do the congruence subgroups), say

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.

It is easy to check this action is well defined. By direct computation, we have

$$\gamma \cdot \gamma' \cdot (\tau) = (\gamma\gamma') \cdot \tau$$

and

$$Im(\gamma \cdot \tau) = \frac{Im(\tau)}{|c\tau + d|^2} > 0 \ i.e.\ \gamma \cdot \mathbb{H} \subset \mathbb{H}$$

In addition, if $\forall \tau \in \mathbb{H}$, $\gamma(\tau) = \gamma'(\tau)$, then γ and γ' differ from an element in $\{\pm I\}$. Thus the action of $SL_2(\mathbb{Z})$ induces an action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$.

Definition 7.1.3. *Write* $j(\gamma, \tau) = c\tau + d$. *For* $k \in \mathbb{Z}$, *define the weight-k operator* $[\gamma]_k$ *acting on function* $f : \mathbb{H} \to \mathbb{C}$ *as*

$$f[\gamma]_k(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau) = (c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d})$$

Lemma 7.1.4. $\forall\ \gamma,\ \gamma' \in SL_2(\mathbb{Z}),\ \tau \in \mathbb{H}$

(1) $j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau))j(\gamma', \tau)$

(2) $[\gamma\gamma']_k = [\gamma]_k[\gamma']_k$

(3) $\frac{d\gamma(\tau)}{d\tau} = j(\gamma, \tau)^{-k}$

These are all due to easy calculations, so the proof is omitted.

Definition 7.1.5. *Let Γ be a congruence subgroup, $k \in \mathbb{Z}$. $f : \mathbb{H} \to \mathbb{C}$ is called a weakly modular function of weight k with respect to Γ if f is meromorphic on $\mathbb{H}$ and*

$$f[\gamma]_k = f\ \forall \gamma \in \Gamma.\ i.e.\ f(\gamma\tau) = (c\tau + d)^k f(\tau)\ \forall \gamma \in \Gamma$$

Suppose f is a weakly modular function of weight k with respect to Γ. Then

$$\begin{aligned}\Gamma > \Gamma(N) &\Rightarrow \begin{pmatrix}1 & N\\ 0 & 1\end{pmatrix} \in \Gamma\\ &\Rightarrow f(\tau + N) = f(\tau)\\ &\Rightarrow \exists\ the\ minimal\ h\ \in \mathbb{N}\ s.t.\ f(\tau + h) = f(\tau)\\ &\Rightarrow f(\tau) = g(e^{\frac{2\pi i\tau}{h}})\ for\ some\ g \in \mathcal{M}(0 < |z| < 1)\end{aligned}$$

Definition 7.1.6. *Near the origin, g has the Laurent expansion*

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

then

$$f(\tau) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{2\pi in\tau}{h}}$$

We say f is holomorphic at ∞ if $a_n = 0\ \forall\ n < 0$. In this case, we write $f(\infty) = a_0$.

Definition 7.1.7. (1) *The holomophic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k with respect to Γ if f is a weakly modular function and is homolophic at ∞.*
(2) *f is called a cusp form if f is a modular form and f vanishes at ∞.*

It is clear that the modular forms of weight k form a vector space. We denote this vector space as $M_k(\Gamma)$. similarly, the cusp forms form a subspace $S_k(\Gamma)$ of $M_k(\Gamma)$.

Proposition 7.1.8. *Assume k is odd and $-I \in \Gamma$. Then $\forall$ modular form f of weight with respect to Γ is 0.*

Proof. Let $\gamma = -I$. Then $\forall \tau \in \mathbb{H}$,

$$f(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau)) = -f(\tau)$$

which forces $f(\tau)$ to be 0. □

7.2 Modular forms and Eisenstein series for $\Gamma_1(N)$

The action of $SL_2(\mathbb{Z})$ can be extended to $Q \cup \{\infty\}$ by setting $\gamma \cdot \infty := \frac{a}{c}$ and $\frac{x}{0} = \infty$ whenever $x \neq 0$.

Definition 7.2.1. *The elements of the obit space $\Gamma_1(N)\backslash(Q \cup \{\infty\})$ are called cusps of $\Gamma_1(N)$. Since there are only one cusp of $SL_2(\mathbb{Z})$ and the index of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$ is finite, the cusps of $\Gamma_1(N)$ are finite.*

Lemma 7.2.2. *Let $v : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a function, $k > 2$. Then*

$$G_k^v(\tau) := \sum_{m,n \in \mathbb{Z}^2 \backslash \{0\}} \frac{v([m])}{(m\tau + n)^k}$$

is a modular form of weight k with respect to $\Gamma_1(N)$.

Proof. First, G_k^v converges to a holomorphic function on $\mathbb{H}$ (see [12, LEMMA 3.7]). $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tilde{a}N + 1 & b \\ 0 & \tilde{d}N + 1 \end{pmatrix} \in \Gamma_1(N)$,

$$G_k^v(\gamma\tau) = (c\tau + d)^k \sum_{m,n} \frac{v([m])}{((am + cn)m\tau + (bm + dn))^k} = (c\tau + d)^k \sum_{x,y} \frac{v([dx - cy])}{(x\tau + y)^k}$$

Since $[dx - cy] = [(\tilde{d}N + 1)x] = [x]$, we get $G_k^v(\tau) = (c\tau + d)^k G_k^v(\tau)$. □

Proposition 7.2.3. *Let ζ_N be the N-th unit root.*

$$G_{k,N} := -\frac{1}{(2\pi i)^k} \sum_{m,n} \frac{\zeta_N^m}{(m\tau + n)^k}$$

is a modular form of weight k with respect to $\Gamma_1(N)$. This modular form is called Eisenstein series.

Proof. It follows from Lemma 7.2.2 immediately by letting $v([m]) = \zeta_N^m$. □

To compute the Fourier expansion of the Eisestein series, we need the following lemma.

Proposition 7.2.4. *For all even integer $k > 0$,*

$$B_k = \frac{(-1)^{\frac{k}{2}+1} 2k!}{(2\pi)^k} \zeta(k)$$

Proof. See [4, page 279]. □

Lemma 7.2.5. *For every $k > 3$,*

$$\sum_{n \in \mathbb{Z}} \frac{1}{m\tau + n} = \frac{(2\pi i)^k}{(k-1)!} \sum_{p=1}^{\infty} p^{k-1} e^{2\pi i m\tau p}$$

Proof. Consider the expansion of $z\sin z$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

The all zeros are $\{k\pi\}_{k\in\mathbb{Z}}$. Let the partial sum

$$S_m = \sum n = 0^m \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

Denote the set of its zeros as $C_m = \{c_n\}$. By the fundamental theorem of algebra, we have S_m is of the form

$$\prod_{n=1}^{m} (1 - \frac{z^2}{c_n^2}).$$

Let $m \to \infty$. Then $S_m \to z\sin z$, $C_m \to \{k\pi\}_{k\in\mathbb{Z}}$. Thus, formally we have

$$z\sin z = z \prod_{n=1}^{m} (1 - \frac{z^2}{(n\pi)^2})$$

i.e.

$$\sin\pi z = \pi z \prod_{n=1}^{m} (1 - \frac{z^2}{(n\pi)^2}).$$

After taking the logarithm, we have

$$\ln(\sin\pi z) = \ln\pi z + \sum_{n=1}^{m} \ln(1 - \frac{z^2}{n^2}).$$

Then take the derivative and use the Euler formula, we get

$$\frac{1}{z} + \sum_{n=1}^{\infty} (\frac{1}{z-d} + \frac{1}{z+d}) = \pi\cot\pi z = (-\pi i)\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}} = (-\pi i)(2\sum_{n=0}^{\infty} e^{2\pi izn} - 1).$$

Take $(k-1)^{th}$ derivative, we have

$$(-1)^{k-1}(k-1)! \sum_{d} \frac{1}{(z+d)^k} = -2\pi i \sum_{n=0}^{\infty} (2\pi in)^{k-1} e^{2\pi izn}$$

i.e.

$$\sum_{d} \frac{1}{(z+d)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=0}^{\infty} n^{k-1} e^{2\pi izn} \quad .$$

□

Now, compute the Fourier expansion directly

$$
\begin{aligned}
G_{k,N}(\tau) &= -\frac{1}{(2\pi i)^k}\left(\sum_{c=0,d}\frac{1}{(m\tau+n)^k}+\sum_{c\neq 0,d}\frac{1}{(m\tau+n)^k}\right)\\
&= -\frac{1}{(2\pi i)^k}\left(2\zeta(k)+2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{\zeta_N^m}{(m\tau+n)^k}\right)\\
&= -\frac{1}{(2\pi i)^k}2\frac{(2\pi i)^k}{(k-1)!}\left(2\zeta(k)+\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}d^{k-1}e^{2\pi i\tau dm}\right) \quad \textit{by the Lemma } 7.2.5\\
&= -\sum_{q=1}^{\infty}\left(\sum_{d|q}d^{k-1}\frac{\zeta_N^{\frac{q}{d}}}{(k-1)!}\right)e^{2\pi i q\tau}-\frac{2\zeta(k)}{(2\pi i)^k} \quad \textit{by letting } q=dm\\
&= -\sum_{q=1}^{\infty}\left(\sum_{d|q}\left(\frac{q}{d}\right)^{k-1}\frac{\zeta_N^d}{(k-1)!}\right)e^{2\pi i q\tau}+\frac{B_k}{k!} \quad \textit{by the Proposition } 7.2.4
\end{aligned}
$$

Actually, we can extend the the definition of Eisenstein series to the case $k = 1, 2$ by letting

$$
G_{k,N} := -\sum_{q=1}^{\infty}\left(\sum_{d|q}\left(\frac{q}{d}\right)^{k-1}\frac{\zeta_N^d}{(k-1)!}\right)e^{2\pi i q\tau}+\begin{cases}\dfrac{1+\zeta_N}{2(1-\zeta_N)} & if\ k=1,\\ \dfrac{B_k}{k!} & if\ k>1,\end{cases}
$$

Remark 7.2.6. *Usually, the Eisenstein series is defined to be*

$$
\sum_{m,n}(m\tau+n)^{-k}
$$

This Eisenstein series is a modular form with respect to $SL_2(\mathbb{Z})$. More generally, the orthogonal complement of $S_k(\Gamma)$ in $M_k(\Gamma)$ under the Pertersson inner product (see [12, LEMMA 4.32]) is called Eisenstein space. The elements of Eisenstein space can be called Eisenstein series.

The ingredients in this subsection are used to define so-called elliptic genus of level N in the later sections. firmation/acceptance email.

8 Genera

The dimension of the first cohomology group of a surface tell us how many holes does this surface have, which is called "genus". "Genus" maps a surface to an integer i.e. the number of the holes. More generally, genera are maps from the collection of manifolds to some ring. This will tell us some topological data about the manifold and even number theory.

8.1 Genera associated to formal power series

Definition 8.1.1. *We say a manifold M bounds or M is bounded if $\exists$ a compact, oriented manifold N of dimension $n+1$ s.t. $\partial N = M$.*

Definition 8.1.2. *Two manifolds M, N are called cobordant if $M \coprod (-N)$ is bounded. Here "$-$" means the orientation reversed.*

"Corbordant" is an equivalent relation. This definition is natural in topology. For example, two cycles represent the same homology class if they are cobordant.

Definition 8.1.3. *Let Ω^n denote the set of the compact smooth manifold of dimension n, up to cobordant equivalence.*

Let addition be $\coprod$, then $(\Omega, +)$ becomes a finitely generated abelian group. Consider the set

$$\Omega := \bigoplus_{n=0}^{\infty} \Omega^n$$

Since the Cartesian product induces a map

$$\Omega^n \times \Omega^m \to \Omega^{m+n}$$

Ω has a graded ring structure. Moreover, $[M \times N] = (-1)^{mn}[N \times M]$ for $M \in \Omega^m$, $N \in \Omega^n$, and the set of single point is the identity of Ω.

Theorem 8.1.4 (Thom). *$\Omega^n \otimes \mathbb{Q} = 0$ of 4 does not divide n and Ω^{4k} is a finitely generated abelian group with rank equal to the number of partitions of K.*

Due to the fact that $[M \times N] = (-1)^{mn}[N \times M]$ and this theorem, $\Omega \otimes \mathbb{Q}$ becomes a commutative ring.

Theorem 8.1.5. $\Omega \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \cdots]$

Definition 8.1.6. *Let R be an integral domain over $\mathbb{Q}$. A genus is a ring homomorphism $\phi : \Omega \otimes \mathbb{Q} \to R$ with $\phi(1) = 1$.*

There are so many genera. Here is a useful way to construct many genera.

Definition 8.1.7. *let (M, J) be a compact almost complex manifold of real dimension $2n$, R be a ring, and*

$$Q(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

be a normalized formal power series with coefficients $a_k \in R$. The genus associated to Q is defined to be

$$\phi_Q(M) = \int_M \prod_{i=1}^{n} Q(\gamma_i) \in R$$

where γ_i are Chern roots of (M, J).

Remark 8.1.8. (1) *The integral is taken over the terms which can be integrated. i.e. those classes in* $H^{2n}(M)$.

(2) *One have to check* ϕ_Q *is indeed a well-defined genus. This refers to the properties of Pontryjagin classes. For the details, see [7, page 14].*

(3) *In fact,* $\prod Q(\gamma_i)$ *can be expressed in Chern classes. For instance, if* $n = 1, 2, 3$ *respectively, then the integral is respectively taken over*

$$\begin{aligned} Q_1 =& a_1 c_1. \\ Q_2 =& a_2 c_1^2 + (a_1^2 - a_2) c_2. \\ Q_3 =& a_3 c_1^3 + (a_1 a_2 - 3a_3) c_1 c_2 + (a_1^3 + 3a_3 - 3a_1 a_2) c_3. \end{aligned}$$

where c_i *are Chern classes of* (M, J).

Example 8.1.9. (1) *Let*

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots.$$

Then the associated genus is called the Todd genus, which is just the integral of the Todd class defined in the subsection 4.3.

(2) *Let*

$$Q(x) = \frac{\sqrt{x}}{tanh(\sqrt{x})} = \sum_{i=0}^{\infty} \frac{2^{2i} B_{2i} x^i}{(2i)!} = 1 + \frac{x}{3} - \frac{x^2}{45} + \cdots.$$

Then the associated genus is called the L genus.

Hirzebruch's signature theorem says if $\dim(M, J) = 4k$, *then*

$$Sgn(M) = \int_M L(M).$$

(3) *Let*

$$Q(x) = \frac{\frac{\sqrt{z}}{2}}{sinh(\frac{\sqrt{z}}{2})} = 1 - \frac{x}{24} + \frac{7x^2}{5760} - \cdots.$$

Then the associated genus is called the $\hat{A}$ *genus.*

The $\hat{A}$ *genus of a spin manifold is an integer. Moreover, if the dimension of the manifold is 4 mod 8, then it is an even integer. In general, the* $\hat{A}$ *genus is not always an integer.*

By Atiyah-Singer index theorem, the $\hat{A}$ *genus equals to the index of its Dirac operator.*

8.2 Hirzebruch's χ_y-genus

Definition 8.2.1 (Hirzebruch χ_y genus). *The Hirzebruch χ_y is the $\mathbb{Q}[y]$-valued genus associated to the power series*

$$Q(x) = \frac{x}{1-e^{-x}}(1+ye^{-x}) = 1 + y + \sum_{k=1}^{\infty} a_k(y)x^k.$$

where

$$a_k(y) = \sum_{n=0}^{k} \frac{B_n}{n!(k-n)!} + \frac{B_k}{k!}y.$$

Definition 8.2.2 (Another definition of Hirzebruch χ_y genus). *Let M be a compact complex manifold of dimension $2n$. The Hirzebruch χ_y genus can be defined to be*

$$\chi_y(M) := \sum_{p=0}^{n} \chi(M, \Omega_M^p)y^p = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(M)y^p,$$

where $h^{p,q}(M)$ are Hodge numbers.

Before proving these two definitions are equivalent, we see some examples first.

Example 8.2.3. (1) $\chi_0(M) = \chi(M, \mathcal{O}_M)$ *is the arithmetic genus.*

(2) *If M is a Kähler manifold of dimension $4k$, then the Hodge numbers are symmetric. Thus*

$$\chi_1(M) = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(M) = Sgn(M)$$

(see [9, Corollary 3.3.18]).

(3) *If M is a compact Kähler manifold, then*

$$\chi_{-1} = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(M) = \sum_{k=0}^{2n} (-1)^k b_k(M) = e(M)$$

is the Euler number of M.

Lemma 8.2.4. *Let M be a compact complex manifold. γ_i are Chern roots of M. Then*

$$\mathrm{ch}\left(\bigoplus_{p=0}^{n} \Omega_M^p y^p\right) = \prod_{i=1}^{n} (1 + ye^{-\gamma_i})$$

Proof.

$$\begin{aligned}\mathrm{ch}\left(\bigoplus_{p=0}^{n}\Omega_M^p y^p\right) &= \sum_{p=0}^{n}\mathrm{ch}(\Omega_M^p)y^p\\ &= \sum_{p=0}^{n}\mathrm{ch}(\bigwedge^p T^*M)y^p\\ &= \sum_{p=0}^{n}\sum_{i_1,\cdots,i_p} e^{-\gamma_{i_1}-\cdots-\gamma_{i_p}}y^p\\ &= \prod_{i=1}^{n}(1+ye^{-\gamma_i})\end{aligned}$$

□

Proposition 8.2.5. *The two definitions above are equivalent.*

Proof. By Hirzebruch-Riemann-Roch Theorem and the lemma above, we have

$$\begin{aligned}\chi_y(M) &= \sum_{p=0}^{n}\chi(M,\Omega_M^p)y^p\\ &= \chi\left(\bigoplus_{p=0}^{n}\Omega_M^p y^p\right)\\ &= \int_M \mathrm{ch}\left(\bigoplus_{p=0}^{n}\Omega_M^p y^p\right)\mathrm{td}(M)\\ &= \int_M \prod_{i=1}^{n}\frac{x}{1-e^{-\gamma_i}}(1+ye^{-\gamma_i})\end{aligned}$$

□

Corollary 8.2.6. (1) *Let $y=0$. Then we have the arithmetic genus*

$$\chi(M,\mathcal{O}_M) = \int_M \mathrm{td}(M)$$

(2) *Let $y=1$. Then we again have Hirzebruch's signature theorem for Kähler manifold of dimension $4k$:*

$$Sgn(M) = \int_M L(M)$$

(3) *Let $y=-1$. Then we have Chern-Gauß-Bonnet formula for compact Kähler manifold of dimension n:*

$$E(M) = \int_M c_n(M)$$

Example 8.2.7. *Let $M = \mathbb{CP}^2$. We have know that*

$$\chi_y(M) = \int_M Q_2$$

where Q_2 is as in Remark 8.1.8.

By direct computation, we have

$$a_1 = \frac{1}{2} - \frac{y}{2},\ a_2 = \frac{1}{12} + \frac{y}{12}.$$

By Example 4.1.4 (3), we have

$$c_1 = \binom{2+1}{1}x = 3x,\ c_2 = \binom{2+1}{2}x^2 = 3x^2.$$

Thus,

$$\chi_y(M) = \int_M (\frac{1}{12} + \frac{y}{12})(3x)(3x) + ((\frac{1}{2} - \frac{y}{2})^2 - 2(\frac{1}{12} + \frac{y}{12}))(3x^2) = y^2 - y + 1$$

Also, by the another definition, one can easily get the same consequence.

8.3 Elliptic genera of level N

In this subsection, we will introduce an important genus, called elliptic genus of level N, which is the central object in the rest sections.

Start from the Jacobi theta function

$$\vartheta(\tau, z) := \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi(n+\frac{1}{2})z} e^{\pi i (n+\frac{1}{2})^2 \tau},$$

with $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$.

Jacobi theta function is invariant under the action of a discrete subgroup of the Heisenberg group (the group of 3×3 upper triangular matrices).

Definition 8.3.1. *Let $\vartheta'(\tau, z)$ denote $\frac{d}{dz}\vartheta(\tau, z)$. The elliptic genus ϕ_N of level N is defined to be the genus associated with the power series*

$$\Theta_N(x) = \frac{x}{2\pi i}\vartheta'(\tau, 0)\frac{\vartheta(\tau, \frac{x}{2\pi i} - \frac{1}{N})}{\vartheta(\tau, \frac{x}{2\pi i})\vartheta(\tau, -\frac{1}{N})} = 1 + a_1(\tau)x + a_2(\tau)x^2 + \cdots$$

It is denoted as $\phi_N(M)$.

Lemma 8.3.2. *The coefficients $a_i(\tau) = G_{k.N}$, the Eisenstein series defined in Proposition 7.2.3. Thus $a_i(\tau)$ are modular forms of weight i for $\Gamma_1(N)$.*

Proof. See [2, Lemma 2.8]. □

Let M be an almost compact complex manifold of dimension $2n$. According to the definition of the genus associated to a formal power series,

$$\phi_N(M) = \int_M \prod_i \Theta(\gamma_i)$$

The integral is taken over the terms which has the degree $2n$. Thus the coefficients of these terms are modular forms of weight n for $\Gamma_1(N)$ i.e. the elliptic genus of M, $\phi_N(M)$ is a modular form of weight n for $\Gamma_1(N)$.

Lemma 8.3.3. *Let E be a complex vector bundle of rank n over a manifold. If we write*

$$\wedge_t E := \sum_{i=0}^{\infty} (\wedge^i E) t^i \text{ and } S_t E := \sum_{i=0}^{\infty} (S^i E) t^i$$

Then

$$\mathrm{ch}(\wedge_t E) = \prod_{i=1}^{\infty} (1 + te^{\gamma_i}) \text{ and } \mathrm{ch}(S_t E) = \prod_{i=1}^{n} \frac{1}{1 - te^{\gamma_i}}$$

Proof. Since the Chern roots of $\wedge^i E$ is given by $\{\gamma_{j_1} + \cdots + \gamma_{j_i}\}$, we have

$$\mathrm{ch}(\wedge^i E) = \sum_{j_1, \cdots, j_i} e^{\gamma_{j_1} + \cdots + \gamma_{j_i}}.$$

Thus,

$$\mathrm{ch}(\wedge_t E) = \sum_{i=1}^{n} \sum_{j_1, \cdots, j_i} e^{\gamma_{j_1} + \cdots + \gamma_{j_i}} t^i = \prod_{i=1}^{\infty} (1 + te^{\gamma_i}).$$

For $S_t E$, see [7,page 175]. □

Write the elliptic genus of level N as

$$\phi_N(M) = \int_M \prod_i \Theta(\gamma_i) = m(q)^n \int_M \prod_i \tilde{\Theta}(\gamma_i) =: m(q)^n \tilde{\phi}_N(M),$$

where

$$m(q) := \frac{1}{1 - q^{\frac{1}{N}}} \prod_{r=1}^{\infty} \frac{(1 - q^r)^2}{(1 - q^{r + \frac{1}{N}})(1 - q^{r - \frac{1}{N}})}$$

The function $\tilde{\phi}_N(M)$ is called the normalized elliptic genus of level N.

The virtual bundle $R_N(q)$ is defined to be

$$R_N(q) := \wedge_{q^{\frac{1}{N}}} T^*M \otimes \bigotimes_{r=1}^{\infty} \wedge_{-q^{r+\frac{1}{N}}} T^*M \otimes \wedge_{-q^{r-\frac{1}{N}}} TM \otimes S_{q^r} T^*M \otimes S_{q^r} TM$$

Its Chern character can be computed by the Lemma 8.3.3:

$$\mathrm{ch}(R_N(q)) = \prod_{i=1}^{n} (1 - e^{-\gamma_i} q^{\frac{1}{N}}) \prod_{r=1}^{\infty} \frac{(1 - e^{-\gamma_i} q^{r + \frac{1}{N}})(1 - e^{\gamma_i} q^{r - \frac{1}{N}})}{(1 - e^{-\gamma_i} q^r)(1 - e^{\gamma_i} q^r)}.$$

Proposition 8.3.4. *The normalized elliptic genus of level N is the index of the virtual bundle, i.e.*

$$\tilde{\phi}_N(M) = \mathrm{Ind}(R_N(q)).$$

Proof.

$$\begin{aligned}
\tilde{\phi}_N(M) &= \int_M \prod_i \frac{\gamma_i \frac{1-e^{-\gamma_i}\zeta_N}{(1-e^{-\gamma_i})(1-\zeta_N)} \prod_r \frac{(1-e^{-\gamma_i}\zeta_N q^r)(1-e^{\gamma_i}\zeta_N^{-1}q^r)(1-q^r)^2}{(1-e^{-\gamma_i}q^r)(1-e^{\gamma_i}q^r)(1-\zeta_N q^r)(1-\zeta_N^{-1}q^r)}}{\frac{1}{1-\zeta_N}\prod_r \frac{(1-q^r)^2}{(1-\zeta_N q^r)(1-\zeta_N^{-1}q^r)}} \\
&= \int_M \prod_i (1-e^{-\gamma}\zeta_N) \prod_r \frac{(1-e^{-\gamma_i}\zeta_N q^r)(1-e^{\gamma_i}\zeta_N^{-1}q^r)}{(1-e^{-\gamma_i}q^r)(1-e^{\gamma_i}q^r)} \cdot \frac{\gamma_i}{1-e^{-\gamma_i}} \\
&= \int_M \mathrm{ch}(R_N(q)) \cdot \mathrm{td}(M) \\
&= \mathrm{Ind}(R_N(q))
\end{aligned}$$

□

8.4 Rigidity of the elliptic genera

In this section, we will define so-called equivariant elliptic genus of level N, which is similar to the non-equivariant one but it in addition relays on a parameter t. The rigidity theorem says that the equivariant elliptic genus of level N is in fact independent of this parameter. This theorem can construct a bridge between geometry and number theory.

Definition 8.4.1. *Let (M^{2n}, J, S^1) be an almost complex manifold with a circle action. If $\forall\, p, q \in M^{S^1}$, we have*

$$w_1(p) + \cdots + w_n(p) \equiv w_1(q) + \cdots + w_n(q) \ (modN)$$

for some $N \in \mathbb{N}^+$, then the circle action is called N-balanced. The common residue class of $w_1(q) + \cdots + w_n(q)(modN)$ is called the type of the action.

Definition 8.4.2. *The index of an almost complex manifold (M, J) is the largest integer k_0 s.t. modulo torsion elements, $c_1 = k_0$ for some nonzero $\eta \in H^2(M)$.*

Proposition 8.4.3. *Let (M^{2n}, J, S^1) be an almost complex manifold with a circle action preserving J. If $N|k_0$, then the circle action is N-balanced*

Sketch of proof. Case 1: Suppose p, q are in the same component of M^{S^1}. Since $\forall z \in S^1$, $d\mu_z(p)$ is smooth in p, $(w_1(p) + \cdots + w_n(p)$ is also smooth in p. By the connectedness, we have $(w_1(p) + \cdots + w_n(p)$ is constant.

Case 2: Suppose p, q are in the same component of M^{S^1}. Recall that the equivariant Chern class $c_1^{S^1}(p) = (w_1(p) + \cdots + w_n(p))x$ for some $x \in H^2_{S^1}(\{p\})$ by the Lemma 4.2.5. Thus it suffices to show $\forall\ p, q \in M^{S^1}$,

$$\frac{c_1^{S^1}(p) - c_1^{S^1}(q)}{Nx} \in \mathbb{Z}.$$

Since the dimension of M is large enough with respect to the dimension of M^{S^1}, there is a path from p to q and the image of the interval $(0, 1)$ avoids other fixed points. Rotating the path, we get a sphere S. Then to show

$$\frac{c_1^{S^1}(p) - c_1^{S^1}(q)}{Nx} = \int_S \frac{c_1}{N}.$$

Since $\frac{c_1}{N} \in H^2(M)$, the integral must be an integer. □

Definition 8.4.4. *An S^1-equivriant bundle E is said to be rigid if its equivariant index* $\mathrm{Ind}_{S^1}(E)$ *is independent of t, hence it lies in $\mathbb{Z} \subset Z[t, t^{-1}]$.*

If a bundle E is S^1-equivariant, then

$$\mathrm{Ind}_{S^1}(E) = \mathrm{Ind}_{S^1}(E)|_{t=1} = \mathrm{Ind}(E).$$

The virtual bundle

$$R_N(q) := \wedge_{q^{\frac{1}{N}}} T^*M \otimes \bigotimes_{r=1}^{\infty} \wedge_{-q^{r+\frac{1}{N}}} T^*M \otimes \wedge_{-q^{r-\frac{1}{N}}} TM \otimes S_{q^r} T^*M \otimes S_{q^r} TM$$

is a formal power series in q. Thus we can write

$$R_N(q) = \sum_{i=0}^{\infty} R_i q^i.$$

By the Proposition 8.3.4,

$$\phi_N(M^{2n}) = m(q)^n \tilde{\phi}_N(M) = m(q)^n \sum_{i=0}^{\infty} \mathrm{Ind}(R_j) q^i$$

Definition 8.4.5. *The equivariant elliptic genus $\phi_N(M, t)$ of level N is defined to be*

$$\phi_N(M, t) := m(q)^n \sum_{i=0}^{\infty} \mathrm{Ind}_{S^1}(R_j) q^i.$$

It is said to be rigid if $\forall i$, R_i is rigid.

Theorem 8.4.6 (Rigidity of elliptic genus of level N). *Let (M, J, S^1) be a compact almost complex manifold with index k_0 and the circle action preserves J. $\forall\ N \in \mathbb{N}^+$ with $N|k_0$, the elliptic genus of level N is rigid, hence it equals the non-equivariant elliptic genus $\phi_N(M)$. If the type of the action is not zero $(modN)$, then $\phi_N(M) \equiv 0$.*

Proof. See [7, page 169-185]. □

Initially, the rigidity theorem for the case $N = 2$ and on a spin manifold was conjectured by Witten. Finally, Hirzebruch generalized this theorem and get the result for all N and removed the condition that the manifold is spin.

9 Relations of Eisenstein series

The modular forms of a certain weight can form a vector space. Thus, given enough number of modular forms of the same weight, they must be linearly dependent i.e. there is a linear relation among them. However, to find such a explicit relation is not easy in general. The good news is that the rigidity theorem can tell us some of these relations.

Theorem 9.0.1. *Let (M^{2n}, J, S^1) be a compact, connected, almost manifold which is acted effectively by a circle with a non-empty set of isolated fixed points. Suppose $N \in \mathbb{N}$ divides the index k_0. Then, for $k > n$ we have the following relations of products of Eisenstein series*

$$\sum_{I \in P_n(k)} \left(\sum_{p \in M^{S^1}} \frac{m_I(w_1(p), \cdots, w_n(p))}{w_1(p) \cdots w_n(p)} \right) G_{I,N} = 0,$$

where $P_n(k)$ is the set of all partitions of k with at most n parts, $m_I(x_1, \cdots, x_n)$ denotes the monomial symmetric polynomial, and $G_{I,N} = G_{i_1} \cdots G_{i_n}$ for $I = [i_1, \cdots, i_n]$.

Sketch of proof. Consider the equivairant elliptic genus $\phi_N(M,t)$ and let $t = e^{2\pi i z}$. By direct comuptation, we have the coefficient of z^{k-n} of the Taylor expansion of $\phi_N(M,t)$ is exactly

$$\sum_{I \in P_n(k)} \left(\sum_{p \in M^{S^1}} \frac{m_I(w_1(p), \cdots, w_n(p))}{w_1(p) \cdots w_n(p)} \right) G_{I,N}.$$

The rigidity theorem 8.4.6, $\phi_N(M,t) \equiv \phi_N(M,1)$. Thus, when $k > n$, the coefficient of z^{k-n} must be 0. □

Example 9.0.2. *Let $M=\mathbb{CP}^2$. By Example 4.1.4 (3), we have $c_1(\mathbb{CP}^2) = 3x$. Thus the index k_0 of $\mathbb{CP}^2$ is 3. Let $N = 3$. Consider the following S^1-action on M:*

$$s \cdot [z_0 : z_1 : z_2] := [z_0 : s^x z_1 : s^y z_2]$$

for some $x, y \in \mathbb{Z}\backslash\{0\}$. There are three fix points, $p = [1 : 0 : 0]$, $q = [0 : 1 : 0]$ and $r = [0 : 0 : 1]$. Near p, take the canonical chart. Then the action is of the form $s \cdot \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) = \left(s^x \cdot \frac{z_1}{z_0}, s^y \cdot \frac{z_2}{z_0}\right)$, so the weight at p is $w_1(p) = x$, $w_2(p) = y$. The action *near q is of the form* $s \cdot \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right) = \left(s^{-x} \cdot \frac{z_1}{z_0}, s^{y-x} \cdot \frac{z_2}{z_0}\right)$, *so the weight at q is* $w_1(q) = -x$, $w_2(q) = y - x$. *Similarly,* $w_1(q) = x - y$, $w_2(q) = -y$.

By the Theorem 9.0.1, when $k > 2$ *we have the following relation:*

$$\sum_{I \in P_2(k)} \left(\frac{m_I(x,y)}{xy} + \frac{m_I(-x, y-x)}{-x(y-x)} + \frac{m_I(-y, x-y)}{-y(x-y)} \right) G_{I,3} = 0 \quad (*)$$

Let $k = 3,\ x = 1, y = -1$. *Then* $(*)$ *becomes*

$$15G_{4,3} + 12G_{1,3}G_{3,3} + 3G_{2,3}^2 = 0$$

i.e.

$$5G_{4,3} + 4G_{1,3}G_{3,3} + G_{2,3}^2 = 0.$$

Let $k = 4,\ x = 1,\ y = 3$. *Then* $(*)$ *becomes*

$$20G_{5,3} + 0G_{1,3}G_{4,3} - 20G_{2,3}G_{3,3} = 0$$

i.e.

$$G_{5,3} = G_{2,3}G_{3,3}.$$

Similarly, let $k = 5, 6$ *respectively. We have*

$$4G_{1,3}G_{5,3} + 2G_{2,3}G_{4,3} + G_{3,3}^2 + 7G_{6,3} = 0$$

and

$$-G_{2,3}G_{5,3} - G_{3,3}G_{4,3} + 2G_{7,3} = 0.$$

These can also be checked by using the Fourier expansions. Furthermore, we can apply the Theorem 9.0.1 to other manifolds (for example, $\mathbb{CP}^n$) to get more relations of the Eisenstein series.

A Morse theory

Definition A.0.1. *Let* $f \in C^\infty(M)$. *p is called critical if* $\mathrm{d}f\,|_p = 0$

Definition A.0.2. $\mathrm{Hess} f$ *is a tensor of type (0,2) defined by* $\mathrm{Hess} f_p(V, W) = W(\tilde{V}(f))(p)$, *where* $V, W \in T_pM$ *and* $\tilde{V}$ *is a smooth extension of V. In a local coordinate,* $\mathrm{Hess} f|_p = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$

It is easy to show that this definition is independent of the extension of V. Hence it is well-defined.

Definition A.0.3. *A critical point p is called non-degenerate if* $\mathrm{Hess} f_p$ *is non-singular.*

This definition of Hess is different from that defined in Riemannian Geometry in which Hess$f$ is defined as $\nabla^2 f$ so that Hess$f(V,W) = W(V(f)) - (\nabla_W V)(f)$

Lemma A.0.4 (Morse). *Let p be a non-degenerated critical point of f. Then $\exists$ a local coordinate $(U, x_1, \cdots, x_n)$ with $x_i(p) = 0$ s.t.*

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

Proof. In an arbitrary local coordinate (U,$y_1, \cdots, y_n$),

$$f(y) = f(p) + \sum y_i y_j \int_0^1 (1-t) \frac{\partial^2 f}{\partial y_i \partial y_j} \, \mathrm{d}t$$

(see[16], 13). Note that

$$\int_0^1 (1-t) \frac{\partial^2 f}{\partial y_i \partial y_j} \, \mathrm{d}t$$

is symmetric so that it can be normalized. The complete proof can be found in [13], page 6. □

Definition A.0.5. *$f \in C^\infty(M)$ is called a Morse function if*

(1) Each critical point is non-degenerated

(2) The critical points have distinct values.

In some books, the definition of the Morse functions does not require the condition(2). We will see what role does this condition play in the Morse theory later.

Corollary A.0.6. *If f is a Morse function, then* Crit(f) *is discrete where* Crit(f) *is the set of all critical points of f.*

Proof. It follows from the Morse Lemma immediately. □

Here is a question. Given a smooth manifold M, does M admit a Morse function? How general are the Morse functions?

Proposition A.0.7. *Let $M \subset \mathbb{R}^N$ be a submanifold of dimension n. Then for almost every $p \in \mathbb{R}^N$, the function $f_p : M \to \mathbb{R},, x \mapsto \|x-p\|^2$ is a Morse function*

Proof. The derivative of f_p is given by $\mathrm{d}f_{p,x}(v) = 2\langle x-p, v\rangle$. Therefore the critical points occur exactly when T_xM is normal to $x-p$. Choose a local coordinate $(u_1, \cdots, u_n)$ around x. Then

$$\frac{\partial f_p}{\partial u_i} = 2(x-p) \cdot \frac{\partial x}{\partial u_i}, \quad \frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + (x-p) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}).$$

By definition, x is a non-degenerate critical point iff x-p is normal to T_xM and the matrix on the right is non-degenerate. By Sard's Thm, it suffices to show that the $p \in \mathbb{R}^N$

s.t. $x-p$ is normal to T_xM and the matrix on the right is singular, are exactly the critical points of some smooth map.

To find such a map, consider the normal bundle NM of M in $\mathbb{R}^N$. Define the map $NM \to \mathbb{R}^N$, $(x,v) \mapsto x+v$. It can be verified that $p = x+v$ is a critical point iff $2(\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + v\frac{\partial^2 x}{\partial u_i \partial u_j})$ is singular. Hence, this function is what we need. □

Remark A.0.8. *By Whitney embedding theorem, every smooth manifold can be embedded in some Euclidean space. Thus for any smooth manifold, there exists a Morse function on it.*

Corollary A.0.9. *The Morse functions are open and dense in $C^\infty(M)$.*

Although almost every smooth function on a manifold is a Morse function, there are still many functions which are not Morse function. Here are some non-examples:

(1)The "height function" of a horizontal torus. The critical points form two circle which are not discrete, so is not a Morse function. However, it is a so-called Morse-Bott function which is a generalization of Morse functions.

(2)$f(x) = x^3$. Even this usual function is not a Morse function.

(3)$f(x,y) = x^2y^2$. $f^{-1}(0)$ is not a manifold so it is not even a Morse-Bott function.

Definition A.0.10. *Let f be a Morse function. λ in the lemma 2.2.4 is called the index of f at p and it is denoted as $\mathrm{Ind}_f(p)$. Equivalently, $\mathrm{Ind}_f(p)$ is the dimension of the largest subspace of T_pM on which Hess_f is negative define.*

One can easily figure out the indices of the "height function" h at the critical points in the initial example. i.e. $\mathrm{Ind}_h(a_1) = 0$, $\mathrm{Ind}_h(a_2) = 1$, $\mathrm{Ind}_h(a_3) = 1$, $\mathrm{Ind}_h(a_4) = 2$.

Recall that every manifold can be equipped with a Riemannian metric. The torus in the initial example is embedded in $\mathbb{R}^3$ so that we can consider the "height" because it is automatically a Riemannian manifold.

Recall that if f is a smooth function on a Riemannian manifold (M,g), then the gradient $\mathrm{grad}f$ is defined to be $(\mathrm{d}f)_\#$. In a local coordinate, $\mathrm{grad}f = \frac{\partial f}{\partial x_i} g^{ij} \frac{\partial}{\partial x_j}$.

Theorem A.0.11 (Regular interval theorem). *Suppose $f : M \to [a,b]$ be a smooth map on a compact Riemannian manifold with boundary. Suppose that f has no critical points and that $f(\partial M) = \{a,b\}$. Then there is a diffeomorphism*

$$F : f^{-1}(a) \times [a,b] \to M$$

s.t. $\pi = f \circ F$ where π is the projection from $f^{-1}(a) \times [a,b]$ to $[a,b]$.

Sketch of proof. Let $\eta_x(t)$ be the integral curve of $\frac{\text{grad} f}{|\text{grad} f|^2}$. Then $F(x,t) := \eta_x(t)$ is such a function. $\square$

There is also a intuitive explanation. Note that $\text{grad} f_p = 0 \Leftrightarrow p \in \text{Crit}(f)$ and $\text{grad}(f)$ is orthogonal to the level sets. Then each point of each level can flow along $\frac{\text{grad} f}{|\text{grad} f|^2}$ to another level set.

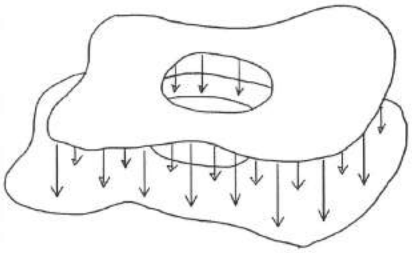

Figure 2: flow along $\frac{\text{grad} f}{|\text{grad} f|^2}$

Corollary A.0.12 (Fundamental theorem 1). *Let M be a compact manifold, and $f : M \to \mathbb{R}$ a Morse function. Let $a < b$ and suppose that $f^{-1}[a,b]$ contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.*

Proof. By the regular interval theorem, there is such F. Since $f^{-1}(a) \times \{a\}$ is a deformation retract of $f^{-1}(a) \times [a,b]$, we see that $f^{-1}(a)$ is a deformation retract of $f^{-1}([a,b])$. We can now paste this deformation retraction with the identity on M^a to obtain the deformation retraction from M^b to M^a.

To prove that M^a is diffeomorphic to M^b we apply the same principle, but we need to be more careful to preserve smoothness during the patching process.

Since $\text{Crit}(f)$ is a closed subset of of the compact M, it is also compact. Therefor there are real numbers c and d with $c < d < a$ so that there are no critical values in $[c,b]$.

By the regular interval theorem there is a natural diffeomorphism F from $f^{-1}([c,b])$ to $f^{-1}c \times [c,b]$, that maps $f^{-1}[c,a]$ diffeomorphically onto $f^{-1}c \times [c,a]$. There is also a diffeomorphism $H : f^{-1}c \times [c,b] \to f^{-1}c \times [c,a]$, and we can insist that it be the identity on $f^{-1}c \times [c,d]$. Thus

$$F^{-1} \circ H \circ F : f^{-1}([c,b]) \to f^{-1}([c,a])$$

is a diffeomorphism that is the identity on $f^{-1}([c,d])$, and thus we can patch it together with the identity on M^d to creat a diffeomorphism from M^b to M^a. $\square$

Corollary A.0.13 (Reeb). *If M is a compact manifold and f is a Morse function on it with only two critical points, then M is homeomorphic to a sphere.*

Theorem A.0.14 (Fundamental theorem 2). *Let $f \in C^\infty(M)$, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contains no critical point of f other than p, for some $\epsilon > 0$. Then $\forall$ sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

The proof is omitted. But one can check this theorem by applying it to the previous example. It is benefit for you to think about what submanifold is attached while passing a critical point in the example.

Recall the Betti numbers are the ranks of the homology groups. Homology groups are related to the CW structure. The fundamental theorem 2 tells us the CW structure of a manifold is related to the critical points of a Morse function. So what is the relation between the Betti numbers and the number of the critical points?

Theorem A.0.15 (Morse inequalities). *Let f be a Morse function on a manifold, b_k be the k^{th} Betti number, and c_k be the number of critical points of index k. Then*

$$(1)\ c_k(f) \geq b_k(M) \quad \forall k$$
$$(2)\ \sum_{k=0}^{n}(-1)^k c_k \geq \sum_{k=0}^{n}(-1)^k b_k$$

(1) *is called weak Morse inequality and* (2) *is called strong Morse inequality.*

Proof. We only prove the weak one. We may regard M as $M^{+\infty}$. When a passes a critical point of index λ, then there is a λ cell attached to M^a by the fundamental theorem 2. Since attaching a λ cell may or may not cause that $b_k(M^a)$ plus 1, we get the weak Morse inequality. □

Corollary A.0.16. *If $c_{\lambda+1} = c_{\lambda-1} = 0$, then $c_\lambda = b_\lambda$ and $b_{\lambda+1} = b_{\lambda-1} = 0$.*

Corollary A.0.17. *Let f be a Morse function on M. Then f has at least as many critical points as the sum of the ranks of the homology groups of M.*

Definition A.0.18. *Let $f \in C^\infty(M)$. A compact connected submanifold $S \subset M$ is said to be a critical submanifold if $S \subset \mathrm{Crit}(f)$ and $\mathrm{Hess}_f|_{\nu^M S}$ is non-degenerate.*

Definition A.0.19. *We say $f \in C^\infty$ is a Morse-Bott function if its critical points are a finite disjoint union of critical submanifids.*

$\forall s \in S$, we can define the index of s as the index of $\mathrm{Hess}_f|_{\nu^M S \times \nu^M S}$. Since the index is locally constant, it can be extended to defining on a component of $\mathrm{Crit}(f)$.

Definition A.0.20. *Let $f \in C^\infty(M)$ is a Morse-Bott function and* $\operatorname{Crit}(f) = \coprod_k S_k$ *with S_k critical submaifolds of f.*

(1) *The Poincaré polynomial is defined to be*

$$P_M(t) := \sum_k b_k(M) t^k$$

(2) *The Morse polynomial is defined to be*

$$M_f(t) := \sum_k t^{\lambda(S_k)} P_{S_k}(t)$$

There is a partial relation $\preceq$ on $\mathbb{Z}[t]$: we say $P \preceq Q$ if $\exists R \in \mathbb{Z}[t]$ s.t.

$$Q = P + (1+t)R$$

Theorem A.0.21 (Morse-Bott inequality). *Let f be a Morse-Bott function on a compact manifold M. Then*

$$M_f \preceq P_M$$

A Morse-Bott function f is called perfect if $M_f = P_M$.

References

[1] M. F. Atiyah: *Elliptic operators and compact groups.* Springer, Lecture notes in Mathematics,401, 1974.

[2] K. Bringmann, A. C. Castro, S. Sabatini, M. Schwagenscheidt, *Rigidity of Elliptic Genera: from Number Theory to Geometry and Back*, arXiv:2001.11072, 2020.

[3] H. Duistermaat, *Symplectic Geometry*, Spring School, June 7-14, 2004, Utrecht.

[4] A. George, *Mathematical methods for physicists* (2nd ed.), Academic Press, Inc (1970), ISBN 978-0120598518

[5] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press (2000).

[6] A. Hatcher, *Vector bundles and K theory*, `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`, 2003.

[7] F. Hirzebruch, T. Berger, and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics 20 (1992), xii+211

[8] D. Husemoller, *Fiber Bundle*, MacGraw - Hill, 1966, Springer-verlag, 1975.

[9] Huybrechets, *Complex Geometry: An Introduction*, Springer Science Business Media (2004).

[10] H. B. Lawson and M. L Michelsohn, *Spin geometry*, Princeton, UK: Univ. Pr. (1998).

[11] Y. Lin and R. Sjamaar,Convexity properties of presymplectic moment maps, arXiv:1706.00520v2, 2017

[12] J. Milne, *Modular functions and modular forms* (v1.31), `https://www.jmilne.org/math/CourseNotes/MF.pdf`, 2017.

[13] J. Milnor, *Morse Theory*, Annals of Mathematics Studies 51 (Princeton University Press, 1963), vi + 153 pp, 24s - Volume 14 Issue 1.

[14] M. Nabil, *Universal Principal Bundles And Classifying Spaces*, `https://www.mims-institut.org/webroot/uploads/papers/MIMS_1532264109.pdf`, 2018.

[15] L. Nicolaescu,*Topics in topology*, `https://www3.nd.edu/~lnicolae/MS.pdf#page83`, 2008.

[16] Frank W. Warner, *Foundations of Diferentiable Manifolds and Lie Groups*, Springer, 1971.

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