硕士学位论文

代数配边和圈类映射

ALGEBRAIC COBORDISM AND CYCLE CLASS MAP

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ALGEBRAIC COBORDISM AND CYCLE CLASS MAP

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摘　要

代数几何的上同调研究领域，圈类映射是重要的。本文将探讨这一主题，并深入分析其各个层面。特别是，Hodge 猜想和 Tate 猜想这两个著名难题与圈类映射紧密相关，这些猜想不仅具有深远的意义，还衍生出多个重要推论。此外，许多代数几何和数论中的问题，如 Birch and Swinnerton-Dyer 猜想，也与代数圈有着深刻的联系。

我们回顾代数圈的基本概念，介绍有理等价、同调等价和数字等价的概念及他们间的关系。有理等价通过考虑代数圈的差值作为更高维圈的边界来比较不同圈; 同调等价则通过 Weil 上同调中的像来关联圈；而数字等价关注的是圈之间的相交情况。理解这些等价关系是研究代数圈及其性质的基础。

由于 Grothendieck 对动机理论的开创性贡献，我们可以将圈类映射视为一种实现。例如，Betti 实现将动机上同调转化为 Betti 上同调，从而恢复了经典的圈类映射。因此，本文还将介绍动机上同调的最新进展，包括 Poincaré 对偶性和 Chow 群兼容性的讨论。动机上同调不仅推广了经典上同调理论，还为研究代数簇的结构及其上同调性质提供了新工具，成为现代代数几何中重要的部分。

即便如此，寻找有效的方法来处理圈类映射仍然是一个挑战。Hodge 猜想之所以困难，很大部分原因在于它连接了代数和拓扑这两种基本数学结构。通过代数配边，可以证明圈类映射可以通过复配边实现。本文将介绍由 Voevodsky 提出的代数配边理论。代数配边提供了一种丰富的结构，能够捕捉代数簇的几何和拓扑信息，是研究圈类映射的强有力工具。

最后，我们将解释 Hodge 理论的核心内容以及圈类映射的具体构造。我们解释为什么圈类映射可以通过复配边进行分解，并讨论这一过程对上同调类代数性的影响。我们将给出一个非代数上同调类的具体例子。那里代数结构与拓扑结构之间的相互作用带来了更多的复杂性。通过深入探讨这些主题，我们旨在揭示代数圈、上同调理论以及更广泛的代数几何领域之间的复杂联系。

关键词：代数圈；代数几何；代数配边

ABSTRACT

In the study of cohomology within algebraic geometry, the cycle class map plays a pivotal role. This paper aims to explore various aspects of this topic in depth. Specifically, two notoriously challenging problems—the Hodge Conjecture and the Tate Conjecture—are closely linked to the cycle class map. These conjectures have profound implications, and we will discuss several significant corollaries. Furthermore, numerous issues in algebraic geometry and number theory, such as the Birch and Swinnerton-Dyer (BSD) Conjecture, are intricately intertwined with algebraic cycles.

We begin with an overview of algebraic cycles. In this section, we will clarify the concepts of rational equivalence, homological equivalence, and numerical equivalence related to algebraic cycles, and elucidate the interrelationships among these equivalence relations. Rational equivalence is a fundamental concept that allows us to compare algebraic cycles by considering their differences as boundaries of higher-dimensional cycles. Homological equivalence, on the other hand, relates cycles through their images in Weil cohomology, while numerical equivalence considers intersections of cycles. Understanding these equivalences provides a robust framework for studying algebraic cycles and their properties.

Building on Grothendieck's groundbreaking insights into the theory of motives, we can interpret the cycle class map as a realisation. For instance, the Betti realisation of motivic cohomology recovers the ordinary cycle class map to Betti cohomology. Consequently, we will introduce recent advancements in motivic cohomology, including discussions on Poincaré duality and compatibility with Chow groups. Motivic cohomology offers a powerful tool for understanding the structure of algebraic varieties and their cohomological properties. It generalizes classical cohomology theories and provides a unified approach to studying algebraic cycles. Additionally, some applications of motivic cohomology will be outlined, highlighting its importance in modern algebraic geometry.

To date, satisfactory methods for studying the cycle class map remain elusive. One reason for the difficulty of the Hodge Conjecture lies in its bridging of two fundamental mathematical structures: algebra and topology. Through the framework of algebraic cobordism, it can be shown that the map from cycles to singular cohomology is realized via complex cobordism. We will introduce the algebraic cobordism theories defined by

Levine-Morel and Voevodsky, respectively, and provide an isomorphism between these two theories. Algebraic cobordism provides a rich structure that captures both geometric and topological information about algebraic varieties, making it a valuable tool in the study of the cycle class map.

Finally, we will delve into fundamental Hodge theories and the construction of the cycle class map. We will explain why the cycle class map can be decomposed through complex cobordism and how this process poses challenges for the algebraicity of cohomology classes. A specific example of a non-algebraic cohomology class will be provided to illustrate these challenges. Similar phenomena also occur in étale cohomology, where the interplay between algebraic and topological structures presents additional complexities. By exploring these topics in depth, we aim to shed light on the intricate connections between algebraic cycles, cohomology theories, and the broader landscape of algebraic geometry.

Keywords: Algebraic cycle; Algebraic geometry; Algebraic cobrdism

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CHAPTER 1 INTRODUCTION

1.1 Background

Pierre Deligne's work on the Weil conjectures[26-27] demonstrated the power of cohomological methods in algebraic geometry. A fundamental object is the cycle class map, which relates to many difficult and important problems in algebraic geometry. Due to Grothendieck's insights into the theory of motives, the role of algebraic cycles has become much more significant. We primarily highlight the failures of the integral Hodge conjecture. Within the development of motivic homotopy theory, we can provide a conceptual explanation for these failures. However, we cannot say much about the p-adic cycle class map to crystalline cohomology since the theory of motives remains far from complete.

The renowned Hodge conjecture posits that every rational Hodge class can be expressed in terms of algebraic cycle. One should note that this is the modified Hodge conjecture. The original Hodge conjecture concerns integral coefficients. British mathematician William Vallance Douglas Hodge conjectured that all integral Hodge classes are algebraic. However, as shown by Atiyah and Hirzebruch[5], this is not true. As an arithmetic analogue, John Tate conjectured that over a finitely generated field k, the cycle class map

$$cl : CH^k(X) \otimes \mathbb{Q}_\ell \to H^{2k}_{\text{ét}}(\overline{X}; \mathbb{Q}_\ell(k))^G$$

is surjective. Similar to the Hodge conjecture, the Tate conjecture also fails to hold when considering integral coefficients[21]. Both the Hodge and Tate conjectures are of significant importance due to their associations with Grothendieck's standard conjectures. Moreover, the integral Hodge conjecture plays a role in addressing the rationality problem of schemes.

The counterexample constructed by Atiyah and Hirzebruch employs the spectral sequence to demonstrate that the image of the integral cycle class map is annihilated by differentials within this spectral sequence

$$H^p(X; KU^q(*)) \Rightarrow KU^{p+q}(X).$$

Subsequently, Burt Totaro discovered that the integral cycle class map can be factored through complex cobordism, leading to a reformulation of the obstructions as follows:

$$CH^k(X) \to MU^{2k}(X^{\mathrm{an}}) \otimes_{MU^*} \mathbb{Z} \to H^{2k}(X^{\mathrm{an}}; \mathbb{Z}).$$

By utilizing the algebraic cobordism theory, this factorization becomes conceptually more transparent. It has been shown that Steenrod operations present an obstruction to cohomology classes being algebraic. Additionally, the Brown-Peterson spectrum allows for the description of an obstruction concerning torsion cohomology classes being algebraic, which is grounded in the framework of algebraic cobordism.

Voevodsky made significant advancements by integrating methods from algebraic topology into the field of algebraic geometry[49,60,76]. His work led to the creation of motivic homotopy theory for schemes, also known as $\mathbb{A}^1$-homotopy theory. Within this innovative framework, each cohomology theory can be represented by a $\mathbb{P}^1$-spectrum (refer to[56]beginning of section 5 and[75]Preface). This new approach enables us to view cohomology theories from a fresh angle, introducing numerous novel cohomology theories, including algebraic cobordism MGL and its various forms. Algebraic cobordism serves as the universal oriented cohomology theory on Sm/S. Voevodsky initially tackled the Bloch-Kato conjecture using algebraic Morava K-theories, which are modifications of MGL. Similar to practices in algebraic topology, it is possible to define the motivic Brown-Peterson spectrum MBP. Ultimately, through the use of MBP and Betti realisation, we are able to establish certain non-algebraic cohomology classes (done by Gereon Quick[63]).

Our overarching goal is to explore the obstructions faced by cohomology classes in being algebraic. Below, we present the main theorems.

The cycle class map is essentially unique:

Theorem 1.1 ([24]): By the universal property of MGL and the Hopkins-Morel-Hoyois theorem, the cycle class map to singular cohomology or étale ℓ-adic cohomology is uniquely determined.

We shall provide an explanation for why the cycle class map of X^{an} factors through complex cobordism:

Theorem 1.2 ([53]): The cycle class map $CH^*(X) \to H^{2*}(X^{an}, \mathbb{Z})$ is indeed the compositions:

$$CH^*(X) \xrightarrow{\sim} \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to MU^{2*}(X^{an}) \otimes_{\mathbb{L}} \mathbb{Z} \to H^{2*}(X^{an}, \mathbb{Z}).$$

The Betti realisation of Voevodsky's motives induces the cycle class map to integral singular cohomology. By combining this with the motivic Brown-Peterson spectrum

MBP, we are able to factor the mod p-cycle class map:

Theorem 1.3 ([63]): The mod p-cycle class map factors through BP:

$$CH^*(X) \to BP^{2*}(X^{an}) \otimes_{BP^*} \mathbb{Z}_{(p)} \to H^{2*}(X^{an}, \mathbb{Z}_{(p)}) \to H^{2*}(X^{an}, \mathbb{Z}/p).$$

Finally, we shall construct a non-algebraic cohomology class in $BP\langle 0\rangle^4(X^{\mathrm{an}}) = H^4(X^{\mathrm{an}}; \mathbb{Z}_{(p)})$. Specifically, the image of this non-algebraic class in $H^2(X^{\mathrm{an}}; \mathbb{Z}/p)$ is non-trivial. Consequently, we obtain a non-algebraic Hodge class.

1.2 Organisation of this thesis

A key focus of this thesis is the study of algebraic cycles. In Chapter 2, we examine foundational concepts related to algebraic cycles and address associated challenges. Chapter 3 delves into motivic cohomology, with a primary emphasis on the theory developed by Suslin and Voevodsky. We elucidate why the category of finite correspondences functions as an additive category and highlight its advantages over general cycles.

In Chapter 4, we revisit essential aspects of complex cobordism before introducing algebraic cobordism in both geometric and abstract contexts. The universal property of algebraic cobordism will be thoroughly explained. Additionally, we explore the connection between algebraic cobordism and motivic cohomology. This exploration leads us to conclude that the ring spectra mapping from the Chow group to the de Rham cohomology spectrum is unique, thereby establishing the uniqueness of the cycle class map.

Chapter 5 examines the construction of the cycle class map to singular cohomology. Its significance is underscored by its relevance to the Hodge conjecture, prompting an introduction to fundamental Hodge theory. Specific examples of non-algebraic classes will be provided.

1.3 Conventions of mathematics

We adopt the ZFC+U framework. For definitions and terminologies in algebraic geometry, we adhere to the conventions established by Alexandre Grothendieck. In addition to "ÉGA" and "ÉGA I 2nd ed.", we also refer to[33-34] for further details.

Let the base scheme S be a finite-dimensional Noetherian scheme throughout, and let Sm/S denote the category of smooth, separated S-schemes of finite type.

CHAPTER 2 BASIC ALGEBRAIC CYCLES

Algebraic cycles represent a classical and foundational concept in algebraic geometry, characterized by their intricate and multifaceted structures. In this chapter, we revisit key notions related to algebraic cycles, including their definition, equivalence relations, fundamental properties of Chow groups, and the theory of Weil cohomology.

2.1 Algebraic cycles

Definition 2.1: An **algebraic cycle** in a scheme X is a formal finite integral linear combination $Z = \sum n_i Z_i$ of integral closed subschemes Z_i of X. If all the Z_i have the same codimension d, we say that Z is a d-codimensional cycle. The group $\mathcal{Z}^d(X)$, which represents the free abelian group of d-codimensional cycles on X.

Remark 2.1: From the definiton, $\mathcal{Z}^*(X_{red}) \simeq \mathcal{Z}^*(X)$. A codimension 1 cycle is called a **Weil divisor**.

For each codimension d integral closed subscheme Z, let $[Z] \in \mathcal{Z}^d(X)$ denote the element corresponding to Z.

Definition 2.2: For any closed subscheme Z of X whose irreducible components Z_i are of codimension d in X. Let y_i be the generic point of Y_i. We define the associate cycle.

$$[Z] := \sum \mathrm{lg}_{\mathcal{O}_{X,y_i}}(\mathcal{O}_{Y,y_i})[Y_i] \in \mathcal{Z}^d(X),$$

where lg means the length function in commutative algebra. The integer $\mathrm{lg}_{\mathcal{O}_{X,y_i}}(\mathcal{O}_{Y,y_i})$ is also called the geometric multiplicity of y_i in Y.

A notion strongly related to the Weil divisor is the Cartier divisor. Let X be a scheme. For an open subset $U \subset X$, we let $\mathcal{O}_{X,\mathrm{reg}} \subset \Gamma(U, \mathcal{O}_X)$ be the subset of regular sections, i.e., those whose restrictions are non-zero divisors in the stalks $\mathcal{O}_{X,x}$ for all $x \in X$. The sheaf $\mathcal{K}_X$ of total rings of fractions of $\mathcal{O}_X$ is the localization of $\mathcal{O}_X$ at $\mathcal{O}_{X,\mathrm{reg}}$.

Definition 2.3: Let X be a scheme. We write Div_X for the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. A **Cartier divisor** D is an element of the group

$$\mathrm{Div}(X) := \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Definition 2.4: Let $D \in \mathrm{Div}(X)$ be a Cartier diviosr. Define the group homomorphism

$$\mathrm{cyc} : \mathrm{Div}(X) \to \mathcal{Z}^1(X), D \mapsto \sum_{Z \in \mathcal{Z}^1(X)} \mathrm{ord}_Z(D)[Z].$$

Example 2.1: Let $f \in \Gamma(\mathbb{A}_k^n, \mathcal{O}_{\mathbb{A}_k^n})$ be a non-unit element, and different from zero and $Y = V(f)$ the equi-codimensional codimension 1 closed subscheme. Then $Y = \mathrm{cyc}(f)$.

Example 2.2: Let $Y = V_+(g) \subset \mathbb{P}_k^n$ be an integrally closed subscheme given by an irreducible homogeneous polynomial g of degree d. Then $x_0^{-d} g$ is a rational function on $\mathbb{P}_k^n$, and

$$\mathrm{cyc}(x_0^{-d} g) = [Y] - d[V_+(x_0)].$$

For a field extension $k \subseteq J$, the base change map sends $\mathcal{Z}^d(X)$ to $\mathcal{Z}^d(X \times_k J)$. This morphism is characterized by mapping each cycle class $[Z] \in \mathcal{Z}^d(X)$ to the cycle associated with $Z \times_k J$. Specifically, given an element $[Z]$ in $\mathcal{Z}^d(X)$, the morphism transforms it into the cycle corresponding to the fiber product $\mathrm{Spec}(J) \times_k Z$ in $\mathcal{Z}^d(\mathrm{Spec}(J) \times_k X)$.

Proposition 2.1: Let $k \subset J$ be an extension of field and X be a smooth k-scheme.

（1）The homomorphism $\mathcal{Z}^d(X) \to \mathcal{Z}^d(Z \times_k J)$ is injective.

（2）If J is Galois over k. Then

$$\mathcal{Z}^d(X) \xrightarrow{\sim} \mathcal{Z}^d(X \times_k J)^{\mathrm{Gal}(J/k)}.$$

Proof: (1) Let Z_1 and Z_2 be different codimension d integral closed subschemes of X. Then the schemes $Z_i \times_k J$ are nonempty and do not intersect. Otherwise, $Z_1 \times_k J \cap Z_2 \times_k J$ descents to a subscheme of X.

(2) Consider a $\mathrm{Gal}(J/k)$-orbit of $\mathcal{Z}^d(X)$. Let Y be their union. Obviously, it is a closed subscheme of X_J. It descends to a subscheme of X. Since classes $[Y]$ form a basis of $\mathcal{Z}^d(X_J)^{\mathrm{Gal}(J/k)}$, it is an isomorphism. ■

For general morphisms of schemes, there is no established definition for the push-forward of cycles. However, this becomes possible for proper morphisms.

Definition 2.5 (Push forward): Let $f : X \to Y$ be a proper morphism and $Z \subset X$ be a cycle. The homomorphism

$$\mathcal{Z}_k(X) \xrightarrow{f_*} \mathcal{Z}_k(Y)$$

is defined as follows: if the dimension of $f(Z)$ equals the dimension of Z, then $f_*(Z) = [k(Z) : k(f(Z))] \cdot f(Z)$; otherwise, it is zero.

Lemma 2.1: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two proper morphisms. Then we have $(g \circ f)_* = g_* \circ f_*$ on cycles.

Proof: Easy. ■

Let X be s scheme and $X_1, \ldots, X_r$ be its irreducible components of X_{red}. If all the X_i has the same dimension N, we obtain the cycle $[X] = \sum_{i=1}^{r} [X_i]$.

Definition 2.6 (Pullback): Consider a flat morphism $f : X \to Y$ of relative dimension n. If Z is a closed integral subscheme within Y and has dimension k, then the inverse image $f^{-1}(Z)$ is equidimensional, possessing a dimension of $k+n$. The pullback $f^*(Z)$ is defined as the class $[f^{-1}(Z)]$, which belongs to the group $Z_{k+n}(X)$.

Theorem 2.1 (Serre): Let A denote a regular ring, and consider p and q as two prime ideals within A. Furthermore, let τ represent a minimal prime ideal of A among the sum of p and q. Then

$$\mathrm{ht}_A(p) + \mathrm{ht}_A(q) \geq \mathrm{ht}_A(\tau).$$

Proof: See[67]Chapter V, part B, Theorem 3. ■

The corresponding geometric form is:

Theorem 2.2 (Serre): For any two closed integral subschemes Z and Y of X, we have

$$\mathrm{codim}_X(Z) + \mathrm{codim}_X(Y) \geq \mathrm{codim}_X(Z \cap Y).$$

Definition 2.7: Let $A = \sum_i n_i [Y_i]$ and $B = \sum_j m_j [Z_j]$ represent two cycles. We define that A and B intersect properly if, for every pair (i, j), the following condition holds:

$$\mathrm{codim}_X(Y_i) + \mathrm{codim}_X(Z_j) = \mathrm{codim}_X(Y_i \cap Z_j).$$

We can define the intersection product $A \cdot B$ when they are intersect properly. Otherwise, we call it an excess intersection.

Let $Z \subset X$ represent a $(k+1)$-dimensional integral closed subscheme, and let r be an element of $K(W)^*$. The associated cycle is expressed as

$$\mathrm{div}(r) = \sum \mathrm{ord}_V(r)[V] \in Z_k(W) \subset Z_k(X),$$

Here, $\mathrm{ord}_V(r)$ denotes the order of vanishing or poles of r along V, and $[V]$ represents the corresponding cycle class in $Z_k(W)$, which is a subset of $Z_k(X)$.

Definition 2.8: A cycle $\alpha \in \mathcal{Z}_k(X)$ is called **rationally equivalent** to 0, written as $\alpha \sim 0$, if $\alpha = 0$ or if there are finitely many $(k+1)$-dimensional closed integral subschemes $W_1, \ldots, W_n \subset X$ together with $r_i \in k(W_i)^*$ such that

$$\alpha = \sum_i \mathrm{div}(r_i).$$

These cycles form a subgroup $\mathcal{Z}_k^{rat}(X) \subset \mathcal{Z}_k(X)$. Equivalently,

$$\mathcal{Z}_{rat}^i(X) = \mathrm{Im}\left(\bigoplus_{Y \in X^{(i-1)}} k(Y)^* \xrightarrow{\mathrm{div}} \bigoplus_{Z \in X^{(1)}} \mathbb{Z}\right).$$

The Chow group $CH^i(X) := \mathcal{Z}^i(X)/\mathcal{Z}_{rat}^i(X)$.

Theorem 2.3: We possess the subsequent conventional properties associated with Chow groups.

（1）For a smooth scheme X, the Chow group $CH(X) = \oplus CH^q(X)$ forms a graded commutative ring under the intersection product. Here, the intersection product maps pairs of cycles to a cycle of the appropriate codimension, preserving the graded structure of the Chow group.

（2）Let $f : X \to Y$ be a proper morphism, there is a induced homomorphism $f_* : CH_k(X) \to CH_k(Y)$.

（3）Consider a flat morphism f from X to Y having a relative dimension of n. Such a morphism naturally leads to an induced homomorphism mapping from the Chow group $CH_k(Y)$ to the Chow group $CH_{k+n}(X)$.

（4）Let $f : X \to X$ be a flat morphism of degree d. Applying the pullback f^* followed by the pushforward f_* results in multiplying elements of $CH_i(Y)$ by the degree d of the morphism.

（5）Consider a closed embedding given by $i : Y \to X$, and let $j : U = X \setminus Y \to X$ be the inclusion map. Then, the sequence

$$CH_q(Y) \to CH_q(X) \to CH_q(U) \to 0$$

is exact.

Example 2.3: If X is a noetherian local factorial scheme, then $CH^1(X) = \mathrm{Pic}(X)$.

（1）For every field k, we have $CH^1(\mathbb{A}_k^n) = \mathrm{Pic}(\mathbb{A}_k^n) = 0$.

（2）For every field k, we have $CH^1(\mathbb{P}_k^n) = \mathrm{Pic}(\mathbb{P}_k^n) = \mathbb{Z}$.

Example 2.4: If F is an extension field of k, and i is an integer greater than 2, the natural homomorphism from $CH^i(X)$ to $CH^i(X_F)$ does not have to be injective, which differs from the case when $i = 1$. While the Chow group of projective space $\mathbb{P}^n$ is torsion-free, the Chow group associated with a Severi-Brauer variety can potentially contain torsion elements.

Example 2.5: Although $\mathcal{Z}^d(X) \xrightarrow{\sim} \mathcal{Z}^d(X_L)^{\mathrm{Gal}(L/k)}$, it is not true for Chow groups that $CH^d(X) \xrightarrow{\sim} CH^d(X_L)^{\mathrm{Gal}(L/k)}$. Let k_s be a separable closure of k. Recalling the

Hochschild-Serre spectral sequence:

$$H^p_{Gal}(Gal(k_s/k); H^q_{\text{ét}}(X \times_k k_s, \mathfrak{F})) \Rightarrow H^{p+q}_{\text{ét}}(X; \mathfrak{F}).$$

Taking $\mathfrak{F} = \mathbb{G}_m$, we have the following exact sequence by standard homological algebra:

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X \times_k k_s)^{Gal(k_s/k)} \to \mathrm{Br}(k) \to \ker(\mathrm{Br}(X) \to \mathrm{Br}(X \times_k k_s)).$$

So, if $\mathrm{Br}(k) \neq 0$, $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X \times_k k_s)^{Gal(k_s/k)}$ are not isomorphic. For example, the Brauer group of $\mathbb{R}$ is $\mathbb{Z}/2 \neq 0$.

We state that two cycles in $\mathcal{Z}^i(X)$ are algebraically equivalent if they are connected by a connected curve.

Clearly, $\mathcal{Z}^i_{rat}(X) \subset \mathcal{Z}^i_{alg}(X)$. They are generally distinct: let X be an elliptic curve and a, b are two distinct points in X.

Theorem 2.4: The group $NS(X) := CH^1(X)/CH^1_{alg}(X) = \mathcal{Z}^1(X)/\mathcal{Z}^1_{alg}(X)$ is the so called Néron-Severi group. The group $NS(X)$ is a finite type $\mathbb{Z}$-module.

Proof: If the base field is $\mathbb{C}$, it is easy. Indeed, the exponential sequence

$$0 \to \mathbb{Z}(1) := 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

gives a lone exact sequence

$$H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \xrightarrow{j} H^2(X, \mathcal{O}_X).$$

Then $NS(X) = \mathrm{Pic}(X)/\ker(c_1) = \ker(j)$ is finitely generated since $H^2(X, \mathbb{Z}(1))$ is.

The general case is deep. See[11]Exp. XIII, Théorème 5.1. ■

2.2 Homological equivalence

The original motivation for Weil cohomology was the Weil conjecture. Let F be a characteristic zero field. One should note that there are many different definitions of Weil cohomology theory in literatures.

For a scheme X within this category, we use d_X to indicate its dimension.

Definition 2.9: A Weil cohomology can be described as a functor

$$H : \mathrm{SmProj}(k)^{op} \to \mathrm{GrVect}_F$$

that satisfy the following conditions:

（1）Each $H^i(X)$ is finite dimensional and concentrate on $0 \leq i \leq 2\dim X$. The F-vector space $H^2(\mathbb{P}^1)$ is one-dimensional and $H^1(\mathbb{P}^1_k) = 0$. Its dual is denoted by $F(1)$ or simply (1) and we call it the **Tate twist**. Given $X \in \mathrm{SmProj}(k)^{op}$, we write

$H^*(X) = \oplus_{r\geq 0} H^{2r}(X)(r)$.

（2）(Trace) There is a functorial trace isomorphism

$$\mathrm{Tr} : H^{2d_X}(X)(d_X) \xrightarrow{\sim} F.$$

It is compatible with product:

$$\begin{tikzcd} H^{2d_X}(X)(d_X) \otimes_F H^{2d}(Y)(d_Y) \arrow[r, "\mathrm{Tr}_X \otimes \mathrm{Tr}_Y"] \arrow[d, "\simeq"] & F \\ H^{2d_X+2d_Y}(X \times_k Y)(d_X + d_Y) \arrow[ur, "\mathrm{Tr}_{XY}"'] & \end{tikzcd}$$

（3）(Künneth formula) For each X and Y, the Künneth formula holds:

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

is an isomorphism.

（4）(cycle class map) For any variety X, there exists a group homomorphism

$$\gamma_X : CH^i(X) \to H^{2i}(X)(i)$$

that fulfills the following properties: 1. $f^* \circ \gamma_Y = \gamma_X \circ f^*$; 2. $\gamma_X(\alpha \cap \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$; 3. $\mathrm{Tr} \circ \gamma_X = \deg$.

Here, f^* denotes the pullback map induced by a morphism f, $\cap$ represents the intersection product in Chow groups.

（5）(weak Lefschetz theorem) Let $h : W \to X$ be a smooth hyperplane. For dimensions i up to $\dim X - 2$, the induced map h^* between the cohomology groups $H^i(X)$ and $H^i(W)$ is bijective. When i equals $\dim X - 1$, this map is injective but not necessarily surjective.

（6）(hard Lefschetz theorem) Let W be a smooth hyperplane section. Introduce the Lefschetz operator L acting on cohomology groups such that for any element $x \in H^i(X)$, it transforms x into $x \cup \gamma_X(W)$ in $H^{i+2}(X)$. It follows that applying L iteratively $\dim X - i$ times results in a mapping from $H^i(X)$ to $H^{2\dim X - i}(X)$, which is an isomorphic correspondence.

Example 2.6: (1) For arbitrary field k, we fix an algebraically closure $\bar{k}$ of k. The étale ℓ-adic cohomology $H^*_{\text{ét}}(X_{\bar{k}}; \mathbb{Q}_\ell)$ is a Weil cohomology. The hard Lefschetz property of ℓ-adic cohomology is highly non-trivial.

(2) If the field k is perfect, one considers the crystalline cohomology $H^*_{cris}(X/W(k))$. In this context, $K = W(k)[1/p]$ comes equipped with the Frobenius

automorphism ϕ. The space defined by

$$H^i_{cris}(X) := H^i_{cris}(X/W(k)) \otimes_{W(k)} K$$

admits a functorial, ϕ-linear, and bijective endomorphism known as the crystalline Frobenius. However, it should be noted that the validity of the hard Lefschetz theorem is contingent upon the Weil conjecture.

Proposition 2.2: Fix a Weil cohomology H, the cycle class map $\gamma : CH^*(X) \to H^{2*}(X)(*)$ is unique.

Proof: From the motivic viewpoint, it is almost automatical. ■

Definition 2.10: A cycle Z is said to be homologically trivial if $\gamma_X(Z) = 0$.

Obviously:

Proposition 2.3: Every rational cycle equivalent to zero is also homologically equivalent to zero.

2.3 Numerical equivalence

Definition 2.11: Let $Z = \sum_i a_i P_i$ be a zero dimensional cycle. The degree of Z is $\deg(Z) := \sum a_i [k(P_i) : k]$.

Definition 2.12: For $Z \in \mathcal{Z}^i(X)$, we say Z is numerially equivalent to zero if for every $W \in \mathcal{Z}^{d-i}(X)$ such that $Z \cap W$ is proper we have $\deg(Z \cap W) = 0$.

We have a chain of inclusions:

$$\mathcal{Z}^i_{rat}(X) \subset \mathcal{Z}^i_{alg}(X) \subset \mathcal{Z}^i_{hom}(X) \subset \mathcal{Z}^i_{num}(X).$$

The following chain of inclusions of subgroups of the Chow groups is a consequence:

$$CH^i_{num}(X) \subset CH^i_{hom}(X) \subset CH^i_{alg}(X) \subset CH^i(X).$$

The dimension of a rational Chow group could be infinity. Here is an example:

Example 2.7 (Clemmaens): The upshot is that Griffiths group $CH^i_{hom}(X)/CH^i_{alg}(X)$ can contain an infinite cyclic subgroup. Let

$$Y = V_+(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + (\sum_{j=0}^{4} a_j z_j)^5) \subset \mathbb{P}^4_{\mathbb{C}}$$

for general $a_0, \dots, a_4 \subset \mathbb{C}$. Then $CH^2_{hom}(Y)/CH^2_{alg}(Y)$ constains an infinite cyclic subgroup. Clemens demonstrated that, when tensoring over $\mathbb{Q}$, the Griffiths group of a generic quintic threefold in $\mathbb{P}^4_{\mathbb{C}}$ possesses an infinite-dimensional structure.

But the dimension of CH^i_{num} is always finite.

CHAPTER 3 MOTIVIC COHOMOLOGY

The theory of mixed motives as imagined by A. Grothendieck (some would say invented or discovered) must offer the universal framework in which to deal with the cohomology of schemes. According to this philosophy, the other cohomological theories are only so many incarnations, so many realisations, of the notion of motives.

3.1 Finite correspondence

We provide an in-depth exposition of the theory of finite correspondences developed by Andrei Suslin and Vladimir Voevodsky. This theory forms the cornerstone of Voevodsky's groundbreaking construction of mixed motives. The framework introduced by Suslin and Voevodsky employs the category of finite correspondences, which offers a refined and precise representation of general cycles.

Finite correspondence theory plays a pivotal role in algebraic geometry, providing a novel perspective on the relationships between algebraic varieties. By introducing the concept of finite correspondences, researchers can more accurately describe mappings between algebraic varieties, thereby facilitating the resolution of complex problems. The application of the category of finite correspondences has made previously intractable issues more approachable.

Moreover, finite correspondence theory is closely intertwined with motivic homology theory. Voevodsky's pioneering work utilized the category of finite correspondences to construct motivic homology theory, significantly advancing the fields of algebraic K-theory, algebraic cycle theory, and motivic cohomology. Recent research in this area has yielded substantial progress, as detailed in relevant literature (e.g.,[20,70]).

In summary, the theory of finite correspondences developed by Suslin and Voevodsky is not only a vital component of modern algebraic geometry but also provides powerful tools for related fields. Through an in-depth exploration of this theory, we gain a deeper understanding of the intrinsic connections between algebraic varieties and their broader implications within the mathematical framework.

Fix a regular scheme S. To simplify the formulas to come, the following conventions will be adopted: For schemes X and Y in Sm/S, we set $X \times_S Y := XY$. For schemes X, Y, and Z in Sm/S, we denote the canonical projection morphism $p^{Y}_{XYZ} : XYZ \to Y$.

Definition 3.1: Let X and Y be schemes within the category Sm/S. A **finite correspondence** from X to Y can be described as a cycle $\alpha = \sum_i n_i x_i$ in the fiber product $X \times_S Y$, where for each index i such that $n_i \neq 0$, the component x_i projects to a generic point of Y via the natural projection map. The groupe of finite S-correspondences from X to Y is denoted by $Cor_S(X, Y)$

A correspondence will be denoted by $\alpha : X \twoheadrightarrow Y$.

Example 3.1: Now, we consider a morphism $f : X \to Y$ in Sm/S.

（1）Given that X/S is separated, the graph Γ_f of f is a closed subscheme in $X \times_S Y$. Furthermore, the composition morphism from Γ_f to $X \times_S Y$ and then to X is an isomorphism. Consequently, the cycle associated with Γ_f, denoted as $\langle \Gamma_f \rangle_{XY}$, constitutes a finite correspondence.

（2）Assume that f is a finite pseudo-dominant morphism. The composition $\Gamma_f \to X \times_S Y \to Y$ is isomorphic to f, implying that it retains the properties of being finite and pseudo-dominant. Consequently, the cycle $\langle \Gamma_f \rangle_{X/Y}$ is a finite correspondence. This correspondence will be denoted as ${}^t f$ and referred to as the transpose of f.

Now, let's examine two finite correspondences:

$$\alpha : X \twoheadrightarrow Y, \quad \beta : Y \twoheadrightarrow Z.$$

We want to define the product of composition of β with α by the following formula:

$$\beta \circ \alpha := p^{XZ}_{XYZ_*}(p^{YZ_*}_{XYZ}(\beta) \cdot p^{XY_*}_{XYZ}(\alpha)).$$

Two problems arise in this formula: (1) Is the intersection on the right side proper? (2) Does the support of the cycle XZ, as obtained in the latter member, exhibit finiteness and pseudo-dominance over X? Fortunately, the answes are yes. These are key properties of finite correspondence. The composition is defined without modulo equivalence relation unlike ordinary cycles. That is why we need standard conjectures in Grothendieck's construction of motives.

Remark 3.1: Unfortunately, certain anticipated properties of Voevodsky's motives lead us back to the standard conjectures[9].

Proposition 3.1: Here are some fundamental properties:

（1）Consider three finite correspondences:

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} T \ .$$

Then $c \circ (b \circ a) = (c \circ b) \circ a$.

(2) Consider a morphism $f : X \to Y$ in Sm/S and $\langle \Gamma_f \rangle$ the finite correspondece from X to Y associated with Γ_f. Then, for every finite correspondence $b : Y \twoheadrightarrow X$, the pullback of b along $f \times_S Z$ is equal to

$$b \circ \langle \Gamma_f \rangle = (f \times_S Z)^*(b).$$

(3) For any finite correspondence $a : X \twoheadrightarrow Y$ and any morphism $g : Y \to Z$ in Sm/S, we have

$$\langle \Gamma_g \rangle_{YZ} \circ a = (g \times_k X)_*(a).$$

Proof: (1) Denote $a' = p^{XY_*}_{XYZT}(a)$, $b' = p^{YZ_*}_{XYZT}(b)$ and $c' = p^{ZT_*}_{XYZT}(c)$. Then, using the formula for change of basis and the formula for projection, along with the compatibility of the flat basis change with the intersection product, we obtain the equality:

$$c \circ (b \circ a) = p^{XT}_{XYZT_*}(c' \cdot (b' \cdot a'))$$

and

$$(c \circ b) \circ a = p^{XT}_{XYZT_*}((c' \cdot b') \cdot a').$$

The associativity of the intersection product concludes.

(2) Results from the definition of the pullback by a morphism in Sm/S and the fact that

$$p^{XY_*}_{XYZ}(\langle \Gamma_f \rangle) = \langle \Gamma_{f \times_S Z} \rangle$$

where $\Gamma_{f \times_S Z}$ is the graph of the Z-morphism $f \times_S Z$.

(3) We can assume that $a = \langle U \rangle_{XY}$ for an integral scheme U. Similarly, we can assume by additivity that Y is integral.

It is a question to prove an equality between cycles of XZ. We can therefore to prove it locally at each point of XZ. We thus come back to the case where S, X, Y, Z are affine schemes with respective rings A, B, C and D.

If I (resp. J) is the defining ideal of U in $XY = \mathrm{Spec}(BC)$ (resp. $YZ = \mathrm{Spec}(CD)$), we set $M = BC/I$ (resp. $N = CD/J$). We obtain $a = \mathcal{Z}^m_{XY}(M)$ and $\langle \Gamma_g \rangle = \mathcal{Z}^n_{YZ}(N)$. Then

$$\langle \Gamma_g \rangle_{YZ} \circ a = p^{XZ}_{XYZ_*}(\mathcal{Z}^{n+m}(N \otimes^L_B M)).$$

Given that the map $\Gamma_g \to Y$ is an isomorphism and $M|_B$ is a rank-1 free B-module, it follows that $M|_B$ is flat. Thus $N \otimes^L_B M = N \otimes_B M$, which is translated to the formula

$$\langle \Gamma_g \rangle_{YZ} \circ a = p^{XZ}_{XYZ_*}(\langle \Gamma_g \times_Y U \rangle_{XYZ}).$$

Now, $\Gamma_g \times_Y U$ is supported in $\Gamma_g \times_S X$. Through the isomorphism $\epsilon : \Gamma_g \to Y$, the restriction of the morphism p^{XZ}_{XYZ} to $\Gamma_g \times_S X$ correspondent to the morphism $g \times_S X$. The formula to prove is therefore derived from the formula above given the isomorphism ϵ.

■

Points (2) and (3) of this proposition show that we can define the identity correspondence of a scheme X of Sm/S as the cycle $\langle \Delta_{X/S} \rangle$, since $\Delta_{X/S}$ is the graph of the morphism of scheme 1_X.

Definition 3.2: The category **finite correspondences** $Cor(S)$ associated with Sm/S has objects that are smooth S-schemes of finite type. The morphisms in this category consist of finite S-correspondences between these schemes.

The naturalness of the change of base by a morphism Sm/S shows that the association $f \mapsto \langle \Gamma_f \rangle$ defines a functor

$$\gamma : Sm/S \to Cor(S).$$

The category $Cor(S)$ admits finite sums, as does the category Sm/S, and the functor γ commutes with finite sums.

Proposition 3.2: The category $Cor(S)$ is additive.

Proof: The finite correspondences from X to Y form a group $Cor(X, Y)$ and the composition is additive by definition. The category $Cor(S)$ admits an initial object, the empty scheme $\emptyset$. It is also the final object because there is a unique finite correspondence $X \twoheadrightarrow \emptyset$ the cycle associated with this closed subscheme $X \times_k \emptyset = \emptyset$.

Let us consider two schemes X and Y in Sm/S and let $Z = X \sqcup Y$ be the disjoint union. We consider the standard open and closed immersions given by $i : X \to Z$ and $j : Y \to Z$. Additionally, we can establish finite correspondences as follows:

$$p : Z \twoheadrightarrow X, \quad q : Z \twoheadrightarrow Y$$

These are constructed by setting $p = \langle \Delta_{X/S} \rangle_{ZX}$ and $q = \langle \Delta_{Y/S} \rangle_{ZY}$. The following relationships are easily to verify:

$$pi = 1_X, \quad qj = 1_Y, \quad qi = 0, \quad pj = 0, \quad ip + jq = 1_Z.$$

These show that Z is both the product and the sum of X and Y in the category $Cor(S)$. ■

So, we can enrich Sm/S to an additive category. One spirit of the theory of motives is making Sm/S into an Abelian category.

We would like to make the product in Sm/S a monoïdale structure in $Cor(S)$. On objects, this tensor product will correspond to the product of the S-schemes. Consider finite correspondences:

$$a : X \twoheadrightarrow Y, \quad b : Z \twoheadrightarrow W.$$

The tensor product of a and b must be a finite correspondence of the form $XZ \to YW$. It is defined by the formula:

$$a \otimes_S^{tr} b = p_{XYZW}^{XY_*}(a) \cdot p_{XYZW}^{ZW_*}(b).$$

As in the case of the composition, we can justify that this formula does indeed make sense.

Based on the commutative (resp. associative) property of the intersection product of cycles, the tensor product exhibits symmetry (resp. associativity). It is also bifunctorial:

Proposition 3.3: Let us consider the finite S-correspondences:

$$a : X \to Y, \quad b : Y \to Z, \quad c : X' \to Y', \quad d : Y' \to Z'.$$

Then

$$(b \circ a) \otimes_S^{tr} (d \circ c) = (b \otimes_S^{tr} d) \circ (a \otimes_S^{tr} b).$$

The following proposition is therefore obtained:

Proposition 3.4: The category $Cor(S)$ with the tensor product

$$[X] \otimes_S^{tr} [Y] = [X \times_S Y]$$

for schemes X and Y in Sm/S and the product of the correspondences is symmetric monoïdale. The unit of the monoïdale structure is the object $[S]$. The graph functor $\gamma : Sm/S \to Cor(S)$ is also monoïdal, where Sm/S is provided with its monoïdale structure defined by the product of the S-schemes.

Proof: All that remains is to prove the assertion concerning the functor γ. Given the morphisms $f : X \to Y$ and $g : Z \to W$ in Sm/S, we must show

$$[\Gamma_f] \otimes_S^{tr} [\Gamma_g] = [\Gamma_{f \times_S g}].$$

This is easy by the Tor formula and the fact that the projection morphism $\Gamma_f \to S$ is smooth, therefore flat. One may see[20]section 9.2 for details. ■

Proposition 3.5: Let $p : X \to S$ represent a finite étale morphism, and let $\delta : X \to X \times_S X$ denote the corresponding diagonal morphism.

Then, $[X]$ is strongly self-dual. Specifically, when considering schemes X, Y, and Z within the category Sm/S, this results in a canonical bijection between $Cor_S(X \times_S Y, Z)$ and $Cor_S(Y, X \times_S Z)$.

Suslin and Voevodsky gave an elementary explaination of finite correspondence. We set a perfect base field k.

If Y is smooth quasi-projective over $\mathrm{Spec}(k)$, we can consider for all $n \geq 0$ the k-scheme $\mathrm{Sym}^n Y := Y^n/S_n$ where the quotient by the symmetric group S_n is defined in[36]V.2. The k-scheme $\mathrm{Sym}^\infty Y := \sqcup_{n\geq 0} \mathrm{Sym}^n Y$ is a k-scheme in commutative monoïdes.

Theorem 3.1: For any $X \in Sm/k$, the set $\mathrm{Hom}_k(X, \mathrm{Sym}^\infty Y)$ is a monoïde of which we can denote $\mathrm{Hom}_k(X, \mathrm{Sym}^\infty Y)^+$ the group completion. If we denote p as the characteristic exponent of k. Then there is an isomorphism

$$\mathrm{Hom}_{Cor(k)}(X, Y) \otimes \mathbb{Z}[1/p] \simeq \mathrm{Hom}_k(X, \mathrm{Sym}^\infty Y)^+ \otimes \mathbb{Z}[1/p].$$

Proof: This is[69]Theorem 6.8. ■

3.2 Completely decomposed topology: Nisnevich topology

The Zariski topology exhibits undesirable behavior with respect to transfers, which poses challenges in certain algebraic contexts. Conversely, it possesses excellent properties concerning cohomological dimension and compactness, unlike the étale topology, which often requires more intricate considerations in these areas. The Nisnevich topology serves as an intermediate framework that combines the favorable attributes of both the Zariski and étale topologies. By integrating the simplicity of the Zariski topology with the refined descent properties of the étale topology, the Nisnevich topology addresses many of the limitations inherent in each individual topology.

Consequently, the Nisnevich topology plays a pivotal role in V. Voevodsky's construction of mixed motives, a foundational concept in modern algebraic geometry and motivic cohomology. This topology provides the necessary tools to bridge gaps between classical algebraic geometry and more abstract constructions, enabling significant advancements in understanding algebraic cycles and their associated cohomology theories.

The development of the Nisnevich topology arose from the limitations of étale descent in algebraic K-theory. While étale descent is powerful in many contexts, it fails to adequately address certain issues in K-theory, particularly those involving transfers and local-global principles. To overcome these shortcomings, the Nisnevich topology was introduced as a refinement that preserves desirable properties while addressing specific deficiencies.

Voevodsky further advanced this field by introducing the concept of a completely decomposed structure, which axiomatizes the properties of the Zariski, Nisnevich, and cdh topologies. This framework provides a unified approach to studying various topologies and their interactions, facilitating deeper insights into algebraic varieties and their cohomological properties. By formalizing the relationships between these topologies, Voevodsky laid the groundwork for groundbreaking developments in motivic homotopy theory and related fields.

In summary, the Nisnevich topology represents a crucial advancement in algebraic geometry, combining the strengths of the Zariski and étale topologies while addressing their respective limitations. Its applications extend beyond mixed motives, influencing numerous areas of modern mathematics, including algebraic K-theory, motivic cohomology, and homotopy theory.

Definition 3.3: Consider a Noetherian scheme denoted by X. We define X_{Nis} as the category comprising étale, separated, and finite type schemes over X. The class of morphisms in $X_{\text{ét}}$, represented by $Cov(X)$, consists of families $\mathcal{U} = (U_i \xrightarrow{f_i} X)_{i\in I}$ where each f_i is an étale morphism of finite type. Specifically, for any point $x \in X$, there exists an index $i \in I$ along with a point $u \in U_i$ such that $x = f_i(u)$, and the induced map between residue fields $\kappa(x) \to \kappa(u)$ is an isomorphism.

Proposition 3.6: For any Noetherian scheme X, the data of $(Cov(X))_{X\in Sm/S}$ constitutes a pre-topology on Sm/S. The topology generated by this pre-topology is called the **Nisnevich topology** on Sm/S, the resulting site Sm/S_{Nis} and called the big Nisnevich site of S. Moreover, if X is a Noetherian scheme, $(Cov(Y))_{Y\in X_{Nis}}$ constitutes a pretopology on X_{Nis}. The corresponding site is denoted by X_{Nis}, and it is called the small Nisnevich site of X.

By definition, Nisnevich topology standards between the Zariski topology and étale topology. In particular, it is less fine than the canonical topology, which means that the representable presheaves are Nisnevich sheaves.

Definition 3.4: Let X be an S-scheme and x a point of X. A neighborhood of x in the Nisnevich topology is an étale X-scheme U and a point u of U projecting onto x and such that the induced morphism between the residual fields $\kappa(u)$ and $\kappa(x)$ is an isomorphism.

We have a notion of morphisms of Nisnevich neighborhoods and we will denote V_x^h the category of Nisnevich neighborhoods of x. The stalk of a Nisnevich sheaf F on S at the point x is

$$F_x = \operatorname*{colim}_{U\in(V_x^h)^{op}} F(U)$$

and the opposite category of V_x^h is filtered.

We can see that for any Nisnevich sheaf F on Sch/S and any point x of an S-scheme X, we have a functorial isomorphism $F_x \simeq F(\mathrm{Spec}(\mathcal{O}^h_{X,x})) = F(X^h_x)$. The proof is almost the same as étale topology.

Consider a field denoted by k. The subsequent proposition establishes that, for category of étale k-schemes of finite type, the Nisnevich topology aligns with the Zariski topology. It is important to note that this equivalence does not extend to the étale topology.

Proposition 3.7: Let $F \in PSh(\mathrm{Spec}\, k_{Nis})$. The following are equivalent:

(1) $F \in Sh(\mathrm{Spec}\, k_{Nis})$.

(2) $F(\emptyset)$ is a singleton and the obvious map $F(X \sqcup Y) \to F(X) \times F(Y)$ is bijective for all objects X and Y in $\mathrm{Spec}\, k_{Nis}$;

(3) For any $X \in \mathrm{Spec}\, k_{Nis}$, the canonical morphism $F(X) \to \prod_{Y \in \pi_0(X)} F(Y)$ is bijective.

Proof: Conditions (2) and (3) are clearly equivalent, and (1) obviously implies (2). To show that (3) implies (1), it is sufficient to show that for any $X \in \mathrm{Spec}\, k_{Nis}$, any covering X for the topology of $\mathrm{Spec}\, k_{Nis}$ contains the covering associated with the inclusion of the connected components of X in X, which results immediately from the decomposition of a k-algebra into the product of separable finite extensions of k and from the definition of the Nisnevich topology. ■

Definition 3.5: Let Y be a Noetherian scheme, we call the **elementary Nisnevich covering** of Y the data of an open immersion $U \xrightarrow{j} Y$ and an étale morphism $V \xrightarrow{p} Y$, with V quasi-compact, such that, if we denote $F = Y - U$, the schemes morphism $V \times_Y F_{red} \to F_{red}$ by a change of basis is an isomorphism.

It can be seen that the notion of the elementary Nisnevich covering of a Noetherian scheme is stable by change of base.

Theorem 3.2: Let X be a Noetherian scheme (resp. S a Noetherian scheme). Let F be a presheaf on X_{Nis} (resp. on Sm/S_{Nis}), the following two properties are equivalent:

(1) F is a sheaf on X_{Nis} (resp. on Sm/S_{Nis});

(2) $F(\emptyset)$ is a singleton and for any elementary Nisnevich covering O, $F(O)$ is Cartesian.

Proof: The case concerning the large Nisnevich sites is formally the result of the case of the small Nisnevich sites, so we will consider only the latter.

Easy direction: (1) to (2). According to Yoneda lemma, it is sufficient to show that the diagram of elementary Nisnevich covering is coCartesian in $Sh(X_{Nis})$, for any elementary Nisnevich covering $(U \xrightarrow{j} Y; V \xrightarrow{p} Y)$ of $Y \in X_{Nis}$. This assertion is verified after passing through the stalk at $\operatorname{Spec} K \xrightarrow{x} X$ of X a values in a field. For all $V \in X_{Nis}$, the stalk of the sheaf V on X_{Nis} is identified with the set $\operatorname{Hom}_X(\operatorname{Spec} K, V) = \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K, \operatorname{Spec} K \times_X V)$, i.e. the set of sections of the $\operatorname{Spec} K$-scheme $V \times_X \operatorname{Spec} K$. We deduce that we can assume that $x : \operatorname{Spec} K \to X$ is an isomorphism. Moreover, it is clear that Y can be assumed to be connected. In this situation, Y is the spectrum of a field, so two cases are possible: either $U = \emptyset$ and in this case $V \to Y$ is an isomorphism, or $U = Y$. But, in these two cases, the desired diagram is tautologically coCartesian, hence the result. Conversely, we refer to[60]. ■

One of the major differences between Nisnevich topology and étale topology is that the latter escapes the formalism of cd-structures (see[79]) and the results derived from them. The essential property that these topologies share is that they have local Henselian rings as local rings.

3.3 Voevodsky's mixed motives and motivic cohomology

Suslin-Voevodsky's motivic cohomology is independent of motives. But to define Betti realisation latter, we begin with Voevodsky's motives.

We let $K^b(Cor(S))$ be the homotopy category of bounded cochain complex with values in $Cor(S)$ since $Cor(S)$ is an additive category. We represent V_S as the smallest thick triangulated subcategory within $K^b(Cor(S))$ that includes complexes of the following structure:

（1）For any smooth S-scheme X, let $p : \mathbb{A}^1_X \to X$ denotes the canonical projection of the affine line on X,

$$\dots 0 \to [\mathbb{A}^1_X] \xrightarrow{p} [X] \to 0 \dots$$

（2）For any distinguished Nisnevich square:

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "j"'] & X \end{tikzcd},$$

the complex

$$\dots 0 \to [V] \xrightarrow{\binom{g}{-k}} [Y] \oplus [Z] \xrightarrow{(j,f)} [X] \to 0 \dots .$$

Definition 3.6: We introduce $DM_{gm}^{eff}(S) := K^b(Cor(S))/V_S$. It is called the **derived category of effective geometric motives**.

We denote $M_S(X)$ the object of $DM_{gm}^{eff}(S)$ represented by the complex equal to X concentrated in degree 0, and we will call it the motive of X. We have thus defined a functor

$$M_S : Cor(S) \to DM_{gm}^{eff}(S).$$

Definition 3.7: Let $f : Y \to X$ be a morphism within the category Sm/S. We define the relative motive associated with f as the object in $DM_{gm}^{eff}(S)$ that is represented by the cone complex of the map from $[Y]$ to $[X]$. It is denoted by $M_S(p)$.

Definition 3.8: Consider a morphism $p : X \to S$ in $Cor(S)$. The preceding definition will be considered in the following cases:

（1）If $s : S \to X$ represents an S-point of X, we define $\tilde{M}_S(X, s)$ to be equal to $M_S(s)$. Additionally, we refer to (X, s) as a pointed S-scheme, and $\tilde{M}_S(X, s)$ is termed the reduced motive corresponding to (X, s).

（2）Let Z be a closed subscheme of X. Define $U = X \setminus Z$, endowed with its natural open subscheme structure within X. The canonical inclusion map from U to X is denoted by $j : U \to X$. We set $M_{S,Z}(X) := M_S(j)$ and we call it the motive of X with support in Z.

Definition 3.9: A presheaf with transfer is an Abelian presheaf F on $Cor(S)$. We say that F is a sheaf with transfer if $F \circ \gamma$ is a Nisnevich sheaf.

The category of sheaves with transfer is denoted by $Sh^{tr}(S)$, where the morphisms consist of natural transformations between these sheaves.

Proposition 3.8: Let X be an S-scheme in Sm/X. Then the functor $Y \mapsto Cor_S(Y, X)$ is an étale sheaf with transfer.

For a scheme X in Sm/S, we denote $\mathbb{Z}_S^{tr}(X)$ the sheaf with transfers $Y \mapsto Cor_S(Y, X)$.

It should be remembered that for a regular scheme S, the category Cor_S is symmetric monoïdale.

Proposition 3.9: There exists a unique closed symmetric monoïdale structure on $Sh^{tr}(S)$ for which the functor

$$\mathbb{Z}_S^{tr} : Cor(S) \to Sh^{tr}(S)$$

is symmetric monoïdale.

Proof: The tensor product of a closed monoïdale category is right exact: the assertion of uniqueness therefore results from the fact that the essential image of $\mathbb{Z}_S^{tr}$ is generated in the Abelian category $Sh^{tr}(S)$.

The structure of the generators in $Sh^{tr}(S)$ enables us to express any sheaf with transfers uniquely in the form

$$F = \operatorname*{colim}_{X/F} \mathbb{Z}_S^{tr}(X)$$

where X/F travels the category of arrows $\mathbb{Z}_S^{tr}(X) \to F$ in $Sh^{tr}(S)$.

For sheaves with transfers F and G, we set

$$F \otimes_S^{tr} G := \operatorname*{colim}_{X/F, Y/G} \mathbb{Z}_S^{tr}(X \times_S Y).$$

This definition is functionalial in F and G. It is immediately clear that this defines a symmetric tensor product on $Sh^{tr}(S)$. Iternal hom $\underline{\mathrm{Hom}}_S(F, G)$ is defined as the sheaf with transfers

$$X \mapsto \mathrm{Hom}_{Sh^{tr}(S)}(\mathbb{Z}_S^{tr}(X) \otimes_S^{tr} F, G).$$

According to Yoneda's lemma, for any scheme X in Sm/S,

$$\mathrm{Hom}_{Sh^{tr}(S)}(\mathbb{Z}_S^{tr}(X), \underline{\mathrm{Hom}}_{Sh^{tr}(S)}(F, G)) = \mathrm{Hom}_{Sh^{tr}(S)}(\mathbb{Z}_S^{tr}(X) \otimes_S^{tr} F, G).$$

■

Example 3.2: Let H^* represent Betti cohomology, de Rham cohomology in characteristic 0, or ℓ-adic cohomology. The presheaf defined by $X \mapsto H^n(X)$ forms a presheaf with transfers on S.

Example 3.3: An important non-example of presheaf with transfers is algebraic K-theory,[58]EXAMPLE2.7.

Example 3.4: V. Voevodsky introduced a new Grothendieck topology on the category of S-schemas in his thesis. The main virtue of this h-topology is that it "trivializes" transfers: this topology sees only the S-morphisms of schemes, systematically substituting the latter for finite correspondences. This property means that the h-sheaves in abelian groups are canonically presheaves with transfers.

We will define a morphism of S-schemes $g : X \to Y$ as a **topological epimorphism** if it is surjective and a subset of Y is open if and only if its inverseimage under g is open in X. Given that this property is not preserved under base change, it becomes essential to focus on **universal topological epimorphisms**. Specifically, these are morphisms of S-schemes $g : X \to Y$ such that for any Y-scheme Z, the induced morphism $X \times_S Z \to Z$ remains a topological epimorphism.

The h-topology is a Grothendieck topology derived from a pre-topology where the coverings of an S-scheme X consist of finite families $(X_i \to X)_{i\in I}$ of S-morphisms of finite type, such that the morphism

$$\coprod_i X_i \to X$$

is a universal topological epimorphism. This topology is notably finer than the flat topology.

Letter from Grothendieck to Faltings: treat a universal homeomorphism of schemes as the algebro-geometric analog of weak equivalence. Grothendieck: X and Y are finite type schemes whose étale cohomology and étale π_1 are equivalent implies universal homeomorphism.

It is immediately apparent that surjective open morphisms and surjective closed morphisms are topological epimorphisms. Universally open morphisms of finite type, e.g. finite flat morphisms, and surjective morphisms, as well as proper and surjective morphisms, are therefore coverings for h-topology. For this reason, the abstract blow-ups are in covers for h-topology.

Abelian h-sheaves possess a canonical structure of presheaf with transfers, i.e. there is a canonical additive functor O_h^{tr} making the triangulated

$$\begin{tikzcd} Sh_h(S) \arrow[d] \arrow[dr, "O_h^{tr}"] & \\ PSh(S) & PSh^{tr}(S) \arrow[l] \end{tikzcd}$$

commute. The sheaves of locally constant abelian groups for the étale topology are canonically provided with a structure of presheaves with transfers.

An interesting example of sheaf with transfer is the Kähler differential for smooth schemes. We can develop algebraic de Rham cohomology theory (of course in characteristic zero) in the framework of h-topology[43]. This approach is much easier than Deligne's or Hartshorn's.

Definition 3.10: We denote $DM^{eff}(S)$ the $\mathbb{A}^1$-localisation of $D(Sh^{tr}(S))$.

Definition 3.11: Let F be a sheaf with transfers on S. The Suslin complex of F is constructed as a complex of sheaves with transfers, which corresponds to the simplicial object given by $\underline{\mathrm{Hom}}_{\mathrm{Sh}^{tr}(S)}(\mathbb{Z}_S^{tr}(\Delta_S^*), F)$. We denote it by $C_*^{sing}(F)$.

By definition, for any $X \in Sm/S$, $\Gamma(X, C_n^{sing}(F)) = F(X \times_S \Delta^n)$. In certain contexts, we adopt a cohomological notation for the Suslin complex. More precisely, we define:

$$C^n_{\mathrm{sing}}(F) := C_{-n}^{sing}(F) = \underline{\mathrm{Hom}}_{Sh^{tr}(S)}(\mathbb{Z}_S^{tr}(\Delta_S^{-n}), F).$$

Definition 3.12: To any smooth k-scheme X, we define the **motivic complex** $M(X) := C^*_{sing}(\mathbb{Z}^{tr}_k(X))$. It's called the motive of X over k.

Let us examine the closed immersion $s : \mathrm{Spec}(k) \to \mathbb{G}_m$ associated with the unit element of the group scheme $\mathbb{G}_m$. It is a monomorphism split by projection. We denote $\mathbb{Z}^{tr}_k(\mathbb{G}_m/1)$ the cokernel sheaf with transfers of the morphism

$$s_* : \mathbb{Z} = \mathbb{Z}^{tr}(\mathrm{Spec}(k)) \to \mathbb{Z}^{tr}_k(\mathbb{G}_m).$$

We therefore deduce morphism of sheaves on Sm/S

$$\eta : \mathbb{Z}^{tr}_k(\mathbb{G}_m) \to \mathbb{Z}^{tr}_k(\mathbb{G}_m) \to \mathbb{Z}^{tr}_k(\mathbb{G}_m/1)$$

where the first arrow is derived from the graph morphism.

Definition 3.13: We define the **Tate motive** $\mathbb{Z}(1)$ as the motivic complex $C^*_{sing}(\mathbb{Z}^{tr}_k(\mathbb{G}_m/1))[-1]$.

Definition 3.14: If X is a smooth scheme over S, we let $M(X)$ represent the object in $DM^{eff}(S)$ that corresponds to $\mathbb{Z}^{tr}_S(X)$.

Consider an integer $n > 0$. We denote $\mathbb{Z}^{tr}_S(\mathbb{G}_m^{\wedge,n})$ the sheaf with transfers on S obtained as the cokernel of the canonical morphism:

$$\bigoplus_{i=1}^{n} \mathbb{Z}^{tr}_S(\mathbb{G}_m^{n-1}) \to \mathbb{Z}^{tr}_S(\mathbb{G}_m^{n})$$

sum of the morphisms induced by the closed immersions of the form $\mathbb{G}_m^{i-1} \times \{1\} \times \mathbb{G}_m^{n-i} \to \mathbb{G}_m^n$.

Definition 3.15: The Tate motive $\mathbb{Z}_S(n)$ is defined as the object of $DM^{eff}(S)$ represented by the complex of sheaves with transfers concentrated in degree n equal to $\mathbb{Z}^{tr}_S(\mathbb{G}_m^{\wedge,n})$ in degree n.

Definition 3.16: Given a smooth scheme X over S and a pair of integers (i, n) where $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, its **effective motivic cohomology** in bidegree (i, n) is the abelian group:

$$\mathrm{Hom}_{DM^{eff}(S)}(M_S(X), \mathbb{Z}_S(n)[i]).$$

According to definitions, the functor $\mathbb{Z}^{tr}_S$ induces a unique triangulated functor:

$$DM^{eff}_{gm}(S) \to DM^{eff}(S).$$

This functor is fully faithful.

Definition 3.17: The category of **geometric motives**, denoted as $DM_{gm}(S)$, is constructed by first formally inverting the Tate motive $\mathbb{Z}_S(1)$ within the category $DM^{eff}_{gm}(S)$

and subsequently taking the pseudo-abelian envelope.

Definition 3.18: Given a regular scheme S and a pair of integers (i, n) belonging to $\mathbb{Z}$, we introduce the **motivic cohomology** of S at degree (i, n) as the abelian group represented by:

$$H^{n,i}(S) = \mathrm{Hom}_{DM_{gm}(S)}(\mathbb{Z}_S, \mathbb{Z}_S(i)[n]).$$

Theorem 3.3: Here are some fundamental but difficult properties of motivic cohomology.

（1）Let X be a smooth scheme over k. The motivic cohomology groups of X can be identified with the higher Chow groups. For any integers p and q, the following isomorphism holds:

$$H^p(X, \mathbb{Z}(q)) \simeq CH^q(X, 2q - p).$$

Notably, when $p = 2q$, this simplifies to:

$$H^{2q}(X, \mathbb{Z}(q)) \simeq CH^q(X).$$

While we have not provided a formal definition of higher Chow groups here, we utilize the aforementioned isomorphism as the defining property of these groups (if you accept). This isomorphism is natural in obvious sense.

（2）Let X be a smooth scheme over k. For integers p and q, the motivic cohomology $H^p(X, \mathbb{Z}(q))$ vanishes in the following scenarios: (1). If q is negative. (2). When p exceeds $2q$. (3). Whenever p is greater than $q + \dim X$.

（3）(Suslin-Kelly) Let k be an algebraically closed field, m an invertible integer in k, $\pi_X : X \to \mathrm{Spec}(k)$ a separated morphism of finite type and let $j \leq 0$. Then there is an isomorphism

$$H^{n+2j}_{\text{ét,c}}(X; \Lambda(j)) \simeq \mathrm{Hom}_{\mathbb{Z}}(CH_j(X, n; \Lambda), \mathbb{Q}/\mathbb{Z}).$$

When X is smooth and equi-dimensional $d = \dim X$, it's

$$\mathrm{Hom}_{\mathbb{Z}}(H^{2(d-j)-n}(X, \Lambda(d-j)), \mathbb{Q}/\mathbb{Z}) \simeq H^{n+2j}_{\text{ét},c}(X, \Lambda(j)).$$

（4）Consider a field L. For each non-negative integer q, there is a product preserving isomorphism:

$$K_q^M(L) \simeq H^q(L, \mathbb{Z}(q)).$$

Additionally, for any positive integer m, this isomorphism leads to an induced isomorphism:

$$K_q^M(L)/mK_q^M(L) \simeq H^q(L, \mathbb{Z}/m\mathbb{Z}(q)).$$

（5）Consider $q \in \mathbb{N}$. Let A be an abelian group whose torsion is prime to the characteristic exponent of the field k. In the derived category of étale sheaves on Sm/k, the image of the motivic complex $A(q)$ is equivalent to the twisted constant sheaf $A(q)$. Notably, if $A = \mathbb{Z}/m\mathbb{Z}$, then the image of $\mathbb{Z}/m\mathbb{Z}(q)$ in this derived category is given by $\mu_m^{\otimes q}$.

Theorem 3.4 (Voevodsky-Rost): Let $m \geq 1$. Let k be a field in which the integer m is invertible. For any integer $q \geq 0$, the Galois symbol (one may see[32]section 4.6 for a construction)

$$K_q^M(k)/mK_q^M(k) \to H_{\text{ét}}^q(k, \mu_m^{\otimes q})$$

is an isomorphism. Equivalently, The obvious morphism

$$H^{p,q}(X, \mathbb{Z}/m\mathbb{Z}) \to H_{\text{ét}}^{p,q}(X, \mathbb{Z}/m\mathbb{Z})$$

is an isomorphism for $p \leq q$. (Moreover, these conditions imply that this morphism is injective for $p = q + 1$.)

Proof: Highly non-trivial. To prove it, Voevodsky invented motivic homotopy theory, motivic Steenrod algebra,… ■

In[10]section 5.10, Beĭlinson demanded that motivic cohomology form a Poincaré duality theory with support as defined in[13]. A key component of this framework is the Gysin map, which serves as a fundamental input to the theory.

Proposition 3.10: Let $f : Y \to X$ be a projective morphism of relative codimension d in Sm/k. There exists a functorial morphism in $DM_{gm}(k)$:

$$f^! : M(X) \to M(Y)(d)[2d].$$

This morphism $f^!$ is referred to as the Gysin morphism.

Proof: See[23]section 2. ■

Theorem 3.5 (Poincaré duality): Let X and Y be two projective schemes in Sm/k. There exists an isomorphism:

$$\mathrm{Hom}_{DM_{gm}(k)}(M(Y)(i)[j], M(X)) \simeq \mathrm{Hom}_{DM_{gm}(k)}(M(Y) \otimes M(X), \mathbb{Z}(d_X - i)[2d_X - j]).$$

By setting $Y = \mathrm{Spec}(k)$, we obtain:

$$H^{2d_X - j, d_X - i}(X) \simeq H_{j,i}(X).$$

Proof: The proof employs only standard methods in the context of algebraic topology. We indicate the construction of Poincaré duality.

There is the so-called slant product

$$/ : H^{p,q}(X \times_k Y) \otimes H_{i,j}(Y) \to H^{p-i,q-j}(X)$$

$$(a,b) \mapsto a/b := M(X)(j) = M(X) \otimes \mathbb{Z}(j) \xrightarrow{1\otimes b} M(X) \otimes M(Y)[-i] \xrightarrow{a} \mathbb{Z}(q)[p-i].$$

Let p be the structure map of X. The co-fundamental calss of X is

$$[X]^* := \Delta_! p^*(1) \in H^{2d_X,d_X}(X \times_k X).$$

We define the map $D : H_{p,q}(X) \to H^{2d_X-p,d_X-q}(X)$ by setting $D(a) := [X]^*/a$, ■

As permitted in the abstract, we review some recent developments in motivic cohomology. To the best of the author's knowledge, there has been no significant progress regarding the abelian category of mixed motives. It is commonly believed that the pursuit of understanding motives is a formidable challenge.

There are two new constructions of motivic cohomology[14,30], which aim to generalise motivic cohomology to non-smooth schemes.

By restricting Elmanto and Morrow's construction to smooth schemes, one obtains the classical motivic complexes. However, Bouis' construction requires an additional $\mathbb{A}^1$-localisation. Both constructions provide limited discussion on cycles, which are fundamentally important in algebraic geometry.

CHAPTER 4 ALGEBRAIC COBORDISM

Algebraic cobordism has two origins. One is Voevodsky's early methods[49,74] to prove the Milnor conjecture or, more generally, the norm-residue theorem. Although Voevodsky's ultimate proofs[77,81] did not appeal to algebraic cobordism, see[37]. The other origin is Levine and Morel's attempt to construct a bordism theory in algebraic geometry[53]. In fact, Levine and Morel's algebraic cobordism forms the geometric counterpart of Voevodsky's construction.

We discuss the construction of algebraic cobordism after recalling complex cobordism. We will give relations between algebraic cobordism, motivic cohomology, and algebraic K-theory and mention some calculations of algebraic cobordism.

4.1 Review complex cobordism

There are two descriptions of complex cobordism. An abstract and quick description is through the complex cobordism spectrum MU.

Let $\xi^n : EU(n) \to BU(n)$ be the universal complex bundle. Let $MU(n)$ be the Thom space of ξ^n. We have the following pullback diagram:

$$\begin{tikzcd} EU(n) \oplus \mathbb{C} \arrow[r] \arrow[d, "i_n^*(\xi^{n+1}) = \xi^n \oplus 1_{\mathbb{C}}"'] & EU(n+1) \arrow[d, "\xi^{n+1}"] \\ BU(n) \arrow[r, "i_n"'] & BU(n+1) \end{tikzcd}$$

where $1_{\mathbb{C}}$ is the trivial vector bundld. Abstract property of Thom space gives the map

$$\Sigma^2 MU(n) = \mathrm{Th}(\xi^n \oplus 1_{\mathbb{C}}) \to \mathrm{Th}(\xi^{n+1}) = MU(n+1).$$

Definition 4.1: The complex cobordism spectrum MU is the colimit

$$MU := \operatorname{colim} MU(n)$$

with transitions map $\Sigma^2 MU(n) \to MU(n+1)$.

As indicated by its name, MU also possesses a geometric interpretation.

Definition 4.2: Let M and N denote two almost complex manifolds, and let X represent any topological space. Two mappings $f : M \to X$ and $g : N \to X$ are said to be bordant if there exists an almost complex manifold W of dimension $(n+1)$, with boundary

$\partial W = M \sqcup N$, such that the union of maps $f \sqcup g$ can extend to W.

Definition 4.3: Denote by $\Omega_n^U(X)$ the bordism class of n-manifolds to X.

Disjoint union makes $\Omega_n^U(X)$ into an abelian group.

Theorem 4.1 (Pontryagin-Thom): The geometric bordism is isomorphic to the abstract bordism:

$$\Omega_n^U(X) \xrightarrow{\sim} \pi_n(MU \wedge X_+).$$

An important feature of MU is its universal property. To describe it, we need the notion of formal group law[50]section 4.4. Lazard showed there is a universal one $\mathbb{L}$. Moreover, Lazard showed there is an isomorphism:

$$\mathbb{L} \cong \mathbb{Z}[x_1, x_2, \dots],$$

where $|x_i| = 2i$. This result is deep. We refer to[62]Theorem 6.8 for a modern abstract proof.

A primary origin of formal group laws stems from cohomology theory.

Definition 4.4: A ring spectrum E is complex orientable if the induced map

$$i^* : \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2 \simeq \mathbb{CP}^2) \simeq \pi_0(E)$$

is surjective. A complex orientation is a choice of an element $x^E \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that $i^*(x^E) = 1$.

Example 4.1: Two examples:

（1）The Eilenberg-MacLane spectrum $H\mathbb{Z}$ is complex oriented. Let

$$x^{H\mathbb{Z}} \in H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$$

be the generator.

（2）The complex K-theory spectrum KU. Its orientation is given by

$$[\xi^1] - 1 \in \widetilde{KU}^0(\mathbb{CP}^\infty) \cong \widetilde{KU}^2(\mathbb{CP}^\infty).$$

Example 4.2: Complex cobordism spectrum MU is also a complex oriented cohomology theory. It is well-know that $MU(n)$ is homotopy equivalent to $BU(n)/BU(n-1)$. In particular, $MU(1) \simeq BU(1) \simeq \mathbb{CP}^\infty$ which defines an element $x^{MU} \in \widetilde{MU}^2(\mathbb{CP}^\infty)$. Since $MU(0) = S^0$, the map $\Sigma^2 MU(0) \to MU(1)$ corresponds to the map $i : S^2 \to \mathbb{CP}^\infty$ which generates the group $\pi_2(\mathbb{CP}^\infty)$. Therefore $i^*(x^{MU}) = 1$.

Proposition 4.1: Let E be a oriented spectrrum, where the complex orientation is denoted by x. Then

（1）$E^*(\mathbb{CP}^\infty) \simeq \pi_*(E)[[x]]$.

（2）$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \pi_*(E)[[x_1, x_2]]$, where $x_i = p_i^*(x)$ for $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ the i-th projection.

The multiplication map $\mathbb{CP}^\infty$

$$m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty.$$

determines the formal group law $\mu^E(x, y) := m^*(x)$.

The homomorphism

$$h : \mathbb{L} \to \pi_*(MU)$$

was shown by Daniel Quillen to actually be an isomorphism[64]:

Theorem 4.2 (Quillen): The ring homomorphism $h : \mathbb{L}_* \to \pi_*(MU)$ is a ring isomorphism and $\mathbb{L}_n \cong MU^{-2n}(pt)$.

Proposition 4.2: Let E be an orientable spectrum. Fix an orientation y^{MU} of MU. Let $g : MU \to E$ be a map. The function $g \mapsto g(y^{MU})$ induces a complex orientation $y \in \tilde{E}^2(\mathbb{CP}^\infty)$. Moreover, the function is a bijection. Hence each complex orientation of E comes from a unique ring spectrum map $MU \to E$ in the stable homotopy category.

4.2 Brown-Peterson spectrum

An important variation of MU is the Brown-Peterson spectrum BP. To define it, we need the notion of localisation. We refer to the relevant parts in[8,57].

Consider the p-localisation $MU_{(p)}$ of MU at a prime number p. According to some general facts, $MU_{(p)}$ is still an $\mathbb{E}_\infty$-ring spectrum.

Theorem 4.3 (Quillen): Daniel Quillen proved the following famous theorem:

（1）There is a unique homotopy idempotent ring spectra map

$$e_p : MU_{(p)} \to MU_{(p)}.$$

（2）There is a ring spectrum BP and two maps of ring spectra

$$\pi : MU_{(p)} \to BP, \quad \ell : BP \to MU_{(p)}$$

that satisfy $\ell \circ \pi \simeq e$ and $\pi \circ \ell \simeq id_{BP}$.

（3）There are elements $v_i \in BP_{2p^i-2}$ such that $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

Proof: See[64] or[50]Theorem 4.6.7. The spectrum BP is defined as

$$BP := \mathrm{im}(e) = \mathrm{hocolim}(MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} \ldots).$$

Note that $\pi_0(BP) \cong \pi_0(MU)$. ■

Definition 4.5: The n-th truncated BP is $BP\langle n\rangle := BP/(v_{n+1}, v_{n+2}, \dots)$.

One can show that $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$.

4.3 Interlude: motivic homotopy theory

Voevodsky's algebraic cobordism MGL mirrors the complex cobordism spectrum MU[76]. To achieve this, it is necessary to consider the motivic homotopy category. Recently, Annala, Hoyois, and Iwasa extended Voevodsky's construction to a non-$\mathbb{A}^1$-invariant version[2].

Voevodsky proposed to investigate the homotopy theory of schemes. He established a homotopy theory for the category Sm/S. The fundamental approach involves emulating constructions from algebraic topology by substituting the interval $[0, 1]$ with the affine line $\mathbb{A}^1$.

During his time at Harvard, Voevodsky developed his motivic homotopy theory, which led to a novel construction of the Atiyah-Hirzebruch spectral sequence modulo certain conjectures (later resolved by Levine). Additionally, he proved the Bloch-Kato conjecture (étale comparison theorem), for which he was awarded the Fields Medal.

Definition 4.6: Let τ be Nisnevich or étale topology. The ∞-category $H_\tau(S)$, which consists of τ-motivic spaces, is defined as the full subcategory of $PSh(Sm/S)$ that includes only those $\mathbb{A}^1$-invariant τ-hypersheaves.

We let $H_{\tau,*}(S)$ be the category of motivic spaces (pointed). Here, $*$ signifies the terminal object within the category $H_\tau(S)$. Then, there is an adjunction

$$(-)_+ : H_\tau(S) \rightleftarrows H_{\tau,*}(S) : u.$$

It is important to observe that $H_\tau(S)$ possesses a symmetric monoidal structure, which can be extended to endow $H_{\tau,*}(S)$ with a monoidal structure.

The stable motivic homotopy quasi-category is constructed by formally inverting $(\mathbb{P}^1_S, \infty)$ in $H_{\tau,*}(S)$:

Definition 4.7: The **stable motivic homotopy category** $SH_\tau(S) := H_\tau(S)[(\mathbb{P}^1_S, \infty)^{-1}]$. When τ is Nisnevich topology, we simply denote it by $SH(S)$.

Remark 4.1: See[65] for why the above constructions all make sense.

Remark 4.2: The Zariski topology is too coarse to meet our requirements. For instance, one of the major drawbacks of the Zariski topology in the context of our applications is that, for a closed immersion $X \xrightarrow{i} Y$ between two smooth S-schemes, this immersion is not locally isomorphic to a closed immersion of the form $(\mathbb{A}^n \times \{0\}) \cap U \to \mathbb{A}^{n+m} \cap U$, where U is an open subset of $\mathbb{A}^{n+m}$. However, such a description exists if U is only étale over $\mathbb{A}^{n+m}$.

Furthermore, we prefer a topology for which the cohomological dimension is bounded above by the Krull dimension. The étale topology is also not suitable because the cohomological dimension of the small étale site of the prime spectrum of a field is generally not zero, being isomorphic to the Galois cohomology. One may see[45]Exposé XVIII for further development of étale cohomological dimension. Consequently, we choose the Nisnevich topology, of which we will recall the definition and some essential properties, particularly the Nisnevich descent which will be very useful. Another crucial reason for not employing the étale topology is that algebraic K-theory fails to fulfill étale descent.

In the category $SH(S)$, the following equivalence holds: $\mathbb{P}^1_k \simeq S^1 \wedge \mathbb{G}_m$. Within the framework of $SH(S)$, both S^1 and $\mathbb{G}_m$ possess invertibility. The operation $(1) \wedge -$ is referred to as the Tate twist. We also establish $\Sigma_{\mathbb{P}^1_k} = S^{2,1} \wedge -$. For every scheme X in the category Sm/S, there corresponds a motivic spectrum denoted by $\Sigma^\infty_{\mathbb{P}^1_S} X_+$.

Theorem 4.4: Consider a closed embedding i from Z into X, where both are smooth schemes over S. The associated normal bundle is represented as $N_{X/Z}$. This setup induces a canonical isomorphism in $H_*(S)$:

$$\mathrm{Th}(N_{X/Z}) \simeq X/(X - Z).$$

Proof: This theorem looks like the tubular neighborhood theorem in differential geometry. Its proof is a little complicated, see[60]Theorem 2.23 on page 115. By[3]Remark 2.5., it is only valid in $\mathbb{A}^1$-invariant $H_*(S)$. ■

Definition 4.8: Let $\mathbb{E} \in SH(S)$. For each $X \in Sm/S$, the cohomology theory associated with $\mathbb{E}$ is $\mathrm{Hom}_{SH(S)}(X, \mathbb{E}(q)[p])$.

If $\mathbb{E}$ is moreover a ring spectrum, $\mathbb{E}^{i,j}(X)$ has an obvious ring structure for each smooth scheme X.

Example 4.3: Here are some spectra in $SH(k)$ representing useful theories.

（1）(Motivic cohomology) For any abelian group A, there is a spectrum MA represents motivic cohomology defined by Voevodsky. One may see[42]Section 4 or[68] for details. There is also a Dold-Thom theorem like construction of MA, see[80]section 3 and[4]section 5.1.

（2）(Algebraic K-theory) For any regular and separated Noetherian scheme S, there exists a spectrum $\mathcal{K} \in SH(S)$ representing algebraic K-theory. One may see[18,76] for a proof. Since X is smooth, we don't need to distinguish Quillen K-theory or Thomason-Trobaugh K-theory[72]3.9. Corollary.

（3）(Étale cohomology) We assume k is separably closed and $\ell \in k^{\times}$. There is a spectrum $\mathbb{HZ}_{\ell} \in SH(k)$ and canonical isomorphisms of groups:

$$\mathbb{HZ}_{\ell}^{p,q}(X) \cong H^p_{\text{ét}}(X;\mathbb{Z}_{\ell}(q)) := \lim_{v} H^p_{\text{ét}}(X;\mathbb{Z}/\ell^v\mathbb{Z}(q)).$$

There is also a spectrum $\mathbb{HQ}_{\ell} \in SH(k)$ representing ℓ-adic cohomology:

$$\mathbb{HQ}_{\ell}^{p,q}(X) \cong H^p_{\text{ét}}(X;\mathbb{Q}_{\ell}(q)) := H^p_{\text{ét}}(X;\mathbb{Z}_{\ell}(q)) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}.$$

If the base scheme k is not a separately closed field, we get the spectrum representing Jannsen's continuous étale cohomology[48] or pro-étale cohomology[12]Proposition 5.6.2.

In fact, Cisinski and Déglise gave an axiomatic approach in[19]. They showed that algebraic de Rham cohomology and Berthelot's rigid cohomology are both representable in $SH(k)$. One may see[22] for further development.

（4）(Deligne-Beĭlinson cohomology with real coefficients) Holmstrom and Scholbach[40]section 3 constructed a spectrum H_D such that

$$H^n_D(X,\mathbb{R}(p)) \cong [\Sigma^{\infty}_{\mathbb{P}^1_k} X_+, H_D(p)[n]]_{SH(k)}$$

for any smooth k-scheme X. Navarro[61]Appendix extended it to non-smooth schemes over an arithmetic field.

（5）(Absolute Hodge cohomology with real coefficients) In[15], Bunke, Nikolaus, and Tamme constructed a spectrum H_{abs} such that

$$H^{2i-n}_{abs}(X,\mathbb{R}(i)) \cong [\Sigma^{\infty}_{\mathbb{P}^1_k} X_+, H_{abs}(i)[n]]_{SH(k)}$$

for any smooth $\mathbb{C}$-scheme X.

（6）(Hermitian K-theory) Hornbostel constructed a spectrum[41] in $SH(k)$ representing Hermitian K-theory when 2 is invertible. When 2 is not invertible, see Calmès, Harpaz, and Nardin's work[16] for a through discussion.

Definition 4.9: A ring spectrum $\mathbb{E} \in SH(k)$ is oriented if there exists a class $c_{\mathbb{E}} \in \mathbb{E}^{2,1}(\mathbb{P}^{\infty}_k)$ that pull back to the class $\Sigma^{\infty}_{\mathbb{P}^1_k}\operatorname{Spec}(k)_+ \wedge \mathbb{P}^1_k \xrightarrow{1\wedge id} \mathbb{E} \wedge \mathbb{P}^1_k$ in $\mathbb{E}^{2,1}(\mathbb{P}^1_k)$ via the inclusion $\mathbb{P}^1_k \to \mathbb{P}^{\infty}_k$.

Example 4.4: Many familiar theories are orientable in our sense. They are motivic cohomology, algebraic K-theory, absolute Hodge cohomology, Deligne-Beĭlinson cohomology, algebraic and analytic de Rham cohomology, and rigid cohomology. Later, we will see that algebraic cobordism provides a universal oriented cohomology theory.

Proposition 4.3: Given an oriented motivic spectrum $\mathbb{E}$, and considering a vector bundle V over X with rank $n+1$, we identify a distinguished class c within $\mathbb{E}^{2,1}(\mathbb{P}(V))$. This leads to the equivalence $\mathbb{E}^{*,*}(\mathbb{P}(V))$ being isomorphic to the quotient $\mathbb{E}^{*,*}(k)[c]/(c^{n+1})$, structured as a left module over $\mathbb{E}^{*,*}(k)$.

Proof: Their proofs are almost the same as the situations in algebraic topology. We only explain the construction of c.

In[60]Section 4, Morel and Voevodsky constructed a functorial isomorphism

$$\mathrm{Pic}(X) \to \mathrm{Hom}_{H_*(k)}(X, B\mathbb{G}_m).$$

They even proved $B\mathbb{G}_m \simeq \mathbb{P}_k^\infty$ in $H_*(k)$,[60]Proposition 3.7 on page 138. We have the first Chern class

$$c_1 : \mathrm{Pic}(X) \to [X, \mathbb{P}_k^\infty]_{H_*(k)} \xrightarrow{\Sigma^\infty_{\mathbb{P}^1_k,+}} [\Sigma^\infty_{\mathbb{P}^1_k,+}X, \Sigma^\infty_{\mathbb{P}^1,+}\mathbb{P}_k^\infty]_{SH(k)} \xrightarrow{c_{\mathbb{E}}} [\Sigma^\infty_{\mathbb{P}^1_k,+}X, \mathbb{E}(1)[2]]_{SH(k)}.$$

The $c \in \mathbb{E}^{2,1}(\mathbb{P}(V))$ is choosed as the first Chern class of $\mathcal{O}_{\mathbb{P}(V)}(-1)$. ■

Proposition 4.4: Let $\mathbb{E}$ be an oriented motivic spectrum. For any $X \in Sm/k$. We have un isomorphisme $\mathbb{E}^{*,*}(X \times_k \mathbb{P}_k^n) \cong \mathbb{E}^{*,*}(X)[c]/(c^{n+1})$ of $\mathbb{E}^{*,*}(k)$-module.

Proof: Considering the homotopy equivalence between $\mathbb{P}_k^n/\mathbb{P}_k^{n-1}$ and $S^{2n,n}$ within the stable motivic homotopy category $SH(k)$, $\mathbb{P}_k^n$ qualifies as a finite cell complex according to the definition provided in[29]Definition 8.1. Given that $\mathbb{E}^{*,*}(\mathbb{P}_k^n)$ constitutes a free module over the base ring $\mathbb{E}^{*,*}(k)$, the Tor-spectral sequence from[29] allows us to establish the isomorphism. ■

The map

$$\sigma : \mathbb{P}_k^\infty \times_k \times \mathbb{P}_k^\infty \to \mathbb{P}_k^\infty$$

determines a fomral group law $F(x,y) := \sigma^*(c)$.

4.4 Voevodsky's algebraic cobordism

Definition 4.10: Consider the Grassmannian scheme $\mathrm{Gr}(m,n)$ of m-dimensional planes in $\mathbb{A}^n$ and its universal bundle $\gamma_{m,n}$. By taking the colimit as n approaches infinity, we obtain the infinite Grassmannian scheme $\mathrm{Gr}(m,\infty)$ equipped with the universal bundle γ_m. The natural embeddings from $\mathrm{Gr}(m,\infty)$ into $\mathrm{Gr}(m+1,\infty)$ give rise to mappings from $E_1 \oplus \gamma_m$ to γ_{m+1}, where E_1 denotes the trivial bundle of rank one.

The **algebraic cobordism spectrum** $MGL = (\mathrm{Th}(\gamma_0), \mathrm{Th}(\gamma_1), \dots)$. Its structure maps are

$$\mathbb{P}^1 \wedge \mathrm{Th}(\gamma_m) \simeq \mathrm{Th}(E_1 \oplus \gamma_m) \to \mathrm{Th}(\gamma_{m+1}).$$

Proposition 4.5 (Bachmann-Hoyois): The spectrum MGL is equivalent to the homotopy colimit $\mathrm{colim}_{(X,\xi)} \mathrm{Th}_X(\xi)$ where X ranges over Sm/k and virtual vector bundle $\xi \in K_0(X)$ has rank zero.

Proof: This is[7]Theorem 16.13. ■

Motivated by this proposition, Annala, Hoyois, and Iwasa defined the non-$\mathbb{A}^1$-invariant algebraic cobordism:

Definition 4.11 (non-$\mathbb{A}^1$-invariant MGL): We work over non-$\mathbb{A}^1$-localised motivic homotopy theory in a moment. The non-$\mathbb{A}^1$-invariant algebraic cobordism is defined by

$$MGL := \mathrm{colim}\, \mathrm{Th}_X(\xi)$$

where X ranges over smooth derived k schemes and $\xi \in K_0(X)$ has rank zero.

Proposition 4.6: Algebraic cobordism spectrum MGL is orientable.

Proof: We prove $\mathbb{P}_k^\infty \simeq \mathrm{Th}(\gamma_1)$ first. Consider the closed immersion

$$l_n : \mathbb{P}_k^{n-1} \to \mathbb{P}_k^n.$$

The normal bundle of l_n is $\mathcal{O}_{\mathbb{P}_k^{n-1}}(-1)$ on $\mathbb{P}_k^{n-1}$. The open complementary $\mathbb{P}_k^n - \mathbb{P}_k^{n-1} \cong \mathbb{A}_k^n$ is contractible in $SH(k)$. Using 4.4, one gets

$$(\mathbb{P}_k^n, 1) \simeq \mathbb{P}_k^n/(\mathbb{P}_k^n - \mathbb{P}_k^{n-1}) \simeq \mathrm{Th}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)).$$

This isomorphism is functorial concerning the inclusion of l_n. Therefore, one takes the homotopy colimit over n to get the isomorphism.

So, we have a natural map

$$c_{MGL} : \Sigma_{\mathbb{P}_k^1}^\infty \mathbb{P}_k^\infty \to \Sigma_{\mathbb{P}_k^1}^\infty \mathrm{Th}(\gamma_1) \to MGL(1)[2].$$

By construction, the restriction of c_{MGL} to $(\mathbb{P}_k^1, \infty)$ corresponds up to $\mathbb{P}_k^1$-desuspension to the unit

$$\Sigma_{\mathbb{P}_k^1}^\infty \mathrm{Spec}(k) = \Sigma_{\mathbb{P}_k^1}^\infty \mathrm{Th}(\gamma_0) \to MGL.$$

Therefore, c_{MGL} is an orientation of MGL. ■

The algebraic cobordism MGL has a similar universal property to MU.

Theorem 4.5 (Gabriele Vezzosi): Let $\mathbb{E} \in SH(k)$ be a ring spectrum. Then the following sets are in bijective correspondence:

（1）Orientation c of $\mathbb{E}$,

（2）morphisms of ring spectra $\psi : MGL \to \mathbb{E}$, by the map

$$\psi \mapsto \psi_*(c_{MGL})$$

where $\psi_* : MGL^{*,*} \to \mathbb{E}^{*,*}$ is the induced map.

（3）The orientations of $\mathbb{E}$ correspond uniquely to isomorphisms of formal group laws defined on $\mathbb{E}^{2*,*}(k)$.

This theorem says nothing about the homotopy ring $MGL_{2*,*}$. It was hypothesized that, for any regular local scheme S, the natural homomorphism

$$\bigoplus_{i=-\infty}^{\infty} MU^{2i} \to \bigoplus_{i=-\infty}^{\infty} MGL^{2i,i}(S)$$

is an isomorphism[76]CONJECTURE 1 on page 601. Hoyois proved the conjecture partially:

Theorem 4.6 (Hopkins-Morel-Hoyois): Consider a field k and let p denote its characteristic exponent. The map from $\mathbb{L}[1/p]$ to the motivic cobordism ring $MGL_{2*,*}(k)[1/p]$ is an isomorphic map.

Proof: This is the main result of[42]. ■

As an application, Marc Levine proved[55] the isomorphism between Ω^* and $MGL^{2*,*}$:

Corollary 4.1: Let k be a field over $\mathbb{Q}$ and $X \in Sm/k$. There is a ring isomorphism

$$\Omega^*(X) \simeq MGL^{2*,*}(X).$$

Proof: We refer to[54] for the construction of Ω^*. ■

This isomorphism also gives a geometric explanation of MGL.

Let $a_1, a_2, \ldots$ be generators of the Lazard ring. A direct consequece of the Hopkins-Morel-Hoyois theorem is

$$MGL/(a_1, a_2, \ldots)[1/p] \simeq M\mathbb{Z}[1/p].$$

So, we can recover motivic cohomology from algebraic cobordism.

Proposition 4.7: Let $\mathbb{E} \in SH(k)$ be a $\mathbb{Z}[1/p]$-linear ring spectrum. Then the following two properties are equivalent.

（1）$\mathbb{E}$ has additive formal group law.

（2）There exists a morphism (dependent on the orientation of $\mathbb{E}$) of motivic ring spectra:

$$\sigma : M\mathbb{Z} \to \mathbb{E}.$$

When these conditions are fulfilled, the additive orientation c on $\mathbb{E}$ is unique and is the image under σ of the canonical orientation on $M\mathbb{Z}$.

Proof: The implication from (2) to (1) is obvious.

Conversely. The orientation c of E corresponds to a morphism of ring spectra

$$\psi : MGL \to \mathbb{E}.$$

This map induces a morphism of formal group law. The map ψ_* maps all generators $a_{ij}, (i,j) \neq (1,0), (0,1)$ of the Lazard ring to zero. Thus ψ induces the morphism σ. ■

If we denote $\mathbb{E}$ as the spectrum corresponding to de Rham cohomology or ℓ-adic cohomology, one can find the cycle class map of Chow groups is uniquely determined. Now, let us say some calculations of MGL.

Proposition 4.8: Let k be any field.

（1）Along the diagonal line $\pi_{n,n}(MGL) \cong K^M_{-n}(k)$ is the Milnor K-theory. For $p < q$ or $2p < q$, on gets $\pi_{p,q}(MGL) = 0$.

（2）If X is a smooth scheme over k. Then rational algebraic cobordism and rational motivic cohomology coincide:

$$MGL^{*,*}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M\mathbb{Q}^{*,*}(X) \otimes_{\mathbb{Z}} \mathbb{L}.$$

An important property of algebraic cobordism is its isomorphism between Chow rings.

Theorem 4.7: Let k be an extension of $\mathbb{Q}$. For arbitrary smooth projective scheme X over k. We have

$$\Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\sim} CH^*(X).$$

Recall that we have defined the spectrum BP in algebraic topology. There is also a motivic version BP. Let $MGL_{(p)}$ be the localisation of MGL. The localisation

$$L : MGL \to MGL_{(p)}$$

is a map of ring spectra. It induces an orientation $x_{(p)}$ of $MGL_{(p)}$.

Just like stable homotopy theory, there is an idempotent map

$$e_{(p)} : MGL_{(p)} \to MGL_{(p)}.$$

Definition 4.12: For each prime number p, the **motivic Brown-Peterson spectrum** $MBP \in SH(k)$ is

$$MBP := \text{hocolim}(MGL_{(p)} \xrightarrow{e_{(p)}} MGL_{(p)} \xrightarrow{e_{(p)}} MGL_{(p)} \to \dots).$$

According to the motivic Landweber exactness theorem, MBP can be equivalently characterized as $MGL_{(p)}/(x)$, where x represents any regular sequence within $\mathbb{L}$ that generates the ideal defining the p-typical formal group law.

Definition 4.13: Let $\{g_i\}$ denote the generators of $\mathbb{L}$. We define the sets $T = \{g_i : i \neq p^k-1, k \geq 1\}$ and $T\langle n\rangle = \{g_i : i \neq p^k-1, 1 \leq k \leq n\}$. The notation MBP is introduced to represent the quotient $MGL_{(p)}/T$, and we define the truncated motivic Brown-Peterson spectrum $MBP\langle n\rangle$ as the quotient $MGL_{(p)}/T\langle n\rangle$.

By construction, there is a tower of motivic spectra

$$MBP = MBP\langle\infty\rangle \to \cdots \to MBP\langle n\rangle \to MBP\langle n-1\rangle \to \cdots \to MBP\langle 0\rangle$$

Proposition 4.9: Let F be a field in which $p \in F^\times$ is a prime element. There exists an isomorphism

$$MBP\langle 0\rangle \xrightarrow{\sim} M\mathbb{Z}_{(p)}.$$

Proof: This is just a reformulation of[42]Theorem 7.12.. ■

A significant application of algebraic cobordism is the existence of Rost varieties, which are crucial in Voevodsky's proof of the Bloch-Kato conjecture regarding Galois symbols (not the conjecture about Riemann zeta function). Here, we present their definition.

Definition 4.14: Let $d \geq 0$. We define $s_d : K_0(X) \to H^{2d,d}(X,\mathbb{Z})$ as the additive natural transformation (for any $X \in Sm/k$) that is uniquely determined by the property that if $\mathcal{L}$ is an algebraic line bundle, then $s_d([\mathcal{L}]) = c_1(\mathcal{L})^d$. Furthermore, if X is a projective smooth scheme of dimension d over k, we denote $s_d(X) := s_d([TX])$, where TX represents the tangent bundle of X.

For details on the definition of s_d, see[78]section 14.

Definition 4.15: Let $n \geq 0$. Let X be a connected smooth projective scheme on k. We say that X is a ν_n-variety if X is of dimension $d = \ell^n - 1$ and that $\deg s_d(X) \not\equiv 0 \bmod \ell^2$ where $\deg : H^{2d,d}(X) \simeq CH^d(X) \to \mathbb{Z}$ is the degree map.

Remark 4.3: Within the theories of motivic cohomology and finite correspondence, the usual degree map of algebraic cycles is

$$\mathbb{Z}(d)[2d] \otimes (\mathbb{Z} \to \mathbb{Z}^{tr}(Y)^*).$$

Definition 4.16: Let $X \in Sm/k$. We define X as a $\nu_{\leq q-1}$-variety if: it is a ν_{q-1}-variety, and for every $0 \leq i \leq q-2$, there exists a morphism $X_i \to X$, where X_i is a ν_i-variety.

Definition 4.17: Let $a = (a_1, \ldots, a_q)$ be a tuple of elements of $k^\times$. A splitting variety for the symbol $\{a_1, \ldots, a_q\}$ is a connected variety $X \in Sm/k$ such that the image of $\{a_1, \ldots, a_q\}$ is zero in $K_q^M(k(X))/\ell K_q^M(k(X))$.

Definition 4.18: Let k be a field with characteristic zero. Suppose $\{a_1, \ldots, a_q\} \neq 0$ in $K_q^M(k)/\ell K_q^M(k)$. An ℓ-generic splitting variety for the symbol $\{a_1, \ldots, a_q\}$ is defined as a splitting variety X such that for any finite extension E of k whose degree is prime to ℓ, there is a k-morphism from Spec E to X.

Example 4.5: Let $q = 2$. Assume that $\mu_\ell \subset k$, and select an ℓ-th primitive root of unity ξ. For $a, b \in k^*$, the central simple algebra $A_\xi(a, b)$ admits an associated Severi-Brauer variety X. Specifically, X is a projective, smooth, geometrically integral k-variety that is isomorphic to $\mathbb{P}_k^{\ell-1}$ if and only if A is not divisible. Furthermore, it is demonstrated that X serves as an ℓ-generic splitting variety for $\{a, b\} \in K_2^M(k)/\ell K_2^M(k)$.

Definition 4.19: Let k be a perfect field. The following are equivalent:

（1）There are no non-trivial finite extensions of k that have a prime degree with respect to ℓ;

（2）For any finite extension of k, its degree must be a power of ℓ;

（3）Gal$(\bar{k}/k)$ is a pro-ℓ-group.

If these equivalent conditions are satisfied, we say that k is ℓ-special.

Example 4.6: Let $\bar{k}$ denote an algebraic closure of k. By applying Zorn's lemma, there exists a k-subfield k' of $\bar{k}$ such that $\ell \nmid [k' : k]$ and k' is maximal with respect to this property. It is clear that k' satisfies the ℓ-special condition.

Theorem 4.8 (Rost): Let k be a field of characteristic zero that contains an ℓ-th root of unity, denoted by ξ. Consider an element $a = \{a_1, \ldots, a_q\} \neq 0$ in $K_q^M(k)/\ell K_q^M(k)$, where $q \geq 2$.

（1）(norm variety) For the symbol $\{a_1, \ldots, a_q\}$, there is an ℓ-generic splitting variety X_a that is smooth, projective, geometrically connected, with a dimension given by $\ell^{q-1} - 1$.

When the field k is ℓ-special, any smooth and projective ℓ-generic splitting variety X with dimension $\ell^{q-1} - 1$, corresponding to the symbol $\{a_1, \ldots, a_q\}$, exhibits the following characteristics:

（2）The variety X is a geometrically connected $v_{\leq q-1}$-variety.

（3）Let $X \in Sm/k$. We denote $\check{C}(X)$ the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$, the simplicial morphisms being defined simply by conceiving X^{n+1} as the "functions of scheme" $\{0, \ldots, n\} \to X$. The canonical map $H_{-1,-1}(\check{C}(X)) \to k^\times$ is injective, where $H_{-1,-1}((X)) := \mathrm{Hom}_{DM_-^{eff}(k)}(\mathbb{Z}, M(\check{C}(X))(1)[1])$

The variety in this theorem is the **Rost variety**.

Remark 4.4: Levine and Pandharipande found some applications of algebraic cobordism in Donaldson-Thomas theory[54]. Recently, Annala, Hoyois, and Iwasa recovered algebraic K-theory from algebraic cobordism[2]. Are there any additional notable applications in the field of algebraic geometry?

CHAPTER 5 CYCLE CLASS IN HODGE THEORY

In this chapter, we commence with an introductory overview of fundamental Hodge theory. Following this, we proceed to a detailed analysis of the cycle class map, particularly its Betti realisation within the context of Voevodsky motives. Additionally, we expound on the importance of the cycle class map and establish criteria for identifying when cohomology classes are algebraic. Finally, we present examples of non-algebraic cohomology classes.

5.1 Basic Hodge theory

The Hodge decomposition exists for general compact Kähler manifolds. For our purpose, we focus on smooth projective complex schemes. An important theorem for us is the famous GAGA theorem. This theorem date back to Serre. We refer to[38] for a modern account.

We refer to[44]Chapters 2,3,4,[43] and[39] for background on de Rham cohomology and singular cohomology.

For a smooth scheme $X \in Sm/\operatorname{Spec}(\mathbb{C})$, its de Rham complex relative to $\operatorname{Spec}(\mathbb{C})$ is denoted by $\Omega^*_{X/\mathbb{C}}$. Its filtration bête $\Omega^{\geq p}_{X/\mathbb{C}}$ is:

$$\Omega^{\geq p}_{X/\mathbb{C}} : 0 \to \cdots \to 0 \to \Omega^{p}_{X/\mathbb{C}} \to \Omega^{p+1}_{X/\mathbb{C}} \to \dots .$$

Let $\psi_p : \Omega^{\geq p}_{X/\mathbb{C}} \to \Omega^*_{X/\mathbb{C}}$ be the canoncial inclusion. The Hodge filtration F is given by $F^p H^i(X, \Omega^*_{X/\mathbb{C}}) := \operatorname{Im}(\psi_p)$.

Theorem 5.1 (Grothendieck): Let Y be a smooth projective scheme over $\operatorname{Spec}(\mathbb{C})$, and denote by Y^{an} the associated complex analytic space. There exists a functorial isomorphism

$$H^i_{\text{Zar}}(Y, \Omega^*_{Y/\mathbb{C}}) \simeq H^i(Y^{\text{an}}, \underline{\mathbb{C}}),$$

and furthermore,

$$F^p H^i_{\text{Zar}}(Y, \Omega^*_{Y/\mathbb{C}}) \simeq F^p H^i(Y^{\text{an}}, \mathbb{C}).$$

In this context, we consider a smooth projective scheme Y defined over $\operatorname{Spec}(\mathbb{C})$. The corresponding complex analytic space is denoted by Y^{an}. We establish a functorial isomorphism between the Zariski hyper-cohomology of de Rham complex on Y and the singular cohomology of its associated analytic space (complex). Additionally, we show that this isomorphism respects the Hodge filtration, specifically:

$$F^p H^i_{\mathrm{Zar}}(Y, \Omega^*_{Y/\mathbb{C}}) \simeq F^p H^i(Y^{\mathrm{an}}, \mathbb{C}).$$

Proof: A key step is the hypercohomology spectral sequence

$$E_2^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow H^{p+q}(X, \Omega^*_{X/\mathbb{C}}).$$

By the GAGA principle, the E_2-page are isomorphic to $E_2^{p,q}(X^{an}) = H^q(X^{an}, \Omega^p_{X^{an}})$. So $H^i(X, \Omega^*_{X/\mathbb{C}}) \simeq H^i(X^{an}, \Omega^*_{X^{an}})$. Finally, the Poincaré lemma gives isomorphisms

$$H^i(X, \Omega^*_{X/\mathbb{C}}) \simeq H^i(X^{an}, \Omega^*_{X^{an}}) \simeq H^i(X^{an}, \mathbb{C}).$$

■

Theorem 5.2 (Hodge decomposition): Consider X as a smooth projective scheme over Spec($\mathbb{C}$). The rational Betti cohomology associated with the analytic space X^{an} possesses a Hodge structure pure of weight k, which is functorial.

$$H^k_{Zar}(X, \Omega^*_{X/\mathbb{C}}) \simeq H^k(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{a+b=k} H^{a,b}(X) = \bigoplus_{a+b=k} H^b(X, \Omega^a_X).$$

Proof: This theorem holds considerable significance, and its proof can be found in[17]Chapter 3. The demonstration of this theorem relies on the Kähler form over X^{an} along with analytical methodologies. Deligne and Illusie demonstrated that the hypercohomology spectral sequence degenerates using purely algebraic techniques. However, their approach did not yield a direct sum decomposition. The challenge of finding a proof that relies exclusively on algebraic geometry remains an open problem.

To provide further context, it is important to note that the Kähler form plays a crucial role in complex geometry, defining the metric structure on manifolds and underpinning many complex geometric and topological properties. In their work, Deligne and Illusie ingeniously utilized tools from algebraic geometry, particularly through the introduction of spectral sequences to address intricate cohomological issues. Although their method successfully proved the degeneration of the spectral sequence, it did not offer an intuitive direct sum decomposition, highlighting the need for further investigation.

Moreover, the significance of this problem lies in its exploration of the deep connections between algebraic geometry and complex geometry. While Deligne and Illusie's work represents a significant advancement, the quest for a fully algebraic-geometric proof of the direct sum decomposition remains challenging. This pursuit not only tests existing tools and techniques but also drives researchers to explore new mathematical frameworks and theories.

In summary, the importance of this theorem extends beyond its immediate result, as it sheds light on the yet-to-be-fully-understood relationship between algebraic and complex geometries. Future research may bring breakthroughs that could ultimately resolve this open problem. ■

The word **functorial** in the theorem means that standard map of cohomology groups induces morphism of Hodge structures. However, this sentence only make sense when we consider Tate twist. Suppose $Y \subset X$ is a smooth closed subscheme with $\mathrm{codim}_X Y = r \geq 1$. One has the Gysin map

$$H^{i-2r}(Y^{an}, \mathbb{Q}) \to H^i(X^{an}, \mathbb{Q}).$$

This is not an interset morphism of Hodge structures, as the only morphisms between pure Hodge structures of different weights are trivial (zero) morphisms. To remedy this, one consider the twisted map

$$H^{i-2r}(Y^{an}, \mathbb{Q}(-r)) \to H^i(X^{an}, \mathbb{Q}).$$

Definition 5.1: Let $\phi : H^{2p}(X(\mathbb{C}), \underline{\mathbb{Z}}) \to H^{2p}(X(\mathbb{C}), \underline{\mathbb{C}})$ be the map induced by $\underline{\mathbb{Z}} \to \underline{\mathbb{C}}$. The group $H^{p,q}(X, \mathbb{Z})$ of **integral Hodge classes** of (p,p)-type is $\{x \in H^{2p}(X, \mathbb{Z}) : \phi(x) \in H^{p,p}(X, \mathbb{C})\}$. Similarly, the group $H^{p,p}(X, \mathbb{Q})$ of **rational Hodge classes** of (p,p)-type is $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$.

A major and important source of Hodge classes is the cycle class. There are many constructions. We will write down all the constructions as much as possible. In our case, we will consider singular cohomology group of X^{an}:

$$CH^r(X) \to H^{2r}(X^{an}, \mathbb{Z}(r)).$$

To define it, we need resolution of singularities.

Definition 5.2: Given a locally Noetherian reduced scheme X, we call a morphism $f : X' \to X$ such that X' is regular and f is proper and birational a **resolution of singularities** of X. When such a morphism exists, we say that we can solve the singularities of X.

The following famous theorem is well-known:

Theorem 5.3 (Hironaka,Temkin): For any reduced excellent scheme Y whose residue fields have characteristic zero, it is possible to achieve a resolution of singularities. Additionally, a resolution of singularities can also be obtained for any quasi-excellent Noetherian scheme Z of characteristic zero.

Recalling the exact sequence of sheaves:

$$0 \to 2\pi i =: \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1.$$

The cycle class map $CH^1(X) \to H^2(X^{an}, \mathbb{Z}(1))$ is the boundary map $H^1(X^{an}, \mathbb{Z}(1)) \simeq CH^1(X) \to H^2(X^{an}, \mathbb{Z}(1))$ associated with the sequence. We call it the **first Chern class**.

Following Grothendieck (For the case of K_0, refer to[35]; for higher K_n, consult[83]section 11 in Chapter V.), there are theories of Chern classes in both algebraic and analytic de Rham cohomology. Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1. One can show $c_1(\mathcal{L}) = c_1(\mathcal{L}^{an})$ under the isomorphism of Grothendieck by explicit cocycle construction. Using the projective bundle theorem and splitting principle, one can further show that

$$c_k(\mathcal{E}) = c_k(\mathcal{E}^{an})$$

for any vector bundle $\mathcal{E}$ under Grothendieck's comparison theorem. By the universal property of algebraic K-theory, there is a Grothendieck Chern class map

$$c_k : K_0(X) \to H^{2k}(X^{an}, \mathbb{Q}(k)).$$

Since X is smooth projective, there is a surjective map

$$\gamma : \mathcal{Z}(X) \to K_0(X), \quad Z \mapsto [\mathcal{O}_Z].$$

The cycle class of a cycle $Z \in \mathcal{Z}^k(X)$ is $c_k(\gamma(Z))$. According to[11]formula (4.4) on page 674, or the Riemann-Roch theorem without denominators, the cycle class is

$$[Z] = \frac{(-1)^{p-1}}{(k-1)!} c_k(\mathcal{O}_Z).$$

Remark 5.1: There exists an alternative definition of the cycle class map that does not rely on algebraic K-theory, as described in[46]Chapter X. For the case of algebraic de Rham cohomology, one may refer to[39]Section 7 in Chapter II. Similarly, these methods can be extended to the étale cycle class map provided that étale homology is appropriately defined, as demonstrated in[52]THÉORÈME (7.2).

In fact, the cycle class $[Z]$ is in $H^{2k}(X, \mathbb{Z})$. Let $Z \subset X$ be a codimension p cycle. It determines a **cycle class**

$$[Z^{an}] \in H^{2k}(X^{an}, \mathbb{Z}(k))$$

as follows: let $\mu : \tilde{Z} \to X$ be a resolution of singularities. By Poincaré duality, the linear function

$$H^{2n-2k}(X^{an}, \mathbb{Z}(n-k)) \to \mathbb{Z}, \quad \alpha \mapsto \frac{1}{(2\pi i)^{n-k}} \int_{\tilde{Z}^{an}} \mu^*(\alpha)$$

is represented by a unique class $\xi \in H^{2k}(X^{an}, \mathbb{Z}(k))$ with property that

$$\frac{1}{(2\pi i)^{n-k}} \int_{\tilde{Z}^{an}} \mu^*(\alpha) = \frac{1}{(2\pi i)^n} \int_{X^{an}} \xi \cup \alpha.$$

The class $[Z^{an}]$ is of type (k, k). Indeed, if $\alpha \in H^{2n-2k}(X^{an}, \mathbb{Z}(n-k))$ is of $(n-i, n-j)$-type with $i \neq j$, then either i or j is strictly greater than p, and $\int_{\tilde{Z}^{an}} \mu^*(\alpha) = 0$. Thus a cohomology class of a codimension k algebraic cycle gives rise to a (k, k)-Hodge class.

Lefschetz showed the cycle class for codimension 1 is surjective:

Theorem 5.4 (Lefschetz): The cycle class map

$$cl : CH^1(X) \to H^2(X, \mathbb{Z}(1))$$

is surjective on integral Hodge classes $H^{1,1}(X, \mathbb{Z})$.

Proof: The exponential short exact sequence of sheaves

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

induces a long exact sequence

$$\cdots \to H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X) \xrightarrow{cl} H^2(X^{an}, \mathbb{Z}(1)) \xrightarrow{f} H^2(X^{an}, \mathcal{O}_X) \to \ldots$$

The map f is identified with the composition

$$H^2(X^{an}, \mathbb{Z}(1)) \to H^2(X^{an}, \mathbb{C}) \to H^{0,2}(X) \cong H^2(X, \mathcal{O}_X)$$

So $\ker f$ is exactly the set of integral Hodge classes. A class $\alpha \in H^2(X, \mathbb{Z}(1))$ which maps to 0 in $H^2(X, \mathcal{O}_X)$ has $\alpha^{0,2} = 0$ in the Hodge decomposition. But then it also has $\alpha^{2,0} = \overline{\alpha^{0,2}} = 0$, and thus it is of type $(1, 1)$ hence a Hodge class. ■

Hodge conjectured that

$$cl : CH^p(X) \to H^{p,p}(X, \mathbb{Z})$$

is also surjective whenever $p \geq 2$ based on Lefschetz's result. However, Hodge's original conjecture is not true in general. See[5,73] and our latter discussion.

The millennium problem is:

Conjecture 5.1: The rational Hodge classes are algebraic, i.e. the map

$$cl_{\mathbb{Q}} : CH^p(X) \otimes \mathbb{Q} \to H^{p,p}(X, \mathbb{Q})$$

is surjective for every p.

There are still examples of integeral Hodge conjectures. Voisin showed that all integral Hodge classes on cubic fourfold are algebraic[82]Theorem 18:

Theorem 5.5: Let X be a cubic fourfold. Then the cycle class map

$$cl : CH^r(X) \to H^{r,r}(X, \mathbb{Z})$$

is surjective.

The Hodge conjecture is renowned for its formidable complexity, and our current understanding of it remains limited. However, its significance transcends its difficulty, lying in its profound implications for algebraic geometry and related fields.

The importance of rational Hodge classes arises from their deep connection to Grothendieck's standard conjectures through the Hodge conjecture. The failure of the integral Hodge conjecture presents significant challenges in establishing the rationality of smooth projective schemes. This failure highlights the intricate relationship between Hodge theory and algebraic cycles, underscoring the need for further research into these fundamental questions.

To provide additional context, the Hodge conjecture posits a bridge between the topological and algebraic aspects of complex algebraic varieties. It asserts that every Hodge class on a smooth projective variety over $\mathbb{C}$ can be expressed as a linear combination of algebraic cycles with rational coefficients. While this conjecture has been verified in certain special cases, its general proof remains elusive.

Moreover, the implications of the Hodge conjecture extend beyond its immediate statement. For instance, if the Hodge conjecture holds true, it would imply the validity of the standard conjectures over fields of characteristic zero. These conjectures, formulated by Grothendieck, are central to the theory of motives and have far-reaching consequences for the structure of algebraic varieties. In particular, they suggest that the category of motives over $\mathbb{C}$ would possess a rich and well-behaved structure, forming a semi-simple Tannakian category.

In summary, the Hodge conjecture not only represents a major open problem in mathematics but also serves as a cornerstone for understanding deeper connections between algebraic geometry and other branches of mathematics. Its resolution would provide significant insights into the nature of algebraic cycles and the structure of motives, thereby advancing our knowledge of complex algebraic varieties.

Proposition 5.1 (Claire Voisin): If a smooth projective $\mathbb{C}$-scheme X is birationally equivalent to projective space $\mathbb{P}^n$, then the integral Hodge conjecture is valid for codimensions $2n-2$ and 4.

An important cohomology theory in the Hodge theory is the Deligne-Beĭlinson cohomology. Its definition is simple, but its properties are complicated.

Definition 5.3: Let $\Omega_X^{*<n} := 0 \to \Omega_X^0 \to \Omega_X^1 \to \cdots \to \Omega_X^{n-1} \to 0$ be the truncated de Rham complex and A be a subring of $\mathbb{C}$. The algebraic Deligne complex $A(n)_D$ is

$$0 \to A(n) \to \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{n-1} \to 0.$$

The Deligne-Beĭlinson cohomology with coefficients R is defined as the hypercohomology

$$H_D^i(X; R(n)) := H_{Zar}^i(X; R(n)_D).$$

According to Beĭlinson, Deligne-Beĭlinson cohomology is extension of mixed Hodge structures. One important aspect of Deligne-Beĭlinson cohomology is the following exact sequence.

Theorem 5.6: The Deligne-Beĭlinson cohomology fits into an exact sequence:

$$0 \to J^k(X) = \mathrm{Ext}_{MHS}(\mathbb{Z}, H^{2k-1}(X; \mathbb{Z}(k))) \to H_D^{2k}(X; \mathbb{Z}(k)) \to H^{k,k}(X; \mathbb{Z}) \to 0.$$

Proof: We refer to[31]Section 7.8. ■

In light of Beĭlinson's profound conjectures on L-functions, the study of Deligne-Beĭlinson cohomology presents significant challenges.

The Hodge conjecture can be equivalently formulated using étale motivic cohomology and Deligne-Beĭlinson cohomology as follows.

Theorem 5.7 (Rosenschon et Srinivas): The Hodge conjecture is equivalent the generalized cycle class map

$$c_L^{p,q} : H_L^p(X; \mathbb{Z}(q)) \to H_D^p(X; \mathbb{Z}(q))$$

is injective on torsion parts.

Proof: See[66]section 5. ■

In Hodge theory, a key concept is that of the absolute Hodge class. To put it simply, an absolute Hodge class refers to a Hodge class that remains unchanged across various embeddings to $\mathbb{C}$.

Let $\sigma : k \to \mathbb{C}$ be an embedding of fields. Each smooth projective scheme X over $\mathrm{Spec}(k)$ base change to a scheme $X \times_{k,\sigma} \mathbb{C}$ over $\mathbb{C}$ along σ. Thus we can consider Hodge decomposition of $X \times_{k,\sigma} \mathbb{C}$.

Definition 5.4: A class $\alpha \in H_{dR}^{2p}(X; \Omega_{X/k})$ is called a Hodge class relative to σ if it is a

Hodge class under the base change map

$$H^{2p}_{dR}(X; \Omega_{X/k}) \to H^{2p}_{dR}(X \times_{k,\sigma} \mathbb{C}; \Omega_{X \times_{k,\sigma} \mathbb{C}/\mathbb{C}}).$$

The class α is an absolute if it is a Hodge class relative to every σ.

Remark 5.2: The homotopy type of $(X \times_{k,\sigma} \mathbb{C})(\mathbb{C})$ is dependent on σ according to Serre.

We can also define absolute Hodge class in term of étale ℓ-adic cohomology.

Definition 5.5: Let $\sigma : \bar{k} \to \mathbb{C}$ be an algebraic closure of k inside $\mathbb{C}$. We say a cohomology class $\alpha \in H^{2p}_{\text{ét}}(X \times_k \bar{k}; \mathbb{Q}_\ell(p))$ is a Hodge class relative to σ if its image under the Artin isomorphism

$$H^{2p}_{\text{ét}}(X \times_k \mathrm{Spec}(\bar{k}); \mathbb{Q}_\ell(p)) \simeq H^{2p}_{Betti}(X \times_{k,\sigma} \mathrm{Spec}(\mathbb{C}); \mathbb{Q}_\ell(q))$$

is a Hodge class.

Remark 5.3: This is not Deligne's definition in[25]. He considered étale cohomology and de Rham cohomology simultaneously.

The Hodge conjecture implies every Hodge class is absolute. Deligne proved it for abelian varieties unconditionally[25].

5.2 Obstruction to cohomology classes being algebraic

The universality of MU yields a natural transformation

$$\mu : MU^*(-) \to H^*(-, \mathbb{Z}).$$

When X is a complex manifold, the image of a class $f : Z \to X$ in $MU^*(X)$ by μ is $\mu([f]) = f_*(1) \in H^*(X, \mathbb{Z})$ where f_* is the Gysin morphism.

Totaro[73]section 3 notes that the integral cycle class map

$$\mathcal{Z}^k(X) \to H^{2k}(X, \mathbb{Z})$$

facotrs through complex cobordism

$$\mathcal{Z}^k(X) \to (MU^*(X) \otimes_{MU^*} \mathbb{Z})^{2k} \to H^{2k}(X, \mathbb{Z}).$$

Through factorization, it is evident that any class in $H^{2k}(X, \mathbb{Z})$ that does not belong to the image of the map

$$MU^{2k}(X) \otimes_{MU^*} \mathbb{Z} \to H^{2k}(X, \mathbb{Z})$$

cannot possess an algebraic nature.

Remark 5.4: Generally, the map

$$MU^*(X) \otimes_{MU^*} \mathbb{Z} \to H^*(X)$$

is neither injective nor surjective.

We can interpret the above factorisation using algebraic cobordism. Indeed, the following diagram is commutative:

$$\begin{tikzcd}
\Omega^*(X) \arrow[r] \arrow[d] & MU^{2*}(X^{an}) \arrow[d] \\
\Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \arrow[r] \arrow[d, "\simeq"'] & MU^{2*}(X^{an}) \otimes_{\mathbb{L}} \mathbb{Z} \arrow[d] \\
CH^*(X) \arrow[r] & H^{2*}(X^{an}, \mathbb{Z})
\end{tikzcd}$$

Horizontal arrows are natural maps by the universal property of Ω^* and CH^*. So Totaro's factorisation is the following compositions:

$$CH^*(X) \xrightarrow{\sim} \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to MU^{2*}(X^{an}) \otimes_{\mathbb{L}} \mathbb{Z} \to H^{2*}(X^{an}, \mathbb{Z}).$$

Proposition 5.2: Let X be a complex manifold. If $k \geq 1$. The image of the morphism $\mu : MU^k(X) \to H^k(X, \mathbb{Z})$ given by the universal property of MU is killed by integral cohomological operations.

Proof: An integral operation of degree $k > 0$ is given by a map of spectra

$$f : H\mathbb{Z} \to \Sigma^k H\mathbb{Z}.$$

For $k > 0$, the group $H\mathbb{Z}^k H\mathbb{Z}$ is finite[51]. In particular, it is torsion. The morphism μ is induced by a spectra map $\nu : MU \to H\mathbb{Z}$. The morphism

$$H^k(H\mathbb{Z}, \mathbb{Z}) \to H^k(MU, \mathbb{Z})$$

induced by ν sends $[f]$ to $[f \circ \nu]$. According to[50]Proposition 4.4.4, the group $H^k(MU, \mathbb{Z})$ is torsion free. Hence $[f \circ \nu]$ is trivial. ■

So, the cohomology operation can detect whether a cohomology class is algebraic. Let us review Steenrod operation. We assume all spaces are pointed.

Definition 5.6: Cohomology operations are a collection of transformation of cohomological functors:

$$f : \tilde{H}^*(-, \mathbb{F}_p) \to \tilde{H}^*(-, \mathbb{F}_p).$$

Definition 5.7: The mod-p Steenrod algebra $\mathcal{A}_p = \mathcal{A}_p^*$ is the $\mathbb{F}_p$-algebra of cohomology operations

$$\tilde{H}^*(-,\mathbb{F}_p) \to \tilde{H}^*(-,\mathbb{F}_p).$$

Milnor found a distinguished family of operations Q_i in[59], we called them **Milnor operations** nowadays.

Definition 5.8: We define Q_0 to be the Bockstein. Inductively, we define

$$Q_{i+1} := P^{p^i} Q_i - Q_i P^{p^i}.$$

Proposition 5.3: The following properties of Q_i are proved in[59].

(1) Q_i has degree $2p^i - 1$.

(2) $Q_i^2 = 0$ and $Q_i Q_j = -Q_j Q_i$. That is all Q_i generate an exterior algebra under composition.

(3) Q_i are derivations.

Proposition 5.4 (Olivier Benoist): Let X be a smooth projective scheme over $\mathbb{C}$. Fix $\alpha \in \mathcal{A}_p$ and x a reduction mod p of a cycle class.

(1) If $\deg(\alpha)$ is odd, then $\alpha(x) = 0$.

(2) If $\deg(\alpha)$ is even, then $\alpha(x)$ is a mod p reduction of a Hodge class.

By the Landweber exactness theorem, the BP cohomology of a space X is given by $MU^*(X) \otimes_{MU^*} BP^*$. A simple calculation shows

$$BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)} = MU^*(X) \otimes_{MU^*} \mathbb{Z}_{(p)}.$$

Hence the mod p-cycle class map factors through BP:

$$CH^*(X) \to BP^{2*}(X^{an}) \otimes_{BP^*} \mathbb{Z}_{(p)} \to H^{2*}(X^{an}, \mathbb{Z}_{(p)}) \to H^{2*}(X^{an}, \mathbb{Z}/p).$$

Recalling the fibre sequence of BP: $\Sigma^{2p^n-2} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \xrightarrow{\rho^n_{n-1}} BP\langle n-1\rangle$. By[71]Proposition 4-4, the following diagram is commutative:

$$\begin{tikzcd}
BP\langle n\rangle^k(X) \arrow[r, "\rho^n_{n-1}"] \arrow[dr, "\rho^n_{-1}"'] & BP\langle n-1\rangle^k(X) \arrow[r, "q_n"] \arrow[d, "\rho^{n-1}_{-1}"] & BP\langle n\rangle^{k+2p^n-1}(X) \arrow[d, "\rho^n_{-1}"'] \\
 & H^k(X;\mathbb{F}_p) \arrow[r, "\pm Q_n"'] & H^{k+2p^n-1}(X;\mathbb{F}_p)
\end{tikzcd}$$

where the top row is by the definition reason and the two vertival arrows are induced by

$$BP\langle n\rangle \to BP\langle 0\rangle = H\mathbb{Z}_{(p)} \to H\mathbb{Z}/p.$$

Combining with the mod p-cycle class map, we get the following diagram:

$$\begin{tikzcd}
 & BP^{2k}(X^{an}) \arrow[d] \arrow[dr, "f"] & & \\
 & BP\langle n\rangle^{2k}(X^{an}) \arrow[r, "\rho^n_{n-1}"'] & BP\langle n-1\rangle^{2k}(X^{an}) \arrow[r, "q_{n*}"] \arrow[d, "\rho^{n-1}_{-1}"'] & BP\langle n\rangle^{2k+2p^n-1}(X^{an}) \arrow[d, "\rho^n_{-1}"] \\
CH^k(X) \arrow[uur, "g"] \arrow[rr, "cl_p"'] & & H^{2k}(X^{an};\mathbb{Z}/p) \arrow[r, "\pm Q_n"'] & H^{2k+2p^n-1}(X^{an};\mathbb{Z}/p)
\end{tikzcd}$$

From the given diagram, we can infer that if $x \in H^{2k}(X;\mathbb{Z}/p)$ satisfies $Q_n(x) \neq 0$, then x cannot be algebraic. To elaborate, if x were algebraic, it would be represented as $cl_p(y)$. Consequently, this would imply $x = cl_p(y) = \rho^{n-1}_{-1} fg(y)$. However, it is known that $q_n(fg(y)) = 0$. Therefore, we have $Q_n(x) = \rho^n_{-1} q_n(fg(y)) = 0$, which contradicts the initial condition $Q_n(x) \neq 0$.

Remark 5.5: The non-trivial point is the map g. It relies on the theory of algebraic cobordism.

Composing different q_n yields a commutative diagram up to sign

$$\begin{tikzcd}
\Sigma BP\langle 0\rangle \arrow[r, "q_0"] & \Sigma^{|Q_0|+|Q_1|} BP\langle 1\rangle \arrow[r, "q_2"] & \dots \arrow[r, "q_n"] & \Sigma^{2\sum_{i=0}^{n} p^i - n-1} BP\langle n\rangle \arrow[dd, "q_{n+1}"] \\
H\mathbb{F}_p \arrow[u, "q_0"] \arrow[d, "Q_{n+1}Q_n\cdots Q_0"] & & & \\
\Sigma^{2\sum_{i=0}^{n+1} p^i - n-2} H\mathbb{F}_p & & & \Sigma^{2\sum_{i=0}^{n+1} p^i - n-2} BP\langle n+1\rangle \arrow[lll, "\rho^{n+1}_{-1}"]
\end{tikzcd}$$

since $\rho^n_{-1} q_n = Q_n \rho^n_{-1}$ up to a sign.

So, we deduce the following commutative diagram:

$$\begin{tikzcd}
BP\langle n\rangle^{k+2\sum_{i=0}^{n} p^i - n-1}(X) \arrow[d, "\rho^{n+1}_n"'] & \\
BP\langle n\rangle^{k+2\sum_{i=0}^{n} p^i - n-1}(X) \arrow[r, "q_{n+1}"] & BP\langle n+1\rangle^{k+2\sum_{i=0}^{n+1} p^i - n-2}(X) \arrow[d, "\rho^{n+1}_{-1}"] \\
H^k(X;\mathbb{F}_p) \arrow[u, "q_n\cdots q_0"'] \arrow[r, "Q_{n+1}\cdots Q_0"'] & H^{k+2\sum_{i=0}^{n+1} p^i - n-2}(X;\mathbb{F}_p)
\end{tikzcd}$$

From the diagram we can read that if $x \in H^k(X;\mathbb{F}_p)$ and $Q_{n+1} \dots Q_0(x) \neq 0$, then $q_n \dots q_0(x) \notin \mathrm{Im}(\rho^{n+1}_n)$. We will construct such an x.

5.3 Betti realisation

Let Y be a complex analytic space. Denote by $AnSm/Y$ the category whose objects are smooth analytic spaces over Y. This category is endowed with the standard topology. Let $SH(AnSm/Y, \mathrm{Sp})$ represent the category of hypersheaves of spectra on $AnSm/Y$ with respect to this topology.

Definition 5.9: Let $\mathbb{D}^1$ be a complex disc. We define $SH_{\mathbb{D}^1}(AnSm/X, \mathrm{Sp})$ be the left Bousfield localisation of $SH(AnSm/X, \mathrm{Sp})$ with respect to $\{\mathbb{D}^1_Y \otimes A \to Y \otimes A\}$ with $Y \in AnSm/X$ and $A \in \mathrm{Sp}$.

We denote $Ouv(X)$ the set of open subspace of X. We denote by $\ell_X : Ouv(X) \to AnSm/X$ the obvious inclusion that provides a pair of adjoint functors

$$\ell_X^* : \mathrm{PSh}(Ouv(X), \mathrm{Sp}) \rightleftarrows \mathrm{PSh}(AnSm/X, \mathrm{Sp}) : \ell_{X_*}.$$

It is trivially descent to

$$\ell_X^* : D(X, \mathrm{Sp}) := SH(Ouv(X), \mathrm{Sp}) \rightleftarrows SH(AnSm/X, \mathrm{Sp}) : \ell_{X_*}.$$

Ayoub proved the following theorem[6]Théorème 1.8:

Theorem 5.8 (Joseph Ayoub): The category $D(X, \mathrm{Sp})$ of hypersheaves of spectra is equivalent to $SH_{\mathbb{D}^1}(AnSm/X, \mathrm{Sp})$.

In particular, when $X = pt$ is a single point, this theorem provides an equivalence of categories

$$SH_{\mathbb{D}^1}(AnSm/X, \mathrm{Sp}) \simeq \mathrm{Sp}.$$

We will define Betti realisation after we review analytification of schemes. For details, see[36].

Theorem 5.9: The functor Φ that maps an analytic space $\mathfrak{X}$ to the set of morphismes of ringed spaces of $\mathbb{C}$-algebras $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{X}, X)$ is represented by an analytic space X^{an} along with a morphism $\phi : X^{\mathrm{an}} \to X$. The space X^{an} is referred to as the analytic space associated with X.

If $|X^{an}|$ is the underlying set of X^{an}, ϕ induces a bijection between $|X^{an}|$ and $X(\mathbb{C})$.

Proof: This is classical. See[36]Exposé XII. Théorème et définition 1.1. ■

Example 5.1: Let $f : Y \to X$ be an étale morphism. Then $f^{an} : Y^{an} \to X^{an}$ is a local isomorphism.

We have an analytic functor

$$An_X : Sm/X \to AnSm/X^{an}.$$

The functor An_X induces an adjunction of hypersheaves of spaces:

$$An_X^* : \mathrm{Sh}_{Nis}(Sm/X) \rightleftarrows \mathrm{Sh}(AnSm/X^{an}) : An_{X_*}.$$

We have $An_X^*(\mathbb{A}_Y^1) = \mathbb{A}_{Y^{an}}^{1,an}$ and $\mathbb{A}_{Y^{an}}^{1,an}$ is homotopy equivalent to $\mathbb{D}^1$ as topological spaces since we care about homotopy type. Thus we have an adjunction

$$An_X^* : SH(X) \rightleftarrows SH_{\mathbb{D}^1}(AnSm/X^{an}) : An_{X_*}.$$

Definition 5.10: The **Betti realisation** functor of $SH(X)$ is the following two compositions

$$SH(X) \to SH_{\mathbb{D}^1}(AnSm/X^{an}) \to D(X, \mathrm{Sp}).$$

Remark 5.6: One may see[28]Definition 4.4. or[20]Chapter 17 for more general construction.

One has canonical homotopy equivalences $\mathrm{Betti}_X(S^1) = S^1$ and $\mathrm{Betti}_X(\mathbb{G}_m) = S^1$. Then we have canonical morphism $E^{p,q}(X) \to (\mathrm{Betti}(E))^p(X(\mathbb{C}))$.

Example 5.2: The following two examples show that Betti realisation gives classical theories.

（1）From the constructions, we know that $\mathrm{Betti}(MGL) = MU$, $\mathrm{Betti}(MBP) = BP$, and $\mathrm{Betti}(MBP\langle n\rangle) = BP\langle n\rangle$.

（2）The Betti realisation of the motivic Eilenberge-Mac Lane spectrum $M\mathbb{Z}$ is Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ by Dold-Thom theorem and Voeovdsky's construction of $M\mathbb{Z}$.

For completeness, we describe the Betti realisation of Voevodsky motives. The analytic functor

$$An : Sm/\mathbb{C}_{Nis} \to Top, \quad X \mapsto X(\mathbb{C})$$

is continuous. It induces a functor

$$An^s : Sh_{Nis}(Sm/\mathbb{C}) \to Sh(Top).$$

Proposition 5.5: If X is smooth projective, the image $An^s(\mathbb{Z}^{tr}(X))$ is the sheaf associated with the presheaf

$$U \mapsto \mathrm{Hom}(U, \coprod_{d\geq 0} S^d X(\mathbb{C}))^+$$

where the scheme $S^d X$ is the symmetric power of X.

Proof: We explain notations. The symmetric power $S^d X := X^d/\Sigma_d$ where X^d is d times the fibre product of X and symmetric group Σ_d acts (permuting the facotrs) on

$X^d = X \times \cdots \times X$. By[80]Proposition 3.5,

$$\mathbb{Z}_{eff}^{tr}(X)(Y) = \mathrm{Hom}_{Sch/\mathbb{C}}(Y, \coprod_{d\geq 0} S^d(X))$$

for any semi-normal S. Combination with $\mathbb{Z}^{tr}(X) = a_{Nis}(\mathbb{Z}_{eff}^{tr}(X)^+)$ we get

$$\mathbb{Z}^{tr}(X)(Y) = \mathrm{Hom}_{Sch/\mathbb{C}}(Y, \coprod_{d\geq 0} S^d(X))^+.$$

As the functor An^s maintains the property of representable functors, it transforms, for all positive integers d and for any smooth scheme X, the sheaf X^d into the sheaf $X^d(\mathbb{C})$. Moreover, the functor An^s commutes with any colimits: it sends the sheaf represented by the smooth ind-scheme $\coprod_{d\geq 0} S^d X$ to the sheaf represented by the space $\coprod_{d\geq 0} S^d(X(\mathbb{C}))$. Since the functor is compatible with algebraic structures, this proposition can be deduced from this. ■

Composing the functor An^s with the exact functor forgetful of transfers, we obtain the right exact functor

$$\Phi : Sh_{Nis}(Cor(\mathbb{C})) \to Sh(Top, Ab).$$

Composing with the Suslin complex functor C_* (3.11), we obtain a right exact functor

$$\Psi : Sh_{Nis}(Cor(\mathbb{C})) \to C^-(Sh(Top, Ab)).$$

Proposition 5.6: For arbitrary smooth quasi-projective smooth X over Spec($\mathbb{C}$), the image $\Psi(\mathbb{Z}^{tr}(X))$ is the complex computing the singular cohomology of X^{an}.

Proof: See[69]Section 8. ■

We deduce from Ψ the topological realisation functor

$$t_{\mathbb{C}} : DM^{-,eff}(\mathbb{C}) \to D(\mathbb{Z}).$$

Example 5.3: Recall the well-known calculations of de Rham cohomology groups of $\mathbb{G}_m$ are $H_{dR}^0(\mathbb{G}_m) = k$, $H_{dR}^1(\mathbb{G}_m) = k[dt/t]$, and 0 otherwise. According to Grothendieck, there is a functorial isomorphism given by integral:

$$H_{dR}^i(X) \xrightarrow{\sim} H_B^i(X) \otimes_{\mathbb{Q}} \mathbb{C}, [\omega] \mapsto \int \omega.$$

Hence, the image of $\mathbb{Z}(1) = M(\mathbb{G}_m)[-1]$ is the complex computing the reduced singular homology of $\mathbb{C}^*$, shifted by -1. So, $t_{\mathbb{C}}(\mathbb{Z}(1)) = 2\pi i\mathbb{Z}$ centered in degree 0, by the residue theorem.

Since the functor $t_{\mathbb{C}}$ is monoïdal and sends the motive $\mathbb{Z}(1)$ to an invertible object, it extends to a functor to $DM^-(\mathbb{C})$ and the results of this paragraph can be summarised as

Theorem 5.10: There exists a symmetric monoidal topological realisation functor

$$t_{\mathbb{C}} : DM^{-}(\mathbb{C}) \to D(\mathbb{Z})$$

which for the motive $M = M(X)$ associated with a smooth quasi-projective scheme X on $\mathbb{C}$ allows us to represent the singular cohomology of X^{an}, i.e.

$$H^p(X^{an};\mathbb{Z}) = \mathrm{Hom}_{D^{-}(\mathbb{Z})}(t_{\mathbb{C}}(M(X)), \mathbb{Z}[p]).$$

The compatibility with the product requires $t_{\mathbb{C}}(\mathbb{Z}(q)) = (2\pi i)^q\mathbb{Z}$. So, the Betti realisation of motivic cohomology $H^{p,q}(X;\mathbb{Z})$ is $H^p(X^{an};\mathbb{Z}(q))$. This fascinates us to study the Hodge theory through motives.

If the scheme X is defined over Spec($\mathbb{R}$), the analytic variety of the complex points $(X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C})(\mathbb{C})$ is provided with an action of the complex conjugation τ. Following this action in the previous construction, it is shown that the topological realisation functor is factored into a diagram

$$\begin{tikzcd} DM^{-}(\mathbb{R}) \arrow[r, "t_{\mathbb{C},\tau}"] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"'] & D(\mathbb{Z}^{\sigma}) \arrow[d] \\ DM^{-}(\mathbb{C}) \arrow[r, "t_{\mathbb{C}}"'] & D(\mathbb{Z}) \end{tikzcd}$$

where the category $\mathbb{Z}^{\sigma}$ refers to abelian groups equipped with an involution. The morphism on the right is induced by disregarding the involution structure.

The Tate motive $\mathbb{Z}(1)$ is real and on its realisation $t_{\mathbb{C}}(\mathbb{Z}(1))$, the involution is induced by the change of orientation of S^1 in $\mathbb{C}^*$ and acts by multiplication by -1.

For any embedding $\sigma : k \to \mathbb{C}$, the extension of the scalars $\sigma_{\mathbb{C}} : DM^{-}(k) \to DM^{-}(\mathbb{C})$ composed with the topological realisation, defines a functor

$$t_{\sigma} : DM^{-}(k) \to D(\mathbb{Z}), \quad M \mapsto t_{\mathbb{C}} \circ \sigma_{\mathbb{C}}(M) =: M_{\sigma}(\mathbb{C}).$$

We then set $H_{\sigma}(M, q) = \mathrm{Hom}_{D(\mathbb{Z})}(M_{\sigma}(\mathbb{C}), (2\pi i)^q\mathbb{Z})$.

If the motivic complex $M = M(X)$ is the motive of a smooth quasi-projective scheme, the Betti realisation coincides with the singular cohomology groups

$$H^p_{\sigma}(M(X), \mathbb{Z}(q)) = H^p(X^{an}, (2\pi i)^q\mathbb{Z}).$$

5.4 Non-algebraic cohomology classes

The Betti realisation functor induces natural transformations

$$H^{p,q}(X;\mathbb{Z}) \to H^p(X^{an};\mathbb{Z}(q))$$

and

$$MBP\langle n\rangle^{p,q}(X) \to BP\langle n\rangle^{p}(X^{an}).$$

Since $H^{2p,p}(X) = CH^{p}(X)$, we denote by cl_n the natural map

$$cl_n : MBP\langle n\rangle^{2i,i}(X) \to BP\langle n\rangle^{2i}(X^{an}).$$

It is a little complicate to check why the Betti realisation gives the cycle class map. We omit it. The methods in[47] would go through.

We get the following obvious commutative diagram induced by universal property of MBP and Betti realisation:

$$\begin{tikzcd} MBP\langle n\rangle^{2i,i}(X) \arrow[r, "cl_n"] \arrow[d, "\rho^n_{-1,M}"'] & BP\langle n\rangle^{2i}(X^{an}) \arrow[d, "\rho^n_{-1}"] \\ H^{2i,i}(X;\mathbb{F}_p) \arrow[r, "cl"'] & H^{2i}(X^{an};\mathbb{F}_p) \end{tikzcd}$$

Theorem 5.11 (Gereon Quick): For any integer $n \geq 0$, there is a smooth projective scheme X and a $b_n \in BP\langle n\rangle^{2\sum_{i=0}^{n} p^i+2}(X^{an})$ such that b_n is not contained in

$$cl_n : MBP\langle n\rangle^{2\sum_{i=0}^{n} p^i+2,\sum_{i=0}^{n}+1}(X) \to BP\langle n\rangle^{2\sum_{i=0}^{n} p^i+2}(X^{an}).$$

When $n = 0$, the map cl_n is

$$cl : M\mathbb{Z}_{(p)}^{4,2}(X) = CH^2(X;\mathbb{Z}_p) \to H\mathbb{Z}_{(p)}^{4}(X^{an}) = H^4(X^{an};\mathbb{Z}_{(p)}).$$

So, we get a non-algebraic Hodge class.

Proof: The scheme X is the so-called Godeaux-Serre variety. Based on Proposition (6.6) from[5], given any finite group G and any integer n greater than or equal to 3, one can find a projective smooth scheme X such that its associated analytic space X^{an} exhibits n-equivalence to the space $K(\mathbb{Z},2) \times BG$. We treat the case $p = 2$.

We take the group $G = (\mathbb{Z}/2)^{n+3}$. Let X be the Godeaux-Serre variety associated with G. Let

$$\phi : X^{an} \to BG \times K(\mathbb{Z},2) \to BG$$

where the first map is the $2\sum_{i=0}^{n+1} p^i + 1$-equivalent. Considering the first map is k-equivalent, each non-zero element within $H^i(G;\mathbb{Z}/2)$ gets transformed into a non-zero element in $H^i(X^{an};\mathbb{Z}/2)$. By[1]Theorem 4.4 on page 66, we have

$$H^*(G;\mathbb{Z}/2) = \mathbb{F}_2[x_1,\dots,x_{n+3}], |x_i| = 1.$$

According to[63]Lemma 3.2, the element

$$Q_n \dots Q_0(x_1 \dots x_{n+3}) \in H^{2\sum_{i=0}^n p^i+2}(G;\mathbb{Z}/2)$$

is non-trivial. Thus

$$y_n := \phi^* Q_n \dots Q_0(x_1 \dots x_{n+3}) = Q_n \dots Q_0(\phi^*(x_1 \dots x_{n+3})) \in H^{2\sum_{i=0}^n p^i+2}(X^{an};\mathbb{Z}/2)$$

is also non-trivial. We define

$$b_n := q_n \dots q_0(\phi^*(x_1 \dots x_{n+3})) \in BP\langle n\rangle^{2\sum_{i=0}^n p^i+2}(X^{an}).$$

Its image in $H^{2\sum_{i=0}^n p^i+2}(X^{an};\mathbb{Z}/2)$ is $\pm y_n$. By the discussion after 5.5,

$$b_n \notin (\mathrm{Im} : BP\langle n+1\rangle^{2\sum_{i=0}^n p^i+2}(X^{an}) \to BP\langle n\rangle^{2\sum_{i=0}^n p^i+2}(X^{an})).$$

Observing the following commutative diagram

$$\begin{tikzcd}
BP^*(X^{an}) \arrow[r] \arrow[rrd, "\rho_{-1}"'] & BP\langle n+1\rangle^*(X^{an}) \arrow[r, "\rho_n^{n+1}"] & BP\langle n\rangle^*(X^{an}) \arrow[d, "\rho_{-1}^n"] \\
 & & H^*(X^{an};\mathbb{Z}/2)
\end{tikzcd}$$

we can conclude that $b_n \notin \mathrm{Im}(cl_n)$. If p is an odd prime number. The argument is almost the same. ■

CONCLUSION

Our primary contribution lies in:

（1）Elucidating some crucial properties of algebraic cycles. Presenting examples of the complexity of Chow groups.

（2）Supplying a straightforward application of the motivic homotopy theory. Highlighting the connection between the Hodge Conjecture and the theory of motives.

（3）There should also be analogous applications in étale cohomology. A possible issue is whether the isomorphism

$$MGL^{2*,*}(X) \otimes_{\mathbb{L}} \mathbb{Z} \simeq CH^*(X)$$

can be generalized to fields of positive characteristic. Precisely through this isomorphism, we decomposed les morphismes classes de cycle in the case of characteristic 0. If the aforementioned isomorphism can be realised in fields of positive characteristic, then we could obtain a counterexample to the integral Tate Conjecture.

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