硕士学位论文

对球面非稳定 v_n 周期同伦群的谱序列计算

SPECTRAL SEQUENCE CALCULATION FOR UNSTABLE v_n-PERIODIC HOMOTOPY GROUPS OF SPHERES

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摘 要

计算拓扑空间的同伦群是代数拓扑学中一个重要且具挑战性的问题。本文回顾了对这一问题的研究历史，展示了目前对这一问题的研究进展，并对其未来的可能进展进行了展望。若非特别说明，本文中所有的讨论均在 p-局部的意义下进行，其中 p 为给定素数。

从稳定同伦情形出发，本文先介绍了该问题在这一情形下的相关研究，并给出了稳定同伦群中 v_n-周期同伦群的严格定义。对于非稳定情形，本文给出了非稳定 v_n 周期同伦群的定义。并利用 Bousfield-Kuhn 函子将对这类同伦群的计算转化为对稳定同伦群的计算。简要而言，对给定的自然数 n，Bousfield-Kuhn 函子是一个从带基点的拓扑空间范畴到谱范畴的函子，使得其像的稳定同伦群同构于原像的 v_n 周期同伦群。该函子也和对于 Morava K-理论 $K(h)$ 的局部化相容。出于可计算性的考虑，一般在实际计算中，我们只会考虑进行 $K(h)$-局部化之后的 Bousfield-Kuhn 函子的像的稳定同伦群。

本文进而给出了目前对这一问题的详细计算过程。简要而言，我们需要先对上述函子的 Goodwillie 塔进行消解。Goodwillie 塔可以视作对函子的 Taylor 展开，其能将函子展开到不同的 n-切除层。在特定的情况下，可以求出消解后得到的每一层纤维的完备 Morava E-同调群。对这些同调群的计算依赖于对每一层纤维进行基于拓扑 André-Quillen 理论的进一步消解。该理论可用于构造 Goodwillie 塔的纤维的几何实现，且其构造过程允许我们用 Koszul 复形刻画每一层的信息。将上面的同调群作为同伦不动点谱序列的输入，可以求出该谱序列的 E_2-页。这一谱序列收敛于我们所需要的谱的同伦群。目前已知的最好结果在本文中被列出。

最后，本文给出了目前对该问题的两个可能的进一步研究的方向。其一是利用代数几何中的一个对偶，设法减少谱序列中出现的非平凡元素个数，以避免难以计算的微分的出现。另一个则是利用椭圆曲线和拓扑模形式等工具，改进对其中出现的完备 Morava E-同调的计算方法。这一改进也可减少谱序列中出现的非平凡元素个数。

关键词：非稳定同伦群；色展同伦论；谱序列

ABSTRACT

Calculating the homotopy groups of topological spaces is a fundamental and challenging problem in algebraic topology. This thesis reviews the research history of this problem, presents the current progress, and outlines possible future developments. Unless otherwise specified, all discussions in this thesis are conducted in the p-local case for a prime p.

Starting from the stable case, we review the history of this problem and provide a rigorous definition of the v_n-periodic homotopy groups in the stable homotopy and the unstable homotopy. Then, we propose a method to compute unstable homotopy groups by applying the Bousfield-Kuhn functor. It is a functor from Top_* to Sp, such that the stable homotopy groups of a spectrum in the target is isomorphic to the v_n-periodic homotopy groups of the space in the source. Here Top_* is the category of pointed topological spaces and Sp is the category of spectra.

This thesis also provides a detailed account of the current computational process for this problem. Firstly, we need to resolve the functor by the Goodwillie tower, which expands the functor into n-excisive layers. Under specific conditions, we can compute these layers' completed Morava E-homology groups. The computation of these homology groups relies on further resolution based on the topological André-Quillen theory, which can be seen as a geometric realization of the Goodwillie tower. Inputting these homology groups to the homotopy-fixed-point spectral sequence, we can compute the E_2-page of this spectral sequence, which converges to the homotopy groups of the spectrum we need. We list the best results we know on this problem in this thesis.

Finally, this thesis outlines two possible directions for further research on this problem. One is to utilize a duality in algebraic geometry to reduce the number of nontrivial entries appearing in the spectral sequence, thus avoiding the appearance of difficult differentials. The other direction is to improve the computational method for the completed Morava E-homology using tools such as elliptic curves and topological modular forms. This improvement can also reduce the number of nontrivial entries appearing in the spectral sequence.

Keywords: Unstable homotopy group; Chromatic homotopy theory; Spectral sequence

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CHAPTER 1 INTRODUCTION

Studying periodicity in homotopy groups is a vital problem within homotopy theory. Compared to the non-periodic parts, we have more tools to detect the periodic components in homotopy groups. Computing the periodic part of homotopy groups can enhance our understanding of computationally challenging objects, such as the unstable homotopy groups of spheres. This thesis summarizes and outlines the research methods and progress on the computation of the unstable v_2-periodic homotopy groups of spheres and suggests potential improvements to existing computational methods.

In this thesis, unless otherwise stated, the spectra or spaces we discuss are p-local. Additionally, I have to use some undefined symbols in this introduction chapter due to space constraints. These symbols will be explained in subsequent chapters.

We begin with the rational homotopy case. For a simply connected space X. Quillen introduced two methods to describe its rational homotopy type in[1]. One of them is the rational differential graded Lie algebra $\mathcal{L}(X)$ and the other one is the rational cocommutative differential graded coalgebra $\mathcal{C}(X)$. The rational homotopy groups of X are given by the homology of $\mathcal{L}(X)$ and the rational homology is given by the homology of $\mathcal{C}(X)$. The differential graded Lie algebra $\mathcal{L}(X)$ can be obtained by taking the derived primitives of the differential graded coalgebra $\mathcal{C}(X)$. In[2], Sullivan reformulated this theory for finite type X. In this situation, the minimal model $\Lambda(X)$ of the differential graded algebra is given by the dual of $\mathcal{C}(X)$. The underlying graded commutative algebra of the minimal model $\Lambda(X)$ is free, and the rational homotopy groups of X are recovered from the dual of the indecomposable of $\Lambda(X)$:

$$\pi_*(X)_{\mathbb{Q}} \cong (Q\Lambda(X))^{\vee}.$$

Thus the rational homotopy groups of a space can be computed by taking the dual of the derived indecomposables of a commutative algebra model of its rational cochains.

A similar result stands for every prime p, that is, the unstable p-adic homotopy type of a simply connected finite type space is similarly encoded in its $\bar{\mathbb{F}}_p$-valued singular cochains. However, to get an equation which is similar to the above one, we need to find some other localizations of unstable homotopy groups. The v_n-periodic homotopy groups are good candidates. The rational homotopy groups can be seen as v_0-periodic homotopy groups.

We need to define it explicitly before we study unstable v_n-periodic homotopy groups. That is our main target in Chapter 2. We will begin with the stable case.

In the perspective of chromatic homotopy theory, the v_n-periodic homotopy groups are induced by type n self-maps $f : \Sigma^t V \to V$ on the homotopy groups of spectra that survive under the action of homotopy colimits. It is denoted as $v_n^{-1}\pi_*(X)$ because we can prove that the v_n-periodic homotopy groups of X are independent of the specific choice of the type n self-map. To simplify computations, we also apply $K(n)$-localization on the spectrum X, where $K(n)$ is the Morava K-theory.

The v_n-periodic homotopy groups can be defined similarly in the unstable case. By applying the Bousfield-Kuhn functor Φ_n on the space X, we can transform the computation back to the stable case. A functor from Top_* to Sp (actually $Sp_{T(n)}$), which is called the Bousfield-Kuhn functor, allows us to calculate $v_n^{-1}\pi_*(X)$ by $\pi_*(\Phi_n X)$. Similar to the stable case, this operator also has a $K(n)$-local version $\Phi_{K(n)}$.

Roughly speaking, the Bousfield-Kuhn functor (with coefficient V) encodes the $Map_*(\Sigma^{nt}V, X)$ for each $n \geq 0$ into a spectrum. Then, the "colimit" of such a functor can give us the Bousfield-Kuhn functor. This process is similar to getting $T(n)$ in the stable case. This process also makes the object we need to deal with into a spectrum, which we have enough tools to compute its homotopy groups.

Next, we will show how to compute the $\pi_*(\Phi_{K(n)}(X))$ in Chapter 3. To further calculate Φ_n, we apply the Goodwillie tower on it to resolve the unstable spheres into some well-known spectra. When X is an odd-dimensional sphere and $n = 2$,[3] shows that this resolution divides $\Phi_{K(n)}X$ into spectra. These spectra consist of the Steinberg summands of classifying spaces of the additive groups of vector spaces over $\mathbb{F}_p$. The completed Morava E-homology of such spectra can be computed. To accomplish this computation, we need to use the topological André-Quillen (co)homology as a model for the Goodwillie tower and apply Koszul duality to compute the required completed Morava E-homology groups.

To be specific, the Goodwillie tower provides filtration of the target space. The topological André-Quillen (co)homology also decides a tower. The comparison map can connect these towers. That is,

$$c^{S_K} : \Phi(X) \to TAQ_{S_K}(S_K^{X_+})$$

. Here Φ is an abbreviation of $\Phi_{K(n)}$. This map constructs a level-wise equivalence of the tower's filtration. To compute each layer of these towers, we need to introduce the Morava E-theory Dyer-Lashof algebra as a model. The Koszul complex helps compute this model. In a nutshell, the Morava E-homology of the resolution of Goodwillie tower is isomorphic to the dual of the Koszul resolution of the Morava E-theory Dyer-Lashof algebra.

The above process can be synthesis into a spectral sequence, that is,

$$Ext^s_{\Delta^q}(\tilde{E}^q(S^q), \bar{E}_t) \Rightarrow E_{q+t-s}\Phi(S^q).$$

Rezk proposed this method in[4], and the specific computations were completed in[5]. In[3], more technical methods were used to complete similar computations but with weaker results.

Using the Morava E-homology described above as an input for the $E(n)$-version Adams-Novikov spectral sequence①, we can obtain an E_2-page of a spectral sequence which convergent to $\pi_*(\Phi(X))$. This spectral sequence is:

$$H^s_c(\mathbb{G}; E_t\Phi(S^q))^{Gal} \Rightarrow \pi_{t-s}\Phi(S^q).$$

Here the $\mathbb{G}$ means the Morava stabilizer group of prime p and height n.

Currently, the best results for the v_2-periodic homotopy groups of unstable spheres are as follows[3]:

Theorem 1.1 (Wang, Theorem 5.4.12): For a prime number $p \geq 5$, $H^*(\mathbb{G}, E_2\Phi_{K(2)}S^3)$ is a vector space over $\mathbb{F}_p$, with a set of basis of 12 different classes with no possible differentials in the ANSS. Therefore, the homotopy groups $\pi_*(\Phi_{K(2)}S^3)$ is a free module over $\mathbb{F}_p[\xi]/\xi^2$ with the above generators.

We will briefly introduce the latest progress on this problem in the last chapter. Further research on this problem currently needs some help. Since $\Phi_{K(n)}X$ is not a ring spectrum in general, dealing with possible non-trivial differentials in the final spectral sequence is challenging. There are currently two approaches to overcoming this difficulty.

The first method is to "switch" the order of spectral sequences as follows. To explain this, we need to summarize our method of computation with the following diagram:

$$Goodwillie \quad tower \xrightarrow{(-)^{GLn(\mathbb{F}p)}} H^*_c(G_n, E_{n*}(\Phi_{K(n)}(S^q))) \xrightarrow{(-)^{D^\times}} \pi_*(\Phi_{K(n)}(S^q)).$$

① denote as ANSS later

The first spectral sequence means we are obtaining the information of completed Morava E-homology from the space we need with the help of the Goodwillie tower and the resolution that the tower provides. The second spectral sequence means we can recover the information of homotopy groups from the completed Morava E-homology for each filtration. The result of such switching is shown as follows. The ? in the middle of the equation is an object that we need to investigate further.

$$Goodwillie \quad tower \xrightarrow{(-)^{D^{\times}}} \quad ? \quad \xrightarrow{(-)^{GLn(\mathbb{F}p)}} \pi_*(\Phi_{K(n)}(S^q)).$$

Its motivation is a duality in the field of algebraic geometry that we will introduce in the last chapter of this thesis.

The second improvement is changing the completed Morava E-theory into the Morava E-theory. This change can reduce the number of elements in the E_2 -page of the ANSS. The progress of this method will also be introduced in the last part of this thesis.

This thesis is organized as follows: The first chapter provides an overview of the content. The second chapter sets up the problem which we will study strictly. After setting up, the third chapter outlines the current computational methods we used in previous research; finally, the fourth chapter briefly discusses possible improvements to the computational methods for this problem.

CHAPTER 2 SETTING UP OF THE PROBLEM

In this chapter, we provide the explicit definition of the v_n-periodic homotopy groups at first. Then we introduce the Bousfield-Kuhn functor and its application in the calculation of the unstable v_n-periodic homotopy group. We also discuss the $K(n)$-localization of this problem for computational purposes.

2.1 The history of v-periodic homotopy group

In the E_2-page of the Adams spectral sequence of the low stems, some structures appear periodically at the top of the table.① This naturally raises a question. Does a periodic structure appear in the homotopy groups of spheres?

In[7], Mahowald and Davis use the self-map to construct v-periodic elements in stable homotopy groups:

Definition 2.1 (Mahowald & Davis): Let X be a finite complex. A periodic operator is $v \in [\Sigma^i X, X]$ such that $v^k \neq 0 \in [\Sigma^{ik} X, X]$ for all $k > 0$. A class $\alpha \in [X, Z]$ is v-periodic if $\alpha \circ v^k \neq 0$ for all k. A class $\beta \in [S^j, W]$ is v-periodic if, for some skeleton $X^{(t-1)}$ of X, β can be decomposed as:

$$S^t \xrightarrow{i} X/X^{(t-1)} \xrightarrow{\bar{\beta}} \Sigma^{t-j} W.$$

and for all such β and all $k \geq 0$,

$$\Sigma^{ik} X \xrightarrow{v^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\bar{\beta}} \Sigma^{t-j} W.$$

is essential. Here Z, W are spectra, and the corresponding group are in the meaning of stable maps.

With the above definition, we can identify these periodic elements in homotopy groups. In particular, a v-periodic element of $\pi_*(S^0)$ gives rise to an infinite family of nonzero elements of $\pi_*(S^0)$ by choosing for each k the first cell on which $\overline{\beta p v^k}$ is essential.

Naturally, we are wondering whether the choice of X and v influence the v-periodic elements in a $\pi_*(Z)$ or not, as well as if X has to satisfy some restrictions to support a

① The study of the periodic phenomenon in the classical Adams spectral sequence can be found in[6], this periodic property also induces a periodic property in the homotopy group of the sphere spectrum.

self-map. To answer these questions, we need to introduce the chromatic perspective. Roughly speaking, it gives us a filtration of self-maps.

The study of chromatic homotopy theory gives us more information about the self-map. In[8], Devinatz-Hopkins-Smith proved the nilpotence theorem which describes a restriction for X to support a permanent self-map f:

Theorem 2.1 (Devinatz & Hopkins & Smith): Denote MU_* as the complex bordism theory. It also decides a homology theory. A self-map f is called stably nilpotent iff iterations of $\overline{MU}_*(f)$ are trivial. The remaining maps are called periodic.

However, the MU is too big. As a result, we always deal with some "localization" versions of it, such as BP, $E(n)$ and $K(n)$, to simplify the problem. In this thesis, the Morava K-theory will be used frequently. We will give an introduction of it in detail as follows.

Morava K-theory was first developed in the research of complex oriented bordism theory MU and the formal group laws related to it. If the readers are interested in the history of it, they can refer to[9].

By considering the classifying map $m : \mathbb{C}P_\infty \wedge \mathbb{C}P_\infty \to \mathbb{C}P_\infty$, a formal group law F_{MU} is decided. Its p-local part decided a spectrum BP, which also decided a formal group law F_{BP}.

For a formal group law F, we define the addition as $f +_F g = F(f(x), g(x))$. For given $n > 0$,

$$[n]_F(x) := \underbrace{x +_F \cdots +_F x}_{n}$$

With the above definition, we can give the explicit structure of F_{BP}

Theorem 2.2 (Hazewinkel): For a prime p, an isomorphism of $\mathbb{Z}_{(p)}$-algebras

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$$

exists. We can choose the generators $v_i \in BP_{2(p^i-1)}$ to be the coefficients of x^{p^i} in the following series

$$[p]_{F_{BP}}(x) = \sum_{i>0} v_i x^{p^i}$$

The height of a formal group law ① is determined by the power of the leading term in the series expansion of $[p]_F(x)$ modulo p. the height of F is defined to be ∞ if $[p]_F(x) = 0$. Two relations $\theta_n(v_n) = 1$ and $\theta_n(v_i) = 0$ otherwise decide a ring homomorphism

① over a commutative $\mathbb{F}_p$-algebra A

$\theta_n : BP^* \to A$, we can define $F_n(x, y) = (\theta_n) * F_{BP}$. According to the above theorem, it follows that F_n has height n. By the Landweber exact functor theorem[10], F_n decides a complex-oriented cohomology theory. This cohomology theory is $K(n)$.

For geometry construction, we can get Morava K-theory by killing generators in BP. Some other relevant spectra are also defined here:

$$BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots v_n]$$

$$P(n) \cong \mathbb{F}_p[v_n, v_{n+1}, \cdots]$$

$$k(n) \cong \mathbb{F}_p[v_n]$$

The spectrum $k(n)$ serves as the (-1)-connected version of the spectrum $K(n)$ in Morava K-theory. By utilizing $k(n)$, we can define $K(n)$ as follows:

$$K(n) = holim[\Sigma^{-2i(p^n-1)}k(n) \to k(n)]$$

The homotopy limit of $BP\langle n\rangle$ can be used to define the Morava E-theory $E(n)$ in a similar way.

In conclusion, we can summarize the above properties into a theorem:

Theorem 2.3: For any prime p, and for all integers $n \geq 1$, there exists a multiplicative, $2(p^n - 1)$-periodic, and complex-oriented cohomology theory denoted by $K(n)^*(-)$, with the coefficient ring given by

$$K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$$

where v_n has degree $|v_n| = 2(p^n - 1)$, and its associated formal group law $F_n(x, y)$ satisfies the relation

$$[p]_{F_n}(x) = v_n x^{p^n}.$$

Furthermore, if p is odd, the product on $K(n)^*(-)$ is commutative; for $p = 2$, it is non-commutative.

Morava K-theories are significantly intertwined with BP-theory and complex cobordism through various intermediate spectra. The computations regarding $K(n)$ also yield valuable insights into the stable homotopy groups of spheres, as demonstrated in[11] and[12].

Theorem 2.4 (Miller): Suppose N is a BP_*BP-comodule in which every element is I_n-torsion and v_n acts bijectively on N. A natural isomorphism exists, which is

$$Ext^*_{BP_*BP}(BP_*, N) \cong Ext^*_{\Sigma(n)_*}(E(n)_*, E(n)_* \otimes_{BP_*} N).$$

Theorem 2.5 (Miller): The natural projection $BP_* \to K(n)_*$ induces an isomorphism

$$Ext^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^*_{K(n)_*K(n)}(K(n)_*, K(n)_*).$$

What's more, $K(n)$ has the unique property. So we can define it in an axiom way. This theorem ① in[13] shows the axiom definition of the Morava K-theory.

Theorem 2.6 (Ravenel): A series of homology theories $K(n)_*$ exist for $n \geq 0$ and any prime p, which have the following properties.②

- $K(0)_*(X)$ coincides with $H_*(X;\mathbb{Q})$ which is torsion-free.
- $K(1)_*(X)$ is one of the $p-1$ isomorphic summands of mod p complex K-theory.
- $K(0)_*(pt.) = \mathbb{Q}$ and $K(n)_*(pt.) = \mathbb{Z}/(p)[v_n, V_n^{-1}]$ for $n \geq 1$. The dimension of v_n is $2p^n - 2$. This ring is a graded field. What's more, $K(n)_*(X)$ is a module over $K(n)_*(pt.)$.
- There exists a Kunneth isomorphism

$$K(n)_*(X \coprod Y) \cong K(n)_*(X) \otimes_{K(n)_*(pt.)} K(n)_*(Y).$$

- X is a p-local finite CW-complex. $\overline{K(n)_*}(X)$ vanishes implies that $\overline{K(n-1)_*}(X)$ vanishes
- If X is a p-local finite CW-complex as above, we have

$$\overline{K(n)_*}(X) = K(n)_*(pt.) \otimes \overline{H_*}(X;\mathbb{Z}/(p))$$

for n large enough. For simply connected and not contractible X, it is non-trivial.

Next, we need to give an explicit definition of the self-map we need to study. That is the v_n-self map, which is the self map of space (or spectrum) we are concerned with in this thesis. Roughly speaking, they are non-nilpotent, graded by the chromatic level's self-maps which can be detected by the Morava K-theory. In particular, the Morava K-theory can detect a specific part of those permanent self-maps by the periodic theorem proved by Hopkins-Smith in[14]:

Definition 2.2 (Hopkins & Smith): A p-local finite complex X is defined to be type n iff $\overline{K(n)_*}(X)$ is nontrivial and it is the smallest n. In particular, X has type ∞ if it is contractible.

Theorem 2.7 (Hopkins & Smith): For p-local type n finite CW-complexes X and Y, we have

- There exists a self-map $f : \Sigma^{d+p}X \to \Sigma^d X$ for some $p \geq 0$ satisfies the condition

① It can also be considered as a definition of $K(n)$

② We follow the standard practice of omitting p from the notation.

that $K(m)_*(f)$ is trivial for $m > n$ and $K(n)_*(f)$ is an isomorphism. (Such a map is called a v_n map, which we will discuss later.) When $n = 0$, we have $d = 0$, and for $n > 0$, d is a multiple of $2p^n - 2$.

• For a continuous mapping $h : X \to Y$, with both X and Y have been suspended sufficiently times to serve as targets for v_n-maps. Assume $g : \Sigma^e Y \to Y$ represents a self-map as previously described. Consequently, there exist positive integers p and q, satisfying $dp = eq$, ensuring the commutativity of the subsequent diagram up to homotopy. This is the uniqueness property of v_n maps.

$$\begin{array}{ccc} \Sigma^{dp}X & \xrightarrow{\Sigma^{dp}h} & \Sigma^{dp}Y \\ \downarrow{\scriptstyle f^p} & & \downarrow{\scriptstyle g^q} \\ X & \xrightarrow{h} & Y \end{array}$$

•

Now, we can give an explicit definition of the periodic part in the stable homotopy group.

Definition 2.3 (Rezk): The v_n-homotopy group (with coefficient V) for a spectrum X and a type n spectrum V as well as the self-map v:

$$v_n^{-1}\pi_*(X;V) := v^{-1}[\Sigma^* V, X]_{Sp}.$$

We can prove that the choice of v doesn't influence the v_n-homotopy group of X.[13]

2.2 Bousfield localization and stable v_n-periodic homotopy group

If we use the above definition directly, we will find that determining whether a map $f : X \to Y$ induces a v_n-periodic homotopy isomorphism is hard. So we hope to find a simple method to judge this. Recall that in p-adic homotopy theory and rational homotopy theory, the isomorphism between homology groups can induce isomorphism between homotopy groups. Since $K(0)_*$ is the rational homology theory, we have a natural conjecture: Does the $K(n)_*$-isomorphism decide the $v_n^{-1}\pi_*$-isomorphism? If not, is there any other spectrum that can decide it?

To answer this question, we can transform isomorphisms between homotopy groups into isomorphisms between homology groups. Then, with the help of the Bousfield equivalence, we can deal with the problem by chromatic homotopy theory in which we have enough mature tools. The Bousfield localization and the Bousfield equivalence are introduced as follows.

Aiming at calculating $K(n)_*(X)$ as well as other generalized (co)homology theories, Bousfield localization was developed in[15] and[16]. This tool enables us to simplify the X into $L_E X$, an E-local spectrum, without changing $E_*(X)$. Bousfield localization can be done functorially.

Bousfield localization is a special situation of localization over a spectrum E:

Definition 2.4 (Bousfield): Let E_* be a generalized homology theory. A space (or a spectrum) Y is called E_*-local if for any map $f : X_1 \to X_2$ satisfies that $E_*(f)$ is an isomorphism, the map

$$[X_1, Y] \xleftarrow{f^*} [X_2, Y]$$

is also an isomorphism.

An E_*-localization of a space or spectrum X is a map η from X to an E_*-local space or spectrum Z which we usually denote as $L_E X$. It has a property that $E_*(\eta)$ is an isomorphism.

The property of E_*-local is stable under the inverse limit, fiber sequence and smash product. However, it is not stable under the homotopy inverse limit. In[17] and[18], Bousfield proved that for any homology theory E_* and any space or spectrum X, the localization $L_E X$ exists and it can be constructed functorially.

For a ring spectrum E, its localization is simple:

Theorem 2.8 (Bousfield): For ring spectrum E and any spectrum X, $E \wedge X$ is E_*-local.

Since $K(n)$ is a ring spectrum, we can easily give the definition of $L_{K(n)}$

Next, we can consider when two different spectra E and F induce the same localization functor. This question leads to the Bousfield equivalence:

Definition 2.5 (Bousfield): Two spectra E and F are Bousfield equivalent the following equivalence holds. That is, for each spectrum X, the smash product $E \wedge X$ is contractible iff $F \wedge X$ is contractible. As a result, these relations decide equivalence classes of spectra. The Bousfield equivalence class of E is denoted by $\langle E \rangle$.

We will list some definitions and properties of Bousfield equivalence which are shown in[13]:

Proposition 2.1 (Ravenel):

- If the contractibility of $E \wedge X$ implies the same property of $F \wedge X$ for each spectrum X, we denote it as $\langle E \rangle \geq \langle F \rangle$.
- $\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$.
- $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$

- If there is a spectra F satisfies that $\langle E\rangle \vee \langle F\rangle = \langle \mathbb{S}\rangle$ and $\langle E\rangle \wedge \langle F\rangle = \langle pt.\rangle$, we called such F a complement of $\langle E\rangle$, denoted as $\langle E\rangle^c$.
- The operations $\wedge$ and $\vee$ have distributive laws:

$$(\langle X\rangle \wedge \langle Y\rangle) \vee \langle Z\rangle = (\langle X\rangle \vee \langle Z\rangle) \wedge (\langle Y\rangle \vee \langle Z\rangle)(\langle X\rangle \vee \langle Y\rangle) \wedge \langle Z\rangle = (\langle X\rangle \wedge \langle Z\rangle) \vee (\langle Y\rangle \wedge \langle Z\rangle)$$

- The localization functors L_E and L_F coincide iff $\langle E\rangle = \langle F\rangle$
- For $\langle E\rangle \leq \langle F\rangle$, $L_E L_F = L_E$ stands. As a result, we can find a natural transformation $L_F \to L_E$.

Bousfield equivalence helps elucidate why we often analyze the p-component of homotopy groups separately. Let $S^0_{\mathbb{Q}}$ represent the rational sphere spectrum, $S^0_{(p)}$ denote the p-local sphere spectrum, and $S^0/(p)$ stand for the mod p Moore spectrum. Then we observe:

- $\langle S^0/(p)\rangle = \langle S^0\mathbb{Q}\rangle \vee \langle S^0/(p)\rangle$
- $\langle S^0\rangle = \langle S^0\mathbb{Q}\rangle \vee \bigvee_p \langle S^0/(p)\rangle$
- $\langle S^0/(p)\rangle \wedge \langle S^0\mathbb{Q}\rangle = \langle pt.\rangle$
- $\langle S^0/(p)\rangle \wedge \langle S^0/(q)\rangle = \langle pt.\rangle$ (The orthogonal property)

For MU, there is a similar result. Any readers interested in this can refer to Chap 7.3 of[13].

What's more, the class invariance theorem shows that the type of p-local spectrum decided its class in Bousfield equivalence, which implies that the choice of type n spectrum V and self-map v doesn't infect the result of localization.

Theorem 2.9 (Class invariance theorem): Consider X and Y as p-local finite CW-complexes with types m and n respectively. Then we have

$$\langle X\rangle < \langle Y\rangle, \quad m > n;$$

$$\langle X\rangle = \langle Y\rangle, \quad m = n.$$

.

By definition, for given (V, v), $v_n^{-1}\pi_*$-isomorphism is equivalent to $T(n)_*$-isomorphism if $T(n)$ is independent of the choices of V and v. The spectrum $T(n)$ is defined as:

Definition 2.6: For a type n p-local spectrum V with a v_n-self-map v,

$$T(n) = v_n^{-1}V := hocolim(V \xrightarrow{v} \Sigma^{-k}V \xrightarrow{v} \Sigma^{-2k}V \xrightarrow{v} \cdots).$$

This spectrum is independent of the choices of V and v in the meaning of homotopy equivalence due to the class invariance theorem.

The following part shows the relation between $K(n)$ and $T(n)$, which pulls this problem back to the field of the chromatic homotopy theory.

Bousfield localization of $E(n)$ can be used to give the chromatic filtration of spectrum X. We use L_nX to represent $L_{E(n)}X$ and C_nX to denote the fiber of the map $X \to L_nX$.

With the following theorem, we can calculate $BP_*(L_nX)$ in terms of $BP_*(X)$

Theorem 2.10 (Localization theorem): For any spectrum X, $BP \wedge L_nY = Y \wedge L_nBP$. In particular, if $v_{n-1}^{-1}BP_*(Y) = 0$, then $BP \wedge L_nY = Y \wedge v_n^{-1}L_nBP$.

Then we can define the chromatic tower and chromatic filtration of X:

Definition 2.7: The chromatic tower for a p-local spectrum X is the inverse system

$$L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow \cdots X.$$

The chromatic filtration of $\pi_*(X)$ is given by the subgroups

$$ker(\pi_*(X) \to \pi_*(L_nX))$$

The localization theorem ensures the convergence of the above inverse system which is proved in[13]. What's more, the monochromatic layers which are the fibers

$$M_nZ \to L_nZ \to L_{n-1}Z$$

satisfy that $M_nZ \in L_nSp := \oplus_{i=0}^{n} K(i)$-local spectra.

In[19], Mark Behrens and Charles Rezk defined an analogue of the chromatic tower with $T(n)$. We denote this tower as the $T(n)$-version chromatic tower.

Definition 2.8 (Rezk): The $T(n)$-version chromatic tower for a p-local spectrum X corresponds to such an homotopy limit

$$L_0^fX \leftarrow L_1^fX \leftarrow L_2^fX \leftarrow \cdots X.$$

Whose monochromatic layers can be defined as the fibers

$$M_n^fZ \to L_n^fZ \to L_{n-1}^fZ$$

satisfy that $M_n^fZ \in L_n^fSp := \oplus_{i=0}^{n} v_i^{-1}\pi_*$-local spectra.

Let M_n^fSp, M_nSp be the subcategories of L_n^fSp, L_nSp, there are pairs of functor

$$(-)_{T(n)} : Ho(M_n^fSp) \leftrightarrows Ho(sp_{T(n)}) : M_n^f,$$

$$(-)_{K(n)} : Ho(M_nSp) \leftrightarrows Ho(sp_{K(n)}) : M_n.$$

It leads to that there are some strong relation between $T(n)$ and $K(n)$. In[13], Ravenel showed that $\langle T(n)\rangle \geq \langle K(n)\rangle$. The inverse direction corresponds to the famous telescope conjecture raised by Ravenel. That is, are $\langle T(n)\rangle = \langle K(n)\rangle$ hold for every n? This conjecture is true for $n = 0, 1$, but for $n \geq 2$, this conjecture is supposed to be false. The case of $n = 0$ is trivial. In[20] and[21], the case of $n = 1$ was proved. For $n \geq 2$, we believe that this conjecture would fail at an early time. But the disproof was not completed until the 6th, June, 2023.① However, we have few tools to calculate $T(n)_*(X)$. As a result, we just use $T(n)_*$ for some abstract proof. If we aim at calculating the v_n-periodic homotopy group, we have to consider $L_{K(n)}X$② instead of $L_{T(n)}X$ although some classes of v_n-periodic homotopy group may be killed in the $K(n)$-localization.

2.3 Unstable v_n-periodic homotopy group and Bousfield-Kuhn functor

Now, we can move our steps to the unstable range. for any finite type n complex V, It supports a v_n-self map:

$$v : \Sigma^{k(N_0+1)}V \to \Sigma^{kN_0}V$$

for some $N_0 \gg 0$ due to the periodicity theorem.

Therefore, for any $X \in Top_*$, the unstable v_n-periodic homotopy group (with coefficient V) can be defined as[22]:

$$v_n^{-1}\pi_*(X;V) := v^{-1}[\Sigma^* V, X]_{Top_*}$$

for $n > 0$. The k-periodicity ensures that it can be defined on any $* \in \mathbb{Z}$.

Since we only have enough tools to compute stable homotopy groups, we need to pull the unstable v_n-periodic homotopy group back to the stable range. The tool we use is the Bousfield-Kuhn functor $\Phi_n : Top_* \to Sp_{T(n)}$. This functor allows us to calculate $v_n^{-1}\pi_*(X)$ by $\pi_*(\Phi_n(X))$.

We begin with the Bousfield-Kuhn functor with coefficient.③ $\Phi_V(X)$ can be constructed as a t-periodic spectrum. Such a spectrum is decided by its 0^{th} space which is defined as a direct limit:

$$Map_*(V, X) \to Map_*(\Sigma^t V, X) \to Map_*(\Sigma^{2t} V, X) \to \cdots$$

① https://www.uio.no/studier/emner/matnat/math/MAT9580/v23/beskjeder/disproof-of-the-telescope-conjecture.html

② This corresponds to the chromatic homotopy theory.

③ This part mainly refers to https://www.math.ias.edu/ lurie/ThursdayFall2017/Lecture6-BousfieldKuhn.pdf

In this definition, we need explicit V and v. To get rid of these, we define $\mathcal{C}_t$ as the ∞-category where objects consist of spaces V equipped with a v_n-self map $v : \Sigma^t V \to V$ at first.

For $t > 0$ an integer, the map $(V, v) \to \Phi_V$ we mentioned above determines a functor of ∞-categories

$$\Phi_{\bullet} : \mathcal{C}_t^{op} \to Fun(Space_*, Sp)$$

By sending (V, v) to (V, v^s), we get a functor $\mathcal{C}^t \to \mathcal{C}^{st}$ which fits into the following diagram.

$$\begin{array}{ccc} \mathcal{C}_s & \longrightarrow & \mathcal{C}_{st} \\ & \searrow^{\Phi_{\bullet}} \quad {}^{\Phi_{\bullet}}\swarrow & \\ & Fun(S_*, Sp) & \end{array}$$

By sending (V, v) to $(\Sigma V, \Sigma v)$, we get a functor $\mathcal{C}^t \to \mathcal{C}^t$ which fits into the following diagram.

$$\begin{array}{ccc} \mathcal{C}_s & \longrightarrow & \mathcal{C}_{s+1} \\ & \searrow^{\Phi_{\bullet}} \quad {}^{\Sigma\Phi_{\bullet}}\swarrow & \\ & Fun(S_*, Sp) & \end{array}$$

With these observations, we see that we can construct a functor $\mathcal{C}' \to Fun(Space_*, Sp)$ to merge the information of the functors $\Phi_{\bullet}$. The $\mathcal{C}'$ can be derived from the ∞-categories $\mathcal{C}_t$. This process is finished by taking a colimit along the transition functions we described earlier. Specifically, we define $\mathcal{C}'$ as the direct limit of the sequence:

$$\mathcal{C}_{1!} \to \mathcal{C}_{2!} \to \mathcal{C}_{3!} \to \cdots$$

where the maps from $\mathcal{C}_{(m-1)!}$ to $\mathcal{C}_{m!}$ for each m are induced by $(V, v) \to (\Sigma V, \Sigma(v^m))$. We will abuse notation by denoting this functor as $\Phi_{\bullet} : \mathcal{C}' \to Fun(Space_{,} Sp)$ as well. We can prove that the ∞-category $\mathcal{C}'$ is equivalent to a full subcategory of Sp. This category is generated by those type $\geq n$ finite spectrum, denoted as $Sp_{\geq n}^{fin}$. Therefore, the functor $\Phi_{\bullet}$ can be regarded as $\Phi_{\bullet} : Sp_{\geq n}^{fin} \to Fun(Space_*, Sp)$. An informal but inspired description is given as follows. When E represents a finite type n spectra, we can find an integer k satisfies that $\Sigma^k E$ becomes homotopy equivalent to $\Sigma^{\infty} V$, where V denotes a finite type n space. This space is naturally equipped with a v_n-self map. In this context, Φ_E can be expressed as $\Sigma^k \circ \Phi_V$.

By consider $\Phi_n(X) = \lim_{\overleftarrow{E \to S^0}} \Phi_E(X)$, we can get the Bousfield-Kuhn functor. This functor is unique and satisfies the following properties.

Proposition 2.2: We have the following properties:

- The spectrum $\Phi_n(X)$ is $T(n)$-local for every pointed space X,
- We can construct an equivalence between $\Phi_E(X)$ and $\Phi_n(X)^E$. This is a functional equivalence which depends on $E \in Sp_{\geq n}^{fin}$ and $X \in Space_*$.
- $\Phi_n(X)$ is left exact.
- If X is a spectrum, then $\Phi_n \Omega^\infty X = L_{T(n)} X$ (It shows that the unstable situation is compatible with the stable situation.)

In particular, describing $\Phi_n(X)$ as $\lim \Phi_{E_K}(X)$ (in the meaning of a homotopy limit) is more convenient. To be specific, we can construct such a direct system of type n spectra:

$$E_0 \to E_1 \to E_2 \to \cdots$$

This direct system is cofinal among all finite type n spectra with a map to S^0.

The v_n-periodic homotopy equivalence $f : X \to Y$ leads to a spectrum homotopy equivalence $\Phi_n(X) \to \Phi_n(Y)$. So this conversion from space to spectrum preserves all of the information of the v_n-periodic homotopy group. Actually, we can factor $\Phi_n(X)$ out as follows

$$Space_* \xrightarrow{M_n^f} Space_*^{v_n} \to Sp.$$

where $Space_*^{v_n}$ is the category of pointed spaces which support v_n-self map. The functor $\Phi : Space_*^{v_n} \to Sp$ admits a left adjoint $\Theta : Sp \to Space_*^{v_n}$

The $K(n)$-localization of Φ_n is denoted as $\Phi_{K(n)} := L_{K(n)} \Phi_n$. We are more concerned about this part since we have enough tools to calculate it instead of $T(n)$-local spectrum.

CHAPTER 3 INGREDIENTS OF CALCULATION.

This chapter introduces the current computational process for this problem. Firstly, we resolve the Bousfield-Kuhn functor by the Goodwillie tower. Then we restrict our view to some specific conditions, such as $n = 2$ and the odd-dimensional sphere S^{2m+1} for computation of the complete Morava E-cohomology groups of each fiber of the Goodwillie tower. The computation is finished by the application of the topological André-Quillen theory which can be seen as a realization of the Goodwillie tower. Finally, these cohomology groups can be input into the homotopy fixed point spectral sequence which converges to the homotopy groups of the spectrum we need. The best result we know of this problem is listed at the end of this chapter.

3.1 The Goodwillie tower

Now we need to find some methods to calculate the $\pi_*(\Phi_n X)$, or its $K(n)$-localization. The $K(n)$-localization is denoted as $\Phi_{K(n)} := L_{K(n)}\Phi_n$. The tool we need is the Goodwillie tower.

To explain it, we need some ideas in the rational homotopy theory. In rational homotopy theory, the information of rational homotopy type can be encoded by a rational cocommutative differential graded ① coalgebra $\mathcal{C}(X)$ and a rational d.g. Lie algebra $\mathcal{L}(X)$. Suillivan connected them by the minimal model $\Lambda(X)$ [2] with the following equation.

$$\pi_*(X)_{\mathbb{Q}} \cong DQ\Lambda(X)$$

For a simply connected finite space, the unstable p-adic homotopy type can be similarly described within its $\overline{\mathbb{F}_p}$-valued singular cochains. However, the p-adic analogue of the mentioned equation does not hold. As a result, people discovered other localization of unstable homotopy groups, which satisfies the above equation or its analogue. That leads to the unstable chromatic homotopy theory. So there is a tight connection between rational homotopy and v_n-periodic homotopy groups.

Now we talk more about the Lie algebra structure. The algebra structure of $\pi_*(X)$ is decided by the homotopy group of X as well as the Lie algebra structure decided by the Whitehead product. In rational homotopy theory, the structure is simplified to the

① We will denote differential graded as d.g. later

Lie algebra structure only. What's more, Quillen's work on rational homotopy theory[1] reveals that:

Theorem 3.1 (Quillen): There is an equivalence of homotopy theories:

{Connected d.g. Lie algebras over $\mathbb{Q}$ } $\simeq$ {Simply connected pointed rational spaces}

Furthermore, For a simply connected pointed rational space X, which corresponds to a Lie algebra g_* under this equivalence. The Lie algebra $(\pi_{*+1}(X), [\bullet, \bullet])$ can be identified with the homology of g_*.

The lower central series filtration of g_*

$$\cdots \subseteq g_*^{(4)} \subseteq g_*^{(3)} \subseteq g_*^{(2)} \subseteq g_*^{(1)}$$

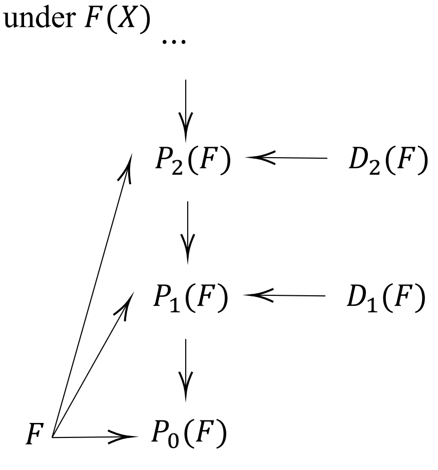
decides a tower:

$$\cdots X_4 \to X_3 \to X_2 \to X_1 \to X_0 \simeq *$$

This tower is useful because we can use this to calculate $\pi_*(F_n)$, where F_n is the fiber of $X_n \to X_{n-1}$. The Goodwillie tower is a refinement of this picture which works well for integers.

In general, the Goodwillie tower describes the homogeneous degree d part for each $d \geq 0$ of a functor $F : \mathcal{C} \to \mathcal{D}$.

Theorem 3.2 (Goodwillie,[23]): Given a homotopy functor $F : \mathcal{C} \to \mathcal{D}$ there exists a natural tower of fibrations under $F(X)$



such that

- $P_d F$ is d-excisive.
- $e_d : F \to P_d F$ is the universal weak natural transformation to a d-excisive functor

The Goodwillie tower has a commutative property with the Bousfield-Kuhn functor with some minor restrictions. That is,

$$P_k \Phi_n \simeq \Phi_n P_k Id.$$

It is also true for $\Phi_{K(n)}$. So we just need to study the Goodwillie tower of Id to understand the properties of Φ_n.

However, since the Bousfield-Kuhn functor has the commutative property with Goodwillie tower as we mentioned before, we can only consider the Goodwillie tower of Id. The theorem of Goodwillie can be rewritten as follows.

Theorem 3.3 (Goodwillie): Let X be a simply connected pointed space. We can prove that X is equivalent to the homotopy limit of a tower

$$\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X)$$

with the following features:

- The unit map $X \to \Omega^\infty \Sigma^\infty X$ coincide with $X \to P_1(X)$.
- Each homotopy fiber $D_n X = fib(P_n(X) \to P_{n-1}(X))$ is an infinite loop space.

This spcae can be described as $\Omega^\infty((\Sigma^\infty X)^n \wedge \mathcal{O}(n))_{h\Sigma_n}$. What's more, the $\mathcal{O}(n)$ has an explicit definition.

- $\mathcal{O}(t)$ is the $\partial_t(F)$, which can be defined as

$$\Omega^\infty \partial_n(F) \simeq co \lim_{k_1,\cdots,k_n} \Omega^{k_1+\cdots+k_n} cr_n(F)(S^{k_1},\cdots,S^{k_n})$$

Here the cr_n is a functor $cr_n : Space_*^n \to Space_*$ defined by a formula

$$cr_n(F)(X_1,\cdots,X_n) = tfib\left(S \to \left(\bigvee_{i\notin S\subseteq[n]} X_i\right)\right)$$

The $tfib$ is the total fiber, which is defined as

$$tfib(X) := fib\left(\mathcal{X}(\emptyset) \to \lim_{\emptyset\neq S\in P(I)} \mathcal{X}(S)\right)$$

The $\mathcal{X}$ is the I-cube, which is a functor from $P(I) \to Space_*$ and $P(I)$ is the set of subsets of a finite set I.

If we apply the Bousfield-Kuhn functor on the Goodwillie tower, we have

$$\cdots \to \Phi_n P_4(X) \to \Phi_n P_3(X) \to \Phi_n P_2(X) \to \Phi_n P_1(X) \simeq L_{T(n)}\Sigma^\infty X$$

and the homotopy fiber $D_n(X)$ turns to:

$$\Phi_n D_k X = L_{T(n)}((\Sigma^\infty X)^k \wedge \mathcal{O}(t))_{h\Sigma_k}$$

Functor Φ_n commute with infinite homotopy limits for a sphere X. In this context, the tower stabilizes ($\Phi_n D_k X \simeq 0$, for $k \gg 0$.)

Come back to the Goodwillie tower of Id. In computational use, the above tower gives us a Bousfield-Kan spectral sequence which we will introduce in the next section:

$$E_2^{s,t} : \pi_s((\Sigma^\infty X)^t \wedge \mathcal{O}(t))_{h\Sigma_t} \Rightarrow \pi_s X.$$

The $\mathcal{O}(t)$ is

$$\partial_n(Id) \simeq D(\Sigma_\infty \Delta_n)$$

The explicit definition of Δ_n can be found in ①

In particular, when X is S^K, the sphere of dimension k, we have the following spectral sequence:

$$E_2^{s,t} : \pi_s(\Sigma^{tk}\mathcal{O}(t))_{h\Sigma_t} \Rightarrow \pi_s(S^k).$$

For more details about the general Bousfield-Kan spectral sequence the readers can refer to[24] or Lurie's lecture notes.

In[5], Zhu calculate the completed E-homology of $\Phi_2(S^{2m+1})$. With the help of the homotopy fixed point spectral sequence and this result, we can calculate the target homotopy group. That is,

$$E_2^{s,t} = H^s(G_n, \hat{E}_*(\Phi_2(S^{2m+1}))).$$

We will discuss such spectral sequences in the following section.

3.2 Bousfield-Kan spectral sequence

To ensure everything we consider in this section is connective, we restrict our view to $sSet$. Consider a tower of fibrations:

$$\cdots Y_s \xrightarrow{p_s} Y_{s-1} \xrightarrow{p_{s-1}} Y_{s-2} \cdots$$

for $s \geq 0$, where $Y := \lim_{\leftarrow} Y_s$, and F_s denotes the fiber of p_s.

By acting π_* on it and rolling it into a spectral sequence, we have:

Definition 3.1: The Bousfield-Kan spectral sequence is

$$E_1^{s,t} = \pi_{t-s} F_s \Rightarrow \pi_{t-s} Y$$

This spectral sequence is particularly useful when applied to the Tot tower. In[25], the author demonstrates how this method can be utilized to derive the homotopy fixed point spectral sequence.

① https://www.math.ias.edu/ lurie/ThursdayFall2017/Lecture11-Derivatives.pdf

Definition 3.2: For a cosimplicial object $X \in sSet$, we define its totalization as

$$Tot(X^\bullet) = sSet(\Delta^\bullet, X^\bullet)$$

and

$$Tot_n(X^\bullet) = sSet(sk_n\Delta^\bullet, X^\bullet)$$

Here $Tot_n(X^\bullet) \to Tot_{n-1}(X^\bullet)$ is a fiber in the Reedy model structure.

In a simplicial category $\mathcal{C}$, $C, D \in \mathcal{C}$ and a simplicial resolution $X^\bullet \to C$, $Hom(X^\bullet, D)$ constitutes a cosimplicial object. Utilizing the cosimplicial object, we can get a spectral sequence which can be used to obtain information about $sSet(C, D)$.

Let G be a group, and X be a spectrum with a G-action. The homotopy fixed points of X are

$$X^{hG} := F((EG)_+, X)^G$$

or in another word, the fixed point of the G-equivariant maps $(EG)_+ \to X$.

According to the above discussion, a simplicial resolution of $(EG)_+$ can be made by the bar construction, which produces a cosimplicial object. This object can be inserted into the Bousfield-Kan spectral sequence.

Specifically, consider $EG = B^\bullet(G, G, *)$ with disjoint basepoint and a map form it to X, we have

Theorem 3.4: For a spectrum X with a G-action, there exists a spectral sequence, denoted as the homotopy fixed-point spectral sequence:

$$E_2^{s,t} = H^s(G, \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG})$$

3.3 Topological André-Quillen (co)homology

Calculating the completed E-homology of $\Phi_2(S^{2m+1})$ needs the help of the topological André-Quillen (co)homology. This section focuses on clarifying the details of it, referred to as TAQ, which mainly refers to[19].

Firstly, we will introduce the origin and motivation behind the definition of TAQ. For unstable homotopy types, we hope to view them as stable homotopy types with extra structure. For a category $\mathcal{C}$, the aim is to embed its homotopy category into the homotopy category of algebras over some spectra. Specifically, it can be written in this form:

$$U : Ho(\mathcal{C}) \leftrightarrows Ho(Alg_?(S(\mathcal{C}))) : C$$

which satisfies that $X \simeq CU(X)$. Here the ? means an algebra structure we don't decide it at now. In this context, ?-algebra models are considered as unstable homotopy types of $\mathcal{C}$. For example, for rational pointed spaces, which stabilization is rational spectra. Since $Ho(Sp_{\mathbb{Q}}) \simeq Ho(Ch_{\mathbb{Q}})$, we have the algebra models that can be chosen as either commutative coalgebras or Lie algebras.

Moreover, we hope the adjunction pair we mentioned above supports a category equivalence:

$$Ho\{X \in \mathcal{C} \quad s.t. \quad X \simeq CU(X)\} \simeq Ho\{A \in Alg_{?}(Sp(\mathcal{C})) \quad s.t. \quad A \simeq CU(A)\}.$$

Ideally, we hope U to be a fully faithful functor. As the canonical method, homotopy descent can be employed to achieve this.

TAQ can be viewed as a model for the algebra of the operad on the unstable homotopy category. To be specific, it can be seen as the functor U above. Using Koszul duality, we can construct TAQ precisely. However, there are some slight differences between the canonical way. So we need to establish its relationship with the model obtained through homotopy descent.

Subsequently, we will study the comparison map, which serves as a tool connecting TAQ and the Goodwillie tower. Through this map, we can regard TAQ as a model for the Goodwillie tower. The proof involves reducing the equivalence of the comparison map to each level of the tower's filtration. Additionally, we can construct a more computational model for TAQ: The Morava E-theory Dyer-Lashof algebra.

The proof mentioned above requires imposing the $\Phi_{K(n)}$-good condition on the spaces we are studying. Thus, we need to explain this condition and specify its closure under certain operations.

To set up the topological André Quillen (co)homology, I will list those definitions that we need to use in the following chapter.

The first one is the algebra over an operad. Let M denote a closed symmetric monoidal category with a monoidal unit I, and let X be any object within this category. An canonical or tautological operad, denoted as $Op(X)$, exists where its n^{th} component corresponds to the internal hom $M(X^{\otimes n}, X)$; the identity of this operad is represented by the map

$$1_X : I \to M(X, X)$$

and the operad multiplication is given by the composite

$$M(X^{\otimes k}, X) \otimes M(X^{\otimes n_1}, X) \otimes \cdots \otimes M(X^{\otimes n_k}, X) \longrightarrow M(X^{\otimes n_1+\cdots+n_k}, X)$$

Consider any operad $\mathcal{O}$ within M. An algebra over $\mathcal{O}$ entails an object X endowed with an operad map $\xi : \mathcal{O} \to Op(X)$. Alternatively, the information regarding an $\mathcal{O}$-algebra is conveyed by a series of mappings

$$\mathcal{O}(k) \otimes X^{\otimes k} \to X$$

These maps define an action of $\mathcal{O}$ through finitary operations on X, alongside compatibility conditions that link the operad multiplication with the act of incorporating k finitary operations on X into a k-ary operation. These conditions also ensure compatibility with actions performed by permutations.

The next one is the algebra over a monad. Let (T, η, μ) be a monad on a category $\mathcal{C}$. An algebra over T consist of

- an object A of $\mathcal{C}$,
- a morphism $a : T(A) \to A$,

such that

- $id_A = a \circ \eta$,
- $a \circ \mu = a \circ T(a)$.

After listing those definitions, we also need to list some techniques to set up the TAQ. Homotopy descent is one of them. The theory of homotopy descent, as proposed by Hess and Lurie, offers a natural candidate solution for the fully faithful functor U. Specifically, the adjunction provides a comonad $\Sigma^\infty_\mathcal{C}\Omega^\infty_\mathcal{C}$ on $Sp(\mathcal{C})$. The spectrum $\Sigma^\infty_\mathcal{C} X$ serves as a coalgebra for this comonad for $X \in \mathcal{C}$. Consequently, we interpret the functor $\Sigma^\infty_\mathcal{C}$ as a refinement to the functor

$$U : Ho(C) \to Ho(Coalg_{\Sigma^\infty_\mathcal{C}\Omega^\infty_\mathcal{C}})$$

When the functor is an equivalence, it is equivalent to the assertion that this adjunction is comonadic. However, this equivalence only holds between two appropriate subcategories of these two categories. Moreover, to fully utilize this equivalence, we require a clear definition and idea of what it entails to be a $\Sigma^\infty_\mathcal{C}\Omega^\infty_\mathcal{C}$-coalgebra.

For instance, If we set $C = Top_*$. A map

$$X \to C(\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty X)$$

exist. The comonadic cobar construction is denoted as $C(-,-,-)$ in the above equation.

In particular, we have a cobar construction

$$C(\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty X) = Tot(QX \Rightarrow QQX \Rightarrow \cdots)$$

which can be considered as a functor called the Bousfield-Kan Q-completion of X. If X is nilpotent, the above map is an equivalence. What's more, for such spaces, its information in unstable homotopy can be reconstructed from the $\Sigma^\infty\Omega^\infty$-comonad structure of $\Sigma^\infty X$ as we expected.

Arone, Klein, Heuts, and others, give a partial description of the $\Sigma^\infty\Omega^\infty$-coalgebra structure. That is, $\Sigma^\infty\Omega^\infty$ can be described as the free commutative coalgebra functor for connected spaces.

The next technique we need is the Koszul duality. For a commutative ring spectrum R and an operad in Mod_R $\mathcal{O}$ ①, the category of $\mathcal{O}$-algebras is denoted as $Alg_\mathcal{O} = Alg_\mathcal{O}(Mod_R)$. An equivalence of $\mathcal{O}$-algebras is a morphism in the category of $\mathcal{O}$-algebras such that its underlying map of spectra is an equivalence. According to the above hypotheses, there exists a free-forgetful adjunction:

$$\mathcal{F}_\mathcal{O} : Mod_R \leftrightarrow Alg_\mathcal{O} : U$$

the functor $\mathcal{F}_\mathcal{O}$ is a free $\mathcal{O}$-algebra generated by X, which has an explicit form:

$$\mathcal{F}_\mathcal{O}(X) = \bigvee_i (\mathcal{O}_i \wedge_R X^{\wedge_R i})_{\Sigma_i}$$

We also denote the associated monad on Mod_R as $\mathcal{F}\mathcal{O}$, abusing notation. Therefore, $\mathcal{O}$-algebras are equivalent to $\mathcal{F}\mathcal{O}$-algebras.

$$Alg_\mathcal{O} \simeq Alg_{\mathcal{F}_\mathcal{O}}.$$

There is a natural transformation of monad since $\mathcal{O}$ is reduced.

$$\epsilon : \mathcal{F}_\mathcal{O} \to Id$$

For an $\mathcal{O}$-algebra A, the coequalizer of ϵ and the structure map of $\mathcal{F}_\mathcal{O}$ decides its module of indecomposable QA:

$$\mathcal{F}_\mathcal{O}(A) \rightrightarrows A \to QA.$$

which has a right adjoint

$$Q : Alg_\mathcal{O} \leftrightarrows Mod_R : triv$$

Equipping X with $\mathcal{O}$-algebra structure maps for a R-module X, we can achieve a $\mathcal{O}$-algebra

① All operads $\mathcal{O}$ discussed in this report are assumed to be reduced, meaning $\mathcal{O}_0 = *$ and $\mathcal{O}_1 = R$.

$trivX$:

$$\mathcal{O}_1 \wedge_R X = R \wedge_R X \xrightarrow{\approx} X$$

$$\mathcal{O}_n \wedge_R X^n \xrightarrow{*} X, \quad n \neq 1$$

Here, we can give the explicit definition of the topological André-Quillen homology of A: the left-derived functor

$$TAQ^{\mathcal{O}}(A) := \mathbb{L}QA.$$

It can be realized as the monadic bar construction which can be computed effectively:

$$TAQ^{\mathcal{O}}(A) \simeq B(Id, \mathcal{F}_{\mathcal{O}}, A).$$

Applying the R-linear dual on $TAQ^{\mathcal{O}}$, we can get the Topological André-Quillen cohomology:

$$TAQ_{\mathcal{O}}(A) := TAQ^{\mathcal{O}}(A)^{\vee}$$

The $TAQ^{\mathcal{O}}$ have two important properties:

• $TAQ^{\mathcal{O}}$ is excisive. (Takes homotopy pushout squares to homotopy pullback squares.)

• $TAQ^{\mathcal{O}}(\mathcal{F}_{\mathcal{O}}(X)) \simeq X$.

In other words, TAQ encode a "cell structure" in an $\mathcal{O}$-algebra.

With the help of the TAQ, we can prove such theorem:

Theorem 3.5 (Rezk): There is an equivalence of categories

$$Ho(Sp(Alg_{\mathcal{O}})) \simeq Ho(Mod_R).$$

with leads the following pair of functors

$$\Sigma^{\infty}_{Alg_{\mathcal{O}}} : Ho(Sp(Alg_{\mathcal{O}})) \leftrightarrows Ho(Mod_R) : \Omega^{\infty}_{Alg_{\mathcal{O}}}$$

by

$$\Sigma^{\infty}_{Alg_{\mathcal{O}}} A \simeq TAQ^{\mathcal{O}}(A),$$

$$\Omega^{\infty}_{Alg_{\mathcal{O}}} X \simeq trivX.$$

Corollary 3.1 (Rezk): The spaces of the $TAQ_{\mathcal{O}}$-spectrum are

$$\Omega^{\infty}\Sigma^{n} TAQ_{\mathcal{O}}(A) \simeq \underline{Alg}_{\mathcal{O}}(A, triv\Sigma^{n} R).$$

The last technique we need is the divided power coalgebras. $\Sigma^{\infty}_{Alg_{\mathcal{O}}}\Omega^{\infty}_{Alg_{\mathcal{O}}}$-coalgebra can be understood as a divided power coalgebra over the Koszul dual $B\mathcal{O}$ according to[26] and[27].

To be precise, the Koszul dual of $\mathcal{O}$ is a symmetric sequence obtained by forming the bar construction according to the composition product.

$$B\mathcal{O} := B(1_R, \mathcal{O}, 1_R) = |1 \Leftarrow \mathcal{O} \Leftarrow \mathcal{O} \circ \cdots|$$

$B\mathcal{O}$ admits a cooperad structure, which is proved by Ching in the paper.

Define $Comm = Comm_R$ as

$$Comm_i = *, \quad i = 0, \quad R, \quad i \geq 1.$$

For $R = \mathbb{S}$, $\mathcal{O} = Comm_S$, Ching proved that

$$BComm_S \simeq (\partial_* Id_{Top_*})^\vee.$$

The RHS of this equation is the duals of the Goodwillie derivatives of Id.

What's more, for an R-Module X, we have

$$\Sigma^\infty_{Alg_\mathcal{O}} \Omega^\infty_{Alg_\mathcal{O}} X \simeq TAQ^\mathcal{O}(trivX) \simeq B(Id, \mathcal{F}_\mathcal{O}, trivX) \simeq \mathcal{F}_{B\mathcal{O}} X.$$

For connective R, $\mathcal{O}$ and X, we can observe that

$$\mathcal{F}_{B\mathcal{O}} X \simeq \prod_i (B\mathcal{O}_i \wedge_R X^{\wedge_R i})$$

As a result, if we consider the homotopy category, the information included in a $\Sigma^\infty_{Alg_\mathcal{O}} \Omega^\infty_{Alg_\mathcal{O}}$-coalgebra C can be expressed by a collection of coaction map:

$$\psi_i : C \to (B\mathcal{O}_i \wedge_R C^{\wedge_R i})_{\Sigma_i}.$$

According to[26] and[28], we have

$$TAQ^\mathcal{O} : Ho(Alg_\mathcal{O}) \leftrightarrows Ho(d.p.Coalg_{B\mathcal{O}}) : C.$$

and according to[27], we have

Theorem 3.6 (Ching): For connective R and $\mathcal{O}$, the functors we mentioned above induce an equivalence between categories

$$Ho(Alg^{\geq 1}_\mathcal{O}) \to Ho(d.p.Coalg^{\geq 1}_{B\mathcal{O}}).$$

Now, we need to consider how to encode an unstable homotopy type $X \in M^f_n Top_*$ by something we mentioned above. This information can be encoded in $T(n)$-local $Comm$-algebra $S^X_{T(n)}$ ① according to the following consequence of Hopkins:

① the $S_{T(n)}$-valued cochains

Theorem 3.7 (Hopkins): For an unstable type n complex V, we have

$$S_{T(n)}^{V} \simeq triv(V^{\vee}).$$

which comes from $S_{T(n)}^{V}$ is an infintie loop object of Alg_{Comm}.

With the corollary we mentioned above, we have

$$\underline{Alg}_{Comm}(S_{T(n)}^{X}, S_{T(n)}^{\Sigma^{*}V}) \simeq \Omega^{\infty}\Sigma^{*}TAQ_{S_{T(n)}}(S_{T(n)}^{X}) \wedge V^{\vee}.$$

By acting $S_{T(n)}^{-}$ on $v_n^{-1}\pi_*(X;V) \cong [\Sigma^{*}V, M_n^f(X)]_{Top_*}$ as well as applying the definition of the Bousfield-Kuhn functors of V, we have

$$c_X^V : \Phi_V(X) \to TAQ_{S_{T(n)}}(S_{T(n)}^{X}) \wedge V^{\vee}$$

Then taking homotopy colimit on both sides as we did for the Bousfield-Kuhn functors, we have such natural transformation:

$$c_X : \Phi_n(X) \to TAQ_{S_{T(n)}}(S_{T(n)}^{X})$$

called comparison map. Naturally, we have a $K(n)$ version:

$$c_X^{K(n)} : \Phi_{K(n)}(X) \to TAQ_{S_{K(n)}}(S_{K(n)}^{X})$$

The main theorem of[29] said that the comparison map $c_X^{K(n)}$ is an equivalence if X is a finite $\Phi_{K(n)}$-good space, especially for the case that X is a sphere.

For such special cases, we have some easier proof as the paper of Behrens and Rezk says. The idea of proof can be sketched as follows.

Notice that both $\Phi_{K(n)}$ and $TAQ_{S_{K(n)}}(S_{K(n)}^{(-)})$ are functors from Top_* to $Sp_{K(n)}$. $\Phi_{K(n)}$ fits the Goodwillie tower, and $TAQ_{S_{K(n)}}(S_{K(n)}^{(-)})$ fits a tower

$$TAQ_R(A) \to \cdots F_K TAQ_R(A) \to F_{k-1}TAQ_R(A) \to \cdots.$$

according to[30]. This tower has such properties:

Theorem 3.8 (Kuhn): The fibers of the tower are given by

$$s^{-1}Lie_k \wedge_{k\Sigma_k} (A^{\wedge_R k})^{\vee} \to F_K TAQ_R(A) \to F_{k-1}TAQ_R(A).$$

for proper A. ①

Notice that in the Goodwillie tower of Id in Top_*, the k^{th} fiber of this tower is

$$\Phi_{K(n)}D_k Id(X) \simeq (s^{-1}Lie_k \wedge_{k\Sigma_k} X^{\wedge k})_{K(n)} \to \Phi_{K(n)}P_k Id(X) \to \Phi_{K(n)}P_{k-1}Id(X).$$

The "coincidence" of the fiber of such two towers is equivalent on each fiber inducing an equivalence between towers for X finite. If we want to prove the comparison map is an

① Refer to Kuhn's paper for details.

equivalence, we just need to show it induces equivalences on each layer of these towers. We need the Dyer Lashof algebra to finish this part.

We use E_n to denote the n^{th} Morava E-theory spectrum. For the completed Morava E-homology, we denote it as $E_n^{\wedge}$, to be $\pi_*(E_n \wedge Z)K(n)$ for a spectra Z. In[31], the second author introduced a monad

$$\mathbb{T} : Mod_{(E_n)_*} \to Mod_{(E_n)_*}.$$

leads to a new algebraic structure called $\mathbb{T}$-algebra. It can be considered on the completed E-homology as a $Comm$-algebra. A $\mathbb{T}$-algebra refers to an algebra over the Morava E-theory Dyer Lashof algebra Λ_n. As a result, the functor $\mathbb{T}M$'s value is the free Λ_n-algebra on M for an $(E_n)_*$-module M. ①

What we need is such a theorem:

Theorem 3.9 (Rezk): If $(E_n^{\wedge})_*Z$ is flat over $(E_n)_*$, then the natural transformation

$$\mathbb{T}(E_n^{\wedge})_*Z \to (E_n^{\wedge})_*\mathcal{F}_{Comm}Z$$

induce an isomorphism

$$(\mathbb{T}(E_n^{\wedge})_*Z)_m^{\wedge} \xrightarrow{\cong} (E_n^{\wedge})_*\mathcal{F}_{Comm}Z$$

Here the m is the unique maximal ideal of $(E_n)_0$.②

With the relationship between $\mathcal{F}_{Comm}$ and TAQ, we can apply a Basterra spectral sequence to such an equation. Then we have such spectral sequence:

$$AQ_{\mathbb{T}}^{*,*}((E_n^{\wedge})_*A; (K_n)_*) \Rightarrow (K_n)_*TAQ_{S_{K(n)}}(A).$$

which leads to some explicit calculation.

For the proof of the main theorem, we can prove the equivalence for QX:

Theorem 3.10 (Rezk): Consider all N-fold suspension spaces X such that $(E_n^{\wedge})_*X$ is free and finitely generated over $(E_n)_*$. In this context, there exists a comparison map

$$(\Sigma^{\infty})_{K(n)} \simeq \Phi_{K(n)}(QX) \xrightarrow{c_{QX}^{K(n)}} TAQ_{SK(n)}(S_{K(n)}^{QX})$$

is an equivalence.

This theorem is established by demonstrating that a $K(n)$-homology isomorphism is induced by the comparison map, given that they are $K(n)$-local. Consequently, it becomes imperative to compute the $K(n)$-homology for each spectrum involved in the equation above.

① For additional details, see[19].

② In[29] we have a stronger version as the Theorem 7.7 in the origin paper.

The $K(n)$-homology of RHS can be calculated by:

Theorem 3.11 (Rezk):

$$(E_n^{\wedge})_* S_{K(n)}^{QX} \to \hat{\mathbb{T}}\tilde{E}_n^* X$$

induce an isomorphism of $\mathbb{T}$-algebras if X satisfies the hypotheses of the above theorem.

and the spectral sequence we mentioned above.

Finally, for X is a sphere, we can apply the above theorem to the Bousfield-Kan cosimplicial resolution. However, since $Q^k X$ is not finitary in general, we need some tools to avoid it. According to the Snaith splitting in[32], we have

$$P_{p^n}(Q^{s+1})(X) \simeq QY^s$$

The space Y^s meets the finitary hypotheses.

What's more,

$$\Phi_{K(n)}(X) \to \Phi_{K(n)} P_k Id(X)$$

is an equivalence for q odd and $k = p^n$, or q even and $k = 2p^n$. We can use the above equivalence in such a case. With the properties of the Goodwillie tower, we can prove the comparison map induces an equivalence for odd sphere S^{2m+1}.

The comparison map is deemed equivalent for X that is $\Phi_{K(n)}$-good. A space is considered $\Phi_{K(n)}$-good if the map

$$\Phi_{K(n)}(X) \to holim P_k(\Phi_{K(n)})(X)$$

is an equivalence.

We will list some important properties of $\Phi_{K(n)}$-good spaces. This properties are also shown in[19].

- A finite space X qualifies as $\Phi_{K(n)}$-good iff the comparison map is an equivalence.
- The product of finite $\Phi_{K(n)}$-good space remains $\Phi_{K(n)}$-good
- The $\Phi_{K(n)}$-good spaces satisfy 2-out-of-3 property in a fiber sequence of finite spaces that is $K(n)$-cohomologically Eilenberg-Moore.
- The special unitary groups $SU(k)$ and symplectic groups $Sp(k)$ are $\Phi_{K(n)}$-good.

The details of such properties and the method of proving the general case can refer to the origin paper of Behrens and Rezk,[33],[34] and[35] for the Arone-Ching approach, and[28] for the Heuts approach.

3.4 Generalized Adams spectral sequence

The last ingredient we need to introduce is the (generalized) Adams spectral sequence. Adams spectral sequence is the most important tool for us to calculate the homotopy group. It gives us a method to extract the homotopy information from the mod p cohomology, or other cohomology theory. If we use the ordinary mod p cohomology, we get the classical Adams spectral sequence. If we use the BP, we get the ANSS. For other cohomology theories, there is some spectral sequence, but the above two spectral sequences are widely used.

We will begin with the classical Adams spectral sequence.

Theorem 3.12 (Adams,[36]): Let X be a spectrum with a finite dimension of $H^*(X)$. There is a spectral sequence

$$E_*^{*,*}, d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

such that

$$E_2^{s,t} = Ext_{\mathcal{A}_p}^{s,t}(H^*(X), \mathbb{Z}_p) \Rightarrow \pi_*(X) \otimes \mathbb{Z}_{(p)}$$

Here $\mathcal{A}_p$ is the mod p Steenrod algebra.

What's more, Adams spectral sequence is multiplicative if X is a ring spectrum.

Adams spectral is induced by the Adams resolution, which is a tower such that each fiber is the wedge of some copies of Eilenberg-MacLane space (with some possible suspension).

$$\begin{array}{ccccccccc} X & \xleftarrow{=} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ & & K_0 & & K_1 & & K_2 & & \end{array}$$

We can roll it into an exact couple and get a spectral sequence. From the algebra perspective, it is a (minimal) resolution of $H^*(X)$ as an $\mathcal{A}_p$-module. That also leads to the computer computation of E_2 page of classical Adams spectral sequence.

If we repeat the same thing for other cohomology theories, we get the generalized Adams spectral sequence. For a given cohomology theory E_* (with some mild restrictions, such as $E_*(E)$ has a E-comodule structure), we have

Definition 3.3: An E_*-Adams resolution for X is a diagram

$$\begin{array}{ccccccccc} X & \xleftarrow{=} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ & & K_0 & & K_1 & & K_2 & & \end{array}$$

such that for all $s \geq 0$ the following conditions hold

- X_{s+1} constitutes the fiber of f_s.
- $E \wedge X_s$ is a retract of $E \wedge K_s$, meaning there exists a map $h_s : E \wedge K_s \to E \wedge X_s$ such that $h_s(E \wedge f_s)$ is the identity map of $E \wedge X_s$. In particular, $E_*(f_s)$ is a monomorphism.
- K_s is a retract of $E \wedge K_s$.
- $Ext^{t,u}(E_*(K_s)) = \pi_u(K_s)$ when $t = 0$ and is otherwise 0.

Here the explicit definition of Ext and the restrictions of E_* at here can refer to Appendix A1 and Chapter 2.2 of[37].

This spectral sequence can detect the E-component of $\pi_*(X)$. That is,

Definition 3.4: An E-completion $\hat{X}$ of X is a spectrum characterized by the following properties:

- There exists a map $X \to \hat{X}$ that results in an isomorphism in E_*-homology.
- $\hat{X}$ possesses an E_*-Adams resolution $\{\hat{X}_s\}$ with $\lim \hat{X}_s = pt$.

The above resolution induces a spectral sequence:

Theorem 3.13: An E_*-Adams resolution for X leads to a natural spectral sequence $E_*^{*,*}$ with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that $E_2^{s,t} = Ext(E_*(X)) \Rightarrow \pi_*(\hat{X})$

If we let $E = BP$, we get the ANSS.

Finding an analogue of the minimal resolution of the comodule is difficult. So we use cobar construction to get the canonical E_*-Adams resolution. It is useful for proof, but for calculation, the cobar complex is too big to compute. Some other tools have been developed to solve this problem. Such as the May spectral sequence and lambda algebra. However, it's not relative to this article. Readers interested in this topic can read Chapter 3 of[37]. The details of calculating the ANSS can be found in Chapter 4.

3.5 The methods of the explicit calculation

After the above preparation, we can describe the known methods of calculating $\pi_*(\Phi_{K(n)}(X))$. There are two known approaches. One is introduced in[3] by Wang, and another one is introduced in[4] by Behrens and Rezk. These two approaches coincide expect the methods of calculating the completed E_n-homology of $\Phi_{K(n)}(X)$.

By considering $E(n)$-Adams spectral sequence of $\Phi_{K(n)}(X)$, we have

$$Ext_{E_{n*}E_n}(E_{n*}, E_n(\Phi_{K(n)}(X))) \Rightarrow \pi_*(\Phi_{K(n)}(X)).$$

Here the E_n means E_n-homology. Since $\Phi_{K(n)}(X)$ is $K(n)$-local, according the Appendix A of[38], we can transform it into the following form:

$$H_c^*(G_n, E_n(\Phi_{K(n)}(X))) \Rightarrow \pi_*(\Phi_{K(n)}(X)).$$

Here G_n is the Morava stabilizer group. We call the original form $K(n)$-local $E(n)$-Adams spectral sequence, and the second form as a special case of homotopy fixed point spectral sequence.①

Then, we need to calculate the E_n-homology of $\Phi_{K(n)}(X)$. We can give a resolution of $\Phi_{K(n)}(X)$ by Goodwillie tower. By acting E_{n*} on the Goodwillie tower, we can get $E_n(\Phi_{K(n)}(X))$ by the Atiyah-Hirzebruch spectral sequence if we know $E_n(D_k(\Phi_{K(n)}(X)))$ for each k as well as the attaching map of Goodwillie tower. The differentials of that spectral sequence can be calculated by representing those generators in $H_c^*(G_n, \mathbb{F}_p)$. Here, we need to consider X as S^m where m is odd because we know enough information of $D_k(\Phi_{K(n)}(S^m))$. In this situation, the attaching map of Goodwillie tower is decided by the James-Hopf map②.

In Wang's approach, he proved that the $D_k(\Phi_{K(n)}(S^m))$ is homotopy equivalent to some spectrum related to the Steinberg summands of $B\mathbb{F}_p^n$. That is,

$$D_{p^t}S^k \simeq \Omega^\infty\Sigma^{k-t}L(t)_k$$

while $D_mS^k \simeq *$ for $m \neq p^t, t \in \mathbb{N}$.

Since we can get $E_{n*}(X)$ by $BP_*(X) \otimes_{BP_*} E_{n*}$, the remaining work is calculating the BP-homology of $L(t)_k$. The E_2-page of Adams spectral sequence of $BP_*(L(t))$ is calculated in[39] if we know the ordinary cohomology of $L(K)$ and it can be calculated by analysing the base of ordinary cohomology of $L(k)$ represented by $\beta^{\epsilon_1}P^{i_1}\beta^{\epsilon_2}P^{i_2}\beta^{\epsilon_3}P^{i_3}\cdots$ admissible. However, if we need the full comodule structure of it, we need the v_n-hidden extension in this spectral sequence.

As a result, we need the BP-cohomology of $L(t)$, which can be calculated by Koszul

① Some papers directly use the homotopy fixed point spectral sequence to describe this, but I prefer to treat it as a special case of the ANSS.

② Only for spheres.

complex ① $BP^*(L(t)_k)$ can be describe by $BP^*(L(t))$ and Dickson-Mui generators.②

Then, using this E_n-homology as an input, we get the E_2 page of the above spectral sequence. In particular, for $n = 2$ and $p \geq 5$, there is no possible nontrivial differential in the ANSS, so we get its unstable v_n-periodic homotopy group.[3]

Another method is given by Mark Behrens and Charles Rezk in[4]. They constructed a natural transformation from pointed spaces to $K(n)$-local spectra called the comparison map.

$$C^{S_K} : \Phi_{K(n)}(X) \to TAQ_{S_K}(S_K^{X_+})$$

This transformation relates $\Phi_{K(n)}(X)$ to the topological André-Quillen cohomology. For X is an odd sphere, the comparison map is an equivalence. Some works have been done to study these spaces such as[19]. But in this article, we will not discuss this.

Ching's work[40] shows that $TAQ_{S_K}(S_K^{X_+})$ has the structure of an algebra over the operad formed by Goodwillie derivatives $\partial_*(Id)$. This can be regarded as a topological analogue of the Lie operad. As a result, we can see $TAQ_{S_K}(S_K^{X_+})$ as a Lie algebra model for the unstable v_n-periodic homotopy type of X (or in a short way, an analogue of $\mathcal{L}(X)$).

Since Dyer-Lashof algebra Δ^q can be used to construct a form of André-Quillen cohomology, we can relate the André-Quillen cohomology with the Koszul resolution of Δ^q. It was finished by constructing a bar construction model for Kuhn's filtration on topological André-Quillen cohomology, in which layers of this filtration are equivalent to the spectra $L(k)_q$. Then we can show the Morava E-homology of the spectrum $L(k)_q$ is isomorphic to to the dual of k^{th} term of the Koszul resolution for Δ^q. In spectral sequence's way, this can be described as

$$Ext^s_{\Delta^q}(\tilde{E}_{n,t}(S^q), \bar{E}_{n,t}) \Rightarrow E_{n,q+t-s}(\Phi_{K(n)}(S^q))$$

for q odd. The attaching map of Goodwillie tower can be studied simultaneously in this approach. This approach is used by[5] for calculating $E_{n*}(\Phi_{K(n)}(S^{2m+1}))$. The result of this calculation can be found in this thesis.

Finally, we can get the main theorem in[3]:

Theorem 3.14 (Wang, Theorem 5.4.12): For a prime number $p \geq 5$, $H^*(\mathbb{G}_2^1, E_2\Phi_{K(2)}S^3)$ is a vector space over $\mathbb{F}_p$, with a set of basis of 12 different classes

① Koszul complex can be seen as a special type of bar construction. Behrens and Rezk show that the information of $L(n)$ is encoded by some Koszul complex in their paper.

② The $L(t)_k$ can be seen as the result of unstable filtration for k odd. This filtration can be defined by the powers of D_n, the n^{th} Dickson-Mui generator.

with no possible differentials in the ANSS. Therefore, the homotopy groups $\pi_*(\Phi_{K(2)}S^3)$ is a free module over $\mathbb{F}_p[\xi]/\xi^2$ with the above generators.

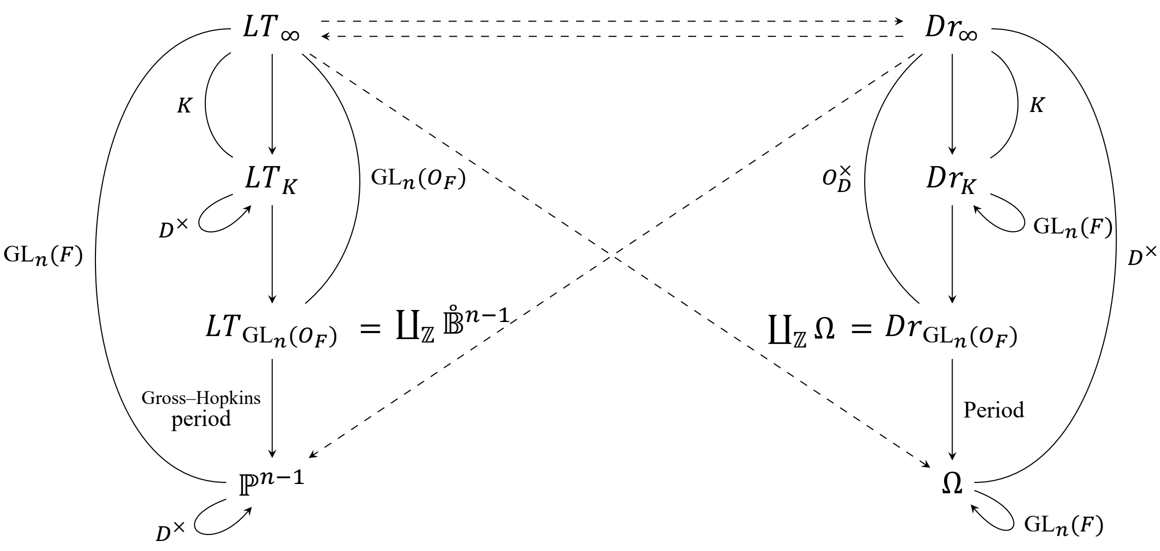
CHAPTER 4 CONCLUSION AND FURTHER PROGRESS

In this thesis, We explain how to transform the calculation of unstable v_n-periodic homotopy groups to the stable range by the Bousfield-Kuhn functor, as well as provide some accessible methods of explicit calculation for unstable v_n-periodic homotopy groups. The following sections will also explore new methods for further development of this problem. We hope that this paper will serve as a helpful introduction and reference for readers who are interested in this topic. We also hope that it will stimulate further research on this fascinating topic.

In the above calculation, we found that there are too many potential nontrivial differentials in the ANSS for most cases. We have few tools to deal with them since $\Phi_{K(n)}(S^{2m+1})$ is not a ring spectrum. As a result, calculations using these approaches are only available in some cases. As a result, we need to develop a new approach to fix this problem.

4.1 The Algebraic Geometry approach

As we mentioned in the introduction, a possible improvement is to “switch” the order of spectral sequence. The following duality inspires it in algebra geometry:



The original method corresponds to the left side of the above diagram. The Koszul complex encodes the information of each odd sphere, and the dimension of these spheres induces a filtration. This information can be seen as a sheaf on $\mathcal{L}\mathcal{J}_{G_n}$ for each Morava stabilizer group. By acting homotopy fixed point spectral sequence on that, we can get the (completed) E_n-homology for $\Phi_n(S^q)$. This process has a dual version on the right side. However, the explicit meaning of this duality needs further study.

4.2 The elliptic curve approach

Another method is changing the completed Morava E-theory into the Morava E-theory. This change can reduce the number of elements in the E_2-page of the $E(n)$-based ANSS.

With a modular description of the Goodwillie tower and Morava E-homology of symmetric groups, we can turn the calculation of the Morava E-homology of the Steinberg summands into an algebraic calculation about the topological modular form. This calculation is equivalent to computing the cohomology of a Hopf algebroid, in which we know all of its relations.

This work has outlined a general framework, but the proofs require further refinement and scrutiny, along with arranging them in the correct logical sequence. The relevant computations need to be verified, too. Under the guidance of the author's supervisor, the study of the computational aspects of this work and verifying the existing computational results are being pushed forward. The computation part of this problem has been transformed into computing the cohomology of a certain Hopf algebroid. Furthermore, the author plans to conduct additional computational experiments based on this framework to gather more information.

CONCLUSION

Our main contributions are:

• Explaining how to transform the calculation of unstable v_n-periodic homotopy groups to stable range by the Bousfield-Kuhn functor.

• Providing some accessible methods of explicit calculation for unstable v_n-periodic homotopy groups.

• Exploring some new methods for further development of this problem.

We hope that this paper will serve as a useful introduction and reference for readers who are interested in this topic. We also hope that it will stimulate further research on this fascinating topic.

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

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