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本科生毕业设计（论文）

题　　目： **类西塔函数：拉马努金的例子、兹维格斯的构造以及其在组合学中的应用**

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Thesis Title: **Mock theta functions: Ramanujan's examples, Zwegers's constructions and combinatorial applications**

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Mock theta functions: Ramanujan's examples, Zwegers's constructions and combinatorial applications

Zeyang Ding

（Department of Mathematics　Tutor：Yifei Zhu）

[摘要]：本毕业论文是一篇关于兹维格斯博士论文的阅读报告，该论文在数学领域取得了突破性成果，发现了类西塔函数的内在特征。论文还将参考其他文献资料，包括 Zagier 的 Bourbaki 讲座、Bringmann 等的最新的专著以及 Serre 的经典数论教材，以进一步加深对这一开创性研究及其背景与发展动态的理解。论文共分为四个主要章节。在第一章中，我们将深入研究模形式的定义和性质, 特别是是其微分算子，并探讨其在数学背景下的各种应用。这些基础知识将为更深入地研究类西塔函数奠定基础。在第二章中，我们将介绍拉马努金的 17 个类西塔函数的例子，并对它们进行分析，以获得对理解论文后续内容至关重要的见解和观察。第三章将重点介绍兹维格斯的创新性工作，遵循他的博士论文的结构和内容。我们将讨论他在确定类西塔函数内在特征方面的方法以及他的发现在数学领域带来的更广泛的影响。最后，在第四章中，我们将探讨基于兹维格斯研究的各种应用，特别是组合学中某些计数统计量的生成函数。通过研究他的发现的实际应用，我们可以更好地理解他的工作的重要性及其对未来数学研究的潜在影响。

[关键词]：模形式、类西塔函数

[**ABSTRACT**]: This thesis aims to provide a comprehensive reading report on Sander Zwegers's Ph.D. dissertation, which made a breakthrough in the field of mathematics by unearthing the missing intrinsic characterization of Ramanujan's famous mock theta functions. The thesis will also draw upon additional reference materials, including Zagier's Bourbaki lecture, Bringmann and others' latest monograph, and Serre's classic number theory textbook to further enhance the understanding of this groundbreaking research.

The thesis consists of four main chapters. In the first chapter, we will delve into the definition and properties of modular forms, especially differential operators of them, exploring their various applications in mathematical contexts. This foundational knowledge will set the stage for a more in-depth examination of mock theta functions.

In the second chapter, we will introduce Ramanujan's seventeen original examples of mock theta functions from the background of history and analyze them to gain insights and observations that will be crucial for understanding the subsequent material in the thesis.

The third chapter will focus on the innovative work of Sander Zwegers, following the structure and content of his Ph.D. dissertation. We will discuss his approach to identifying the missing intrinsic characterizations of mock theta functions and the implications of his findings for the broader mathematical community.

Lastly, in the fourth chapter, we will explore various applications based on Zwegers's research, specifically the generating functions of combinatorial varieties. By examining the practical uses of his discoveries, we can better appreciate the significance of his work and its potential impact on future mathematical research.

[**Key words**]: Modular forms, mock theta functions

Contents

1. Introduction

Ramanujan's mock theta functions are a class of mathematical functions that have garnered significant interest due to their mysterious properties and connections to various areas of mathematics. Introduced by the legendary Indian mathematician Srinivasa Ramanujan in the early 20th century, mock theta functions were part of his final and enigmatic work before his untimely death in 1920. Initially, these functions appeared as q-series with peculiar properties, exhibiting some similarities to theta functions and modular forms. However, their true nature and the underlying mathematical structure remained elusive for many years. Ramanujan provided a list of 17 examples of mock theta functions without a formal definition, which led to intrigue and confusion among mathematicians for decades. Despite the uncertainty surrounding mock theta functions, they have since been connected to various fields of mathematics, such as number theory, combinatorics, and mathematical physics. It wasn't until the 21st century, with Sander Zwegers's work, that a deeper understanding of mock theta functions was achieved by connecting them to the theory of modular forms, specifically to harmonic Maass forms. This discovery paved the way for a better comprehension of their analytic properties and generated new research directions. Today, Ramanujan's mock theta functions continue to be a subject of active research and a source of fascination for mathematicians, as they unlock further insights into their properties, applications, and connections to other mathematical objects.

In Zwegers's exposition on mock theta functions, Sander Zwegers provides a comprehensive overview of the rich and intriguing history of these mathematical functions, their connections to other areas of mathematics, and their applications in various fields. Beginning with the origin of mock theta functions, attributed to the renowned Indian mathematician Srinivasa Ramanujan, Zwegers delves into the mysterious nature of these functions, which were left largely unexplored for decades. This groundbreaking discovery allowed for a better understanding of the analytic properties of these functions and opened up new avenues of research in areas like number theory, combinatorics, and mathematical physics. Additionally, the exposition delves into the applications of mock theta functions in various fields, such as quantum topology, statistical mechanics, and representation theory.

2. Modular Forms

Since mock theta function has a close connection with modular forms, we will introduce basic definition and properties about modular forms. Moreover, we will give some application to help readers better understand modular forms.

2.1 Definition and Properties of Modular Forms

To begin with, we define the upper half complex plane:

$$\mathfrak{H} := \{z = a + bi \in \mathbb{C} : b > 0\}$$

The special linear group is as usual:

$$SL(2, \mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \right\}$$

We get Möbius transformations (or fractional linear transformations) with this group acting on $\mathfrak{H}$:

$$\forall \gamma \in SL(2, \mathbb{R}), \quad \gamma(z) := \frac{az + b}{cz + d}, \quad \forall z \in \mathfrak{H}$$

One can easily check that:

$$\forall \gamma \in SL(2, \mathbb{R}), \quad z \in \mathfrak{H}, \quad \gamma(z) \in \mathfrak{H}$$

Notice that the matrices $\pm\gamma$ act in the same way on $\mathfrak{H}$, so we can work instead with the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm 1$.

Definition 2.1. For any given $\Gamma \subseteq SL(2, \mathbb{Z})$, we define modular forms with weight k $f : \mathfrak{H} \to \mathbb{C}$ satisfying:

- f is holomorphic in $\mathfrak{H}$.
- f is required to be bounded as $z \to$i∞.
- $f\left(\frac{az+b}{cz+d}\right) = (cz + d)^k f(z), \quad \forall z \in \mathfrak{H}, \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$(This is always called the modular transformation property)

Remark. *The weight k is typically a positive integer and only the zero function can satisfy the second condition for the odd k. If k is negative, this will contradict with the second item. Assuming k is zero, the modular form will be invariant in its fundamental domain, which is the trivial case. When k is odd, one can check that:* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \Gamma \subseteq SL(2, \mathbb{Z}) \quad \Rightarrow$ $f\left(\frac{-z+0}{0z-1}\right) = (0z - 1)^k f(z) \quad \Rightarrow f \equiv 0$. *Now consider the function space consisting of modular forms of weight* k *defined on* Γ*, denoted as* $M_k(\Gamma)$*. Then*

$$M_*(\Gamma) := \oplus_k M_k(\Gamma)$$

forms a graded ring: the ring operations are defined as ordinary addition and multiplication, with the additive identity being 0 and the multiplicative identity being the constant function

1 (with weight 0). Next, we verify the closure property:

$$f_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_k(z), \quad f_l\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f_l(z)$$
$$f_k \cdot f_l = (cz+d)^{k+l} f_k(z) f_l(z)$$

It can be seen that a modular form of weight k multiplied by a modular form of weight l will yield a modular form of weight $k+l$. In fact $M_k(\Gamma)$ is finite-dimensional, and that $M_*(\Gamma)$ is generated by a finite number of generators, as we will see below.

Proposition 2.2. $f\left(\left(\gamma\gamma'\right)(z)\right) = f\left(\gamma\left(\gamma'(z)\right)\right), \quad \forall \gamma, \gamma' \in \Gamma$

Proof. The proof is easy:

$$Assuming: \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \gamma' = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$$\gamma\gamma' = \begin{bmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{bmatrix} \in \Gamma$$

$$\begin{aligned} f\left(\left(\gamma\gamma'\right)(z)\right) &= f\left(\frac{\left(aa_1 + bc_1\right)z + ab_1 + bd_1}{\left(ca_1 + dc_1\right)z + cb_1 + dd_1}\right) \\ &= \left(\left(ca_1 + dc_1\right)z + cb_1 + dd_1\right)^k f(z) \end{aligned}$$

$$\begin{aligned} f\left(\gamma\left(\gamma'(z)\right)\right) &= f\left(\gamma\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right)\right) \\ &= f\left(\frac{a\frac{a_1 z + b_1}{c_1 + d_1} + b}{c\frac{c_1 z + d_1}{a_1 z + b_1} + d}\right) \\ &= f\left(\frac{a\left(a_1 z + b_1\right) + b\left(c_1 z + d_1\right)}{c\left(a_1 z + b_1\right) + d\left(c_1 z + d_1\right)}\right) \\ &= f\left(\frac{\left(aa_1 + bc_1\right)z + ab_1 + bd_1}{\left(ca_1 + dc_1\right)z + cb_1 + dd_1}\right) \\ &= \left(\left(ca_1 + dc_1\right)z + cb_1 + dd_1\right)^k f(z) \end{aligned}$$

$$f\left(\left(\gamma\gamma'\right)(z)\right) = f\left(\gamma\left(\gamma'(z)\right)\right)$$

□

Proposition 2.3. $f(z+t) = f(z), \quad \forall t \in \mathbb{Z}, \quad z \in \mathfrak{S}$

Proof. Assuming the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \Gamma, \quad t \in \mathbb{Z}$$

Then

$$f\left(\frac{1z+t}{0z+1}\right) = (0z+1)^k f(z)$$

This means

$$f(z+t) = f(z)$$

$\square$

Proposition 2.4. *When $k < 0$ or* k *is odd, the dimension of $M_k(\Gamma(1))$ is 0, as we have stated in 2.1; If* k > 0 *is an even integer, we have:*

$$\dim M_k(\Gamma(1)) \leq \begin{cases} \left[\frac{k}{12}\right] + 1 & x \not\equiv 2 \mod 12 \\ \left[\frac{k}{12}\right] & x \equiv 2 \mod 12 \end{cases}$$

Proof. The proof is technical and long, thus we do not show it here, one can see the detail in [Zag08] p10-p12. $\square$

2.2 Examples of modular forms: Eisenstein series

In this part, we will focus on Eisenstein series, which is an important example of modular forms.

We give definition of Eisenstein series in two forms

Definition 2.5. Let z be a complex number with strictly positive imaginary part. Define the holomorphic Eisenstein series $G_k(z)$ of weight k, where k>2 is an even integer, by the following series:

$$G_k(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k} \tag{1}$$

Remark. *One can easily check that the sum is absolutely and locally uniformly convergent. The factor $\frac{1}{2}$ arises since (m,n) and -(m,n) contributes to the same element of the sum.*

We also want to introduce the other normalization

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) \tag{2}$$

since this normalization has rational Fourier coefficients, as we will show below.

Proposition 2.6. *The Fourier expansion of this normalization $\mathbb{G}_k(z)$ is*

$$\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{3}$$

where B_k s.t. $\sum_{k=0}^{\infty} B_k x^k/k! = x/\left(e^x - 1\right)$, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $q = e^{2\pi i z}$.

Proof. We begin with the famous Euler identity:

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \frac{\pi}{\tan \pi z} \quad z \in \mathbb{C}/\mathbb{Z} \tag{4}$$

We can get a Fourier series from the right hand side:

$$\frac{\pi}{\tan \pi z} = \pi \frac{\cos \pi z}{\sin \pi z} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -\pi i \frac{1+2q-q}{1-q}$$
$$= -\pi i \left(1 + \frac{2q}{1-q}\right) = -\pi i \left(1 + 2q \sum_{r=0}^{\infty} q^r\right) = -2\pi i \left(1/2 + \sum_{r=1}^{\infty} q^r\right)$$

Substitute this result into 4, differentiate k-1 times and divide by

$$(-1)^{k-1}(k-1)!$$

to get

$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{\pi}{\tan \pi z}\right) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r$

Then we will get our result if we divide the sum of m into two cases: m=0 and m$\neq$0

$$G_k(z) = \frac{1}{2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^k} + \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{1}{(mz+n)^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}$$
$$= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr}$$
$$= \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right),$$

The last line can just be seen from the generating function of B_k. □

We give some examples of the proposition:

$$\mathbb{G}_4(z) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \cdots,$$
$$\mathbb{G}_6(z) = -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots,$$
$$\mathbb{G}_8(z) = \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots.$$

Remark. *Since we know that Eisenstein series is a modular form and the weight of G_k is just k, we can get some Eisenstein series modular identity:*

$$E_4(z)^2 = E_8(z), \quad E_4(z)E_6(z) = E_{10}(z),$$
$$E_6(z)E_8(z) = E_4(z)E_{10}(z) = E_{14}(z).$$

This is true since the dimension of $M_k(\Gamma(1))$ discussed here is exactly one discussed in 2.4.

Then we can also get:

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120},$$
$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_9(n-m) = \frac{\sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n)}{2640}.$$

combined with the Fourier expansion given in 2.6 and the fact that the leading coefficient of Eisenstein series $\mathbb{G}_k(z)$ is known(B_k is known).

In the above discussion, we restricted ourselves to the case when k>2, since then the series are absolutely convergent, which define modular forms of weight k. However, the Fourier expansion of $\mathbb{G}_2(z)$ defined in 2.6 is holomorphic. Thus in weight 2, we can also define two special Eisenstein series $\mathbb{G}_2(z)$ and $G_2(z)$ by the equation 2,i.e.,

$$\mathbb{G}_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots,$$
$$G_2(z) = -4\pi^2 \mathbb{G}_2(z).$$

Meanwhile, we can get another definition when k=2 by 4.4:

$$G_2(z) = \frac{1}{2}\sum_{n\neq 0} \frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}.$$

But notice the difference that the summation over n first and then over m is not absolutely convergent, thus we cannot interchange the order of summation to get the modular transformation property, which means $G_2(z)$ is not a modular form. However, $G_2(z)$ and $\mathbb{G}_2(z)$ do have some modular properties, as we will show right now.

Proposition 2.7. *For $z \in \mathfrak{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ we have*

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d). \tag{5}$$

Proof. There are many ways to prove this, the famous one is due to Hecke. Since the proof is too long, we do not intend to show it here. One can see [Zag08] p19-20 for details. □

Remark. *The proposition is important, since it contains some transformation properties of quasimodular Eisenstein series. We will use this proposition in the next section.*

2.3 Differential Operators on Modular Forms

Recall the transformation property of the modular forms: $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \forall z \in \mathfrak{H}, \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$

If we specialize this to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which belongs to $\Gamma(1)$, then we see that any modular form on $\Gamma(1)$ satisfies $f(z+1) = f(z)$ for all $z \in \mathfrak{H}$, i.e., it is a periodic function of period 1. It is therefore a function of the quantity $e^{2\pi iz}$, traditionally denoted q; more precisely, we have the Fourier development

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz} = \sum_{n=0}^{\infty} a_n q^n \quad \left(z \in \mathfrak{H}, \quad q = e^{2\pi iz}\right)$$

Then if f is a modular form of weight k with the Fourier expansion like this, we can differentiate to get derivatives

$$Df = f' := \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq} = \sum_{n=1}^{\infty} n a_n q^n \tag{6}$$

(where the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients) satisfies

$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f'(z) + \frac{k}{2\pi i} c(cz+d)^{k+1} f(z). \tag{7}$$

If we had only the first term, then f' would be a modular form of weight $k+2$. The presence of the second term, far from being a problem, makes the theory much richer. To deal with it, we will:

- modify the differentiation operator so that it preserves modularity;
- make combinations of derivatives of modular forms which are again modular;
- relax the notion of modularity to include functions satisfying equations like 7
- differentiate with respect to $t(z)$ rather than z itself, where $t(z)$ is a modular function.

The first and second approaches will be discussed in the two subsections. For the other two subsections, one can check Zagier[] for details.

2.3.1 Derivatives of Modular Forms

As already stated, the first approach is to introduce modifications of the operator D which do preserve modularity. There are two ways to do this, one holomorphic and one not. We begin with the holomorphic one. Comparing the transformation equation 6 with equations 5, we find that for any modular form $f \in M_k(\Gamma_1)$ the function

$$\vartheta_k f := f' - \frac{k}{12} E_2 f$$

sometimes called the Serre derivative, belongs to $M_{k+2}(\Gamma_1)$. (We will often drop the subscript k, since it must always be the weight of the form to which the operator is applied.) A first consequence of this basic fact is the following. We introduce the ring $\widetilde{M}_*(\Gamma_1) := M_*(\Gamma_1)[E_2] = \mathbb{C}[E_2, E_4, E_6]$, called the ring of quasimodular forms on $SL(2, \mathbb{Z})$. (Recall that we introduce the modular forms in the remark of the definition of modular forms) Then we have:

Proposition 2.8. *The ring $\widetilde{M}_*(\Gamma_1)$ is closed under differentiation. Specifically, we have*

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2 E_4 - E_6}{3}, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2}.$$

Proof. Clearly ϑE_4 and ϑE_6, being holomorphic modular forms of weight 6 and 8 on Γ_1, respectively, must be proportional to E_6 and E_4^2, and by looking at the first terms in their Fourier expansion we find that the factors are $-1/3$ and $-1/2$. Similarly, by differentiating 5 we find the analogue of 7 for E_2, namely that the function $E_2' - \frac{1}{12}E_2^2$ belongs to $M_4(\Gamma)$. It must therefore be a multiple of E_4, and by looking at the first term in the Fourier expansion one sees that the factor is $-1/12$. $\square$

We now turn to the second modification of the differentiation operator which preserves modularity, this time, however, at the expense of sacrificing holomorphy. For $f \in M_k(\Gamma)$ (we now no longer require that Γ be the full modular group Γ_1) we define

$$\partial_k f(z) = f'(z) - \frac{k}{4\pi y} f(z) \tag{8}$$

where y denotes the imaginary part of z. Clearly this is no longer holomorphic, but from the calculation

$$\frac{1}{\Im(\gamma z)} = \frac{|cz+d|^2}{y} = \frac{(cz+d)^2}{y} - 2ic(cz+d) \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})\right)$$

and 6 one easily sees that it transforms like a modular form of weight $k+2$, i.e., that $(\partial_k f)|_{k+2}\gamma = \partial_k f$ for all $\gamma \in \Gamma$. Moreover, this remains true even if f is modular but not holomorphic, if we interpret f' as $\frac{1}{2\pi i}\frac{\partial f}{\partial z}$. This means that we can apply $\partial = \partial_k$ repeatedly to get non-holomorphic modular forms $\partial^n f$ of weight $k+2n$ for all $n \geq 0$. (Here, as with ϑ_k, we can drop the subscript k because ∂_k will only be applied to forms of weight k; this is convenient because we can then write $\partial^n f$ instead of the more correct $\partial_{k+2n-2} \cdots \partial_{k+2}\partial_k f$.)

For example, for $f \in M_k(\Gamma)$ we find

$$\begin{aligned}
\partial^2 f &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k+2}{4\pi y}\right)\left(f' - \frac{k}{4\pi y}f\right) \\
&= f'' - \frac{k}{4\pi y}f' - \frac{k}{16\pi^2 y^2}f - \frac{k+2}{4\pi y}f' + \frac{k(k+2)}{16\pi^2 y^2}f \\
&= f'' - \frac{k+1}{2\pi y}f' + \frac{k(k+1)}{16\pi^2 y^2}f
\end{aligned}$$

and more generally, as one sees by an easy induction,

$$\partial^n f = \sum_{r=0}^{n}(-1)^{n-r}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{(k+r)_{n-r}}{(4\pi y)^{n-r}}D^r f \tag{9}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$. The inversion of 9 is

$$D^n f = \sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{(k+r)_{n-r}}{(4\pi y)^{n-r}}\partial^r f, \tag{10}$$

and describes the decomposition of the holomorphic but non-modular form $f^{(n)} = D^n f$ into non-holomorphic but modular pieces: the function $y^{r-n}\partial^r f$ is multiplied by $(cz+d)^{k+n+r}(c\bar{z}+d)^{n-r}$ when z is replaced by $\frac{az+b}{cz+d}$ with $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma$.

The usual way to write down modular forms is via their Fourier expansions, i.e., as power series in the quantity $q = e^{2\pi i z}$ which is a local coordinate at infinity for the modular curve $\Gamma\backslash\mathfrak{h}$. But since modular forms are holomorphic functions in the upper half-plane, they also have Taylor series expansions in the neighborhood of any point $z = x + iy \in \mathfrak{h}$. The "straight" Taylor series expansion, giving $f(z+w)$ as a power series in w, converges only in the disk $|w| < y$ centered at z and tangent to the real line, which is unnatural since the domain of holomorphy of f is the whole upper half-plane, not just this disk. Instead, we should remember that we can map $\mathfrak{H}$ isomorphically to the unit disk, with z mapping to 0 , by sending $z' \in \mathfrak{h}$ to $w = \frac{z'-z}{z'-z}$. The inverse of this map is given by $z' = \frac{z-zw}{1-w}$, and then if f is a modular form of weight k we should also include the automorphy factor $(1-w)^{-k}$ corresponding to this fractional linear transformation (even though it belongs to PSL $(2,\mathbb{C})$ and not Γ). The most natural way to study f near z is therefore to expand $(1-w)^{-k}f\left(\frac{z-zw}{1-w}\right)$ in powers of w. The following proposition describes the coefficients of this expansion in terms of the operator 8.

Proposition 2.9. *Let f be a modular form of weight k and $z = x+ iy$ a point of 5 . Then*

$$(1-w)^{-k}f\left(\frac{z-\bar{z}w}{1-w}\right) = \sum_{n=0}^{\infty}\partial^n f(z)\frac{(4\pi y w)^n}{n!} \quad (|w| < 1). \tag{11}$$

Proof. From the usual Taylor expansion, we find

$$\begin{aligned}(1-w)^{-k} f\left(\frac{z-\bar{z}w}{1-w}\right) &= (1-w)^{-k} f\left(z+\frac{2iyw}{1-w}\right) \\ &= (1-w)^{-k} \sum_{r=0}^{\infty} \frac{D^r f(z)}{r!}\left(\frac{-4\pi yw}{1-w}\right)^r\end{aligned}$$

and now expanding $(1-w)^{-k-r}$ by the binomial theorem and using 9 we obtain 11. □

2.3.2 Rankin–Cohen Brackets and Cohen-Kuznetsov Series

Let us return to equation 7 describing the near-modularity of the derivative of a modular form $f \in M_k(\Gamma)$. If $g \in M_\ell(\Gamma)$ is a second modular form on the same group, of weight ℓ, then this formula shows that the non-modularity of $f'(z)g(z)$ is given by an additive correction term $(2\pi i)^{-1}kc(cz+d)^{k+\ell+1}f(z)g(z)$. This correction term, multiplied by ℓ, is symmetric in f and g, so the difference $[f,g] = kfg' - \ell f'g$ is a modular form of weight $k+\ell+2$ on Γ. One checks easily that the bracket $[\cdot,\cdot]$ defined in this way is anti-symmetric and satisfies the Jacobi identity, making $M_*(\Gamma)$ into a graded Lie algebra (with grading given by the weight +2).

We can continue this construction to find combinations of higher derivatives of f and g which are modular, setting $[f,g]_0 = fg, \quad [f,g]_1 = [f,g] = kfg' - \ell f'g$

$$[f,g]_2 = \frac{k(k+1)}{2} fg'' - (k+1)(\ell+1)f'g' + \frac{\ell(\ell+1)}{2} f''g$$

and in general

$$[f,g]_n = \sum_{\substack{r,s\geq 0 \\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{\ell+n-1}{r} D^r f D^s g \quad (n \geq 0) \tag{12}$$

the nth Rankin-Cohen bracket of f and g.

Proposition 2.10. *For $f \in M_k(\Gamma)$ and $g \in M_\ell(\Gamma)$ and for every $n \geq 0$, the function $[f,g]_n$ defined by 12 belongs to $M_{k+\ell+2n}(\Gamma)$.*

There are several ways to prove this. We will do it using Cohen-Kuznetsov series. If $f \in M_k(\Gamma)$, then the Cohen-Kuznetsov series of f is defined by

$$\widetilde{f}_D(z,X) = \sum_{n=0}^{\infty} \frac{D^n f(z)}{n!(k)_n} X^n \in \mathrm{Hol}_0(\mathfrak{H})[[X]] \tag{13}$$

where $(k)_n = (k+n-1)!/(k-1)! = k(k+1)\cdots(k+n-1)$ is the symbol already used above and $\mathrm{Hol}_0(\mathfrak{H})$ denotes the space of holomorphic functions in the upper half-plane of subexponential growth. This series converges for all $X \in \mathbb{C}$. Its key property is given by:

Proposition 2.11. *If $f \in M_k(\Gamma)$, then the Cohen-Kuznetsov series defined by 13 satisfies the modular transformation equation*

$$\tilde{f}_D\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right) = (cz+d)^k \exp\left(\frac{c}{cz+d}\frac{X}{2\pi i}\right) \tilde{f}_D(z, X). \tag{14}$$

for all $z \in \mathfrak{H}, X \in \mathbb{C}$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Proof. This can be proved in several different ways. One way is direct: one shows by induction on n that the derivative $D^n f(z)$ transforms under Γ by

$$D^n f\left(\frac{az+b}{cz+d}\right) = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \frac{(k+r)_{n-r}}{(2\pi i)^{n-r}} c^{n-r} (cz+d)^{k+n+r} D^r f(z)$$

for all $n \geq 0$ (equation 7 is the case $n = 1$ of this), from which the claim follows easily. Another, more elegant, method is to use formula 9 or 10 to establish the relationship

$$\tilde{f}_D(z, X) = e^{X/4\pi y} \tilde{f}_\partial(z, X) \quad (z = x + iy \in \mathfrak{H}, X \in \mathbb{C}) \tag{15}$$

between $\tilde{f}_D(z, X)$ and the modified Cohen-Kuznetsov series

$$\widetilde{f}_\partial(z, X) = \sum_{n=0}^{\infty} \frac{\partial^n f(z)}{n!(k)_n} X^n \in \mathrm{Hol}_0(\mathfrak{H})[[X]] \tag{16}$$

The fact that each function $\partial^n f(z)$ transforms like a modular form of weight $k + 2n$ on Γ implies that $\widetilde{f}_\theta(z, X)$ is multiplied by $(cz + d)^k$ when z and X are replaced by $\frac{az+b}{cz+d}$ and $\frac{X}{(cz+d)^2}$, and using 15 one easily deduces from this the transformation formula 14. Yet a third way is to observe that $\tilde{f}_D(z, X)$ is the unique solution of the differential equation $\left(X\frac{\partial^2}{\partial X^2} + k\frac{\partial}{\partial X} - D\right) \tilde{f}_D = 0$ with the initial condition $\tilde{f}_D(z, 0) = f(z)$ and that $(cz + d)^{-k} e^{-cX/2\pi i(cz+d)} \tilde{f}_D\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right)$ satisfies the same differential equation with the same initial condition. $\square$

Now to deduce 2.9 we simply look at the product of $\tilde{f}_D(z, -X)$ with $\widetilde{g}_D(z, X)$. 2.11 implies that this product is multiplied by $(cz + d)^{k+\ell}$ when z and X are replaced by $\frac{az+b}{cz+d}$ and $\frac{X}{(cz+d)^2}$ (the factors involving an exponential in X cancel), and this means that the coefficient of X^n in the product, which is equal to $\frac{[f,g]_n}{(k)_n(\ell)_n}$, is modular of weight $k + \ell + 2n$ for every $n \geq 0$.

3. Seventeen original examples in Ramanujan's letter

Ramanujan divided his seventeen examples into four of order 3, ten of order 5, and three of order 7, though he did not indicate what are these orders. We will give most of the seventeen mock theta functions to get some observations.

We denote the mock theta functions of order 3 f, ϕ, ψ and χ.

Definition 3.1.

$$\mathrm{f}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2\cdots(1+q^n)^2},$$
$$\phi(q) = \sum_{n=0}^{\infty} \frac{-q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n}},$$
$$\psi(q) = \sum_{n=1}^{\infty} \frac{-q^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1}}.$$

Two relations among these functions (as well as some further complicated relations involving χ) were proved later by Watson:

Proposition 3.2.

$$2\phi(q) - f(q) = f(q) + 4\psi(q) = \frac{1-2q+2q^4-2q^9+\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}. \tag{17}$$

Proof. The proof is just basic calculation and one can see [Wat36] for some details. □

Notice that the expression on the right-hand side is, up to a factor $q^{-1/24}$, a modular form of weight 1/2. In this 3 order case, we can observe that:

1. there are linear relations among the mock theta functions (here, ϕ = f + 2ψ);
2. they span a space containing a subspace of ordinary modular forms;
3. one must multiply by suitable powers of q to get the correct modular behavior.

For the observation 1, we will also list linear relations in the 5 order case, which is more typical and complicated. Observation 2 and observation 3 inspire Zwegers' work that various known identities from the literature could be interpreted as saying that each of Ramanujan's examples belongs to an infinite families of functions: ”Lerch Sums” (Zagier first called the infinite family of functions ”Lerch Sum”).

There are ten mock theta functions of order 5 of Seventeen original examples in Ramanujan's letter. Following Ramanujan's symbols, we divide ten functions into 2 groups and denote them f_j, ϕ_j, ψ_j, χ_j and F_j with $j \in \{1, 2\}$.

Definition 3.3. The five functions with index j = 1 are given by

$$
\begin{aligned}
f_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)} \\
\phi_1(q) &= \sum_{n=0}^{\infty} q^{n^2}(1+q)\left(1+q^3\right)\cdots\left(1+q^{2n-1}\right) \\
\psi_1(q) &= \sum_{n=1}^{\infty} q^{n(n+1)/2}(1+q)\left(1+q^2\right)\cdots\left(1+q^{n-1}\right) \\
\chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})\cdots(1-q^{2n})}, \\
F_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}
\end{aligned}
$$

and the five with index $j = 2$ are very similar, e.g.,

$$
\begin{aligned}
f_2(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)\cdots(1+q^n)} \\
\chi_2(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})\cdots(1-q^{2n+1})}
\end{aligned}
$$

Proposition 3.4. *linear relations among these functions or between them and classical modular forms (multiplied by suitable powers of q) can be summarized as below:*

$$
\begin{pmatrix} f_1(\sqrt{q}) & f_1(-\sqrt{q}) & \chi_1(q)-2 & \phi_1(-q) & \psi_1(\sqrt{q}) & \psi_1(-\sqrt{q}) & F_1(q)-1 \\ f_2(\sqrt{q}) & -f_2(-\sqrt{q}) & \chi_2(q)\sqrt{q} & -\phi_2(-q)/\sqrt{q} & \psi_2(\sqrt{q}) & -\psi_2(-\sqrt{q}) & F_2(q)\sqrt{q} \end{pmatrix}
$$
$$
= \begin{pmatrix} U_1(q) & V_1(q) & W_1(q) \\ U_2(q) & V_2(q) & W_2(q) \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & -2 & -3 & 1 & -1 & 1 & -1 \end{pmatrix}
$$

where U_j and V_j , multiplied by $q^{-1/120}$ for j = 1 and by $q^{11/120}$ for j = 2, are quotients of classical theta series and only W_1 and W_2 are new mock theta functions not in seventeen original examples.

Proof. One can see [Wat36] for details. □

Finally, we list three order 7 mock theta functions

Definition 3.5.

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})},$$
$$\mathcal{F}_2(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q^{n})(1-q^{n+1})\cdots(1-q^{2n-1})},$$
$$\mathcal{F}_3(q) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(1-q^{n})(1-q^{n+1})\cdots(1-q^{2n-1})}.$$

In fact, mock theta functions of order 7 are much simpler than both order 3 and order 5 mock theta functions, since they are linearly independent.

Proposition 3.6. *Mock theta functions of order 7 form in a natural way a vector space of dimension 3, with no linear relations.*

Proof. For the proof, one can see [Hic88]. □

Remark. *It is the simplicity of these functions that there are no relations, either among these functions or between them and classical modular forms, makes properties of them less apparent. [Zag09] mentioned three properties. One classical observation is from Ramanujan himself that these functions satisfy some asymptotic formulas at roots of unity. Another is some identities relating three order 7 mock theta functions to indefinite theta series and to some special Eisenstein series. One can see [War01] to study these identities. The third property can be briefly stated here, which also shows the core of Zwegers' work. One can check that the Fourier coefficients of order 7 mock theta functions grow rapidly when calculating to high order. However, if we multiply any of the series $\mathcal{F}_j$ by some infinite products, which up to a rational power of q is a modular form, then the coefficients of this product will grow much slowly, which suggests that the series are closely related to modular forms.*

4. Zwegers's thesis

4.1 Lerch sums

In Ramanujan's "lost notebooks" [AB12], he related mock theta functions to some q-hypergeometric series, a typical result being the identity

$$\prod_{n=1}^{\infty}(1-q^n)\cdot f(q) = 2\sum_{n=-\infty}^{\infty}(-1)^n \frac{q^{n(3n+1)/2}}{1+q^n}$$

, where f(q) is the first order 3 mock modular functions mentioned in the last section. Sums appearing on the right-hand side were called by Zwegers "Lerch sums" after famous Czech mathematician M.Lerch. In fact, on page 3 of Ramanujan's lost notebook, he also defines the Appell-Lerch sum

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_{n+1}^2} \tag{18}$$

which is connected to some of his sixth-order mock theta functions by

$$\sum_{n=1}^{\infty} a(n) q^n := \phi(q) \tag{19}$$

Maybe this idea inspires Zwegers to study this sum further. We will study these sums following Zwegers's thesis. In Zwegers's thesis, Lerch sums are written as

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}} \qquad (\tau \in \mathbb{H}, v \in \mathbb{C}, u \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})).$$

We will use the same notations.

Proposition 4.1. *For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ define*

$$\theta(z) = \theta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z + \frac{1}{2})}.$$

Then θ satisfies:

(1) $\theta(z+1) = -\theta(z)$.

(2) $\theta(z+\tau) = -e^{-\pi i \tau - 2\pi i z}\theta(z)$.

(3) *Up to a multiplicative constant, $z \mapsto \theta(z)$ is the unique holomorphic function satisfying (1) and (2).*

(4) $\theta(-z) = -\theta(z)$.

(5) *The zeros of θ are the points $z = n\tau + m$, with $n, m \in \mathbb{Z}$. These are simple zeros.*

(6) $\theta(z;\tau+1) = e^{\frac{\pi i}{4}}\theta(z;\tau)$.

(7) $\theta(\frac{z}{\tau}; -\frac{1}{\tau}) = -i\sqrt{-i\tau}e^{\pi i z^2/\tau}\theta(z;\tau)$.

(8) $\theta(z;\tau) = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n)$, *with $q = e^{2\pi i \tau}$, $\zeta = e^{2\pi i z}$. This is the Jacobi triple product identity.*

(9) $\theta'(0;\tau+1) = e^{\frac{\pi i}{4}}\theta'(0;\tau)$ *and* $\theta'(0;-\frac{1}{\tau}) = (-i\tau)^{3/2}\,\theta'(0;\tau)$.

(10) $\theta'(0;\tau) = -2\pi\eta(\tau)^3$, *with η as in the introduction.*

Proof. The ten proposition is just the basic proposition of unitary theta functions, one can just calculate or refer to any books introducing theta series. ☐

It turns out to be convenient to normalize the Lerch sums. Since the mock theta functions will eventually be expressed as linear combinations of these normalized sums.

Proposition 4.2. *For $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$ and $\tau \in \mathbb{H}$, define*

$$\mu(u, v) = \mu(u, v; \tau) := \frac{e^{\pi i u}}{\theta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}.$$

Then μ satisfies:

(1) $\mu(u+1, v) = -\mu(u, v)$,

(2) $\mu(u, v+1) = -\mu(u, v)$,

(3) $\mu(u, v) + e^{-2\pi i(u-v) - \pi i \tau} \mu(u+\tau, v) = -i e^{-\pi i(u-v) - \pi i \tau/4}$,

(4) $\mu(u+\tau, v+\tau) = \mu(u, v)$,

(5) $\mu(-u, -v) = \mu(u, v)$,

(6) $u \mapsto \mu(u, v)$ *is a meromorphic function, with simple poles in the points $u = n\tau + m$ $(n, m \in \mathbb{Z})$, and residue $\frac{-1}{2\pi i} \frac{1}{\theta(v)}$ in $u = 0$,*

(7) $\mu(u+z, v+z) - \mu(u, v) = \dfrac{1}{2\pi i} \dfrac{\theta'(0)\theta(u+v+z)\theta(z)}{\theta(u)\theta(v)\theta(u+z)\theta(v+z)}$,
for $u, v, u+z, v+z \notin \mathbb{Z}\tau + \mathbb{Z}$,

(8) $\mu(v, u) = \mu(u, v)$.

Proof. The proof of (1), (4) and (5) is trivial. The proof of (2) and (3) is direct by part (2) of Proposition 4.1.

(6) From the definition we see that $u \mapsto \mu(u, v)$ has a simple pole if $1 - e^{2\pi i n\tau + 2\pi i u} = 0$, for some $n \in \mathbb{Z}$. So $u \mapsto \mu(u, v)$ has simple poles in the points $u = -n\tau + m$ $(n, m \in \mathbb{Z})$. The pole in $u = 0$ comes from the term $n = 0$. We see

$$\lim_{u \to 0} u\, \mu(u, v) = \frac{1}{\theta(v)} \lim_{u \to 0} \frac{u}{1 - e^{2\pi i u}} = \frac{-1}{2\pi i} \frac{1}{\theta(v)}.$$

(7) Consider $f(z) = \theta(u+z)\theta(v+z)\left(\mu(u+z, v+z) - \mu(u, v)\right)$. Using (1), (2) and (5) of Proposition 4.1, and (1), (2), (4) and (6) of this proposition, we see that f has no poles, a zero for $z = 0$, and satisfies

$$\begin{cases} f(z+1) = f(z) \\ f(z+\tau) = e^{-2\pi i \tau - 2\pi i(u+v+2z)} f(z). \end{cases}$$

It follows that the quotient $f(z)/\theta(z)\theta(u+v+z)$ is a double periodic function with at most one simple pole in each fundamental parallelogram, and hence constant:

$$f(z) = C(u,v)\theta(z)\theta(u+v+z). \tag{20}$$

To compute C we consider $z=-u$. If we take $z=-u$ in (20) we find

$$f(-u) = C(u,v)\theta(-u)\theta(v) = -C(u,v)\theta(u)\theta(v) \tag{21}$$

by (4) of Proposition 4.1.

By definition we have

$$\begin{aligned} f(-u) &= \lim_{z\to -u}\theta(u+z)\theta(v+z)\left(\mu(u+z,v+z)-\mu(u,v)\right) \\ &= \theta(v-u)\cdot\lim_{z\to 0}\theta(z)\mu(z,v-u) \\ &= \theta(v-u)\cdot\lim_{z\to 0}\frac{\theta(z)}{z}\cdot\lim_{z\to 0}z\mu(z,v-u) = -\frac{1}{2\pi i}\theta'(0), \end{aligned} \tag{22}$$

where we have used (6). Combining (21) and (22) gives the desired result.

(8) Take $z=-u-v$ in (7) and use (5) of Proposition 4.1 to find

$$\mu(-v,-u) = \mu(u,v).$$

The desired result is just this identity combined with (5). □

Remark. *The properties here are we show the proof from (6) to (8) in Zwegers's thesis. The function μ is not quite a Jacobi form but exhibits "mock" behavior. In particular, we will see that there is a nonholomorphic correction term which can be added to make it transform as the Jacobi form.*

Proposition 4.3. *Let μ be as in Proposition 4.2. Then μ satisfies the following modular transformation properties:*

(1) $\mu(u,v;\tau+1) = e^{-\frac{\pi i}{4}}\mu(u,v;\tau)$,

(2) $\dfrac{1}{\sqrt{-i\tau}}e^{\pi i(u-v)^2/\tau}\mu\left(\dfrac{u}{\tau},\dfrac{v}{\tau};-\dfrac{1}{\tau}\right)+\mu(u,v;\tau) = \dfrac{1}{2i}h(u-v;\tau)$,
with h as in Definition 4.4.

Proof. (1) is from (6) of Proposition 4.1 immediately.

(2) Replacing (u,v,z,τ) by $(\frac{u}{\tau},\frac{v}{\tau},\frac{z}{\tau},-\frac{1}{\tau})$ in (7) of Proposition 4.2 and using (7) and (9) of Proposition 4.1 we see that the left-hand side depends only on $u-v$, not on u and v separately. Call it $\frac{1}{2i}\tilde{h}(u-v;\tau)$. Using (1) and (3) of Proposition 4.2 we see that $\tilde{h}$ satisfies the two identities (1) and (2) of Proposition 4.5, so if we can prove that $\tilde{h}$ is a holomorphic function, then we may conclude that $\tilde{h}=h$, as desired.

The poles of both $u \mapsto \mu(u, v)$ and $u \mapsto \mu(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau})$ are simple, and occur at $u \in \mathbb{Z}\tau + \mathbb{Z}$, so the only poles of $u \mapsto \tilde{h}(u - v)$ could be simple poles for $u \in \mathbb{Z}\tau + \mathbb{Z}$. Since this is a function of $u - v$ it has no poles at all, and hence is holomorphic. $\square$

Remark. *For (2) of Proposition 4.3, Zwegers gives another briefer proof, which, however, needs more technic.*

In recent work, Lerch sums have some other applications besides it can serve as tools to study the Ramanujan's mock theta functions. We will show Chan's conjectures about Ramanujan's congruences which are closely connected to Lerch sums. Although these congruences have something to do with Ramanujan's mock theta functions, The study of them can no doubt be independent direction.

We use the q-series notation appearing in the most of the thesis studying the Ramanujan's congruences.

$$\begin{aligned}
(a;q)_0 &:= 1 \\
(a;q)_n &:= \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n \geq 1, \\
(a;q)_\infty &:= \lim_{n\to\infty} (a;q)_n, \quad |q| < 1,
\end{aligned}$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty$$

For any positive integer j, we also use $E_j := (q^j; q^j)_\infty$. Let $x, z \in \mathbb{H}$ with neither z nor xz an integral power of q. Following the definition given by [HM14], Lerch sums can also be written as $m(x, q, z)$

$$m(x,q,z) := \frac{1}{(q, q/z, q; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2} z^{n+1}}{1 - xzq^n}$$

In [BB20], he conjectured that, for any non-negative integer n,

$$a(50n + 19) \equiv a(50n + 39) \equiv a(50n + 49) \mid\equiv 0(\mathrm{mod} 25).$$

Since Ramanujan's mock theta functions are connected with Lerch sums. Chan in the same thesis also considered for any integer p $\geq$ 2 and 1$\leq$j$\leq$p-1 with p and j coprime, the Fourier coefficients of Lerch sums

$$\sum_{n=0}^{\infty} a_{j,p}(n) q^n = \frac{1}{(q^j, q^{p-j}, q^p; q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{pn(n+1)/2 + jn + j}}{1 - q^{pn+j}}$$

and proved that

$$\sum_{n=0}^{\infty} a_{j,p}(pn + (p-j)j) q^n = p \frac{E_p^4}{E_1^3 \left(q^j, q^{p-j}; q^p\right)_\infty^2}$$

which implies different congruences

$$a_{j,p}(pn + (p - j)j) \equiv 0(\text{mod}p)$$

Chan also presented the following conjectural congruences:

$$\begin{aligned}
a_{1,6}(2n) &\equiv 0(\text{mod}2),\\
a_{1,10}(2n) &\equiv a_{3,10}(2n) \equiv 0(\text{mod}2),\\
a_{1,6}(6n+3) &\equiv 0(\text{mod}3),\\
a_{1,3}(5n+3) &\equiv a_{1,3}(5n+4) \equiv 0(\text{mod}5),\\
a_{1,10}(10n+5) &\equiv 0(\text{mod}5),\\
a_{3,10}(10n+5) &\equiv 0(\text{mod}5).
\end{aligned}$$

4.2 The Mordell integral

Lerch sums divided by a theta function ca also be called Zwegers' μ-function. This function can be "completed" giving rise to the function $\mu(z_1, z_2; \tau)$, a prototype for "harmonic Maass-Jacobi forms", which we will mention in the application section. This function is a critical base to build mock modular forms. The completion needs an auxiliary non-holomorphic function arising from Mordell Integral, which is the objective of this section. The function h defined in Definition 4.4 is essentially the function ϕ studied by Mordell: $\phi(x;\tau) = -\frac{1}{2}\tau e^{-\pi i\tau/4+\pi ix}h(x - \frac{\tau}{2} + \frac{1}{2};\tau)$. This integral appeared in the work of L. Kronecker and B. Riemann. However, Mordell was the first to analyze its behavior relative to modular transformations, so Zwegers refers to it as the Mordell integral. In this subsection, we let $\zeta = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$ to simplify the notations.

Definition 4.4. For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ set

$$h(z) = h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau x^2 - 2\pi zx}}{\cosh \pi x}\, dx.$$

Remark. *h is an even function of z.*

Proposition 4.5. *The function h has the following properties:*

(1) $h(z+1) = -h(z) + \frac{2}{\sqrt{-i\tau}}\, e^{\frac{\pi i}{\tau}(z+\frac{1}{2})^2}$,

(2) $h(z+\tau) = -\zeta q^{\frac{1}{2}} h(z) + 2\zeta^{\frac{1}{2}} q^{\frac{3}{8}}$,

(3) $z \mapsto h(z;\tau)$ is the unique holomorphic function satisfying (1) and (2),

Proof. (1) The proof is just basic manipulation

$$\begin{aligned}
h(z) + h(z+1) &= \int_{\mathbb{R}} \frac{e^{\pi i\tau t^2 - 2\pi zt}}{\cosh(\pi t)} \left(1 + e^{-2\pi t}\right) dt = 2\int_{\mathbb{R}} e^{\pi i\tau t^2 - 2\pi t\left(z+\frac{1}{2}\right)} dt\\
&= \frac{2}{\sqrt{-i\tau}} e^{\frac{\pi i}{\tau}\left(z+\frac{1}{2}\right)^2}.
\end{aligned}$$

(2) After shifting $t \mapsto t - i$, we get

$$\zeta^{-1} q^{-\frac{1}{2}} h(z+\tau) = -\int_{\mathbb{R}} \frac{e^{\pi i \tau (t+i)^2 - 2\pi z(t+i)}}{\cosh(\pi(t+i))} dt = -\int_{i+\mathbb{R}} \frac{e^{\pi i \tau t^2 - 2\pi z t}}{\cosh(\pi t)} dt$$

The Residue Theorem gives

$$\begin{aligned} h(z) + \zeta^{-1} q^{-\frac{1}{2}} h(z+\tau) &= \left(\int_{\mathbb{R}} - \int_{i+\mathbb{R}} \right) \frac{e^{\pi i \tau t^2 - 2\pi z t}}{\cosh(\pi t)} dt = 2\pi i \operatorname{Res} \frac{e^{\pi i \tau t^2 - 2\pi z t}}{\cosh(\pi t)} \\ &= 2\zeta^{-\frac{1}{2}} q^{-\frac{1}{8}}. \end{aligned}$$

(3) To establish the uniqueness claim, assume that h_1 and h_2 both have the claimed properties. Then $f := h_1 - h_2$ is an entire function which satisfies

$$\begin{aligned} f(z+1) &= -f(z), \\ f(z+\tau) &= -\zeta q^{\frac{1}{2}} f(z). \end{aligned}$$

According to Liouville's theorem, f is constant. Letting $y \to \infty$ yields that this function is zero function. ☐

The following proposition provides the critical transformation that we shall use to construct modular transformation properties in the next subsection.

Proposition 4.6.

(1) $h(\frac{z}{\tau}; -\frac{1}{\tau}) = \sqrt{-i\tau}\, e^{-\pi i z^2/\tau}\, h(z;\tau)$,

(2) $h(z;\tau) = e^{\frac{\pi i}{4}} h(z;\tau+1) + e^{-\frac{\pi i}{4}} \dfrac{e^{\pi i z^2/(\tau+1)}}{\sqrt{\tau+1}}\, h\Big(\dfrac{z}{\tau+1}; \dfrac{\tau}{\tau+1}\Big)$.

Proof. (1) Let $g(x) = \frac{1}{\cosh \pi x}$. We first compute the Fourier transform $\mathcal{F}g$ of g: Using Cauchy's formula we get

$$\left(\int_{\mathbb{R}} - \int_{\mathbb{R}+i} \right) \frac{e^{2\pi i z x}}{\cosh \pi x} dx = 2\pi i \operatorname*{Res}_{x=i/2} \frac{e^{2\pi i z x}}{\cosh \pi x} = 2e^{-\pi z},$$

but

$$\int_{\mathbb{R}+i} \frac{e^{2\pi i z x}}{\cosh \pi x} dx = \int_{\mathbb{R}} \frac{e^{2\pi i z (x+i)}}{\cosh \pi (x+i)} dx = -e^{-2\pi z} \int_{\mathbb{R}} \frac{e^{2\pi i z x}}{\cosh \pi x} dx,$$

so we find

$$(\mathcal{F}g)(z) := \int_{\mathbb{R}} \frac{e^{2\pi i z x}}{\cosh \pi x} dx = \frac{2e^{-\pi z}}{1 + e^{-2\pi z}} = g(z).$$

Let $f_\tau(x) = e^{\pi i \tau x^2}$, $\tau \in \mathbb{H}$. The Fourier transform of f_τ is given by

$$\mathcal{F} f_\tau = \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}}.$$

We now see

$$\int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 + 2\pi i z x}}{\cosh \pi x}\, dx = \mathcal{F}(f_\tau \cdot g)(z) = (\mathcal{F} f_\tau) * (\mathcal{F} g)(z)$$
$$= \frac{1}{\sqrt{-i\tau}} f_{-\frac{1}{\tau}} * g(z) = \frac{1}{\sqrt{-i\tau}} \int_{\mathbb{R}} \frac{e^{\pi i \frac{-1}{\tau}(z-x)^2}}{\cosh \pi x}\, dx.$$

This identity holds for $z \in \mathbb{R}$. Since both sides are analytic functions of z, the identity holds for all $z \in \mathbb{C}$. If we replace z by iz we get the desired result.

(2) Using (1) and (2) from 4.5, we can show that the right-hand side, considered as a function of z, also satisfies (1) and (2). The equation now follows from (3). □

We have finished all the properties about Mordell Integrals we need in Zwegers's thesis. We will also give some other examples of Mordell Integral like what we do in last subsection. In fact, Ramanujan also studied definite integrals besides many indefinite integrals and recorded modular transformations involving the Mordell integral. We will show some famous conjectural identities remaining unsolved related to Mordell Integrals in his notebooks. This equations were all proved by Choi after Zwegers's thesis.

Use the same notation as 4.1, we can define tenth tenth-order mock theta functions. They appeared in Ramanujan's lost notebook.

Definition 4.7.

$$\phi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} / \left(q; q^2\right)_{n+1}$$
$$\psi(q) := \sum_{n=0}^{\infty} q^{(n+1)(n+2)/2} / \left(q; q^2\right)_{n+1}$$

Proposition 4.8.

$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1+\sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\frac{\pi}{5n}} \psi\left(-e^{-\frac{\pi}{n}}\right)$$
$$= \sqrt{\frac{5+\sqrt{5}}{2}} e^{-\frac{\pi n}{5}} \phi\left(-e^{-\pi n}\right) - \frac{\sqrt{5}+1}{2\sqrt{n}} e^{-\frac{\pi}{5n}} \phi\left(-e^{\frac{\pi}{n}}\right),$$
$$\int_0^\infty \frac{e^{-\pi n x^2}}{\cosh \frac{2\pi x}{\sqrt{5}} + \frac{1-\sqrt{5}}{4}} dx + \frac{1}{\sqrt{n}} e^{\frac{\pi}{5n}} \psi\left(-e^{-\frac{\pi}{n}}\right)$$
$$= -\sqrt{\frac{5-\sqrt{5}}{2}} e^{\frac{\pi n}{5}} \phi\left(-e^{-\pi n}\right) + \frac{\sqrt{5}-1}{2\sqrt{n}} e^{-\frac{\pi}{5n}} \phi\left(-e^{\frac{\pi}{n}}\right).$$

Remark. *Two identities give transformation relations involving Mordell integrals and Ramanujan's tenth-order mock theta functions.*

Proof. The proof is just basic manipulation of analysis. One can find proof in [Cho14] in detail. □

There are also two equations involving a Mordell integral, hypergeometric series and generalized Lambert series.

Proposition 4.9. *Let* $q_1 = e^{-\frac{\pi}{3n}}$ *and* $q = e^{-3\pi n}$*,*

$$\begin{aligned}
&\frac{2}{\sqrt{3}}\int_0^\infty \frac{e^{-\frac{\pi n x^2}{3}}\cos \pi t x}{e^{\frac{2\pi x}{3}}+1+e^{-\frac{2\pi x}{3}}}dx = q^{\frac{1}{18}}\sum_{m=1}^{\infty}\frac{q^{\frac{(2m-1)^2}{6}}}{\left(-e^{\pi t}q^{\frac{1}{3}};q^{\frac{2}{3}}\right)_m\left(-e^{-\pi t}q^{\frac{1}{3}};q^{\frac{2}{3}}\right)_m} \\
&+\frac{e^{-\frac{3\pi t^2}{4n}}q_1^{\frac{1}{2}}}{\sqrt{n}}\sum_{m=1}^{\infty}\frac{q_1^{\frac{3}{2}(2m-1)^2}}{\left(-e^{\frac{\pi i t}{n}}q_1^3;q_1^6\right)_m\left(-e^{-\frac{\pi i t}{n}}q_1^3;q_1^6\right)_m} \\
&=\frac{q^{-\frac{1}{36}}}{\left(q^{\frac{2}{3}};q^{\frac{2}{3}}\right)_\infty}\left\{\sum_{m=1}^{\infty}(-1)^{m+1}q^{\frac{(2m-1)^2}{4}}\left(\frac{1}{1+e^{\pi t}q^{\frac{2m-1}{3}}}+\frac{1}{1+e^{-\pi t}q^{\frac{2m-1}{3}}}-1\right)\right. \\
&\left.+\frac{e^{-\frac{3\pi t^2}{4n}}}{n}\sum_{m=1}^{\infty}(-1)^{m+1}q_1^{\frac{9}{4}(2m-1)^2}\left(\frac{1}{1+e^{\frac{\pi i t}{n}}q_1^{3(2m-1)}}+\frac{1}{1+e^{-\frac{\pi i t}{n}}q_1^{3(2m-1)}}-1\right)\right\}.
\end{aligned}$$

Proof. One can find proof in [Cho14] in detail. The proof which depends on the basic properties of Mordell integral and theta functions is very technical. However it is natural when we see the proof after understanding Zwegers's thesis. □

4.3 Real-analytic Jacobi forms

We are now in a position to define the completed function $\tilde{\mu}$, which we mention in the last subsection. This allows us to find nonholomorphic completions of Ramanujan's mock theta functions. In Zwegers's thesis, he called this function real-analytic Jacobi form.

Definition 4.10. For $z \in \mathbb{C}$ we define

$$E(z) = 2\int_0^z e^{-\pi u^2}du = \sum_{n=0}^{\infty}\frac{(-\pi)^n}{n!}\frac{z^{2n+1}}{n+1/2}$$

One can check that this is an odd entire function of z.

Lemma 4.11. *For* $z \in \mathbb{R}$ *we have*

$$E(z) = sgn(z)\left(1-\beta(z^2)\right),$$

where

$$\beta(x) = \int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du \qquad (x \in \mathbb{R}_{\geq 0}).$$

Proof. It is just basic manipulation. □

Remark. *It is also not difficult to check that*

$$\beta(x) = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi x\right).$$

This can be used to show that specializations of $\tilde{\mu}$ to torsion points have Fourier expansions matching the expansions of harmonic Maass forms in $H_k(\Gamma)$ for some Γ, where

$$\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\} \text{ for some } N \in \mathbb{N}$$

. One can see [Bri+17] Lemma 4.3 for details.

The function E will play another critical role in Zwegers's second family of functions to describe mock theta functions. There he realized the mock theta functions as pieces of harmonic Maass forms, via the theory of indefinite theta functions.

Definition 4.12. Let $u \in \mathbb{C}$ and $\tau \in \mathbb{H}$

$$R(u;\tau) = \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} \left\{sgn(\nu) - E\Big((\nu + a)\sqrt{2y}\Big)\right\} (-1)^{\nu-\frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i \nu u},$$

$y = Im(\tau)$ and $a = \frac{Im(u)}{Im(\tau)}$.

Lemma 4.13. *For all $c, \epsilon > 0$, this series converges absolutely and uniformly on the set $\{u \in \mathbb{C}, \tau \in \mathbb{H} \mid |a| < c, y > \epsilon\}$. The function R it defines is real-analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{2}y^{-1/2}e^{-2\pi a^2 y}\theta(\overline{u}; -\overline{\tau}) \tag{23}$$

and

$$\frac{\partial}{\partial \overline{\tau}} R(a\tau - b;\tau) = -\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y} \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu-\frac{1}{2}}(\nu + a)e^{-\pi i \nu^2 \overline{\tau} - 2\pi i \nu(a\overline{\tau} - b)}. \tag{24}$$

Proof. One can show that R converges locally uniformly on compact sets (in u and τ) by operating the key step to split

$$\begin{aligned} &\text{sgn}(\nu) - E\left(\left(\nu + \frac{1}{a}\right)\sqrt{2y}\right) \\ &= \left(\text{sgn}(\nu) - \text{sgn}\left(\nu + \frac{1}{a}\right)\right) + \text{sgn}\left(n + \frac{1}{a}\right)\beta\left(2\left(n + \frac{1}{a}\right)^2 y\right). \end{aligned}$$

The first summand contributes finitely many terms, and we can use the estimate $0 \leq \beta(x) \leq e^{-\pi x}$ to get the second term is bounded. Since R is the (infinite) sum of real-analytic functions, and the series converges absolutely and uniformly, it is real-analytic. We fix $\tau \in \mathbb{H}$, and determine $u = a\tau - b$ by the coordinates $a, b \in \mathbb{R}$. We see

$$\begin{aligned}
&\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) R(a\tau - b; \tau) \\
&= \left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{sgn(\nu) - E\Big((\nu + a)\sqrt{2y}\Big)\right\} (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu(a\tau - b)} \\
&= -\sqrt{2y} \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} E'\Big((\nu + a)\sqrt{2y}\Big)(-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu(a\tau - b)} \\
&= -2\sqrt{2y} \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} e^{-2\pi(\nu+a)^2 y}(-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu(a\tau - b)} \\
&= -2\sqrt{2y}e^{-2\pi a^2 y} \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\overline{\tau} - 2\pi i\nu(a\overline{\tau} - b)} \\
&= -2i\sqrt{2y}e^{-2\pi a^2 y}\theta(a\overline{\tau} - b; -\overline{\tau}),
\end{aligned}$$

with θ as in Proposition 4.1 and the term-by-term differentiation being easily justified. Since $\frac{\partial}{\partial \overline{u}} = \frac{i}{2y}\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)$, this gives the differential equation (23). Similarly

$$\begin{aligned}
&\frac{\partial}{\partial \overline{\tau}} R(a\tau - b; \tau) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{sgn(\nu) - E\Big((\nu + a)\sqrt{2y}\Big)\right\} (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu(a\tau - b)} \\
&= -\frac{i}{2}\frac{1}{\sqrt{2y}} \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} (\nu + a)\, E'\Big((\nu + a)\sqrt{2y}\Big)\, (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu(a\tau - b)} \\
&= -\frac{i}{\sqrt{2y}} e^{-2\pi a^2 y} \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} (-1)^{\nu-\frac{1}{2}}(\nu + a)\, e^{-\pi i\nu^2\overline{\tau} - 2\pi i\nu(a\overline{\tau} - b)},
\end{aligned}$$

□

Proposition 4.14. *The function R has the following elliptic transformation properties:*

(1) $R(u + 1) = -R(u)$,

(2) $R(u) + e^{-2\pi iu - \pi i\tau}R(u + \tau) = 2e^{-\pi iu - \pi i\tau/4}$,

(3) $R(-u) = R(u)$.

Proof. Part (1) and (3) follows immediately from the fact that E is an odd function. To prove (2), we just need to replace ν by $\nu - 1$. Comparing the formula with the definition of R(u) will give the desired result, since for $\nu \in 1/2 + \mathbb{Z}$

$$\mathrm{sgn}(\nu) - \mathrm{sgn}(\nu - 1) = \begin{cases} 2 & \text{if } \nu = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

□

Proposition 4.15. *R has the following modular transformation properties:*

(1) $R(u;\tau+1)=e^{-\frac{\pi i}{4}}R(u;\tau)$,

(2) $R\left(\frac{u}{\tau};-\frac{1}{\tau}\right)=-\sqrt{-i\tau}e^{-\frac{\pi i u^2}{\tau}}R(u;\tau)+\sqrt{-i\tau}e^{-\frac{\pi i u^2}{\tau}}h(u;\tau)$

Proof. Part (1) is trivial. The left hand side of (2) we call $\tilde{h}(u;\tau)$. Using (1) and (2) of Proposition 4.14 we can see that $\tilde{h}$ satisfies:

$$\begin{cases}\tilde{h}(u)+\tilde{h}(u+1)=\frac{2}{\sqrt{-i\tau}}e^{\pi i(u+\frac{1}{2})^2/\tau},\\ \tilde{h}(u)+e^{-2\pi i u-\pi i\tau}\tilde{h}(u+\tau)=2e^{-\pi i u-\pi i\tau/4}.\end{cases}$$

Part (3) of Proposition 4.5 determines h as the unique holomorphic function with these properties. This reduces the proof to showing that $\tilde{h}$ is a holomorphic function of u. We fix $\tau\in\mathbb{H}$, and determine $u=a\tau-b$ by the coordinates $a,b\in\mathbb{R}$ (this implies $a=\frac{Im(u)}{Im(\tau)}$ as in Lemma 4.13). Since $\frac{\partial}{\partial\overline{u}}=\frac{i}{2y}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)$, we have to show that

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)\tilde{h}(a\tau-b;\tau)=0.$$

According to Lemma 4.13 we have

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau)=-2i\sqrt{2y}e^{-2\pi a^2y}\theta(a\overline{\tau}-b;-\overline{\tau}) \tag{25}$$

We have

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right)=\tau\left(\frac{\partial}{\partial b}+\frac{1}{\tau}\frac{\partial}{\partial a}\right)R\left(a-\frac{b}{\tau};-\frac{1}{\tau}\right).$$

Up to a factor τ this is the same as $\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau)$, with (a,b,τ) replaced by $(b,-a,-\frac{1}{\tau})$. Hence by (25) we find

$$\begin{aligned}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right)&=-2i\tau\sqrt{2y'}e^{-2\pi b^2y'}\theta\left(-\frac{b}{\overline{\tau}}+a;\frac{1}{\overline{\tau}}\right)\\ &=2i\tau\sqrt{2y'}e^{-2\pi b^2y'}\theta\left(-\frac{a\overline{\tau}-b}{\overline{\tau}};\frac{1}{\overline{\tau}}\right),\end{aligned}$$

with $y'=Im(-\frac{1}{\tau})=\frac{y}{\tau\overline{\tau}}$. In the last step we have used (4) of Proposition 4.1.
If we now use (7) of Proposition 4.1, with $z=a\overline{\tau}-b$ and τ replaced by $-\overline{\tau}$, we see that this equals

$$\begin{aligned}&2i\tau\sqrt{2y'}e^{-2\pi b^2y'}\cdot-i\sqrt{i\overline{\tau}}e^{-\pi i(a\overline{\tau}-b)^2/\overline{\tau}}\theta(a\overline{\tau}-b;-\overline{\tau})\\ &\quad=2i\sqrt{2y}\sqrt{-i\tau}e^{-\pi i(a\tau-b)^2/\tau}e^{-2\pi a^2y}\theta(a\overline{\tau}-b;-\overline{\tau}).\end{aligned}\tag{26}$$

Using (25) and (26) we find

$$\begin{aligned}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)&\tilde{h}(a\tau-b;\tau)\\&=\frac{1}{\sqrt{-i\tau}}e^{\pi i(a\tau-b)^2/\tau}\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R\left(\frac{a\tau-b}{\tau};-\frac{1}{\tau}\right)\\&\quad+\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R(a\tau-b;\tau)=0.\end{aligned}$$

We have established the fact that $\tilde{h}$ is holomorphic, and hence equals h. □

Remark. *We show the proof from [Zwe02]. One can see that the proof of it involves many propositions and lemmas we have proved.*

Zwegers' s key result that combines the properties of μ and R to find a function $\tilde{\mu}$ which is no longer holomorphic, but has better elliptic and modular transformation properties than μ, which finally solved the mystery of a modular framework for Ramanujan' s mock theta functions, is encapsulated in the following result which ties together the sequence of observations above.

Theorem 4.16. *We set*

$$\tilde{\mu}(u,v;\tau)=\mu(u,v;\tau)+\frac{i}{2}R(u-v;\tau), \tag{27}$$

then

(1) $\tilde{\mu}(u+k\tau+l,v+m\tau+n)=(-1)^{k+l+m+n}e^{\pi i(k-m)^2\tau+2\pi i(k-m)(u-v)}\tilde{\mu}(u,v)$, *for* $k,l,m,n\in\mathbb{Z}$,

(2) $\tilde{\mu}\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=v(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}}e^{-\pi ic(u-v)^2/(c\tau+d)}\tilde{\mu}(u,v;\tau)$, *for* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, *with* $v(\gamma)=\eta(\frac{a\tau+b}{c\tau+d})/\Big((c\tau+d)^{\frac{1}{2}}\eta(\tau)\Big)$

(3) $\tilde{\mu}(-u,-v)=\tilde{\mu}(v,u)=\tilde{\mu}(u,v)$,

(4) $\tilde{\mu}(u+z,v+z)-\tilde{\mu}(u,v)=\frac{1}{2\pi i}\frac{\theta'(0)\theta(u+v+z)\theta(z)}{\theta(u)\theta(v)\theta(u+z)\theta(v+z)}$, *for* $u,v,u+z,v+z\notin\mathbb{Z}\tau+\mathbb{Z}$,

(5) $u\mapsto\tilde{\mu}(u,v)$ *has singularities in the points* $u=n\tau+m$ $(n,m\in\mathbb{Z})$. *Furthermore we have* $\lim_{u\to0}u\tilde{\mu}(u,v)=\frac{-1}{2\pi i}\frac{1}{\theta(v)}$.

Proof. Compared to propositions above, the theorem is just observation based on all the propositions we have proved. Thus, we omit the proof. One can find detailed proof in [Zwe02]. □

Remark. *There is a very interesting property about this function, the completion of Zwegers' μ-function. All three function in* (27) *have a property that the other two do not have: $\tilde{\mu}$ transforms well (like a Jacobi form), μ is meromorphic and $u, v \mapsto R(u-v)$ depends only on $u-v$.*

Remark. *Parts (1) and (2) of the theorem say that the function $\tilde{\mu}$ transforms like a two-variable Jacobi form of weight $\frac{1}{2}$ and index $\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Therefore Zwegers called this function a real-analytic Jacobi form. The broader study of special real-analytic Jacobi forms, called harmonic Maass-Jacobi forms come to fruition at about the same time that Zwegers wrote his thesis, and we will give some examples of harmonic Maass forms in the next section.*

5. Further examples and applications

5.1 Quasimodular Eisenstein series and Rankin–Cohen Brackets

One application of Zwegers's theory is that it now becomes as easy to prove identities among mock theta functions (or more generally, among mock modular forms) as it previously was for modular forms. For example, the so-called "Mock theta conjectures" for the mock theta functions of order 5, which were stated by Ramanujan in his "Lost Notebook", were proved only in 1988 by D. Hickerson after heroic efforts, but now with the knowledge of the transformation properties of the mock theta functions the proof becomes automatic: one only has to verify that the left- and right-hand sides of the identities become modular after the addition of the same non-holomorphic correction term and that the first few coefficients of the q-expansions agree. We can also give an example combined with the knowledge of the quasimodular Eisenstein series and Rankin–Cohen Brackets we mention in the section above. To begin with, we introduce a family of scalar-valued functions having completions that transform like modular forms of every even integral weight k on the full modular group. The k th function $F_k = F_k(\tau)$ is defined as

$$F_k = \sum_{n\neq 0}(-1)^n \left(\frac{-3}{n-1}\right) n^{k-1}\frac{q^{n(n+1)/6}}{1-q^n} = -\sum_{r>s>0}\left(\frac{12}{r^2-s^2}\right) s^{k-1}q^{rs/6}$$

Then the function

$$f(\tau) = \frac{F_2(q) - 12E_2(\tau)}{\eta(\tau)} = q^{-1/24}\left(1 - 35q - 130q^2 - 273q^3 - 595q^4 - \cdots\right),$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is the usual quasimodular Eisenstein series of weight 2, is a mock modular form of weight $\frac{3}{2}$ on the full modular group with shadow $\eta(\tau)$, and for each integer $n > 0$ the sum of $12F_{2n+2}(\tau)$ and $24^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} [f, \eta]_n$ (where $[f, g]_n$ denotes the n-th Rankin-Cohen bracket, here in weight $\left(\frac{3}{2}, \frac{1}{2}\right)$), is a modular form of weight $2n + 2$

on $\mathrm{SL}(2,\mathbb{Z})$.

As we mention in the final remark in the last section, we will discuss the application of the theory of harmonic Maass form in the combinatorics. Thanks to Zwegers' s thesis, Ramanujan' s q-series are now, nearly 100 years later, understood as examples of weight 1/2 mock modular forms, which we have shown in the last section Zwegers' thesis. Ramanujan coined the term "mock theta function," and it is for this reason that holomorphic parts of harmonic Maass forms are now referred to as mock modular forms.

5.2 Applications of Harmonic Maass Forms in Combinatorics

5.2.1 Definitions and generating functions

Definition 5.1. The rank of a partition is its largest part minus the number of its parts.

Example 5.2. Here we list the partitions of 4 and their associated ranks. The ranks of these partitions modulo 5 cover each residue class exactly once.

Partition	Rank	Rank (mod 5)
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

Definition 5.3. For a partition λ, let $o(\lambda)$ denote the number of ones in λ, and define $\mu(\lambda)$ as the number of parts strictly larger than $o(\lambda)$. Then the crank of λ is defined as

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda)=0 \\ \mu(\lambda)-o(\lambda) & \text{if } o(\lambda)>0 \end{cases}$$

Example 5.4. Here we illustrate the cranks of the partitions of 6. We note that the cranks of these partitions modulo 11 divide the partitions of 6 into 11 groups of equal size (namely one).

Partition	$o(\lambda)$	$\mu(\lambda)$	Crank	Crank (mod 11)
6	0	1	6	6
$5+1$	1	1	$1-1=0$	0
$4+2$	0	2	4	4
$4+1+1$	2	1	$1-2=-1$	10
$3+3$	0	2	3	3
$3+2+1$	1	2	$2-1=1$	1
$3+1+1+1$	3	0	$0-3=-3$	8
$2+2+2$	0	3	2	2
$2+2+1+1$	2	0	$0-2=-2$	9
$2+1+1+1+1$	4	0	$0-4=-4$	7
$1+1+1+1+1+1$	6	0	$0-6=-6$	5

In view of the role of the rank and the crank for partition congruences, it is natural to study their general properties. To this end, we make use of two important generating functions in the variables ζ and q. It turns out that these functions are intimately connected to modular forms and mock modular forms.

Definition 5.5. Let M(m, n) (resp. N(m, n)) be the number of partitions of n with crank (resp. rank) m. Then the two-parameter generating functions may be written as

$$C(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)\zeta^m q^n = \prod_{n=1}^{\infty} \frac{1-q^n}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}$$
$$= \frac{1-\zeta}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-\zeta q^n},$$
$$R(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)\zeta^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q;q)_n (\zeta^{-1}q;q)_n} = \frac{1-\zeta}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-\zeta q^n}.$$

Remark. *Two specializations of the rank generating function should be highlighted. Thanks to a well-known identity arising from counting partitions according to the sizes of their so-called Durfee squares, one easily finds our first example, which states that*

$$P(q) = R(1;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}$$

where P is the partition generating function. A second example is given by Ramanujan's third order mock theta function f, whose definition may be restated as

$$f(q) = R(-1;q).$$

This gives a combinatorial interpretation of the coefficients of f as the number of partitions with even rank minus the number of those with odd rank. As we see below, the modularity properties of these two specializations are not coincidences, but manifestations of the rank generating function's nature as a "mock Jacobi form".

Although the generating functions in 5.5 are nearly identical, their modularity properties are very different. Indeed, these q-series, written as Lambert-type series, are identical apart from differing powers of q in their numerators. The crank generating function is closely related to a Jacobi form, while the rank generating function is essentially a mock Jacobi form. More specifically, in terms of Dedekind's η-function, the θ-function, and the Zwegers μ-function , we have the following lemma (recall that q = e(τ) and ζ = e(z)).

Lemma 5.6. *The crank and rank generating functions can be expressed in terms of Jacobi and mock Jacobi forms as follows:*

i) For $z \notin \mathbb{Z} + \mathbb{Z}\tau$, we have

$$C(\zeta;q) = -\frac{2\sin(\pi z) q^{\frac{1}{24}} \eta^2(\tau)}{\vartheta(z;\tau)}$$

ii) If $z \notin \mathbb{Z} + \mathbb{Z}\tau$, we have

$$R(\zeta;q) = -2\sin(\pi z)\left(\frac{q^{\frac{1}{24}}\eta(3\tau)^3}{\eta(\tau)\vartheta(3z;3\tau)} - q^{-\frac{1}{8}}\zeta^{-1}\mu(3z,-\tau;3\tau) + q^{-\frac{1}{8}}\zeta\mu(3z,\tau;3\tau)\right)$$

Proof. First we consider i). An interpretation of the combinatorial definition of the crank functions leads to (1), which yields the claim using the definitions of η and θ. (The condition given in the statement of i) just avoids the poles arising when θ(z; τ) has a zero.) Now we turn to the proof of ii). We can write

$$\begin{aligned} R(\zeta;q) &= \frac{1-\zeta}{\zeta^{\frac{3}{2}}(q)_\infty} A_3(z,-\tau;\tau) \\ &= \frac{1-\zeta}{\zeta^{\frac{3}{2}}(q)_\infty}\left(A_1(3z,-\tau;3\tau) + \zeta A_3(3z,0;3\tau) + \zeta^2 A_1(3z,\tau;3\tau)\right). \end{aligned}$$

A short computation then shows that

$$\begin{aligned} A_3(3z,0;3\tau) &= \frac{-i\eta^3(3\tau)}{\vartheta(3z;3\tau)} \\ A_1(3z,\pm\tau;3\tau) &= \vartheta(\pm\tau;3\tau)\mu(3z,\pm\tau;3\tau) \\ \vartheta(\pm\tau;3\tau) &= \mp i q^{-\frac{1}{6}}\eta(\tau) \end{aligned}$$

which yields the claim when combined with the expression above for R. □

Using the properties of η and θ, it follows that C is essentially (that is, up to rational powers of ζ and q) a Jacobi form of weight -1/2 and index -1/2. Similarly, using these properties along with those of the Zwegers μ-function, it follows that R is essentially a mock Jacobi form. Using these (mock) modularity properties, one can prove many theorems about congruences and asymptotic properties of crank and rank partition functions, as we shall see in the next few subsections.

5.2.2 Properties of the crank partition function

Since the crank generating function is essentially a Jacobi form, we begin our discussion with this simpler case. We first consider certain crank statistics which sift partitions according to the residues of their cranks modulo t. More precisely, for any integer r and positive integer t, let M(r, t; n) be the number of partitions of n whose crank is congruent to r (mod t):

$$M(r,t;n) := \sum_{m \equiv r (modt)} M(n,n).$$

For fixed r and odd t, Mahlburg proved that these partition functions also satisfy Ramanujan-type congruences. In fact, he showed that such congruences hold for all arithmetic progressions $r(\text{mod}t)$ for any fixed odd t, thereby giving a Dyson-style explanation for partition function congruences of the form ($\mathcal{Q} > 3$)

$$p(An+B) \equiv 0 \pmod{\mathcal{Q}^j}$$

Namely, these congruences follow directly from congruences for $M(r,t;n)$ thanks to the tautological identity

$$p(n) = \sum_{0 \leq r \leq t-1} M(r,t;n)$$

Theorem 5.7. *Suppose that $t \geq 1$ is odd and that $\mathcal{Q} \nmid 6t$ is prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An+B$ such that for every $0 \leq r < t$ we have*

$$M(r,t;An+B) \equiv 0 \pmod{\mathcal{Q}^j}$$

In particular, we have that

$$p(An+B) \equiv 0 \pmod{\mathcal{Q}^j}$$

One can see [Bri+17] for details.

Reformulating the conjecture of Dyson described above in our new notation, note that Andrews' and Garvan's combinatorial explanation for Ramanujan's congruences may be summarized in the following relations, valid for all non-negative integers n :

$$\begin{aligned}
&M(0,5;5n+4) = M(1,5;5n+4) = \cdots = M(3,5;5n+4) = M(4,5;5n+4) \\
&M(0,7,7n+5) = M(1,7;7n+5) = \cdots = M(5,7;7n+5) = M(6,7;7n+5) \\
&M(0,11;11n+6) = M(1,11;11n+6) = \cdots = M(10,11;11n+6).
\end{aligned}$$

The previous theorem demonstrates that the crank also plays a role in infinitely many partition congruences, but in a very different way. There we saw that the crank partition functions satisfy congruences themselves, which in turn imply congruences for $p(n)$. In light of these results, it is very natural to ask about the extent to which there are further identities for the sifted crank statistics, and what is the general theory underlying such identities. Much is now known in this direction, and there is indeed a rich structure of crank identities. For example, for n odd, we have

$$M(0,8;n) + M(1,8;n) = M(3,8;n) + M(4,8;n)$$

and related identities are known for the moduli $5, 7, 8, 9, 10$, and 11.

Such identities are rare. It turns out that cranks are not generally uniformly distributed among residue classes modulo t. In this direction, we have the following conjecture in the special case of cranks modulo 3. If $n \in \mathbb{N}$, then we have

$$\begin{aligned} M(0,3;3n) &> M(1,3;3n), \\ M(0,3;3n+1) &< M(1,3;3n+1), \\ M(0,3;3n+2) &< M(0,3;3n+2) \quad \text{unless } n \in \{1,4,5\}. \end{aligned}$$

This conjecture was proven by Kane[Kan04], using the classical Circle Method. In particular, he determined the asymptotic behavior of $M(0,3;n)$ and $M(1,3;n)$ and explicitly bounded the resulting error terms. More general asymptotics for cranks were determined by Zapata[Zap15].

5.2.3 Properties of the rank partition function

In this subsection, we establish congruences for and inequalities between Dyson's rank functions. To do so, we make use of mock modular forms. Analogous to the sifted crank statistics, if r and t are integers, we let N(r, t; n) be the number of partitions of n whose rank is r(mod t).

Theorem 5.8. *Let t be a positive odd integer, and let $Q \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq r < t$ we have*

$$N(r,t;An+B) \equiv 0 \pmod{\mathcal{Q}^j}$$

SKETCH OF PROOF. Using the usual orthogonality relations of roots of unity gives that for $t \in \mathbb{N}$ and $r \in \mathbb{Z}$

$$\sum_{n=0}^{\infty} N(r,t;n)q^n = \frac{1}{t}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj} R\left(\zeta_t^j;q\right)$$

where for $N \in \mathbb{N}, \zeta_N = e^{2\pi i/N}$. Thus a classical dissection argument then implies that the function

$$\sum_{n=0}^{\infty} \left(N(r,t;n) - \frac{1}{t}p(n)\right) q^{n-\frac{1}{24}}$$

is a mock theta function whose shadow is, up to a multiplicative constant,

$$\sum_{j=0}^{1} \sum_{n \equiv 2r+(-1)^j} {}_{(\bmod 2t)} (-1)^j \left(\frac{12}{n}\right) n q^{\frac{n^2}{24}}$$

We are now able to reduce the problem to one involving classical modular forms. The key observation is that the shadow is always supported on finitely many square classes. Thus, we may sieve out the coefficients to restrict the Fourier expansion to run over appropriate arithmetic progressions away from the support of the shadow. This yields the holomorphic part of a harmonic Maass form (on a higher level congruence subgroup) with vanishing shadow, thus giving a classical weakly holomorphic modular form. To be more precise, the function defined by

$$\sum_{\substack{n\geq 0 \\ \left(\frac{1-2\pi n}{Q}\right)=-1}} \left(N(r,t;n)-\frac{1}{t}p(n)\right) q^{n-\frac{1}{24}}$$

where the sum runs over those n that satisfy $\left(\frac{1-24n}{Q}\right) = -1$, is weakly holomorphic. Now the claim follows as basic manipulation.

Remark. *Similar congruences can be proven for general mock theta functions. This is because twisting operators always annihilate the shadow on certain arithmetic progressions and since the general theory of weakly holomorphic modular forms guarantees the existence of many congruences for classical modular forms. It is expected that essentially all congruences of mock modular forms arise in this manner.*

We now turn to inequalities for ranks. Analogous to the above inequalities for the crank function, using clever combinatorial methods, we have

Proposition 5.9.

$$\begin{aligned} N(0,2;2n) &< N(1,2;2n) && \textit{if } n\geq 1 \\ N(0,4;n) &> N(2,4;n) && \textit{if } 26<n\equiv 0,1 \pmod 4 \\ N(0,4;n) &< N(2,4;n) && \textit{if } 26<n\equiv 2,3 \pmod 4. \end{aligned}$$

Proof. The proof is similar to the one of crank generating functions. □

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