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南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目：一个类挂谷集的构造
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院　　系：数学系
专　　业：数学与应用数学
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2026 年 4 月 29 日

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2026 年 4 月 29 日

Construction of a Kakeya-type Set

Chen Zihan

(Department of Mathematics Thesis Advisor: Yang Tongou)

[ABSTRACT]: This paper is aimed at finishing the construction of a closed subset with Lebesgue measure zero that contains a sphere of every radius in the Euclidean space with dimension greater than or equal to two. The construction is a generalization to what has been done in the plane, by A. S. Bescovitch and R. Rado in 1968. The detailed procedure of construction is similar to that of the so-called Kakeya set. In this procedure, a key lemma given by Lawrence Kosala and Thomas Wolff in 1999 plays the same role as the Kakeya needle set, so that we can follow the so-called standard pattern to construct the expected set. We also finish the proof of this technical lemma, which makes use of maximal separated subsets of the unit sphere and translations of closed annuli. In addition, some basic results related to metric space and Lebesgue measure are also applied.

[Keywords]: Kakeya-type set, Sphere, Lebesgue measure

[摘要]： 本文旨在完成对欧几里得空间中（维数大于等于二）一个具有勒贝格测度为零且对任意正数都包含一个对应半径球面的闭子集的构造。该构造是对 A. S. 贝斯科维奇与 R. 拉多于 1968 年在平面上所做工作的推广。其详细构造过程与所谓的挂谷集的构造类似。在此过程中，Lawrence Kosala 与 Thomas Wolff 于 1999 年给出的一个关键引理起到了与 Kakeya 针集相同的作用，从而使我们能够遵循所谓的标准模式来构造所需的集合。我们同时也完成了这一技术性引理的证明，该证明利用了单位球面的极大分离子集以及闭环带的平移。此外，文中还应用了一些与度量空间和勒贝格测度相关的基本结果。

[关键词]： 类挂谷集；球面；勒贝格测度

Construction of Kakeya-type Sets

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April 30, 2026

Contents

1 Introduction and Motivation

In 1968, Besicovitch and Rado proved that for $d = 2$ there exists a closed set of Lebesgue measure zero in $\mathbb{R}^2$ that contains a circle of every radius $r > 0$ [1]. For general dimensions $d \geq 2$, a construction was later obtained by Kolasa and Wolff in 1999 [2]. Indeed, they proved the following lemma and claimed that the target set could be given via a standard pattern.

Lemma 1.1. *Let* R_0, R_1 *be real numbers such that* $\frac{1}{2} \leq R_0 < R_1 \leq 2R_0 \leq 2$. *For each integer* $n \geq 1$, *let* $M \approx n^{\frac{d-1}{2}}$ *be the cardinality of a maximal* $\frac{1}{C\sqrt{n}}$*-separated subset of* S^{d-1}. *Then any annulus* $\{x \in \mathbb{R}^d \colon R_0 \leq |x - x_0| \leq R_1\}$ *may be divided into* n^M *closed subannuli, each of width* $(R_1 - R_0)n^{-M}$, *which when properly translated yield a compact set* E_n *such that* $m(E_n) \lesssim (R_1 - R_0)/n$. *Moreover,* $E_n \subset \{x \in \mathbb{R}^d \colon 2R_0 - R_1 \leq |x - x_0| \leq R_1\}$.

A notation shall be introduced first. Let f, g be nonnegative functions. By denoting $f \lesssim g$, we mean that there exists $c > 0$ so that $f(\theta) \leq cg(\theta)$ uniformly holds for θ; by denoting $f \approx g$, we mean that both $f \lesssim g$ and $g \lesssim f$ hold. So in the above Lemma 1.1, the relations $M \approx n^{\frac{d-1}{2}}$ and $m(E_n) \lesssim (R_1 - R_0)/n$ hold with coefficients depending only on d.

Kolasa and Wolff did not give an expilcit construction of the target set, so this paper is aimed to construct the expected set. We state the main result here.

Theorem 1.2. *For any* $d \geq 2$, *there is a closed subset* $E \subset \mathbb{R}^d$, *with Lebesgue measure zero, that contains a sphere of every radius.*

The construction of the expected set has the key idea similar to that of the so-called Kakeya set. Recall that the Kakeya set is a compact subset of $\mathbb{R}^2$ with Lebesgue measure zero that contains a unit line segment in every direction. Roughly speaking, one can construct the Kakeya set following the procedure below:

1. First divide an equilateral triangle into many subtriangles, so that in the later construction each of these subtriangles is transformed into a compact set with Lebesgue measure zero that contains a unit line segment with direction in a certain range.

2. For each of these subtriangles, carry out the Perron tree process so that the result set, called the Kakeya needle set, has Lebesgue measure arbitrarily small; in another, the result set can be obtained in an open neighborhood that has Lebesgue measure in the same order.

3. Inductively construct a sequence of open subsets, each of which has closure contained in the previous one and covers a Perron tree, so that they are uniformly bounded and having Lebsegue measure tending to zero. Besides, each of the open sets contain a unit line segment in the direction in given range.

4. Finally, take the intersection of the closure of the open sets, so then we obtain a compact set with Lebesgue measure zero that contains a unit line segment in the direction in given range. Take finite union of theses intersections we obtain the Kakeya set.

We point out that the set E_n given by Lemma 1.1 actually plays the same role as the Kakeya needle set in the above construction. They both provide a way to compress the original set to have arbitrarily small Lebesgue measure, and at the same time, preserve the structure of a certain range of radii of spheres or directions of segments. In another, the above procedure inspires us how to furthermore make Lebesgue measure from being arbitrary small to being zero, taking countable intersections, and how to preserve the structure of the intersection, taking open neighborhoods.

The rest of this paper is organized as follows: in Section 2 we will prove the main result Theorem 1.2, in case of admitting Lemma 1.1, and some basic preliminaries required will also be given; in Section 3 we will give a proof of the key lemma which is pretty technical. At the end of this paper, some results concerning integration in polar coordinates will be given, which are applied through the paper.

2 Proof of Theorem 1.2

2.1 Separability of $\mathbb{R}^d$

Let $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, then we denoted by $d(x, A)$ the distance from the point x to the set A. That is,

$$d(x, A) = \inf\{|x - y| \colon y \in A\}.$$

Generally, if A, B are subset of $\mathbb{R}^d$, then the distance between two sets is

$$d(A, B) = \inf\{|x - y| \colon x \in A, y \in B\}.$$

We recall some basic facts related to separability in $\mathbb{R}^d$ that will be used in the construction.

Proposition 2.1. *Let A, K be nonempety subsets of $\mathbb{R}^d$ with A closed and K compact. If $A \cap K = \varnothing$, then $d(A, K) > 0$.*

Proof. If $d(A, K) = 0$, then for each $n \in \mathbb{N}$ there are $x_n \in A$ and $y_n \in K$ such that $|x_n - y_n| < \frac{1}{n}$. Since K is compact, there is a convergent subsequence $\{y_{n_j}\}_j$ with limit $y \in K$. Then $\{x_{n_j}\}_j$ also converges to y. But A is closed, $y \in A$ and thus $y \in A \cap K$, a contradiction. □

Proposition 2.2. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{R}^d$ satisfying $\inf_{i \neq j} d(A_i, A_j) > 0$, then the union $A = \bigcup_{k=1}^{\infty} A_k$ is closed.*

Proof. Let $\{x_n\}$ be a sequence in A that converges to x. Then $\{x_n\}$ is a Cauchy sequence, so for $0 < \varepsilon < \inf_{i \neq j} d(A_i, A_j)$, there is $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $m, n \geq N$. In particular, $|x_n - x_N| < \varepsilon$ for all $n \geq N$. Since $x_N \in A$, there is $k \in \mathbb{N}$ such that $x_N \in A_k$. We claim that $x_n \in A_k$

for $n \geq N$. Otherwise, for some $j \geq N$ there is $x_j \notin A_k$, but then $d(x_j, A_k) \geq \inf_{i \neq j} d(A_i, A_j) > \varepsilon$, which leads to a contradiction. Now that $\{x_n\}$ is a sequence in A_k except for finitely many terms. Since A_k is compact, $x = \lim x_n \in A_k \subset A$, hence A is closed. □

2.2 Lebesgue Measure of Annuli

For any $\delta > 0$ and any set $E \subset \mathbb{R}^d$, we denote by $N_\delta(E)$ the δ-neighborhood of E. That is,

$$N_\delta(E) = \{x \in \mathbb{R}^d \colon d(x, R) < \delta\},$$

One can easily verifies that for any $\delta > 0$,

1. the δ-neighborhood of a nonempty set is always open;
2. taking δ-neighborhood is exchangable with taking arbitrary union, and in particular,

$$N_\delta(A_1 \cup \cdots \cup A_N) = N_\delta(A_1) \cup \cdots \cup N_\delta(A_N).$$

For a measurable set $E \subset \mathbb{R}^d$ we denote by $m(E)$ its Lebesgue measure. Recall that the Lebesgue measure of a set $E \subset \mathbb{R}^d$ can be given by

$$m(E) = \inf\{m(U) \colon U \supset E, U \text{ open}\}.$$

If E is the closed annulus $A = \{x \in \mathbb{R}^d \colon R_0 \leq |x - x_0| \leq R_1\}$, we can take the open sets to be the δ-neighborhoods of A. Indeed, for any open set $U \supset A$, we have $U^c \cap A = \varnothing$ with U^c closed and A compact, so then $d(U^c, A) > 0$ by Proposition 2.1. Let $0 < \delta < d(U^c, A)$, then $A \subset N_\delta(A) \subset U$. Thus we have

$$m(A) = \inf_{\delta > 0} m(N_\delta(A)).$$

Moreover, this result can be generalized to the case of finite union of annuli.

Proposition 2.3. *Let $E = \bigcup_j A_j$ be a finite union of closed annuli, then*

$$m(E) = \inf_{\delta > 0} m(N_\delta(E)).$$

Proof. For any $\delta > 0$, we have $E \subset N_\delta(E)$, hence $m(E) \leq m(N_\delta(E))$. Let $\varepsilon > 0$, then there is an open set $U \supset E$ such that $m(U) \leq m(E) + \varepsilon$. Since $E = \bigcup_j A_j$, each A_j is a subset of U, so there is $\delta_j > 0$ such that $N_{\delta_j}(A_j) \subset U$ by Proposition 2.1. Now take $\delta = \min_j \delta_j$, then $N_\delta(A_j) \subset N_{\delta_j}(A_j) \subset U$ and

$$N_\delta(E) = \bigcup_j N_\delta(A_j) \subset U.$$

Thus $m(N_\delta(E)) \leq m(U) \leq m(E) + \varepsilon$. □

If A is an open annulus, then $m(A) = m(\overline{A})$ where $\overline{A}$ is the closure of A. This is because $m(S) = 0$ for any sphere S in $\mathbb{R}^d$. Again we can generalize this result to finite unions of open annuli.

Proposition 2.4. *Let* $E = \bigcup_j A_j$ *be a finite union of open annuli, then for* $\delta > 0$,

$$m(N_\delta(E)) = m(\overline{N_\delta(E)}).$$

Proof. It remains to show that the boundary of $N_\delta(E)$ has Lebesgue measure zero. Note that

$$\partial N_\delta(E) \subset \bigcup_j \partial N_\delta(A_j),$$

and that each $\partial N_\delta(A_j)$ is a union of inner sphere and outer sphere of $N_\delta(A_j)$, hence has Lebesgue measure zero. □

2.3 Proof of Theorem 1.2

To begin with, we give a modified version of Lemma 1.1 which can be used in a more flexible way. If we put $R_0 = 1$ and $R_1 = 2$, then Lemma 1.1 states that the annulus $A = \{x \in \mathbb{R}^d \colon 1 \le |x| \le 2\}$ can be divided into finitely many closed sub-annuli, each of which translated properly, yielding a compact set E that has Lebesgue measure arbitrarily small. Then for any pair r, R of radii with $1 \le r < R \le 2$, the annulus $A_{r,R} = \{x \in \mathbb{R}^d \colon r \le |x| \le R\}$, as a subset of A, has its image under this process contained in E. The image is again a finite union of closed translated sub-annuli of $A_{r,R}$ (not necessarily of equal width); also, it is compact and has Lebesgue measure arbitrarily small. In conclusion, we have

Lemma 2.5. *Let* r, R *be real numbers with* $1 \le r < R \le 2$ *and let* $A_{r,R} = \{x \in \mathbb{R}^d \colon r \le |x| \le R\}$ *be the closed annulus. For any* $\varepsilon > 0$, *we can divide* $A_{r,R}$ *into finitely many closed sub-annuli, each translated properly, yielding a compact set* S *that has Lebesgue measure less than* ε.

We now turn to the proof of Theorem 1.2. For the first step, we will construct a compact set with Lebesgue measure zero that contains a sphere of every radius in $[1, 2]$. For convenience, we denote by $S(x, r)$ to be the sphere $\{y \in \mathbb{R}^d \colon |y - x| = r\}$ centered at x and of radius r. We also denote by S^{d-1} the set $\{e \in \mathbb{R}^d \colon |e| = 1\}$, whose elements are called directions.

Step 1:

By Lemma 2.5 there is a compact set $S_1 \subset \mathbb{R}^d$, which is a finite union of translated sub-annuli of the original annulus $A = \{x \in \mathbb{R}^d \colon 1 \le |x| \le 2\}$, having Lebesgue measure $< \frac{1}{2}$. Then we can find $\delta_1 > 0$ such that $m(N_{\delta_1}(S_1)) < \frac{1}{2}$ by Proposition 2.3.

Inductively, let $S_k \subset \mathbb{R}^d$ be a finite union of translated sub-annuli A_j of A, whose δ_k-neighborhood has Lebesgue measure $< 2^{-k}$. Let N be the counting number of these sub-annuli, and for each fixed $j = 1, \dots, N$, we divide A'_j, the translated A_j, into ℓ concentric sub-annuli of equal width, which we denote by $A_j^1, \dots, A_j^\ell$. Now apply Lemma 2.5 to each A_j^m ($m = 1, \dots, \ell$), and take the union S_{k+1}. Since there are finitely many sub-annuli, we can guarantee that S_{k+1} has Lebesgue measure $< 2^{-(k+1)}$. Moreover, in the process of translation, each sub-annulus of A_j^m moves at a distance no more than the width of A_j^m, hence less than δ_k if ℓ is sufficiently large. Thus we can make sure that

$S_{k+1} \subset N_{\delta_k}(S_k)$. Now combining Proposition 2.1 and Proposition 2.3 we can find $\delta_{k+1} > 0$ so that $N_{\delta_{k+1}}(S_{k+1}) \subset N_{\delta_k}(S_k)$ and $m(N_{\delta_{k+1}}(S_{k+1})) < 2^{-(k+1)}$.

For each k, let S'_k be the closure of $N_{\delta_k}(S_k)$. Then S'_k is compact, $m(S'_k) < 2^{-k}$, and $S'_1 \supset S'_2 \supset \cdots$. Take $S = \bigcap_1^\infty S'_k$, then S is compact and $m(S) = 0$. It suffices to show that S contains a sphere of every radius in $[1, 2]$. Fix $r \in [1, 2]$, and for each k let C_k be the set of centers c such that the sphere $S(c, r) \subset S'_k$. By construction, each S_k contains a sphere of radius r, so C_k is nonempty. Also, the C_k's are nested since so are the S'_k's. We claim that C_k is compact, then by nested sets theorem, $\bigcap_1^\infty C_k$ is nonempty, so that there is a point x_r such that $S(x_r, r) \subset S$.

We now prove that each C_k is compact. Obviously C_k is bounded. Let $\{c_n\}$ be a sequence of elements in C_k that converges to c^*, then for any direction $e \in S^{d-1}$, $c_n + re \to c^* + re$. Since $c_n + re \in S'_k$ and S'_k is closed, we obtain that $c^* + re \in S'_k$, hence $S(c^*, r) \subset S'_k$ and $c^* \in C_k$.

□

For the second step, note that we have the composition

$$(0, \infty) = \bigcup_{-\infty}^{\infty} [2^k, 2^{k+1}].$$

Thus we have previously dealt with the case $k = 0$. For general $k \in \mathbb{Z}$, we can make use of dilations.

Step 2:

We denote by E_0 the set given in the first step. For any $k \in \mathbb{Z}$, let E_k be the dilation of E_0 by coefficient 2^k. That is,

$$E_k = 2^k E_0 = \{2^k x \in \mathbb{R}^d \colon x \in E_0\}.$$

Then E_k is still compact and $m(E_k) = 2^k m(E_0) = 0$. Also, E_k contains a sphere of every radius in $[2^k, 2^{k+1}]$. Indeed, let $R \in [2^k, 2^{k+1}]$, then $R/2^k \in [1, 2]$, so there is $x_R \in \mathbb{R}^d$ such that $S(x_R, R/2^k) \subset E_0$. Thue $S(2^k x_R, R) = 2^k S(x_R, R/2^k) \subset 2^k E_0 = E_k$.

□

Recall that the set E_0 in the first step is contained in S'_1, where S'_1 is the closure of S_1's δ_1-neighborhood. By Lemma 1.1 we have $S_1 \subset \{x \in \mathbb{R}^d \colon |x| \le 2\}$. We may require that $\delta_1 \le 1$, so we have $E_0 \subset \{x \in \mathbb{R}^d \colon |x| \le 3\}$, and it follows that $E_k \subset \{x \in \mathbb{R}^d \colon |x| \le 3 \cdot 2^k\}$. To obtain tha target set, one just need to separate them properly in $\mathbb{R}^d$.

Step 3:

We first consider the real line $\mathbb{R}$. For each $k \in \mathbb{Z}$ let $I_k = [0, 2^{k+3}]$. We then translate these intervals so that they are disjoint, and denote by I'_k the translated I_k. More precisely, we can inductively translate the I_k's so that $\inf I'_{k+1} = 1 + \sup I'_k$.

Next we embedd $\mathbb{R}$ into $\mathbb{R}^d$ to be the axis spanned by the direction $e_1 = (1, 0, \dots, 0) \in S^{d-1}$, then we have a sequence of disjoint closed regions, $I'_k \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$. For each k, let B_k be the closed ball of radius 2^{k+2} located in $I'_k \times \mathbb{R}^{d-1}$. Then each E_k can be translated into B_k, denoted by E'_k, so that

$E = \bigcup_{-\infty}^{\infty} E'_k$ is a disjoint union of compact subsets. Since

$$\inf_{i\neq j} d(E_i, E_j) \geq \inf_{i\neq j} d(B_i, B_j) \geq 1,$$

we obtain that E is closed by Proposition 2.2. Besides, E has Lebesgue measure zero since

$$m(E) = \sum_{-\infty}^{\infty} m(E'_k) = \sum_{-\infty}^{\infty} m(E_k) = 0,$$

and contains a sphere of every radius by construction of E'_k. Thus we finish the proof of Theorem 1.2.

□

3 Proof of Lemma 1.1

3.1 Estimate of Length

Let $S_0 = S(C_0, R_0)$ and $S_1 = S(C_1, R_1)$, and assume that $R_0 < R_1$. We also require that $|C_1 - C_0| < R_1 - R_0$, so then

$$\alpha = R_1 - R_0 - |C_1 - C_0| > 0.$$

Let R be a ray emanating from C_1, Q the intersection point of R with S_0, and θ the angle between R and the ray from C_1 through C_0 (see Figure 1). Denote by L the length of the segment C_1Q.

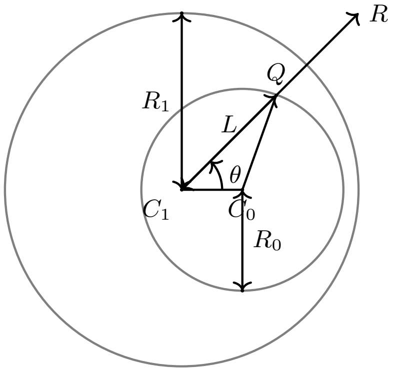


Figure 1: Geometry of two spheres with $|C_1 - C_0| < R_1 - R_0$. The inner sphere S_0 (center C_0, radius R_0) lies entirely inside the outer sphere S_1 (center C_1, radius R_1). The ray R from C_1 at angle θ intersects S_0 at point Q; the distance $|C_1Q|$ is denoted by L.

The following lemma gives an estimate of $R_1 - L$, which will be used later to estimate the volumn.

Lemma 3.1. *Under the above assumptions, if $\theta \ll 1$ and $\theta\frac{|C_1-C_0|}{R_0} \ll 1$, then the length of the segment of R between S_0 and S_1 satisfies*

$$R_1 - L \approx \alpha + \frac{1}{2}\theta^2 |C_1 - C_0| \left(1 + \frac{|C_1 - C_0|}{R_0}\right),$$

with both sides are functions of θ.

Proof. We first compute L, the distance from C_1 to Q. By the law of cosines applied to the triangle with vertices C_1, C_0, and Q, we have

$$R_0^2 = |C_1 - C_0|^2 + L^2 - 2L\,|C_1 - C_0|\cos\theta.$$

Solving this quadratic equation for L yields

$$L = |C_1 - C_0|\cos\theta + R_0\sqrt{1 - \frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta}.$$

So then

$$R_1 - L = R_1 - |C_1 - C_0|\cos\theta - R_0\sqrt{1 - \frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta}.$$

Add and subtract $|C_1 - C_0|$ then we obtain that

$$R_1 - L = \alpha + |C_1 - C_0|(1-\cos\theta) + R_0\left(1 - \sqrt{1 - \frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta}\right).$$

Lower bound. By convexity, the inequality $\sqrt{1-t} \le 1 - \frac{1}{2}t$ holds for $t \ge 0$. Applying this with $t = \frac{|C_1-C_0|^2}{R_0^2}\sin^2\theta$ gives

$$1 - \sqrt{1 - \frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta} \ge \frac{1}{2}\frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta.$$

Note that when θ is sufficiently small, $1 - \cos\theta \ge \frac{1}{4}\theta^2$, $\sin^2\theta \ge \frac{1}{2}\theta^2$, and hence

$$R_1 - L \ge \alpha + \frac{1}{4}\theta^2|C_1 - C_0| + \frac{1}{4}\theta^2\frac{|C_1 - C_0|^2}{R_0},$$

Combining the coefficients, we obtain that

$$\begin{aligned} R_1 - L &\ge \alpha + \frac{1}{4}\theta^2|C_1 - C_0|\left(1 + \frac{|C_1 - C_0|}{R_0}\right) \\ &> \frac{1}{2}\left[\alpha + \frac{1}{2}\theta^2|C_1 - C_0|\left(1 + \frac{|C_1 - C_0|}{R_0}\right)\right]. \end{aligned}$$

Upper bound. For the upper bound, we use the inequality $\sqrt{1-t} \ge 1 - 2t$ for $0 \le t \le 1$. This gives

$$1 - \sqrt{1 - \frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta} \le 2\frac{|C_1 - C_0|^2}{R_0^2}\sin^2\theta.$$

Note that $1 - \cos\theta \le 2\theta^2$ and $\sin^2\theta \le \theta^2$ for $\theta \in \mathbb{R}$. Hence

$$R_1 - L \le \alpha + 2\theta^2|C_1 - C_0| + 2\theta^2\frac{|C_1 - C_0|^2}{R_0},$$

Combining the coefficients, we obtain that

$$\begin{aligned} R_1 - L &\le \alpha + 2\theta^2|C_1 - C_0|\left(1 + \frac{|C_1 - C_0|}{R_0}\right) \\ &< 4\left[\alpha + \frac{1}{2}\theta^2|C_1 - C_0|\left(1 + \frac{|C_1 - C_0|}{R_0}\right)\right]. \end{aligned}$$

Combining the lower and upper bounds, we conclude that

$$R_1 - L \approx \alpha + \frac{1}{2}\theta^2|C_1 - C_0|\left(1 + \frac{|C_1 - C_0|}{R_0}\right).$$

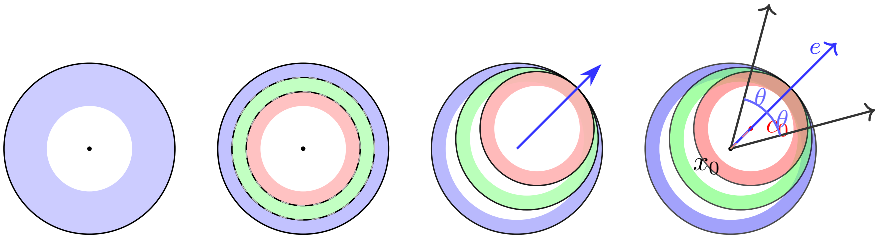
□

3.2 Estimate of Volume

For $x_0 \in \mathbb{R}^d$ and $e \in S^{d-1}$, we denote by $\Gamma_\theta^e(x_0) = \{x_0 + th : t \geq 0,\ h \in S^{d-1},\ \mathrm{dist}_{S^{d-1}}(h, e) \leq \theta\}$, where $\mathrm{dist}_{S^{d-1}}$ denotes the angle between vectors on the unit sphere; that is, $\Gamma_\theta^e(x_0)$ is the cone with vertex x_0, aperture θ, and axis e.

Let $A = \{x \in \mathbb{R}^d : R_0 \leq |x - x_0| \leq R_1\}$ be an annulus with center x_0. The quantity $R_1 - R_0$ is called the *width* of A. In the following we assume that $\frac{1}{2} \leq R_0 < R_1 \leq 2R_0 \leq 2$.

For a fixed integer $n \in \mathbb{N}$ and a fixed direction $e \in S^{d-1}$, we construct a set $\phi_n^e(A)$ as follows: first divide A into n concentric subannuli of equal width $(R_1 - R_0)/n$; number them from A_1 to A_n sorted from outer to inner; translate each subannulus in the direction e relative to the center x_0: the k-th annulus is translated by a distance $(k-1)(R_1 - R_0)/n$. In this way the n translated annuli intersect tangentially along a ray in the direction e (see Figure 2). The resulting set $\phi_n^e(A)$ is called a *figure*, and x_0 is called its *center*. Obviously $m(\phi_n^e(A)) \leq m(A)$.



(a) **(b)** **(c)** **(d)**

Figure 2: Construction of $\phi_n^e(A)$ for $d = 2$, $n = 3$, and e at 45°. (a) The original annulus A; (b) Division into n concentric sub-annuli $A_1, \ldots, A_n$; (c) Translation along e by distances $(i-1)(R_1 - R_0)/n$ yields $\phi_n^e(A)$; (d) Geometric relationship: x_0 is the center of $\phi_n^e(A)$, c_0 is the center of the innermost translated annulus.

Lemma 3.2. *Under the above assumptions, if n is sufficiently large, then*

$$m\left(N_\delta\left(\phi_n^e(A) \cap \Gamma_\theta^e(x_0)\right)\right) \lesssim (R_1 - R_0)\, n^{-\frac{d+1}{2}},$$

where $\delta = 2(R_1 - R_0)/n$ and $\theta = 2/\sqrt{n}$.

Proof. First note that if x_0 is the center of $\phi_n^e(A)$ and c_0 is the center of A_n, then we have

$$\phi_n^e(A) \cap \Gamma_\theta^e(x_0) \subset \Gamma_\theta^e(x_0) \cap \{x : |x - x_0| \leq R_1\} \cap \{x : |x - c_0| \geq R_0\}.$$

For convenience we denote by E the latter set, so it reduces to showing that

$$m\left(N_\delta(E)\right) \lesssim (R_1 - R_0)\, n^{-\frac{d+1}{2}}.$$

Let $\Omega = \Gamma_\theta^e(x_0) \cap S^{d-1}$, and for each $\omega \in \Omega$ let $I_\omega = \{r \geq 0 \colon x_0 + r\omega \in E\}$. Then $|I_\omega| \leq R_1 - L$ where L is the distance from x_0 to the intersection point of the ray $R = \{x_0 + r\omega \colon r \geq 0\}$ with the sphere $|x-c_0|=R_0$. By Lemma 3.1 applied with $C_1=x_0$, $C_0=c_0$, $P_1=R_1$, $P_0=R_0$, and with

$$\alpha=\frac{R_1-R_0}{n},\quad |x_0-c_0|=\frac{(n-1)(R_1-R_0)}{n},$$

we obtain that

$$\begin{aligned}R_1-L&\approx\alpha+\frac{1}{2}\omega^2|x_0-c_0|\left(1+\frac{|x_0-c_0|}{R_0}\right)\\&=\frac{R_1-R_0}{n}+\frac{1}{2}\omega^2\frac{(n-1)(R_1-R_0)}{n}\left(1+\frac{(n-1)(R_1-R_0)}{nR_0}\right).\end{aligned}$$

Since $\omega\le\delta=2/\sqrt{n}$, $n-1\le n$, and $R_1-R_0\le R_0$ (as $R_1\le 2R_0$), we have

$$R_1-L\lesssim\frac{R_1-R_0}{n}+\frac{1}{2}\cdot\frac{4(R_1-R_0)}{n}\cdot 2=C_0\frac{R_1-R_0}{n}.$$

That is, $|I_\omega|\lesssim(R_1-R_0)/n$ for all $\omega\in\Omega$.

In another, the spherical cap Ω has surface measure

$$\sigma(\Omega)=\frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\theta\sin^{d-2}\varphi\,d\varphi\lesssim\int_0^\theta\sin^{d-2}\varphi\,d\varphi.$$

Since $\sin\varphi\le\varphi$, we have

$$\sigma(\Omega)\lesssim\int_0^\theta\varphi^{d-2}\,d\varphi=\frac{\theta^{d-1}}{d-1}\lesssim\theta^{d-1}.$$

But $\theta^{d-1}=(2/\sqrt{n})^{d-1}=2^{d-1}n^{(d-1)/2}$, thus we have $\sigma(\Omega)\lesssim n^{(d-1)/2}$.

We now estimate $m(N_\delta(E))$. For each $x\in N_\delta(E)$, there exists $y\in E$ such that $|x-y|<\delta$. Write $y=x_0+r_y\omega_y$ with $\omega_y\in\Omega$ and $r_y\in I_{\omega_y}$, and $x=x_0+r_x\omega_x$ with $\omega_x\in S^{d-1}$ and $r_x\ge 0$. Then

$$|x-y|^2=r_x^2+r_y^2-2r_xr_y\langle\omega_x,\omega_y\rangle,$$

while

$$|r_x-r_y|^2=r_x^2+r_y^2-2r_xr_y.$$

Since $\langle\omega_x,\omega_y\rangle\le 1$, we have $|r_x-r_y|\le|x-y|<\delta$, hence r_x lies in the δ-neighborhood of I_{ω_y}, which is an interval of length at most $|I_{\omega_y}|+2\delta\lesssim\delta$.

In another, observe that $\omega_x-\omega_y=\frac{1}{r_y}(x-y)-\frac{r_x-r_y}{r_y}\omega_x$, so taking absolute values yields

$$|\omega_x-\omega_y|\le\frac{|x-y|}{r_y}+\frac{|r_x-r_y|}{r_y}\le\frac{|x-y|}{r_y}+\frac{|x-y|}{r_y}=\frac{2|x-y|}{r_y}.$$

Since $|x-y|<\delta$ and $r_y\ge R_0\ge 1/2$, we have $|\omega_x-\omega_y|\le 4\delta/R_0$. Thus ω_x lies in the $4\delta/R_0$-neighborhood of Ω on S^{d-1}, denoted by $\tilde{\Omega}$. That is, $\tilde{\Omega}$ is a spherical cap centered at x_0 with angular radius $\theta+4\delta/R_0$. Therefore,

$$\frac{\sigma(\tilde{\Omega})}{\sigma(\Omega)}=\frac{\int_0^{\theta+4\delta/R_0}\sin^{d-2}\varphi\,d\varphi}{\int_0^\theta\sin^{d-2}\varphi\,d\varphi}.$$

For small angles, $\sin\varphi\approx\varphi$, thus

$$\frac{\sigma(\tilde{\Omega})}{\sigma(\Omega)}\approx\left(\frac{\theta+4\delta/R_0}{\theta}\right)^{d-1}=\left(1+\frac{4\delta}{R_0\theta}\right)^{d-1}.$$

Since $\delta/\theta \lesssim 1/\sqrt{n} \to 0$ as $n \to \infty$, we obtain $\sigma(\tilde{\Omega}) \lesssim \sigma(\Omega)$.

In conclusion, for each $\omega_x \in \tilde{\Omega}$, the set of radial coordinates r_x for which $x_0 + r_x\omega_x$ belongs to $N_\delta(E)$ is contained in some interval J_{ω_x} of length at most $C_r\delta$. Moreover, $r_x \leq R_1 + \delta \leq 3$ (since $R_1 \leq 2$ and δ is small), so $r_x^{d-1} \leq 3^{d-1}$.

Now we integrate in polar coordinates:

$$\begin{aligned} m\left(N_\delta(E)\right) &= \int_{x\in N_\delta(E)} dx \leq \int_{\omega\in\tilde{\Omega}} \int_{r\in J_\omega} r^{d-1}\, dr\, d\sigma(\omega) \\ &\leq \int_{\tilde{\Omega}} \left(\sup_{r\in J_\omega} r^{d-1}\right) \cdot |J_\omega|\, d\sigma(\omega) \leq 3^{d-1} \cdot C_r\delta \cdot \sigma(\tilde{\Omega}). \end{aligned}$$

Combining $\sigma(\tilde{\Omega}) \lesssim \sigma(\Omega)$ and $\sigma(\Omega) \lesssim n^{(d-1)/2}$, we obtain

$$|N_\delta(E)| \lesssim \delta \cdot n^{-\frac{d-1}{2}} = \frac{2(R_1 - R_0)}{n} \cdot n^{-\frac{d-1}{2}} = 2(R_1 - R_0)n^{-\frac{d+1}{2}}.$$

Thus

$$|N_\delta(E)| \lesssim (R_1 - R_0)n^{-\frac{d+1}{2}}.$$

Since $\phi_n^e(A) \cap \Gamma_\theta^e(x_0) \subset E$, we have $N_\delta\left(\phi_n^e(A) \cap \Gamma_\theta^e(x_0)\right) \subset N_\delta(E)$, and hence

$$\left|N_\delta\left(\phi_n^e(A) \cap \Gamma_\theta^e(x_0)\right)\right| \leq |N_\delta(E)| \leq C|E| \lesssim (R_1 - R_0)n^{-\frac{d+1}{2}}.$$

This completes the proof of Lemma 3.2. □

In the recursive construction, we will need to iterate the operation in different directions. For this purpose, we introduce a variant of the previous construction. For a finite union of annuli $A = \bigcup_j A_j$ (with possibly different centers), we define

$$\Phi_n^e(A) = \bigcup_j \phi_n^e(A_j),$$

where ϕ_n^e is the construction described previously. Thus Φ_n^e applies the same translation procedure to each annulus in the union.

Lemma 3.3. *Let $A = \{x \in \mathbb{R}^d : R_0 \leq |x - x_0| \leq R_1\}$ be an annulus, and let $e_k, e_{k+1}, \ldots, e_{k+j}$ be arbitrary directions in S^{d-1}. Then for sufficiently large n (depending only on d),*

$$\Phi_n^{e_{k+j}} \circ \cdots \circ \Phi_n^{e_{k+1}} \circ \Phi_n^{e_k}(A) \cap \Gamma_\theta^{e_k}(x_0) \subset N_\delta\left(\Phi_n^{e_k}(A) \cap \Gamma_\theta^{e_k}(x_0)\right),$$

where $\delta = 2(R_1 - R_0)/n$ and $\theta = 2/\sqrt{n}$.

Proof. Let

$$y \in \Phi_n^{e_{k+j}} \circ \cdots \circ \Phi_n^{e_{k+1}} \circ \Phi_n^{e_k}(A) \cap \Gamma_\theta^{e_k}(x_0).$$

By the definition of the composition, there exists a sequence of points

$$z_k \in \Phi_n^{e_k}(A),\ z_{k+1},\ \ldots,\ z_{k+j} = y$$

where each $z_{\ell+1}$ is obtained from z_ℓ under the construction $\Phi_n^{e_{\ell+1}}$, so that

$$|z_{\ell+1} - z_\ell| \leq \frac{R_1 - R_0}{n^{\ell-k+1}}, \qquad \ell = k, k+1, \dots, k+j-1.$$

Summing these inequalities gives

$$|y - z_k| \leq \sum_{t=0}^{j-1} \frac{R_1 - R_0}{n^{t+1}} \leq (R_1 - R_0) \cdot \frac{1/n}{1 - 1/n} \leq \frac{2(R_1 - R_0)}{n} = \delta.$$

Now the sub-annulus S of $\Phi_n^{e_{k+j}} \circ \cdots \circ \Phi_n^{e_{k+1}} \circ \Phi_n^{e_k}(A)$ containing y has its preimage S' in $\Phi_n^{e_k}(A)$ containing z_k with y correspondent to z_k. Thus S is obtained by moving S' so that $z_k \in S'$ is moved to $y \in S$. Then $S \cap \Gamma_\theta^{e_k}(x_0)$ is removed from $S' \cap \Gamma_\theta^{e_k}(x_0)$ by a distance less than δ. Since $\Phi_n^{e_{k+j}} \circ \cdots \circ \Phi_n^{e_{k+1}} \circ \Phi_n^{e_k}(A)$ is a finite union of these sub-annuli, we have

$$\Phi_n^{e_{k+j}} \circ \cdots \circ \Phi_n^{e_{k+1}} \circ \Phi_n^{e_k}(A) \cap \Gamma_\theta^{e_k} \subset N_\delta(\Phi_n^{e_k}(A) \cap \Gamma_\theta^{e_k}(x_0)).$$

□

3.3 Proof of the Key Lemma

We now prove Lemma 1.1. Recall that we are given R_0, R_1 with $1/2 \leq R_0 < R_1 \leq 2R_0 \leq 2$, and a positive integer n. We will construct a compact set E_n satisfying the required properties.

Proof of Lemma 1.1:

Without loss of generality we may assume that $x_0 = 0$. Inductively we define

$$A_0 = \{x \in \mathbb{R}^d \colon R_0 \leq |x| \leq R_1\}, \quad A_k = \Phi_n^{e_k}(A_{k-1}) \text{ for } k = 1, \dots M.$$

Now take $E_n = A_M$. Then E_n is a union of n^M annuli, each of width $(R_1 - R_0)/n^M$, hence compact. We first claim that $E_n \subset \{x \in \mathbb{R}^d \colon R_0 - (R_1 - R_0) \leq |x| \leq R_1\}$. In fact, for a general annulus $A = \{x \in \mathbb{R}^d \colon r_0 \leq |x - x_0| \leq r_1\}$, we have

$$\begin{aligned}
\phi_n^e(A) &\subset \left\{x \in \mathbb{R}^d \colon r_0 - \frac{n-1}{n}(r_1 - r_0) \leq |x - x_0| \leq r_1\right\} \\
&\subset \left\{x \in \mathbb{R}^d \colon r_0 - (r_1 - r_0) \leq |x - x_0| \leq r_1\right\} \\
&\subset \left\{x \in \mathbb{R}^d \colon R_0 - (R_1 - R_0) \leq |x - x_0| \leq R_1\right\},
\end{aligned}$$

where the last inclusion holds if we in addition restrict that $R_0 \leq r_0 < r_1 \leq R_1$. Thus each subannulus of E_n is contained in $\{x \in \mathbb{R}^d \colon R_0 - (R_1 - R_0) \leq |x| \leq R_1\}$, so is E_n.

It remains to show that $m(E_n)$ has the target upper bound. For this to be reached, we are going to cover E_n by finitely many sets, whose Lebesgue measures have a uniform upper bound.

For each sub-annulus of E_n, it is the image under the composition $\Phi_n^{e_M} \circ \cdots \circ \Phi_n^{e_1}$ of some concentric sub-annulus of the original A_0, so its center is given by

$$(R_1 - R_0) \sum_{j=1}^{M} \frac{i_j}{n^j} e_j,$$

where each i_j is some integer in $[0, n-1]$. So we associate each sub-annulus of E_n to a multi-index $I = (i_1, \dots, i_M)$, called its address. We also denote by $x(I)$ its center. For each $k = 1, \dots, M$, let S_k be the set of indices of the form $(i_1, \dots, i_{k-1}, 0, \dots, 0)$. That is, S_k represents those sub-annuli of E_n that remains fixed under $\Phi_n^{e_k}, \dots, \Phi_n^{e_M}$, so for each $I \in S_k$, $x(I)$ is the center of some figure of A_k, which we denote by $A_k(x(I))$.

If $z \in E_n$, then

1. z is contained in some annulus with address $(i_1, \dots, i_M)$, and thus $z \in A_M(x(i, \dots, i_{M-1}, 0))$;
2. for any $k = 1, \dots, M$ and any $I \in S_k$, $z \in \Phi_n^{e_M} \circ \cdots \circ \Phi_n^{e_{k+1}}(A_k(x(I)))$.

For each $k = 1, \dots, M$ and each $I \in S_k$, consider the set

$$N_{\frac{2(R_1-R_0)}{n^k}}\left(A_k(x(I)) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I))\right).$$

We claim that the collection of all such sets covers E_n. Indeed, let $z \in E_n$ be arbitrary. Choose $I = (i_1, \dots, i_{M-1}, 0) \in S_M$ such that $z \in A_M(x(I))$. If C is sufficiently large, then

$$\mathbb{R}^d = \Gamma^{e_1}_{1/\sqrt{n}}(0) \cup \Gamma^{e_2}_{1/\sqrt{n}}(x(i_1, 0, \dots, 0)) \cup \cdots \cup \Gamma^{e_M}_{1/\sqrt{n}}(x(i_1, \dots, i_{M-1}, 0)).$$

This follows from the fact that consecutive centers are very close, say

$$|x(i_1, \dots, i_k, 0, \dots, 0) - x(i_1, \dots, i_{k+1}, 0, \dots, 0)| \leq (R_1 - R_0)/n^k,$$

and that the cones $\Gamma^{e_k}_{1/\sqrt{n}}$ are sufficiently wide to cover the gaps (this is ensured by the maximal separation of D). Hence there is k such that

$$z \in \Gamma^{e_k}_{2/\sqrt{n}}(x(i_1, \dots, i_{k-1}, 0, \dots, 0)).$$

Along with previous discussion we have

$$z \in \Phi_n^{e_{k+1}} \circ \cdots \circ \Phi_n^{e_M}\left(A_k(x(I))\right) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I)).$$

Now applying 3.3 with $R_1 - R_0 = (R_1 - R_0)/n^{k-1}$ we have

$$z \in N_{\frac{2(R_1-R_0)}{n^k}}\left(A_k(x(I)) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I))\right).$$

We finally turn to the estimate of $m(E_n)$. For each fixed k, the cover sets are indexed by $I \in S_k$, and there are $|S_k| = n^{k-1}$ such indices. Applying Lemma 3.2 with $R_1 - R_0$ replaced by $(R_1 - R_0)/n^{k-1}$ (which is the width of the annuli at stage $k-1$), we obtain

$$m\left(N_{\frac{2(R_1-R_0)}{n^k}}\left(A_k(x(I)) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I))\right)\right) \lesssim \frac{R_1 - R_0}{n^{k-1}} \cdot n^{-\frac{d+1}{2}}.$$

Summing over $I \in S_k$ gives

$$\sum_{I \in S_k} m\left(N_{\frac{2(R_1-R_0)}{n^k}}\left(A_k(x(I)) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I))\right)\right) \lesssim n^{k-1} \cdot \frac{R_1 - R_0}{n^{k-1}} \cdot n^{-\frac{d+1}{2}} = (R_1 - R_0)\, n^{-\frac{d+1}{2}}.$$

By previous discussion we have

$$m(E_n) \leq \sum_{k=1}^{M} \sum_{I \in S_k} m\left(N_{\frac{2(R_1-R_0)}{n^k}} \left(A_k(x(I)) \cap \Gamma^{e_k}_{2/\sqrt{n}}(x(I)) \right) \right).$$

Note that there are $M \approx n^{(d-1)/2}$ terms in the outer sum, we obtain

$$m(E_n) \lesssim M \cdot (R_1 - R_0)\, n^{-\frac{d+1}{2}} \approx n^{\frac{d-1}{2}} \cdot (R_1 - R_0) \cdot n^{-\frac{d+1}{2}} = (R_1 - R_0)\, n^{-1}.$$

$\square$

4 Appendix

4.1 Integration in Polar Coordinates

In this part, we will introduce integration in polar coordinates, so that we can explicitly describe the surface of a sphere. Most of them can be found in [3].

We still denote by S^{d-1} the unit sphere in $\mathbb{R}^{d-1}$. If $x \in \mathbb{R}^d \setminus \{0\}$, the polar coordinates of x are

$$r = |x| \in (0, \infty), \quad x' = \frac{x}{|x|} \in S^{d-1}.$$

The map $\Phi(x) = (r, x')$ is a continuous bijection from $\mathbb{R}^d \setminus \{0\}$ to $(0,\infty) \times S^{d-1}$, whose inverse is $\Phi^{-1}(r, x') = rx'$. Let $m_* = \Phi \# m$ be the pushforward of m by Φ. That is,

$$m_*(E) = m\left(\Phi^{-1}(E)\right)$$

for any Borel set in $(0,\infty) \times S^{d-1}$. We also define the measure $\rho = \rho_d$ on $(0,\infty)$ by $\rho(E) = \int_E r^{d-1}\, dr$ where E is any Borel set of $(0,\infty)$.

Theorem 4.1. *There is a unique Borel measure $\sigma = \sigma_{d-1}$ on S^{d-1} such that $m_* = \rho \times \sigma$. If f is Borel measurable on $\mathbb{R}^d$ and either $f \geq 0$ or $f \in L^1(m)$, then*

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \int_{S^{d-1}} f(rx') r^{d-1}\, d\sigma dr.$$

A direct result of Theorem 4.1 is as follows:

Corollary 4.2. *If f is a measurable function on $\mathbb{R}^d$, either nonnegative or integrable such that $f(x) = g(|x|)$ for some function g defined on $(0,\infty)$, then*

$$\int f(x)\, dx = \sigma(S^{d-1}) \int_0^\infty g(r) r^{d-1}\, dr.$$

We next calculate $m(S^{d-1})$ and $\sigma(S^{d-1})$.

The Lebesgue Measure. Consider the characteristic function $\chi_{S^{d-1}}(x)$ and the function $\delta_1(x)$ that takes value 1 at $x = 1$ and vanishes elsewhere. Then $\chi_{S^{d-1}}(x) = \delta_1(|x|)$. Applying the above corollary we have

$$m\left(S^{d-1}\right) = \int \chi_{S^{d-1}}(x)\, dx = \sigma\left(S^{d-1}\right) \int_0^\infty \delta_1(r) r^{d-1}\, dr.$$

But $\delta_1(x) = 0$ almost everywhere, hence

$$\boxed{m(S^{d-1}) = 0}. \tag{1}$$

By translation and dilation, we obtain that $m(S) = 0$ for any sphere in $\mathbb{R}^d$. This result supports the proof of Proposition 2.4.

As for $\sigma(S^{d-1})$, we need the following integral.

Proposition 4.3. *If $a > 0$, then*

$$I_d = \int_{\mathbb{R}^d} e^{-a|x|^2}\, dx = \left(\frac{\pi}{a}\right)^{d/2}.$$

Proof. For $d = 2$ applying Corollary 4.2 with $f(x) = e^{-a|x|^2}$ and $g(r) = e^{-ar^2}$ we have

$$I_2 = 2\pi \int_0^\infty re^{-ar^2}\, dr = \frac{\pi}{a}.$$

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, there is $e^{-a|x|^2} = \prod_1^d e^{-ax_j^2}$, so Tonelli's theorem shows that $I_d = (I_1)^d$. In particular, $I_1 = (I_2)^{1/2} = (\pi/a)^{1/2}$, hence $I_d = (\pi/a)^{d/2}$. □

The Surface Measure. Applying Proposition 4.3 with $a = 1$, and following the same argument as in the proof, one has

$$\pi^{d/2} = \int_{\mathbb{R}^d} e^{-|x|^2}\, ds = \sigma(S^{d-1}) \int_0^\infty e^{-r^2} r^{d-1}\, dr.$$

Consider the substitution $s = r^2$ we have

$$\pi^{d/2} = \frac{\sigma(S^{d-1})}{2} \int_0^\infty s^{(d/2)-1} e^{-s}\, ds = \frac{\sigma(S^{d-1})}{2} \Gamma\left(\frac{d}{2}\right).$$

Therefore we conclude that

$$\boxed{\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}}.$$

4.2 Maximal Separated Subset

For $\varepsilon > 0$, a subset $D \subset S^{d-1}$ is called ε-separated, if $|u - v| \geq \varepsilon$ for any $u, v \in D$. A maximal ε-separated subset is a ε-separated subset that cannot be enlarged by adding another point, so it satisfies

1. $|u - v| \geq \varepsilon$ for all $u, v \in D$;
2. for every other $\omega \in S^{d-1}$, there is $u \in D$ such that $|\omega - u| < \varepsilon$.

In the following construction, we require that $\varepsilon = \frac{1}{C\sqrt{n}}$ where C is some constant independent of n. Let $D = \{e_1, e_2, \ldots, e_M\} \subset S^{d-1}$ be a maximal ε-separated set.

Lemma 4.4. *If n is sufficiently large, then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 n^{\frac{d-1}{2}} \leq M \leq c_2 n^{\frac{d-1}{2}},$$

where c_1, c_2 are independent of n. In other words, $M \approx n^{(d-1)/2}$.

Proof. Since n is large enough, we have $\varepsilon \ll 1$. In this case, given any pair of points $u, v \in S^{d-1}$ with Euclidean distance $|u - v| < \varepsilon$, the angle θ between the two vectors is approximately equal to $|u - v|$. This is because

$$\begin{aligned} |u - v|^2 &= \langle u - v, u - v \rangle = |u|^2 + |v|^2 - 2\langle u, v \rangle \\ &= 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}, \end{aligned}$$

and so $|u - v| = 2\sin\frac{\theta}{2} \approx 2 \cdot \frac{\theta}{2} = \theta$ when θ is small.

Thus given a point $u_0 \in S^{d-1}$ and a radius $r \lesssim \varepsilon$, the intersection of the open ball $B(u_0, r)$ with S^{d-1} is contained in the cone $\Gamma_\theta^e(x_0) = \{x_0 + th : t \geq 0,\ h \in S^{d-1},\ \mathrm{dist}_{S^{d-1}}(h, e) \lesssim \varepsilon\}$ where $\mathrm{dist}_{S^{d-1}}$ denotes the angle between two vectors.

Upper bound. Since D is ε-separated, the balls (on the sphere) of radius $\varepsilon/2$ centered at points of D are pairwise disjoint. Thus the intersection of each of these balls with S^{d-1} has surface area $\gtrsim \varepsilon^{d-1}$ (for small ε). The total surface area of S^{d-1} is a constant $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. Therefore,

$$M \cdot c_d \varepsilon^{d-1} \leq \omega_{d-1},$$

so $M \lesssim \varepsilon^{-(d-1)} = (C\sqrt{n})^{d-1} = C^{d-1} n^{(d-1)/2}$. Since C is fixed, we have $M \lesssim n^{(d-1)/2}$.

Lower bound. By maximality, D is also an ε-cover: every point of S^{d-1} is within distance ε of some $u \in D$. Hence the balls of radius ε centered at D cover S^{d-1}. Each such ball has area $\lesssim \varepsilon^{d-1}$. Consequently,

$$M \cdot C_d \varepsilon^{d-1} \geq \omega_{d-1},$$

so $M \gtrsim \varepsilon^{-(d-1)} = C^{d-1} n^{(d-1)/2}$. Thus $M \geq c_1 n^{(d-1)/2}$ for some $c_1 > 0$.

Combining both bounds yields the desired estimate. $\square$

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