硕士学位论文

代数几何中的同伦论方法以上同调运算观之

METHODS OF HOMOTOPY THEORY IN ALGEBRAIC GEOMETRY FROM THE VIEWPOINT OF COHOMOLOGY OPERATIONS

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METHODS OF HOMOTOPY THEORY IN ALGEBRAIC GEOMETRY FROM THE VIEWPOINT OF COHOMOLOGY OPERATIONS

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摘 要

本文是一篇关于代数几何中同伦论方法的综述，从概念和计算两个角度，以动机上同调运算（motivic cohomology operation）、特别是幂运算为主线。文章分为两部分：第一部分是某些经典同伦论方法的回顾，侧重于同伦代数和谱序列；第二部分介绍动机同伦论及其相关构造的新进展，侧重于动机同伦论中的乘积结构。

第一部分是文章的第 1 至 5 章。第 1 章回顾同伦论的发展历史，并给出一些在具体几何拓扑问题上应用同伦论的例子。第 2、3 章介绍了无穷范畴及其上面的代数结构，重点讨论如何用纤维范畴上的 Grothendieck 构造刻画乘积结构，并为后文研究动机同伦论中的乘积结构做铺垫。第 4 章介绍系统地构造幂运算的方法，并给出了 mod p 系数上同调幂运算新的直观阐释，直接地展示幂运算内蕴的同伦相容性（homotopy coherence）。第 5 章总结了 Bruner 关于广义 Adams 谱序列和延拓幂（extended power）的工作，并详细解释了广义 Adams 谱序列如何探测同伦运算。

第二部分是文章的第 6 至 9 章。第 6 章介绍 Robalo，Bachmann 和 Hoyois 用第 2、3 章提供的框架简洁地构造非稳定动机同伦范畴和稳定动机同伦范畴的方法，其中主要强调稳定动机同伦范畴的万有性质。第 7 章主要介绍 Bachmann 和 Hoyois 构造的动机同伦论中的范数（norm）及其相关性质，并将其作为动机同伦论中乘积结构的核心框架。第 8 章通过引入等变动机同伦论和 Bachmann，Elmanto 和 Heller 提出的动机余极限的概念，介绍如何从范数结构中诱导出动机延拓幂和动机幂运算，并讨论了它们与 Voevodsky 构造的动机幂运算之间的联系。第 9 章简要地介绍动机上同调和动机 Adams 谱序列，侧重于应用前文讨论的方法，并讨论了未来可能进行研究的方向，包括动机复形上的乘积结构，动机同伦论中的结构化环谱（structured ring spectra）的构造，以及对动机 Adams 谱序列的推广。

关键词：动机同伦论；上同调运算；谱序列

ABSTRACT

This paper is a survey of certain methods of homotopy theory in algebraic geometry, emphasizing how conceptual structures integrate with computational tools. It consists of two parts. The first part (Chapter 1 - Chapter 5) discusses classical homotopy theory, including ∞-categories, higher algebras via stable ∞-categories, power operations, and Adams spectral sequences. The second part (Chapter 6 - Chapter 9) discusses motivic homotopy theory and its multiplicative structure. We first introduce how to use ∞-categories and higher algebra to construct unstable motivic homotopy category and stable motivic homotopy category. Then we introduce Bachmann and Hoyois's construction of norms in motivic homotopy theory and explain how norms lift certain arithmetic phenomena (i.e. norms of Galois field extensions) in algebraic geometry to the homotopy-theoretic level. Based on this, we show how to derive motivic power operations from norms using motivic extended powers and equivariant motivic homotopy theory following the joint work of Bachmann, Elmanto, and Heller. In the end, we apply all the discussed methods to motivic cohomology and motivic Adams spectral sequences and discuss possible future research topics.

Keywords: motivic homotopy theory; cohomology operations; spectral sequences

TABLE OF CONTENTS

INTRODUCTION

Methods of homotopy theory have both conceptual and computational perspectives. These methods turn out to be effective in the study of spaces up to continuous deformation. For example, cohomology operations play an important role in both understanding the abstract properties of spaces and computing their concrete invariants. In Chapter 1, we introduce some historical background about these topics. In particular, power operations are the most important cohomology operations and one perspective we try to show in this paper is that power operations conceptually reflect the multiplicative homotopy coherence (coherence usually means that some specific diagrams commute) depicted by abstract homotopy theory in cohomology, and computationally serve as the foundation of spectral sequence algorithms.

From this viewpoint, the paper is divided into two parts. In the first part, we introduce some basic notions and results about ∞-categories in Chapter 2 and higher algebra via stable ∞-categories in Chapter 3 as the desired categorical framework. Then we review some basic concepts and constructions of power operations in classical homotopy theory and try to reveal how they reflect the multiplicative coherence in Chapter 4.

Next, we study how spectral sequences organize the computations of power operations. More concretely, we study how the power operations that are presented in the E_2-page reflect the multiplicative information of the homotopy classes in the E_∞-page along an Adams spectral sequence following Bruner's work[1] in Chapter 5. We summarize Bruner's work in the following diagram.

$$
\begin{tikzcd}
\text{power operations} \arrow[dd, "\text{displayed}"'] & \mathbb{H}_\infty\text{-structure} \arrow[l, "\text{induces at}"', "\text{cohomology level}"] \arrow[r, "\text{induces at}", "\text{homotopy level}"'] & \text{homotopy operations} \arrow[dd, "\text{displayed}"] \\
 & \boxed{\text{How to convert cohomology operations into homotopy operations}} \arrow[u, Rightarrow, "\text{encoded}"'] \arrow[d, Rightarrow, "\text{presented}"] & \\
\underset{E_2\text{-page}}{\mathrm{Ext}^{s,t}_{E^*E}(E^*Y, E^*X)} \arrow[rr, Rightarrow, "\text{differentials in the Adams spectral sequence}", "\text{converges along the Adams spectral sequence}"'] & & \underset{E_\infty\text{-page}}{[X, Y]^E_{t-s}}
\end{tikzcd}
$$

In the second part, we focus on exploring methods of homotopy theory in algebraic geometry with a theme of motivic power operations. However, algebraic geometry poses some challenges for applying homotopy theory methods directly due to its richer structure and more complicated phenomena. To overcome these challenges, we use ∞-categories to introduce the framework of motivic homotopy theory in Chapter 6. This is a variant of classical homotopy theory that takes into account the arithmetic information encoded by schemes. Motivic homotopy theory allows us to construct and study cohomology theories for schemes that respect both their geometric and arithmetic features.

With this setup, we study norms in motivic homotopy theory following Bachmann-Hoyois's construction[2] in Chapter 7. Norms are generalizations of norms for finite Galois extensions and are the encapsulation of the coherence data in a multiplicative structure which was elusive previously. Moreover, Bachmann and Hoyois defined the notion of normed motivic spectra as an analog of a structured ring spectrum in classical homotopy theory.

Recall that encoding coherence data is also the *raison d'être* of $\mathbb{E}_\infty$-structures and $\mathbb{H}_\infty$-structures in classical homotopy theory. The comparison between the classical homotopy theory and motivic homotopy theory can be shown in the following diagram.

$$\begin{tikzcd}
\underset{\text{based finite-set morphisms}}{\mathcal{F}in_*} \arrow[d, "\text{parametrize}"] & \boxed{\text{intrinsic symmetry}} \arrow[l, "\text{presented as}"'] \arrow[r, "\text{presented as}"] \arrow[d, "\text{encoded in}"] & \underset{\text{finite étale morphisms}}{\mathrm{FEt}} \arrow[d, "\text{parametrize}"'] \\
\underset{\mathbb{E}_\infty\text{-operad action}}{\mathbb{E}_\infty\text{-structure}} \arrow[d, "\text{pass to homotopy}"] & \text{multiplicative coherence data} \arrow[l, "\text{encoded in}"'] \arrow[r, "\text{encoded in}"] & \text{motivic norm structure} \arrow[d, dashed, "\text{take motivic colimits}"'] \\
\underset{\mathbb{E}_\infty\text{-operad homotopy action}}{\mathbb{H}_\infty\text{-structure}} \arrow[d, "\text{Construction 4.13}"] & & \text{motivic extended powers} \arrow[d, dashed, "\text{Construction 8.12}"'] \\
\text{power operations} & & \text{motivic power operations}
\end{tikzcd}$$

In particular, it is effective to use the technique of spans to describe the coherence for $\mathcal{F}in_*$ and FEt, see Appendix C in the joint work of Bachmann and Hoyois[2]. As a part of the diagram, we show also how norms induce motivic power operations on motivic spectra via motivic extended powers following Bachmann-Elmanto-Heller's work[3] in Chapter 8. We introduce how they define a notion of colimit for diagrams in a motivic category indexed by a presheaf of spaces (e.g. an étale classifying space), and study the basic properties of this construction. As a case study, it constructs the motivic analogs of the classical extended and generalized powers, which refine the categorical versions of these constructions as special cases. It also offers more computationally tractable models of these constructions using equivariant motivic homotopy theory.

Finally, we give a brief introduction to motivic cohomology and motivic Adams spectral sequences in Chapter 9. Moreover, we discuss how to apply the methods in previous chapters to motivic cohomology and motivic Adams spectral. Specifically, we focus on three entry points:

- the multiplicative structure on motivic cohomology,
- the theory of motivic structured ring spectra,
- how to generalize motivic Adams spectral sequences following Bruner's work

In particular, We hope that the diagram of Bruner's work has a motivic version so that we can gain more insights into the multiplicative coherence encoded by norms.

CHAPTER 1 HISTORICAL BACKGROUND

In this chapter, we introduce some historical background of cohomology operations and motivic homotopy with concrete examples.

1.1 Classification problems and homotopy theory

Classifying objects up to a specified equivalence relation is central to nearly all of geometry and topology. Many beautiful theorems are solutions to particular classification problems, such as the classification theorem of closed surfaces, or they are motivated by classifications, such as partial solutions to the generalized Poincaré conjecture. Some of the deepest results related to classification problems made essential use of the methods of algebraic topology, translating geometric questions to computations with algebra.

Algebraic topology carries out such translations from a geometric problem into an algebraic problem by taking invariants. The effective working of this type of methods depends on the following two aspects:

- the associated algebraic problem captures the essential features of the geometric problem;
- the associated algebraic problem is sufficiently simple to solve.

Actually, these two aspects are reciprocal to each other: the more geometric information an algebraic problem encodes, the more difficult it is to solve. In this case, homotopy theory plays a central role in reconciling these two aspects. The strategy of homotopy theory to resolve a classification problem is to convert a task of classifying objects (spaces, manifolds, etc.) into a task of classifying related (stable) homotopy classes. This method works effectively because in the first place, with proper set-up, homotopy classes are able to capture sufficient geometric features. Here are some examples.

Theorem 1.1 (Thom[4]): Let G be a subgroup of $\mathrm{GL}(F, k)$ for $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Suppose X is an n-dimensional manifold. Then there is a bijection between the set of cobordism classes of submanifolds of X with a G-structure (on their tangent bundle) and the set of homotopy classes of continuous maps from X to a Thom space MG:

$$\{G\text{-cobordism classes in } X\} \xleftrightarrow{\text{bijection}} [X, MG]$$

Theorem 1.2 (Pontryagin[5]): There is a bijection between the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$ and the set of homotopy classes of continuous maps between spheres:

$$\{k\text{-dimensional framed cobordism classes in } \mathbb{R}^{n+k}\} \xleftrightarrow{\text{bijection}} [\mathrm{S}^{n+k}, \mathrm{S}^n]$$

Theorem 1.3 (Steenrod[6]): Given a topological group G, there is a bijection between the set of isomorphic classes of principal G-bundles over a paracompact space X and the set of homotopy classes of continuous maps from X to a classifying space BG:

$$\{\text{isomorphism classes of principal } G\text{-bundles on } X\} \xleftrightarrow{\text{bijection}} [X, BG]$$

These theorems demonstrate a general principle that classifying geometric objects of a specific type is equivalent to classifying homotopy classes of maps to a corresponding object. The homotopical structure of this “classifying object” largely determines the classification in question.

In addition to the capability of homotopy classes to encode geometric information, there are many tools to effectively address the associated algebraic problems with homotopy classes, making them easier to solve. The most significant ones are homological and cohomological types of machinery with operations, which we discuss in the next section.

1.2 How cohomology operations work

The method of detecting homotopy classes by cohomology operations originated from Steenrod’s work[7]. In this work, Steenrod constructed a device called cup-i products, for $i \geq 1$, as a higher-order analog of cup product to give some results on the classification of homotopy classes of maps from an $(n+1)$-dimensional complex to the n-dimensional sphere. Specifically, Steenrod used cup-i products to derive a family of cohomology operations called Steenrod squares on mod-2 cohomology. These are the first examples of cohomology operations. From the 1950s to the 1960s, Steenrod developed the theory of such cohomology operations[8-11]. On the mod-p cohomology, these cohomology operations along with the Bockstein operations became known as Steenrod operations. They were widely applied to solve various problems in topology and geometry. For example, Borel and Serre proved that S^{2n}, for $n \geq 4$, do not admit an almost complex structure[12]. Around the same time, Thom solved the Steenrod problems of determining when an integral or mod-2 homology class of a finite-dimensional polyhedron can be realized as a manifold [4].

In the previous examples, the crux is to exploit the actions of Steenrod operations on cohomology rings. From this viewpoint, it is natural to use homological methods to analyze these actions. Specifically, mod-p stable cohomology operations form an algebra called the Steenrod algebra $\mathcal{A}_p$. In the 1950s, Adem discovered a set of relations in $\mathcal{A}_p$ [13-14]; Serre showed that Steenrod operations and their Adem relations fully determine the algebra $\mathcal{A}_p$ as generators and relations [15]; Milnor showed the Hopf algebra structure of Steenrod algebras and their duals [16]. In the same period, Adams invented his famous spectral sequences [17] to show the existence of Hopf elements in $\pi_{2n-1}(\mathrm{S}^n)$ and $\pi_{4n-1}(\mathrm{S}^{2n})$ for $n \leq 4$. The significance of the Adams spectral sequences is to exhibit how Steenrod operations detect homotopy classes and illustrate the extent to which the information is detected. In particular, Greenlees explained how the Adams spectral sequences "cure the blindness" of a cohomology theory in his enlightening article [18].

In the 1960s, the monograph by Steenrod and Epstein was published and it gives a comprehensive introduction to cohomology operations [19]. Specifically, the authors presented a systematic method to construct power operations by using transfers and extended powers. It is tempting to think of and desirable in practice that such power operations be applicable to cohomology theories other than mod-p ordinary cohomology and this systematic construction work more generally.

This is indeed the case. In fact, power operations on other generalized cohomology theories led to even deeper results than their ordinary analogues did. For example, Adams and Atiyah constructed power operations in K-theory, called the Adams operations [20]. Equipped with these, Adams solved the problem of vector fields on spheres completely [21], obtaining stronger results than those in [22] with Steenrod squares. Adams and Atiyah also presented an elegant solution to the Hopf invariant one problem [23], which is conceptually much simpler than Adams's proof using secondary cohomology operations on ordinary cohomology [24]. Besides K-theory, tom Dieck constructed power operations in cobordism theory [25]. Quillen then used these operations to show that the cobordism ring $MU^*(X)$ is generated by $\bigoplus_{i\geq 0} MU^i(X)$ as an $MU^*(pt)$-module, and deduced his theorem on formal group laws that the complex cobordism ring is isomorphic to the Lazard ring [26]. As demonstrated above, the structure of cohomology operations is prevalent and carries a wealth of information through various cohomology theories. For deeper investigations, we need a framework to conceptualize cohomology theories in order to exploit this intrinsic structure of power operations. In fact, stable homotopy theory serves as a desired framework.

1.3 Cohomology theories and stable homotopy theory

In[27], E.H. Brown showed that for each generalized cohomology theory h^*, there exists a sequence of spaces $\{E_n\}$ with structure maps $\varepsilon_n : E_n \to \Omega E_{n+1}$ such that $h^n(Y) \cong [Y, E_n]$ and the suspension isomorphisms are induced by the adjoint maps of $\{\varepsilon_n\}$. These spaces with structure maps form a spectrum in the sense of[28] and we say h^* is represented by the spectrum $E = \{E_n, \epsilon_n\}$. For example, ordinary cohomology theory with coefficient ring R is represented by the Eilenberg-MacLane spectrum HR, complex K-theory is represented by the K-theory spectrum KU, and G-cobordism theory is represented by the Thom spectrum MG for a classical group G[29]. In general, the representability theorem indicates that the study of the generalized cohomology theories is equivalent to the study of spectra, which are central objects in stable homotopy theory. Notably, the manipulation of Steenrod operations can be simplified in the context of stable homotopy theory. For example, Rudyak showed how to simplify Thom's method using Steenrod operations[4] by an approach with stable homotopy theory[30]. More applications of stable homotopy theory are documented in May's review[31].

Here, we focus on how power operations are present at the level of spectra. Given a cohomology theory E, its degree-n cohomology operations are natural transformations from E^* to E^{*+n}. By the Yoneda lemma and Brown's representability theorem, $E^*E = [E, E]_{-*}$ is the algebra of cohomology operations on E. If we take $E = H\mathbb{F}_p$, the mod-p Steenrod algebra $\mathcal{A}_p \cong H\mathbb{F}_p^* H\mathbb{F}_p$. Recall that $\mathcal{A}_p$ is generated by mod-p Steenrod operations subject to Adem relations[15], and Steenrod operations are induced by extended powers[19]. Therefore, in order to study power operations in stable homotopy theory, we need to define extended powers for spectra.

In the 1970s, May and his collaborators built a theory of multiplicative $\mathbb{E}_\infty$-structures in spaces and spectra through a series of works[32-34]. Furthermore, May demonstrated that an $\mathbb{E}_\infty$-structure produces power operations[35]. In particular, HR, KU, and Thom spectra are all $\mathbb{E}_\infty$-spectra[34], which illuminates why ordinary cohomology, complex K-theory, and cobordism theory each possess power operations. Conversely, the existence of power operations does not imply the existence of an $\mathbb{E}_\infty$-structure, which means that $\mathbb{E}_\infty$-structures may be too stringent for utilizing power operations. A more suitable structure to supply power operations is an $\mathbb{H}_\infty$-structure, a weaker notion than $\mathbb{E}_\infty$, which was introduced by May in the 1980s[1]. There, May used equivariant half-smash products to define extended powers of ring spectra and then defined the notion of $\mathbb{H}_\infty$-structure in terms of maps related to extended powers. Bruner showed that every $\mathbb{H}_\infty$-ring spectrum admits an associated generalized Adams spectral sequence and explained how an $\mathbb{H}_\infty$-structure converts cohomology operations into homotopy operations, which is the essence of Adams-type spectral sequences. McClure analyzed the connection between $\mathbb{H}_\infty$-structures and power operations and showed that the power operations in mod-p ordinary cohomology, complex K-theory, and cobordism theory coincide with the respective operations derived from $\mathbb{H}_\infty$-structures.

1.4 The homotopy theory of smooth schemes

Besides within topology, algebraic geometry is a field which makes extensive use of cohomological methods. We would naturally expect that the model of homotopy theory and cohomology operations can be modified in a suitable way so that they function well in algebraic geometry. To achieve this, we need to address the following two questions:

- How can we carry out homotopy theory in a general setting beyond topology?
- Cohomology theories in algebraic geometry are defined by sheaves, while cohomology theories in algebraic topology are defined by spectra. How can we generalize the homotopical framework to incorporate these two types of cohomology?

For the first question, Quillen built a framework called homotopical algebra, which distills the essential features for working with homotopy theoretic tools in terms of axiomatic properties possessed by a model category[36]. For the second question, K.S. Brown used sheaves of spectra (or simplicial sets) to generalize sheaf cohomology, with coefficients in a complex of abelian sheaves. These set the stage for performing homotopy theory in algebraic geometry that is compatible with cohomology theories.

In the 1990s, Morel and Voevodsky constructed $\mathbb{A}^1$-homotopy theory of schemes, also called motivic homotopy theory[37]. Under this framework, Voevodsky constructed motivic power operations[38] and these operations led to an elegant solution of the Milnor conjecture[39] and the Bloch-Kato conjecture[40] (the earlier proofs of these conjectures had been given by Voevodsky in the 1990s, but they were lengthy as the framework of motivic homotopy theory had not been well developed at that time). Apart from the settlement of these famous conjectures, the following are more evidences showing why Morel and Voevodsky's approach is a reasonable and fruitful one.

As an analogue of Theorem 1.3, Morel[41], Asok, Hoyois, and Wendt[42] proved that given a smooth affine scheme X over a Noetherian commutative ring of a particular class, isomorphic classes of rank-r algebraic vector bundles over X are in bijection with $\mathbb{A}^1$-homotopy classes of maps from X to the infinite Grassmannian of r-planes.

As an analogue of Brown's representability theorem from Section 1.3, Voevodsky also constructed motivic stable homotopy theory to represent cohomology theories in algebraic geometry[43]. For example, motivic cohomology theory (analogous to singular cohomology theory) is represented by the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}_{mot}$, algebraic K-theory (analogous to complex K-theory) is represented by KGL, and algebraic cobordism theory (analogous to complex cobordism) is represented by MGL.

Moreover, motivic stable homotopy theory is deeply related to classical stable homotopy theory. To be more concrete, let k be a field with an embedding $k \hookrightarrow \mathbb{C}$. There is a realization functor

$$t_{\mathbb{C}} : \mathcal{SH}(k) \to \mathcal{SH}$$

where $\mathcal{SH}(k)$ is the motivic stable homotopy category over k and $\mathcal{SH}$ is the classical stable homotopy category. The striking coincidences are

$$H\mathbb{Z}_{mot} \xmapsto{t_{\mathbb{C}}} H\mathbb{Z} \qquad KGL \xmapsto{t_{\mathbb{C}}} KU \qquad MGL \xmapsto{t_{\mathbb{C}}} MU$$

The properties of the realization functor $t_{\mathbb{C}}$ indicate that motivic (stable) homotopy theory is an adequate version of (stable) homotopy theory in algebraic geometry. It is worth noting that $t_{\mathbb{C}}$ plays a central role in Voevodsky's earlier unpublished proof of the Milnor conjecture[44]. Voevodsky proved a purely topological result on MU and $H\mathbb{Z}/\ell$. He then applied the result and the realization functor $t_{\mathbb{C}}$ to prove a motivic version of the result on motivic cohomology and algebraic cobordism, which is essential to the proof of the Milnor conjecture. Specifically, the proof of the topological results relies on the use of the Steenrod algebra, while the proof of the motivic result relies on a motivic Steenrod algebra and the realization functor $t_{\mathbb{C}}$ which preserves the structure of these algebras. This technique demonstrates the significance and efficacy of the methods of homotopy theory in algebraic geometry via cohomology operations.

This profound connection between classical homotopy theory and motivic homotopy theory has not only advanced the research in algebraic geometry and number theory, but also facilitated the study of classical stable homotopy theory. Isaksen and Dugger constructed motivic Adams spectral sequences in the 2010s and used them to improve the computations of the classical stable stems[45]. More recently, the discovery of a deep relationship between the motivic Adams spectral sequence and the algebraic Novikov spectral sequence has led to great extensions of the computations to higher dimensions[46-48]. Central to this relationship is an element τ in the mod-p cohomology of a point, which serves as a parameter for a deformation between motivic and classical stable homotopy categories. This element featured in Voevodsky's computations with the motivic algebra and its dual.

CHAPTER 2 CATEGORICAL FRAMEWORK FOR HOMOTOPY THEORY

In this chapter, we will introduce ∞-categories as the categorical framework for homotopy theory. ∞-categories are generalizations of ordinary categories that allow for higher-dimensional morphisms and equivalences. In Section 2.1, we introduce how to define ∞-categories by using simplicial sets that satisfy the Kan condition, which is so-called the model of quasi-categories. Meanwhile, we introduce an equivalent definition of ∞-categories using simplicial categories in Section 2.2. From the viewpoint of simplicial categories, higher homotopies are parametrized by Kan complexes as the mapping spaces. The positions of simplices and their faces in the mapping Kan complexes exhibit the coherence among the higher homotopies. The most intuitive illustration refers to Remark 4.2.

In Section 2.3, we introduce the notions of limits, colimits, and adjoint functors in the context of ∞-categories. In particular, we focus on presentable ∞-categories and the adjoint functor theorem in Section 2.4.

2.1 The theory of ∞-categories

Definition 2.1: Let n be a non-negative integer, then the datum of the category $[n]$ consists of

- The set of objects is $\{0, 1, 2, \dots, n\}$,
- The morphism is defined by

$$\mathrm{Hom}_{[n]}(k, j) = \begin{cases} \emptyset & k > j \\ \{\to\} & k \leq j \end{cases}$$

Let Δ denote simplex category, the objects of simplex category are $\{[n]\}_{n \in \mathbb{N} \cup \{0\}}$ and the morphisms are functors.

In other words, Δ is the category of finite ordinals and order-preserving maps.

Definition 2.2 (Simplicial objects): Given a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. Morphisms between simplicial sets are natural transformations. In particular, a simplicial set is an object in $\mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set})$ and we denote the category $\mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets by $\mathrm{s}\mathcal{S}\mathrm{et}$.

Construction 2.3: The following are basic simplicial sets:

(1) For $[n] \in \Delta$, a simplicial set Δ_n defined by

$$\Delta_n([m]) := \mathrm{Hom}_\Delta([m], [n])$$

is called the *standard n-simplex*. By Yoneda lemma, for any simplicial set $X_\bullet$, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta_n, X) \cong X_n$$

where elements in X_n are called *n-simplices* of X.

(2) The *boundary* $\partial\Delta_n$ is defined by

$$\partial\Delta_n([m]) := \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid f \text{ is not surjective}\}$$

(3) For $0 \leq k \leq n$, *the kth n-horn* Λ^n_k is defined by

$$\Lambda^n_k([m]) := \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid f([m]) \cup \{k\} \neq [n]\}$$

In particular, if $0 < k < n$, the horn Λ^n_k is said to be an inner horn.

Remark 2.1: The Yoneda embedding $\Delta \hookrightarrow \mathrm{s}\mathcal{S}\mathrm{et}$ exhibits $\mathrm{s}\mathcal{S}\mathrm{et}$ as a cocompletion of the simplex category Δ. Given any colimit-preserving functor $Q : \Delta \to \mathcal{C}$ targeted at a cocomplete category $\mathcal{C}$ (i.e. a cosimplicial object in $\mathcal{C}$), one has a unique extension $\bar{Q} : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}$ of Q. This refers to the *Yoneda extension* or left Kan extension.

Definition 2.4: Given two simplicial sets X, Y, the simplicial set $\mathrm{Map}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, Y)$ is defined to be

$$\mathrm{Map}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, Y)_n = \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X \times \Delta^n, Y)$$

In this way $\mathrm{s}\mathcal{S}\mathrm{et}$ is enriched by itself.

Proposition 2.1: Given three simplicial sets X, Y, Z, then we have

$$\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, \mathrm{Map}(Y, Z))$$

Definition 2.5 (The nerve of category): Given a small category $\mathcal{C}$, the simplicial set $N(\mathcal{C})$ defined by

$$N(\mathcal{C})_n := \mathcal{F}\mathrm{un}([n], \mathcal{C})$$

which is essentially the set of n-composable morphisms in $\mathcal{C}$. The degeneracy map is given by composition and the face map is given by adding an identity morphism. In particular, $N([n]) \cong \Delta_n$.

Proposition 2.2: The nerve functor $N : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is fully faithful.

Proof: See[49]Tag 002z. ■

Definition 2.6: A *quasi-category* or say $(\infty, 1)$-category $\mathcal{C}$ is a simplicial set such that every inner horn $\Lambda^n_k \to \mathcal{C}$, $0 < k < n$, can be extended to an n-simplex $\Delta^n \to \mathcal{C}$. Elements in $\mathcal{C}_0$ are *objects in* $\mathcal{C}$ and elements in $\mathcal{C}_n$ are called n*-morphisms in* $\mathcal{C}$ for $n > 0$. For convenience, 1-morphisms are simply called morphisms in $\mathcal{C}$. Two morphisms $f, g : c \to d$ are homotopic if there is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma^2 = (g, f, \mathrm{id})$. The *homotopy category* $\mathrm{h}\mathcal{C}$ of $\mathcal{C}$ is a category whose objects are elements in $\mathcal{C}_0$ and whose morphisms are homotopy classes of $\mathcal{C}_1$.

Remark 2.2: The condition of quasi-categories ensures that the homotopy category of a quasi-category is indeed a category. In particular, the composition law and associativity can be displayed by filling inner horns.

Definition 2.7: A morphism $f : x \to y$ in an ∞-category $\mathcal{C}$ is an *equivalence* if $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$.

In this paper, an $(\infty, 1)$-category is simply called an ∞-category.

Definition 2.8: Functors between two ∞-categories $\mathcal{C}, \mathcal{D}$ are morphisms between them as simplicial sets.

Proposition 2.3: Let $\mathcal{D}$ be an ∞-category and $\mathcal{C}$ be a simplicial set, then mapping space $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ is an ∞-category. In this way, the category of ∞-categories is enriched by itself.

Proof: See[49]Tag 0066. ■

Construction 2.9: The standard cosimplicial space $|\Delta^\bullet| : \Delta \to \mathcal{T}\mathrm{op}$ assigns

$$\Delta^n \mapsto \left\{ (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1 \text{ and } x_i > 0 \text{ for all } i \right\}$$

Then for any topological space X, we define the singular simplicial set of X as

$$\mathrm{Sing}_\bullet(X) := \mathrm{Hom}(|\Delta^\bullet|, X)$$

and it forms a functor $\mathrm{Sing}_\bullet : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$. Furthermore, there exists a *geometric realization functor* $| - | : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ as an extension of $|\Delta^\bullet|$ such that we have the adjunction

$$| - | : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftharpoons \mathcal{T}\mathrm{op} : \mathrm{Sing}_\bullet$$

Theorem 2.1: There exists a model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$ such that

（1）cofibrations are monomorphisms between simplicial sets;

（2）weak equivalences are the morphisms whose geometric realization are weak equivalences of topological spaces;

(3) fibrations are the morphisms that have the right lifting property with respect to all horn inclusions, which are so-called *Kan fibrations*.

Furthermore, the adjunction $(|-|, \mathrm{Sing}_\bullet)$ is a Quillen equivalence. This is the so-called *Quillen model for simplicial sets*.

Definition 2.10: A *Kan complex* X is a simplicial set such that every horn $\Lambda^n_k \to X$, $0 \le k \le n$, can be extended to an n-simplex $\Delta^n \to X$. In particular, Kan complexes are exactly fibrant objects in Quillen's model. The category of Kan complexes is denoted by $\mathcal{K}$an.

Definition 2.11: An ∞-*groupoid* is a quasi-category whose morphisms are all isomorphisms in the homotopy category.

Proposition 2.4: An ∞-category is an ∞-groupoid if and only if it is a Kan complex.

2.2 Simplicial categories and ∞-categories

Definition 2.12 (Simplicial category): A simplicial category $\mathcal{C}_\bullet$ is a category enriched by simplicial sets. The corresponding enriched functors are simplicial functors. The category of simplicial categories is denoted by s$\mathcal{C}$at.

Definition 2.13: Given a simplicial category $\mathcal{C}_\bullet$, morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}_\bullet}(X,Y)_0$ are homotopic if there is an 1-simplices in $\mathrm{Hom}_{\mathcal{C}_\bullet}(X,Y)_1$ whose boundary is f and g. Hence we can define the homotopy category $\mathrm{Ho}(\mathcal{C}_\bullet)$ by quotient the homotopy relation.

Construction 2.14 (Simplicial resolution): First we define the simplicial category $C(\Delta^n)$ associated to $[n]$ by

$$\mathrm{Map}_{C[\Delta^n]}(i,j) = \begin{cases} NP_{i,j}, & i \le j, \\ \emptyset, & i > j. \end{cases}$$

where $P_{i,j} := \{I \subset \{i, i+1, \dots, j-1, j\} \mid i,j \in I\}$ is a poset ordered by inclusion (as a category). Then it forms a cosimplicial object $C[\Delta^n]$ in s$\mathcal{C}$at. Furthermore, by using Yoneda extension, there is an adjunction

$$C[-] : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftharpoons \mathrm{s}\mathcal{C}\mathrm{at} : N_\Delta$$

where N_Δ is called the *homotopy coherent nerve functor* and C is called *simplicial thickening*.

Definition 2.15: A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is a *weak equivalence* if

(1) the induced functor $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is essentially surjective;

(2) for any objects $x, y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence of simplicial sets.

which is also called a *Dwyer-Kan equivalence*.

These two conditions mean that a Dwyer-Kan equivalence is homotopically essentially surjective and homotopically fully faithful.

Theorem 2.2: There exists a left proper combinatorial model structure on s$\mathcal{C}$at such that

(1) the Dwyer-Kan equivalences are weak equivalences;

(2) a simplicial category is fibrant if and only if all simplicial mapping spaces are Kan complexes (which is so-called a *locally fibrant simplicial category*).

This refers to the *Bergner model structure*.

Definition 2.16: A morphism $f : X \to Y$ of simplicial sets is a *categorical equivalence* if the induced simplicial functor $C[f] : C[X] \to C[Y]$ is a Dwyer-Kan equivalence.

Theorem 2.3: The exists a left proper combinatorial model structure on s$\mathcal{S}$et such that

(1) cofibrations are monomorphisms;

(2) weak equivalences are categorical equivalences;

(3) fibrations are the morphisms that have the right lifting property with respect to all inner horn inclusions, which are so-called *inner fibrations*.

This model structure refers to the *Joyal model structure*. In particular, fibrant objects in this model category are exactly ∞-categories.

Theorem 2.4: The adjunction $C[-]$: s$\mathcal{S}$et $\rightleftarrows$ s$\mathcal{C}$at: N_Δ is Quillen equivalence between the Joyal model structure and the Bergner model structure.

This theorem tells us that an ∞-category is equivalent to a $\mathcal{K}$an-enriched category and vice versa.

Definition 2.17: *The ∞-category of spaces $\mathcal{S}$* is $N_\Delta(\mathcal{K}\mathrm{an})$.

Definition 2.18: A *marked simplicial set* is a pair $(X, \mathcal{E}_X)$ where X is a simplicial set X and $\mathcal{E}_X \subset X_1$ contains the degenerate ones. A morphism of marked simplicial sets $(X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ means a morphism of simplicial sets $f : X \to$ such that $f(\mathcal{E}_X) \subset \mathcal{E}_Y$. The category of marked simplicial sets is denoted by s$\mathcal{S}$et$^+$.

For any ∞-category $\mathcal{C}$, it gives us a marked simplicial set $\mathcal{C}^{\natural}$ by marking equivalences. The *∞-category of (small) ∞-categories* $\mathcal{C}\mathrm{at}_\infty$ is the homotopy coherent nerve of the full simplicial subcategory of s$\mathcal{S}$et$^+$ of the marked simplicial sets of the $\mathcal{C}^{\natural}$ for some ∞-category $\mathcal{C}$. The ∞-category of locally small ∞-categories is denoted by $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

2.3 Categorical constructions for ∞-categories

Definition 2.19: Let K and L be two simplicial sets. The *join* $K \star L$ of K and L is the simplicial set defined by

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j,\ n \geq 0$$

Roughly speaking, the vertices of $K \star L$ are the union of vertices in K and L. The edges of $K \star L$ consist of the union of edges in K and L separately plus extra edges from x to y for each $(x, y) \in (K_0 \times L_0)$.

Actually, the join construction is a functor $\star : \mathrm{s}\mathcal{S}\mathrm{et} \times \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$. In particular, we have canonical inclusions $K \to K \star L$ and $L \to K \star L$.

Proposition 2.5: For any $i, j \geq 0$, we have $\Delta^i \star \Delta^j \cong \Delta^{i+j+1}$. All these isomorphisms are compatible with obvious inclusions of Δ^i and Δ^j.

Construction 2.20 (Slice simplicial sets): Let $f : K \to X$ be a morphism of simplicial sets. The *slice simplicial set* $X_{/f}$ *of* X *over* f is defined by setting:

- An n-simplex of $X_{/f}$ is a map of simplicial sets $\bar{f} : \Delta^n \star K \to X$ such that $\bar{f}|_K = f$.
- For any order-preserving map $\alpha : [m] \to$, the associated map on $X_{/f}$ is given by

the composite

$$\Delta^m \star K \xrightarrow{\alpha \star \mathrm{id}_K} \Delta^n \star K \xrightarrow{\bar{f}} X$$

Construction 2.21 (Coslice simplicial sets): Let $f : K \to X$ be a morphism of simplicial sets. The *coslice simplicial set* $X_{f/}$ *of* X *under* f is defined by setting:

- An n-simplex of $X_{f/}$ is a map of simplicial sets $\bar{f} : K \star \Delta^n \to X$ such that $\bar{f}|_K = f$.
- For any order-preserving map $\alpha : [m] \to$, the associated map on $X_{f/}$ is given by

the composite

$$K \star \Delta^m \xrightarrow{\mathrm{id}_K \star \alpha} K \star \Delta^n \xrightarrow{\bar{f}} X$$

Proposition 2.6: Let K be a simplicial set, let $\mathcal{C}$ be an ∞-category, and let $p : K \to \mathcal{C}$ be a diagram. Then the simplicial sets $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are ∞-categories.

Proof: See[49]Tag 018F. ∎

Example 2.1: Let $\mathcal{C}$ be an ∞-category with an object x, characterized by a functor $\kappa_x : \Delta^0 \to \mathcal{C}$. Then the ∞-category $\mathcal{C}_{/\kappa_x}$ is called *the over ∞-category above* x and we may denote it by $\mathcal{C}_{/x}$ simply. Dually, $\mathcal{C}_{\kappa_x/}$ is defined as *the ∞-category of objects under* x and is simply denoted by $\mathcal{C}_{x/}$.

Definition 2.22: Given an ∞-category $\mathcal{C}$, an object $x \in \mathcal{C}$ is said to be a *final object*, if the canonical map $\mathcal{C}_{/x} \to \mathcal{C}$ is an acyclic fibration of simplicial sets. Dually, x is an *initial object*, if the canonical map $\mathcal{C}_{x/} \to \mathcal{C}$ is an acyclic fibration of simplicial sets.

Proposition 2.7: Given an ∞-category $\mathcal{C}$ and an object x, x is final if and only if for all $x' \in \mathcal{C}$, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(x', x)$ are acyclic Kan complexes. Dually, x is initial if and only if for all $x' \in \mathcal{C}$, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(x, x')$ are acyclic Kan complexes.

Definition 2.23: Let K be a simplicial set and let $\mathcal{C}$ be an ∞-category. An initial object in $\mathcal{C}_{p/}$ is said to be *colimit* of a diagram $p : K \to \mathcal{C}$. Dually, a *limit* of p is a final object in $\mathcal{C}_{/p}$.

Definition 2.24: An ∞-category $\mathcal{C}$ is *cocomplete* (resp. *complete*) if $\mathcal{C}$ admits a colimit (resp. limit) for any diagram $p : K \to \mathcal{C}$.

Adjoint functors are very important tools in ordinary category theory and they can be characterized in terms of natural transformations called units and counits. Next, we will define adjoint functors for ∞-categories following this formulation. Recall that if for two ∞-categories $\mathcal{C}, \mathcal{D}$, $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ also forms an ∞-category and *natural transformations between functors of* ∞*-categories* are morphisms in $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$.

Definition 2.25: Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. Given a pair of natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$, we say (η, ϵ) are *compatible up to homotopy* if the following conditions are satisfied:

（1）$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F \circ \eta} F \circ G \circ F \xrightarrow{\epsilon \circ \mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F$ is the identity isomorphism id_F.

（2）$G = \mathrm{id}_{\mathcal{D}} \circ G \xrightarrow{\eta \circ \mathrm{id}_F} G \circ F \circ G \xrightarrow{\mathrm{id}_F \circ \epsilon} G \circ \mathrm{id}_{\mathcal{D}} = G$ is the identity isomorphism id_G.

Definition 2.26: Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. (F, G) is *a pair of adjunctions* if there exits a pair of natural transformations $(\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F, \epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}})$ that is compatible up to homotopy. Here η is *the unit of the adjunction* and ϵ is *the counit of the adjunction*. We say that F is a left adjoint of G, G is a right adjoint of F, and denote them by

$$F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$$

Proposition 2.8: Given a pair of adjunctions $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ for ∞-categories, by taking homotopy categories, $(hF, hG) : h\mathcal{C} \rightleftharpoons h\mathcal{D}$ forms a pair of adjunctions for ordinary categories.

Proof: [49]Tag 02EY. ■

Proposition 2.9: Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. G admits a left adjoint functor if and only if for any $X \in \mathcal{C}$, there exists an object $Y \in \mathcal{D}$ with a morphism $X \xrightarrow{u} G(Y)$ such that for any $Z \in \mathcal{D}$, the following composite is a homotopy equivalence between Kan complexes

$$\mathrm{Map}_{\mathcal{D}}(Y, Z) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}(G(Y), G(Z)) \xrightarrow{\circ u} \mathrm{Map}_{\mathcal{C}}(X, Z)$$

Proof: See[49]Tag 02FV. ■

Proposition 2.10: Let $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ be a pair of adjunctions for ∞-categories. Then F preserves colimits and G preserves limits.

Proof: See[49]Tag 02KE. ■

2.4 Presentable ∞-categories and the adjoint functor theorem

Definition 2.27: Let $\mathcal{C}$ be an ∞-category and κ be a cardinal. A $\mathcal{C}$ is said to be κ-*accessible* if $\mathcal{C}$ admits κ-filtered colimits and there is a small subcategory $\mathcal{D}$ such that

（1）For any object c in $\mathcal{C}$, we can canonically write $c = \mathrm{colim}_K F$, where $F : K \to \mathcal{D}$ is a κ-filtered diagram in $\mathcal{D}$.

（2）For each $d \in \mathcal{D}$, the associated functor $\mathrm{Map}_{\mathcal{C}}(d, -) : \mathcal{C} \to \mathrm{s}\mathcal{S}\mathrm{et}$ preserves κ-filtered colimits.

The idea is that an accessible ∞-category is essentially determined by a small subcategory. Roughly speaking, the first condition is about the generation of objects and the second condition is about the generation of morphisms.

Definition 2.28: An ∞-category is *presentable* if it is accessible and admits any small colimits. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subset \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be the subcategory of presentable ∞-categories and left adjoint functors. Similarly, $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ is the subcategory presentable ∞-categories and right adjoint functors.

Definition 2.29 (Simplicial presheaves): Given a small simplicial set K, we define

$$\mathcal{P}(K) := \mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathcal{S})$$

which is called the ∞-*category of simplicial presheaves on* K.

Construction 2.30 (Yoneda embedding for ∞-categories): Given an ∞-category $\mathcal{C}$, we construct the Yoneda embedding

$$y : \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

by setting

$$x \mapsto \mathrm{Map}_{\mathcal{C}[\mathcal{C}]}(-, x)$$

where $C[\mathcal{C}]$ is the associated simplicial category of $\mathcal{C}$. Since $\mathcal{C}$ is an ∞-category, $C[\mathcal{C}]$ is locally fibrant i.e. enriched by $\mathcal{K}$an. Generally, it can be defined for any simplicial set by using Kan's functor s$\mathcal{S}$et $\to \mathcal{K}$an.

Given two ∞-categories $\mathcal{A}$ and $\mathcal{B}$, we denote by

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{A},\mathcal{B}) \subset \mathcal{F}\mathrm{un}(\mathcal{A},\mathcal{B})$$

the full subcategory spanned by the functors preserving colimits. Then the image of $\mathcal{F}\mathrm{un}(\mathcal{P}(\mathcal{A}),\mathcal{B})$ in $\mathcal{F}\mathrm{un}(\mathcal{A},\mathcal{B})$ along the Yoneda embedding is exactly $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{A},\mathcal{B})$.

Theorem 2.5 (Adjoint Functor Theorem): A functor between presentable ∞-categories is a left adjoint if and only if it preserves colimits. It is a right adjoint if and only if it preserves limits and is accessible.

Definition 2.31: If A functor between two ∞-categories admits a fully faithful right adjoint, then it is called a *localization*.

There is another way to perform localization. Let $\mathcal{MC}\mathrm{at}_\infty$ be the ∞-category of " marked" ∞-categories i.e. pairs $(\mathcal{C},\mathcal{W})$ where $\mathcal{C}$ is a ∞-category and $\mathcal{W}$ is a collection of equivalence classes of arrows in $\mathcal{C}$ that contains all the degenerate maps

Theorem 2.6: [50]Proposition 5.5.1.1 An ∞-category $\mathcal{C}$ is presentable if and only if there is a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible, localization of $\mathcal{P}(\mathcal{D})$.

CHAPTER 3 HOMOTOPICAL ALGEBRA VIA HIGHER CATEGORIES

This chapter introduces homotopical algebra via higher categories, which is a method to study homotopy theory using higher categorical structures. The chapter consists of four sections:

- The first section reviews the Grothendieck construction in ∞-categories, which is a way to associate an ∞-category to a functor from an ∞-category to the ∞-category of ∞-categories. This construction allows one to study fibrations and cofibrations in ∞-categories.
- The second section shows how to use the Grothendieck construction to construct symmetric monoidal ∞-categories, which are ∞-categories equipped with a tensor product that satisfies certain coherence conditions. We show that every symmetric monoidal infinity category can be obtained as a Grothendieck construction applied to a suitable functor.
- The third section discusses stable infinity categories and their symmetric monoidal structure. Stable ∞-categories are ∞-categories that have finite limits and colimits and satisfy a triangulated axiom. They are important for studying derived categories and spectra. We show that every stable infinity category has a canonical symmetric monoidal structure given by its tensor product.
- The fourth section explains some results from Robalo's thesis [51] on formal inversion and stabilization. Formal inversion is a process that allows one to invert some objects with respect to the tensor product for a symmetric monoidal ∞-category. Stabilization is a process to realize a formal inversion concretely. This technique will be used to construct the motivic stable homotopy category in Section 6.2.

3.1 Grothendieck constructions for fibered categories

Definition 3.1: Let $p : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. A morphism $f : c_1 \to c_2$ in $\mathcal{C}$ is *p-cocartesian* or a *p-cocartesian lift* of $\alpha = p(f)$ if the following map is an equivalence between ∞-categories

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$$

where $\mathcal{C}_{f/} := (\mathcal{C}_{c_1/})_{f/}$ and $\mathcal{D}_{p(f)/} = (\mathcal{D}_{d_1/})_{p(f)/}$. Dually, f is *p-cartesian* if the induced functor

$$\mathcal{C}_{/f} \to \mathcal{C}_{/c_2} \times_{\mathcal{D}_{p(c_2)/}} \mathcal{D}_{/p(f)}$$

is an equivalence between ∞-categories.

Remark 3.1 (The Grothendieck construction for ordinary categories): Given a functor $p : \mathcal{C} \to \mathcal{D}$, we say $f : c_1 \to c_2$ in $\mathcal{C}$ is *p-cocartesian* if for any $c_3 \in \mathcal{C}$ with $h : c_1 \to c_3$ in $\mathcal{C}$, let $d_i := p(c_i)$, $\alpha = p(f) : d_1 \to d_2$, $p(h) = \gamma : d_1 \to d_3$, and any $\beta : d_2 \to d_3$ such that $\beta \circ \alpha = \beta$, there is a unique $g : c_2 \to c_3$ such that $\beta = p(g)$ and $h = g \circ f$.

$$\begin{tikzcd}
c_1 \arrow[r, "f"] \arrow[d] \arrow[rr, bend right, "\forall h"'] & c_2 \arrow[r, "\exists ! g"] \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\gamma"'] & d_2 \arrow[r, "\forall \beta"] & d_3
\end{tikzcd}$$

In other words, f is p-cocartesian if the following diagram is a cartesian diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(c_2, c_3) \arrow[r, "f^*"] \arrow[d, "p"'] & \mathrm{Map}_{\mathcal{C}}(c_1, c_3) \arrow[d, "p"] \\
\mathrm{Map}_{\mathcal{D}}(p(c_2), p(c_3)) \arrow[r, "p(f)^*"] & \mathrm{Map}_{\mathcal{D}}(p(c_1), p(c_3))
\end{tikzcd}$$

This diagram means that $\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$ is an isomorphism and reveals the insight of the ∞-categorical definition. Next we demonstrate why we need such lifting property.

Note that if $f' : c \to c'$ and $f'' : c \to c''$ are two p-cocartesian arrows with the same target $p(f') = p(f'')$, there is a unique isomorphism $\phi : c \to c''$ in $\mathcal{C}_{p(c')}$ such that $\phi \circ f' = f$".

If $p : \mathcal{C} \to \mathcal{D}$ has the property that for all $c_1 \in \mathcal{C}$ and all morphism α in $\mathcal{D}$, we can lift α to a p-cocartesian morphism $f : c_1 \to c_2$:

$$\begin{tikzcd}
c_1 \arrow[r, dashed, "f"] \arrow[d, "p"'] & c_2 \arrow[d, "p"] \\
p(c_1) \arrow[r, "\alpha"] & p(c_2)
\end{tikzcd}$$

then we cam assign each $c \in \mathcal{C}$ and $\alpha : p(c) \to d$ a p-cocartesian lift. For $\alpha : d_1 \to d_2$, let $f : c_1 \to c_2$ be a p-cocartesian lift of $\alpha : p(c_1) \to d_2$, we have a functor

$$\alpha_! : \mathcal{C}_{d_1} \to \mathcal{C}_{d_2},\ c_1 \mapsto c_2$$

which sends $\phi_1 : c_1' \to c_1''$ to $\phi_2 : c_2' \to c_2''$ in the following diagram

$$\begin{tikzcd}
c_1' \arrow[rr, "\phi_1"] \arrow[rrrd] \arrow[rdd, "p"'] & & c_1'' \arrow[rr] \arrow[ldd, "p"] & & c_2'' \arrow[dd, "p"] \\
& & & c_2' \arrow[ru, dashed, "\exists!\phi_2"] \arrow[dd, "p"] & \\
& d_1 \arrow[rrr, "\alpha"] \arrow[rrd, "\alpha"] & & & d_2 \arrow[ld, equal] \\
& & & d_2 &
\end{tikzcd}$$

Since the functor depends on the choice of lifting, $\beta_! \circ \alpha_! \neq (\beta \circ \alpha)_!$ in general. Nevertheless, there is a unique isomorphism $\beta_! \circ \alpha_! \cong (\beta \circ \alpha)_!$. This is related to the notion of *pseudo-functors* in the context of ordinary categories. However, since the ∞-categorical framework carries homotopy coherence naturally, we do not deal with this subtle issue of the choice of liftings.

Theorem 3.1: [50]Theorem 3.2.0.1 There is an equivalence

$$\mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{C}\mathrm{at}}_\infty) \xrightarrow{\simeq} \mathrm{Cart}_{\mathcal{C}}$$

where $\mathrm{Cart}_{\mathcal{C}} \subset \widehat{\mathcal{C}\mathrm{at}}_{\infty/\mathcal{C}}$ is the subcategory of cartesian fibrations over $\mathcal{C}$ whose morphisms are functors preserving cartesian fibrations. This equivalence is called the *unstraightening functor*. Dually, we have

$$\mathcal{F}\mathrm{un}(\mathcal{C}, \widehat{\mathcal{C}\mathrm{at}}_\infty) \xrightarrow{\simeq} \mathrm{coCart}_{\mathcal{C}}$$

where $\mathrm{coCart}_{\mathcal{C}} \subset \widehat{\mathcal{C}\mathrm{at}}_{\infty/\mathcal{C}}$ is the subcategory of cocartesian fibrations over $\mathcal{C}$ whose morphisms are functors preserving cocartesian fibrations.

Remark 3.2: Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$, we construct a new category $\int F$ that is informally described as follows

- The objects are pairs (c, X) where $c \in \mathcal{C}$ and $X \in F(X)$;
- Given two objects $(c, X), (c', X') \in \int F$, an arrow $(c, X) \to (c', X')$ is a pair (f, α), where $f : c \to c'$ is a morphism in the category and $\alpha : c \to F(f)c'$ is a morphism in $F(c)$."

The corresponding cartesian fibration is given by the natural projection $\int F \to \mathcal{C}$, $(c, X) \mapsto$

c. This construction is called the *Grothendieck construction*.

3.2 Symmetric monoidal ∞-categories

Let $\mathcal{F}in_*$ be the category of pointed finite sets. We specify a finite pointed set of cardinality n by setting

$$\langle n\rangle = \{0, 1, \cdots, n\}$$

whose pointed point is 0. We denote $\langle n\rangle^\circ$ the non-pointed part of $\langle m\rangle$ i.e $\langle m\rangle \setminus \{0\}$. Note that one should distinguish $\langle n\rangle$ and $[n]$ since $[n]$ is ordered while $\langle n\rangle$ is not. We just use these integers to mark elements and ignore their orders.

Definition 3.2: A symmetric monoidal ∞-category is a cocartesian fibration $q: \mathcal{D}^\otimes \to N(\mathcal{F}in_*)$ such that

$$\mathcal{D}^\otimes_{\langle n\rangle} \xrightarrow{\prod \delta_{i!}} \prod_{i=1}^{n} \mathcal{D}^\otimes_{\langle 1\rangle}$$

is an equivalence, where $\delta_i: \langle n\rangle \to \langle 1\rangle$ is defined by

$$\delta_i(j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

This condition is called the *Segal condition*. A symmetric monoidal functor between $q: \mathcal{D}^\otimes \to N(\mathcal{F}in_*)$ and $p: \mathcal{C}^\otimes \to N(\mathcal{F}in_*)$ is a morphism in $\mathrm{coCart}_{N(\mathcal{F}in_*)}$.

Example 3.1: Suppose $\mathcal{D}$ is an ∞-category that admits finite products. We define an associated ∞-category $\mathcal{D}^\times$ whose objects are finite sequences $(X_1, \cdots, X_n)$ and a morphism from $(X_1, \cdots, X_n)$ to $(Y_1, \cdots, Y_k)$ is given by $(\alpha, \{f_i\}_i)$, where $\alpha: \langle k\rangle \to \langle n\rangle$ in $\mathcal{F}in_*$ and

$$f_i: \prod_{j \in \alpha^{-1}(i)} X_j \to Y_i,\ i = 1, \cdots, k$$

Higher morphisms (i.e. n-simplices for $n > 1$) in $\mathcal{D}^\times$ are given by higher morphisms in $N(\mathcal{F}in_*)$. The desired cocartesian fibration $\mathcal{D}^\times \to N(\mathcal{F}in_*)$ is given by

$$(X_1, \cdots, X_n) \mapsto \langle n\rangle$$

Actually, for a symmetric monoidal (ordinary) category $\mathcal{M}$, the symmetric monoidal ∞-category structure for $N(\mathcal{M})$ can be given in an analogous way. The associated category $\mathcal{D}^\times$ refers to the notion of multicategories, which is closely related to operads[52-53].

Remark 3.3: Given a symmetric monoidal ∞ category $p: \mathcal{M}^\otimes \to N(\mathcal{F}in_*)$, the source of the cocartesian category is not the actual underlying ∞-category according to Example

3.1. The actual underlying ∞-category is closer to $\mathcal{M} = \mathcal{M}^{\otimes}_{\langle 1\rangle}$ and the tensor product is given by

$$\otimes : \mathcal{M}^2 \simeq \mathcal{M}^{\otimes}_{\langle 2\rangle} \to \mathcal{M}^{\otimes}_{\langle 1\rangle} = \mathcal{M}$$

which is induced by the map $\langle 2\rangle \to \langle 1\rangle$ sending $1, 2$ to 1.

Definition 3.3: A symmetric monoidal ∞-category $p : \mathcal{M}^{\otimes} \to N(\mathcal{F}in_*)$ is a *presentable symmetric monoidal ∞-category* if the underlying category $\mathcal{M} = \mathcal{M}^{\otimes}_{\langle 1\rangle}$ is presentable and any choice of tensor product bifunctor $\otimes : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ preserving colimits in each variable.

Definition 3.4: A morphism $\alpha : \langle n\rangle \to \langle m\rangle$ is *inert* if for any $i \in \langle m\rangle^{\circ}$, the preimage $\alpha^{-1}(i) \subset \langle n\rangle^{\circ}$ is a singleton. α is *active* if $\alpha^{-1}(j) \subset \langle n\rangle$ is a singleton for each point $j \in \langle n\rangle$.

Definition 3.5: Given a symmetric monoidal ∞-category $p : \mathcal{M}^{\otimes} \to N(\mathcal{F}in)_*$, a *commutative algebra object* of $\mathcal{C}$ is a section $s : N(\mathcal{F}in)_* \to \mathcal{M}^{\otimes}$ sending inert morphisms to cocartesian morphisms. Let $\mathrm{CAlg}(\mathcal{C}^{\otimes}) \subset \mathcal{F}\mathrm{un}_{N(\mathcal{F}in_*)}(N(\mathcal{F}in_*), \mathcal{C}^{\otimes})$ be the ∞-categories of commutative algebras.

Example 3.2: Note that $\mathcal{C}\mathrm{at}_\infty$ admits finite products and by Example 3.1, $\mathcal{C}\mathrm{at}_\infty^{\times}$ forms a symmetric monoidal ∞-category. Then $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty^{\times})$ is exactly the ∞-category of (small) symmetric monoidal ∞-categories. Similarly, $\widehat{\mathcal{C}\mathrm{at}}_\infty^{\times}$ is a symmetric monoidal ∞-category.

Theorem 3.2: Let $\mathcal{C}, \mathcal{D}$ be two presentable ∞-categories. Let $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}) \subset \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ be the full subcategory of right adjoint functors. Then $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ is a presentable ∞-category, and $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ corepresents the functor

$$\mathcal{F}\mathrm{un}^{L,L}(\mathcal{C}\times, \mathcal{D}, -) \subset \mathcal{F}\mathrm{un}(\mathcal{C}\times\mathcal{D}, -) : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

where $\mathcal{F}\mathrm{un}^{L,L}(\mathcal{C}\times, \mathcal{D}, \mathcal{E})$ is the subcategory of functors preserving colimits in each variable.

Proof: [54]Lemma 4.8.1.16,Proposition 4.8.1.17 ■

Therefore, we may take

$$\mathcal{C}\otimes\mathcal{D} \simeq \mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$$

as the tensor product between two presentable ∞-categories and thus $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ forms a symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Theorem 3.3: Let $\mathcal{P} : \mathcal{C} \mapsto \mathcal{P}(\mathcal{C})$ be the functor of taking presheaves. Then

$$\mathcal{C}\mathrm{at}_\infty^{\times} \xrightarrow{\mathcal{P}} \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty^{\otimes}$$

is a symmetric monoidal functor.

Proof: See[54]Corollary. ∎

3.3 Stable ∞-categories and their tensor products

Definition 3.6: Given an ∞-category $\mathcal{C}$, an object $0 \in \mathcal{C}$ is said to be a *zero object* if it is both initial and final. If $\mathcal{C}$ admits a zero object, it is said to be a pointed ∞-category.

Definition 3.7: Let $\mathcal{C}$ be a pointed ∞-category with zero object 0. A *triangle* is a commutative square $\Delta^1 \times \Delta^1 \to \mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

A triangle is *left exact* (resp. *right exact*) if the square is a cartesian (resp. cocartesian). We say a triangle is *exact* if it is both left exact and right exact.

The limits and colimits in ∞-categories should be understood as kinds of homotopy limits and colimits. We may compute them by transporting them in a simplicial category and invoking two-sided (co)bar constructions. For example, the push-out of the following diagram in $\mathcal{S}_*$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

is ΣA. Dually, the pull-back for the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

is ΩB. Therefore a triangle of the following form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

is left exact (resp. right exact) if and only if $A \simeq \Omega B$ ($\Sigma A \simeq B$). To generalize this idea, we have the following definition

Definition 3.8: Given a pointed category $\mathcal{C}$ with zero object 0 that admits finite limits and colimits, we define $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ by sending $X \in \mathcal{C}$ to the colimit of the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

Similarly, we define $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ by sending Y to the limit of the following diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Proposition 3.1: Let $\mathcal{C}$ be a pointed ∞-category. The following statements are equivalent

（1）$\mathcal{C}$ is stable.

（2）$\mathcal{C}$ admits finite colimits and $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence;

（3）$\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence.

Definition 3.9: Let $\mathcal{S}_*^{\mathrm{fin}} \subset \mathcal{S}_*$ be the full subcategory generated by points with finite colimits. A functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ is said to be *excisive* (resp. *reduced*) if F sends cocartesian squares to cartesian squares (resp. sends initial objects to final objects). We denote

$$\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$

the full subcategory of reduced and excisive functors.

Remark 3.4: Actually, for any pointed ∞-category $\mathcal{D}$ that admits finite colimits, we can define excisive or reduced property for any functor $\mathcal{D} \to \mathcal{C}$ in the same way. Note that $\mathcal{S}_*^{\mathrm{fin}}$ is the initial one in pointed ∞-categories with finite colimits. Therefore, we can expect some universal property for $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$.

Definition 3.10: Let $\mathcal{C}$ be an ∞-category that admits finite limits. *The ∞-categories* $\mathcal{S}\mathrm{p}(\mathcal{C})$ *of spectrum objects* in $\mathcal{C}$ is defined as

$$\mathcal{S}\mathrm{p}(\mathcal{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$

If $\mathcal{C} = \mathcal{S}_*$, we just simply denote it by $\mathcal{S}\mathrm{p}$ and call it *the ∞-category of spectra*.

Proposition 3.2: [54]Proposition 1.4.2.16 Let $\mathcal{C}$ be an ∞-category that admits finite limits. The ∞-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$ is stable.

Proposition 3.3: [54]Remark 1.4.2.25 Let $\mathcal{C}$ be an ∞-category that admits finite limits. The functor

$$\mathcal{S}\mathrm{p}(\mathcal{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \lim\{\cdots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}\}$$

induced by evaluation on spheres is an equivalence of ∞-categories. Therefore we may also call $\mathcal{S}\mathrm{p}(\mathcal{C})$ the *stabilization* of $\mathcal{C}$.

Remark 3.5: The idea of stabilization occurred very early, see[55]Section 12. However, there was no homotopy theory for homotopical categories or even categories. More specifically, there was no framework to tell us what homotopy limits for diagrams in the category of small categories are at that time. From this point of view, we are further convinced that the higher categorical theory is meaningful.

Remark 3.6: By taking adjunction, we may write

$$\mathcal{S}\mathrm{p} \simeq \mathrm{colim}\{\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Proposition 3.4: [54]Proposition 1.4.4.4 If $\mathcal{C}$ is a presentable ∞-category, then so is $\mathcal{S}\mathrm{p}(\mathcal{C})$.

Corollary 3.1: The functor $\mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ that evaluates at $§^0$ preserves limits. Thus it admits a left adjoint $\Sigma^\infty : \mathcal{C} \to \mathcal{S}\mathrm{p}(\mathcal{C})$ by using Theorem 2.5.

Proposition 3.5: [54]Corollary 1.4.4.5 For any two presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, if $\mathcal{D}$ is stable, then

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}\mathrm{p}(\mathcal{C}), \mathcal{D}) \xrightarrow{-\circ\Sigma^\infty} \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Proposition 3.6: For every stable

Proposition 3.7: [56]Proposition 5.25 Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{st}$ be the ∞-category of stable presentable ∞-categories with left adjoint functors. $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{st}$ admits a closed symmetric monoidal structure such that

(1) $\mathcal{S}\mathrm{p}$ is the monoidal unit.

(2) The tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ is given by $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$.

(3) The internal hom of $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{P}\mathrm{r}^{\mathrm{L}}_{st}$ is given by $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{D}_1, \mathcal{D}_2)$.

The ∞-category of presentable stable symmetric monoidal ∞-categories can be given by $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{st})$.

Corollary 3.2: Given a presentable ∞-category $\mathcal{C}$, its stabilization $^{(}\mathcal{C})$ can be given by $\mathcal{C} \otimes \mathcal{S}\mathrm{p}$. In this way, we can define smash product for $\mathcal{S}\mathrm{p}(\mathcal{C})$

$$(\mathcal{C} \otimes \mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{S}\mathrm{p}) \to (\mathcal{C} \otimes \mathcal{C}) \otimes (\mathcal{S}\mathrm{p} \otimes \mathcal{S}\mathrm{p}) \to \mathcal{C} \otimes \mathcal{S}\mathrm{p}$$

where the tensor product is in the symmetric monoidal structure for $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

3.4 Generalized stabilization with universal characterization

Definition 3.11: Let $\mathcal{C}$ be a symmetric monoidal ∞-category with tensor product $\otimes$. An object $X \in \mathcal{C}$ is *invertible* if the functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ is an equivalence.

Theorem 3.4: [51]Proposition 2.9 Let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, and let $X \in \mathcal{C}$. There exists a presentable symmetric monoidal ∞-category $\mathcal{C}[X^{-1}]$ with

$$\Sigma_X^\infty : \mathcal{C} \to \mathcal{C}[X^{-1}]$$

in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ such that for any $\mathcal{D} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, the composition

$$\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})(\mathcal{C}[X^{-1}], \mathcal{D}) \xrightarrow{\circ \Sigma_X^\infty} \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})(\mathcal{C}, \mathcal{D})$$

is a fully faithful embedding and its essential image is the full subcategory of functors sending X to an invertible object in $\mathcal{D}$. This is called *formal inversion of* $\mathcal{C}$ *with respect to* X.

Theorem 3.5: Let $\mathcal{C}$ be a presentable symmetric monoidal category and let $X \in \mathcal{C}$. The *stabilization with respect to* X is defined to be

$$\mathrm{Stab}_X(\mathcal{C}) := \mathrm{colim}\{\mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \cdots\}$$

If the cyclic permutation action on $X \otimes X \otimes X$ is homotopic to an identity map by applying $X \otimes -$ several times, then the $\mathrm{Stab}_X(\mathcal{C})$ admits a canonical symmetric monoidal structure and $\mathrm{Stab}_X(\mathcal{C}) \simeq \mathcal{C}[X^{-1}]$.

Proof: See[51]Theorem 2.14, Corollary 2.22. ■

Example 3.3: The ∞-category of pointed spaces $\mathcal{S}_*$ admits a symmetric monoidal structure by considering the smash product. Then we have

$$\mathcal{S}\mathrm{p} \simeq \mathcal{S}_*[(S^1)^{-1}]$$

The motivation for topologists to consider the "negative circle" is about the Alexander duality, the Freudenthal suspension theorem and the Bott periodicity theorem, see[57].

CHAPTER 4 POWER OPERATIONS FOR STRUCTURED ALGEBRAS

This chapter aims to study how we construct power operations and more general Steenrod operations from various multiplicative structures with homotopy coherence.

4.1 Algebraic formalism for Steenrod operations

This section mainly refers to May's algebraic approach to Steenrod operations[35].

Let R be a commutative ring with a unit 1, p be a prime number, and r be a natural number. Let $G \subset \Sigma_r$ be a subgroup and $C_p \subset \Sigma_p$ be the p-Sylow subgroup. Let V be a free $R[\Sigma_r]$ resolution of R, W be a free $R[G]$-resolution of R and $j : W \to V$ be an inclusion induced by $G \subset \Sigma_r$. In particular, if $G = C_p$, then we let W be the *Tate resolution* of R. Let $R[C_p] = R[T]/(T^p)$, where T can be identified as $(12 \cdots p) \in \Sigma_p$ that generates C_p, the Tate resolution W is defined to be

$$W_i = R[C_p]\{e_i\},\ \ d(e_{2i}) = (1 - T)e_{2i-1},\ \ d(e_{2i+1}) = (1 + T + \cdots + T^{p-1})e_{2i}$$

We have the following standard result by computing group cohomology of C_p with coefficient $\mathbb{F}_p$.

Theorem 4.1: The group cohomology algebra is given by

$$\begin{cases} H^*(C_2, \mathbb{F}_2) = \mathbb{F}_2, & |x| = 1, \\ H^*(C_p, \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}[x] \otimes_{\mathbb{F}_p} \mathbb{F}_p[y], & |x| = 1, |y| = 2, \quad p \text{ is an odd prime} \end{cases}$$

where $\Lambda_{\mathbb{F}_p}$ means the exterior algebra over $\mathbb{F}_p$.

Construction 4.1: Given a $\mathbb{Z}$-graded homotopy associated and commutative differential R-algebra K, we let Σ_r act on K^r by permuting factors and let Σ_r act on K trivially.

Definition 4.2: The category $\mathcal{C}(G, R)$ of G-structured R-algebras consists of the following data:

- Objects: A pair (K, θ) where K is a $\mathbb{Z}$-graded homotopy associated and commutative differential R-algebra and $\theta : W \otimes K^r \to K$ is a $R[G]$-equivariant morphisms of complexes such that

  (1) $\theta(e_0 \otimes_R x_1 \otimes_R \cdots \otimes_R x_r) = x_1 \cdots x_r$.

  (2) There exists $R[\Sigma_r]$-morphism $\phi : V \otimes_R K^r \to K$ such that $\theta \sim \phi \circ (j \otimes_R \mathrm{id})$

via a chain homotopy.

- Morphisms: $(K,\theta) \xrightarrow{f} (K,\theta')$ is a morphism of R-complexes such that the following diagram commutes up to G-equivariant chain homotopy

$$\begin{tikzcd} W \otimes K^r \arrow[r, "\theta"] \arrow[d, "1\otimes f^r"'] & K \arrow[d, "f"] \\ W \otimes K'^r \arrow[r, "\theta"] & K' \end{tikzcd}$$

The tensor product $(K,\theta) \otimes_k (K',\theta')$ is defined to be $(K \otimes_k K', \tilde{\theta})$, where $\tilde{\theta}$ is defined to be the following composition

$$W \otimes_R (K \otimes_k K')^r \xrightarrow{\psi \otimes_R \sigma} W \otimes_R W \otimes_k K^r \otimes_k K'^r \xrightarrow{\mathrm{id} \otimes_R \tau \otimes_l \mathrm{id}} W \otimes_k K \otimes_R W' \otimes_k K' \xrightarrow{\theta \otimes \theta'} K \otimes_R K'$$

where $\psi : W \to W \otimes_R W$ is a $R[G]$-homomorphism covering $R \cong R \otimes_R R$, σ is the shuffle permutation, and τ is the transposition.

Construction 4.3: For any $(K,\theta) \in \mathcal{C}(C_p, \mathbb{F}_p)$,

$$\theta_* : H_*(W \otimes_{C_p} K^p) \to H_*(K)$$

encodes all the data to define the mod-p total Steenrod operation for $H_*(K)$. Let $x \in H_q(K)$ be a homology class represented by $u \in K$. We set

$$D_i(x) = \theta_*(e_i \otimes u^p) \in H_{pq+i}(X)$$

In the case of $p = 2$, we define the *algebraic Steenrod squares*

$$\mathrm{Sq}_s : H_q(K) \to H_{q+s}(K),\ x \mapsto \begin{cases} 0 & s < q \\ D_{s-q}(x) & s > q \end{cases}$$

In the case of an odd prime p, we define *algebraic Steenrod operations*

$$\mathrm{P}_s : H_q(K) \to H_{q+2s(p-1)}(K),\ x \mapsto \begin{cases} 0 & 2s < q \\ (-1)^s \nu(q) D_{(2s-1)(p-1)}(x) & 2s \geq q \end{cases}$$

and

$$\beta \mathrm{P}_s : H_q(K) \to H_{q+2s(p-1)-1}(K),\ x \mapsto \begin{cases} 0 & 2s < q \\ (-1)^s \nu(q) D_{(2s-1)(p-1)-1}(x) & 2s \geq q \end{cases}$$

where $\nu(n) = (-1)^j (m!)^\varepsilon$ for $n = 2j + \varepsilon$ and $m = (p-1)/2$.

In the cohomological setting

$$
\begin{aligned}
K^i &:= K_{-i} \\
\mathrm{Sq}^s &:= \mathrm{Sq}_s : H^q(K) \to H^{q+s}(X) \\
\mathrm{P}^s &:= \mathrm{P}_{-s} : H^q(K) \to H^{q+2s(p-1)}(K) \\
\beta\mathrm{P}^s &:= \beta\mathrm{P}_{-s} : H^q(K) \to H^{q+2s(p-1)+1}(K)
\end{aligned}
$$

Theorem 4.2: The algebraic Steenrod operations Sq, P and $\beta\mathrm{P}$ have the following properties

(1) For any $x \in H^q(K)$, we have that

- In the case of $p = 2$, $\mathrm{Sq}^q(x) = x^2$ and $\mathrm{Sq}^n(x) = 0$ for $n > q$.
- In the case of an odd prime p, $\mathrm{P}^s(x) = x^p$ if $2s = q$, and $\mathrm{P}^s(X) = \beta\mathrm{P}^s(x) = 0$ if $2s > q$.

(2) Let β be the Bockstein operation. Then $\mathrm{Sq}^1 = \beta$ in the case of $p = 2$ and $\beta\mathrm{P} = \beta \circ \mathrm{P}$ exactly in the case of an odd prime p.

Proof: See [35]Proposition 2.3, Proposition 2.4, Proposition 2.5. ■

Definition 4.4: A *Cartan object* in $\mathcal{C}(G, R)$ is an object $(K, \theta) \in \mathcal{C}(G, R)$ such that the product $K \otimes K \to K$ can be extended to $(K, \theta) \otimes (K, \theta) \to (K, \theta)$.

Theorem 4.3: [35]Proposition 2.6 Let (K, θ) and (K', θ') be two objects in $\mathcal{C}(C_p, \mathbb{F}_p)$, and let $(x, x') \in H^q(K) \times H^r(K')$. If $p = 2$,

$$
\mathrm{Sq}^r(x \otimes x') = \sum_{i+j=r} \mathrm{Sq}^i(x) \otimes \mathrm{Sq}^j(x').
$$

If p is an odd prime, then

$$
\mathrm{P}^r(x \otimes x') = \sum_{i+j=r} \mathrm{P}^i(x) \otimes \mathrm{P}^j(x'),
$$

and

$$
\beta\mathrm{P}^r(x \otimes x') = \sum_{i+j=r} \beta\mathrm{P}^i(x) \otimes \mathrm{P}^j(x') + (-1)^q \mathrm{P}^i(x) \otimes \beta\mathrm{P}^j(x').
$$

These formulas are called *Cartan formulas*.

Corollary 4.1: [35]Corollary 2.7 If (K, θ) is a Cartan object, then we have the *internal Cartan formulas* i.e. we may replace the tensor product by the multiplication on K.

Remark 4.1: Let U be a $R[\Sigma_{p^2}]$-resolution of R and let $H = C_p \wr C_p \subset \Sigma_{p^2}$ be the p-Sylow subgroup. Note that $H = C_p \wr C_p$ acts on $W \otimes_R W^p$ "diagonally", where one of C_p acts naturally on W and the other C_p acts on W^p by permuting factors. Let $\omega : W \otimes W^p \to U$ be a $R[H]$-homormophism extending the identity $R \to R$. Then the following homotopy commutative diagram demonstrates the Adem relations.

$$\begin{tikzcd}
(W \otimes_R W^p) \otimes_R K^{p^2} \arrow[r, "\omega\otimes_k \mathrm{id}"] \arrow[d, "\text{shuffle}"] & U \otimes_R K^{p^2} \arrow[dr, "\xi"] & \\
& & K \\
W \otimes_R (W \otimes_R K^p)^p \arrow[r, "1\otimes_k \theta^p"] & W \otimes_R K^p \arrow[ur, "\theta"] &
\end{tikzcd}$$

4.2 Application: power operations on the mod p cohomology of spaces

The main goal of this section is to illustrate that given a CW-complex (or a simplicial complex) X, how we can modify $C^*(X, \mathbb{F})$ to make it an object in $\mathcal{C}(C_p, \mathbb{F}_p)$.

Let $\Delta\colon X \to X^m$ be the diagonal map. Note that there is no cellular approximation (or simplicial approximation) of Δ that is Σ_m-equivariant. Here we let Σ_m act on X trivially and act on X^m by permuting factors. We expect to construct a map $\phi : E\Sigma_m \times X \to X^m$ such that ϕ is both Σ_m-equivariant and cellular (or simplicial). This map should be regarded as an *enhanced diagonal map*. Since we require it to be cellular or simplicial, we can manipulate it on the level of chain complexes. Steenrod used the trick of acyclic carrier to show the existence of such a map[8,58]. Now we show some insight of the enhanced diagonal map.

Before we start to construct this map, we need to specify a model of EG. First, we consider the Milnor model for the universal bundle EG[59] using join construction. Recall that the *join* between two spaces X, Y is

$$X \star Y = \{ax + by \mid x \in X,\ y \in Y,\ a, b \in [0,1],\ a + b = 1\}$$

and its topology is given by some induced topology. In Milnor's construction, $EG = G \star G \star G \star \cdots$ with $\mathbb{N}$-copies. The Δ-complex structure of EG is given by

- n-simplicies are marked by ordered $(n+1)$-tuples $[g_0, \cdots, g_n]$ with $g_i \in G$;
- For $0 \le i \le n$, the i-th face of $[g_0, \cdots, g_n]$ is $[g_0, \cdots, \hat{g}_i, \cdots, g_n]$

Another interpretation for EG is taking it as the geometric realization of $E_\bullet G$ where $E_n G = G^{n+1}$, see[31].

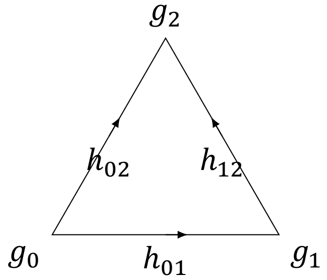
Then by taking simplicial approximation for Δ, we have a simplicial map $\Delta' X \to X^m$, even though Δ' is not Σ_m-equivariant. Nevertheless, we claim that for any $g \in \Sigma_m$, there is chain homotopy from $g\Delta'$ to Δ'. Actually, this homotopy exists if we applying the acyclic carrier theorem (see Appendix A. Theorem A.1) for the carrier $C(\sigma) = \bar{\sigma}^m$ for each simplex σ[58]Section 2. Since $g\Delta' \simeq g\Delta = \Delta \simeq \Delta'$, for any $g_0, g_1 \in G$, there is a homotopy from $g_0 D_0$ to $g_1 D_0$, and we write the homotopy by

$$h_{g_0 g_1}: X \times I \to X^m$$

Then we can define

$$\begin{array}{rcl} (\Sigma_m \star \Sigma_m) \times X & \longrightarrow & X^m \\ (tg_0 + (1-t)g_1, x) & \longmapsto & h_{g_0 g_1}(x, 1-t) \end{array}$$

In other words, we let $[g_0, g_1] \times X \to X^m$ parameterize the homotopy $h_{g_1 g_0}$. In this way, we have construct $(E\Sigma_m)_1 \times X \to X^m$. For a 2-simplex $[g_0, g_1, g_2]$ of $E\Sigma_m$, we can see



$$\begin{array}{lll} [g_0, g_1] & \text{parameterize} & h_{01} = h_{g_0, g_1}: g_0\Delta' \simeq g_1\Delta' \\ [g_1, g_2] & \text{parameterize} & h_{12} = h_{g_1, g_2}: g_1\Delta' \simeq g_2\Delta' \\ [g_0, g_2] & \text{parameterize} & h_{02} = h_{g_0, g_2}: g_0\Delta' \simeq g_2\Delta' \end{array}$$

Note that the join of homotopies $h_{01} * h_{12}$ is from $g_0 D_0$ to $g_2 D_0$. Since both $h_{01} * h_{12}$ and h_{02} are carried by C, the equivariant acyclic carrier theorem promises a homotopy $H: X \times I \times I \to X^m$ from $h_{01} * h_{12}$ to h_{02}. We let the 2-simplex $[g_0, g_1, g_3]$ parametrize this 2-homotopy H. Inductively using the trick of acyclic carrier theorem, we construct a

$$\phi: E\Sigma_m \times X \to X^m$$

Note that Σ_m acts on $E\Sigma_m$ freely by left action. Let Σ_m acts on $E\Sigma_m \times X$ diagonally. Then we can see ϕ is G-equivariant according to our construction. By taking the simplicial chain complexes with coefficient $\mathbb{F}_p$ and using the Eilenberg-Zilber theorem, we have

$$\phi_*: C_*(E\Sigma_m; \mathbb{F}_p) \otimes C_*(X; \mathbb{F}_p) \to C_*(X; \mathbb{F}_p)^m$$

Furthermore, we can replace Σ_m by any transitive subgroup $G \subset \Sigma_m$ to give a similar construction for $EG \times X \to X^m$. By taking $m = p$, and note that

$$C_*(X; \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(C^*(X; \mathbb{F}_p); \mathbb{F}_p)$$

we have

$$\theta : C_*(E\Sigma_p; \mathbb{F}_p) \otimes C^*(X; \mathbb{F}_p)^p \to C^*(X; \mathbb{F}_p)$$

which is given by

$$e \otimes f_1 \otimes \cdots \otimes f_p \mapsto (a \mapsto f_1 \otimes \cdots \otimes f_p(\phi_*(e \otimes a)))$$

Since $E\Sigma_m$ is contractible and its Δ-complex is a free complex, we may take $C_*(E\Sigma_m; \mathbb{F}_p)$ as a $\mathbb{F}_p[\Sigma_m]$-resolution of $\mathbb{F}_p$. In this way, we can regard $(C^*(X; \mathbb{F}_p), \mathbb{F}_p)$ as an object in $\mathcal{C}(C_p; \mathbb{F}_p)$ and the mod p Steenrod operations can be given by Construction 4.3.

Remark 4.2: The homotopy from $h_{01} * h_{12}$ to h_{02} is a kind of inner horn filling, if we look at the triangle, which is a kind of Kan condition. In this way, we can see why quasi-categories require this condition, otherwise homotopies may not be coherent up to higher homotopies, which is the insight of homotopy coherence.

Remark 4.3: The construction $\{E_\bullet\Sigma_m\}_{m\in\mathbb{N}}$ actually forms an $\mathbb{E}_\infty$-operad called the *Barratt-Eccles* operad[60]. Then in the category of CW-complexes with cellular maps, each CW-complex X is an "$\mathbb{E}_\infty$-coalgebra" given by

$$E\Sigma_m \times X \to X^m$$

and thus the cochain complex $C^*(X; \mathbb{F}_p)$ is a natural $\mathbb{E}_\infty$-algebra.

Remark 4.4: The mod-p power operation is more effective than the literally power with respect to the cup product because the former one can detect the Σ_d-action on $[u]^d$, while the later one does not. If we use the language of equivariant cohomology using Borel construction, the following diagram will demonstrate the reason.

$$\begin{array}{ccccc} H^n(X) & \xrightarrow{\quad\mathcal{P}^d\quad} & H^{nd}_{\Sigma_d}(X^d) & \xrightarrow{\Delta^*} & H^{nd}(B\Sigma_d \times X) \\ [u] & & [u^d]_{\Sigma_d} & & \\ n\text{-cocycle class} & & \Sigma_d\text{-equivariant } nd\text{-cocycle class} & & \end{array}$$

4.3 Application: power operations on the cohomology of Hopf algebroid

Definition 4.5: A *Hopf algebroid* over a commutative ring k is a pair of commutative k-algebras (A, Γ) endowed with maps

- $\eta_L : A \to \Gamma$ called left unit or source,
- $\eta_R : A \to \Gamma$ called right unit or target,
- $\Psi : \Gamma \to \Gamma \otimes_A \Gamma$ called coproduct or composition,

- $\epsilon : \Gamma \to A$ called counit or identity,
- $c : \Gamma \to \Gamma$ called conjugation or inverse.

and the data satisfies the following rules:

（1）η_L is flat.

（2）$\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_A$.

（3）$\Gamma \xrightarrow{\Psi} \Gamma \otimes_A \Gamma \xrightarrow{q} \Gamma \otimes_A A \cong \Gamma$ is the identity map, where $q = \mathrm{id}_\Gamma \otimes \epsilon$ or $\epsilon \otimes \mathrm{id}_\Gamma$.

（4）$(\mathrm{id}_\Gamma \otimes \Psi) \circ \Psi = (\Psi \otimes \mathrm{id}_\Gamma) \circ \Psi$.

（5）$c \circ \eta_R = \eta_L$ and $c\eta_L = \eta_R$.

（6）$c \circ c = \mathrm{id}_\Gamma$.

（7）There exists maps such that the following diagram commutes

$$\begin{tikzcd}
\Gamma & \Gamma \otimes_k \Gamma \arrow[l, "c\circ\mathrm{id}"'] \arrow[r, "\mathrm{id}\circ c"] \arrow[d] & \Gamma \\
& \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\epsilon"] \arrow[r, "\epsilon"'] \arrow[u, "\Psi"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

Remark 4.5: (η_L, η_R) exhibits Γ as an A-bimodule and Γ is ab A-comodule.

Remark 4.6: Given a Hopf algebroid $(A, \Gamma, \eta_L, \eta_R, \Psi, \epsilon, c)$ and a commutative k-algebra R, there is a groupoid whose objects are $\mathrm{Hom}_k(A, R)$ and its morphisms are $\mathrm{Hom}_k(\Gamma, R)$. In summary, a Hopf algebroid determines a functor from the category of k-algebras to the category of groupoids.

Definition 4.6: Given a Hopf algebroid (A, Γ), a *right* Γ*-comodule* M is an right A-module M together with an A-linear map $\psi_M : M \to M \otimes_A \Gamma$ such that $(\mathrm{id}_M \otimes \epsilon) \circ \psi_M = \mathrm{id}_M$ and $(\mathrm{id}_M \otimes \Psi) \circ \psi_M = (\psi_M \otimes \mathrm{id}_A) \circ \psi_M$. The category of Γ-comodules is denoted by Comod_Γ.

Remark 4.7: Given a right A-module M, $M \otimes_A \Gamma$ is a right Γ-comodule naturally, which is called the extened comodule of M.

Suppose P, Q are two graded right A-modules, then we define

$$\mathrm{Hom}_R^t(P, Q) := \{f : P^* \to Q^{*+t} \mid f \text{ is } A\text{-linear}\}$$

In this way, $\mathrm{Hom}_A(P, Q)$ is a graded A-module.

If P, Q are right Γ-comodules, then let $\mathrm{Hom}_\Gamma(P, Q)$ be the submodule of $\mathrm{Hom}_A(P, Q)$ consisting of Γ-comodule morphisms.

Theorem 4.4 (Comparison theorem): Given an A-split exact sequence $X = \{0 \to P \to X_0 \to X_1 \to \cdots\}$ of right Γ-comodules and a complex $Y = \{0 \to Q \to Y_0 \to Y_1 \to \cdots\}$ consisting of injective right Γ-comodules, for any Γ-homomorphism $f : P \to Q$, there is a unique chain homotopy class of Γ-homomorphisms $F : X \to Y$ extended f, where $X = \{X_0 \to X_1 \to \cdots\}$ and $Y = \{Y_0 \to Y_1 \to \cdots\}$.

We denote Ext_A^j the j^{th} right derived functor of Hom_Γ with respect to injective and Γ-split comodules resolution.

Definition 4.7: Given two right Γ-comodules, the right Γ-comodule structure on $M \otimes_A N$ is defined to be

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} M \otimes_A \Gamma \otimes_A N \otimes \Gamma \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} M \otimes_A N \otimes_A \Gamma \otimes_A \Gamma \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \phi} M \otimes_A N \otimes_A \Gamma$$

where $\phi : \Gamma \otimes_A \Gamma \to \Gamma$ is the morphism induced by the multiplication in Γ.

Remark 4.8: Note that we can always embed the category of left (or right) A-modules into the category of A-bimodules by setting $a \cdot m = (-1)^{|m||a|} m \cdot a$. Thus the tensor product between two right A-modules can be identified with the tensor product between the associated A-bibmodules. This can be done because A is commutative (in the sense of graded algebras).

Let Γ_R (resp. Γ_L) be a right A-module (resp. left A-module) by forgetting the left A-action induced by η_L (resp. the right A-action induced by η_R). The A-bibmodule structure of Γ and Γ_R are different with this setting. However, given a right A-comodule M, $M \otimes_A \Gamma \cong M \otimes_A \Gamma_R$, see[1]P92. Let $\theta : M \otimes_A \Gamma_R \to M \otimes_A \Gamma$ be the isomorphism (as right Γ-comodules). Specifically, the following diagram commutes.

$$\begin{tikzcd}
 & M \otimes_A \Gamma_R \arrow[dd, "\theta"] \\
M \arrow[ur, "\mathrm{id} \otimes \eta_R"] \arrow[dr, "\psi_M"'] & \\
 & M \otimes_A \Gamma
\end{tikzcd}$$

Let $p : \Gamma_R \to \bar{\Gamma}$ be the cokernel of η_R and given an element $x \in \Gamma$, let $\bar{x} := p(x)$. Define $t : \bar{\Gamma} \to \Gamma_R$ by $\bar{x} \mapsto x - \eta_R \circ \epsilon(x)$. Note that t is well-defined, since if $x = \eta_R(y)$, then $t(\bar{x}) = \eta_R(y) - \eta_R \circ \epsilon \eta_R(y) = 0$. Then for any right Γ-comodule M, we have the following Γ-split short exact sequence

$$p \longrightarrow M \xrightarrow{\mathrm{id} \otimes_A \eta_R} M \otimes_A \Gamma_R \xrightarrow{1 \otimes_A p} M \otimes_A \bar{\Gamma} \longrightarrow 0$$

where a section of $\mathrm{id} \otimes_A \eta_R$ is $\mathrm{id} \otimes_A \epsilon$ and a section of $\mathrm{id} \otimes_A p$ is $\mathrm{id} \otimes_A t$.

Definition 4.8: Let M be a right A-comodule, then the *normalized canonical resolution* $C(\Gamma, M)$ of M is the Γ-split differential graded right Γ-comodule

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots$$

where $C_s = M \otimes_A \bar{\Gamma}^{\otimes s} \otimes_A \Gamma_R$, $d_s = (1 \otimes_A \eta_R) \circ (1 \otimes_A p)$ and a section σ_s of d_s is $(\mathrm{id} \otimes_A t) \circ (\mathrm{id} \otimes_A \epsilon)$. We denote

$$m|a_1|\cdots|a_s|a := m \otimes_A \bar{a_1} \otimes_A \cdots \otimes_A \bar{a_s} \otimes_A a$$

and we assign it *homological degree* s, *internal degree* $t = |m| + \sum |a_i| + |a|$, *bidegree* (s, t) and *total degree* $t - s$.

If N is a right Γ-comodule, the *canonical complex* $C(N, \Gamma, M)$ is defined to be

$$C_{s,t}(N, \Gamma, M) := \mathrm{Hom}^t_\Gamma(N, C_s(\Gamma, M))$$

Proposition 4.1: $\mathrm{Ext}^{s,t}_\Gamma(N, M) \cong H_{s,t}(C(N, \Gamma, M))$.

Proof: See[1]Chapter IV, Proposition 1.2. ■

Definition 4.9: Let $\mathcal{C}$ be the category whose objects are triples (N, Γ, M) such that

（1）(A, Γ) is a Hopf algebroid over k,

（2）M is a commutative unital algebra in Comod_Γ (let $\eta_M : \Gamma \to M$ be the unit),

（3）N is a cocomutative unital coalgebra in Comod_Γ (let $\epsilon_N : \Gamma \to N$ be the counit).

and whose morphism $(N, \Gamma, M) \to (N', \Gamma', M')$ are triples (f, λ, g) such that

（1）λ: $(A, \Gamma) \to (A', \Gamma')$ is a morphism of Hopf algebroids,

（2）$f : M \to M'$ is an λ-equivariant morphism of algebras preserving units,

（3）$g : N' \to N$ is a λ-equivariant morphism of coalgebras preserving counits.

Given a triple $(N, A, M) \in \mathcal{C}$, let ϕ: $M^n \to M$ be the iterated product and Δ: $N \to N^n$ be the iterated coproduct. Then by using the comparison theorem 4.4, we can extend ϕ: $M^n \to M$ to

$$\tilde{\phi}: C(\Gamma, M)^n \to C(\Gamma, M)$$

and this extension is unique up to homotopy. In this way, $C(\Gamma, M)$ is a homotopy associative and commutative differential graded algebra in Comod_Γ. Furthermore, the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Gamma(N, C(\Gamma, M))^n \arrow[r, equal] \arrow[d, "\otimes"] & C(N,\Gamma,M)^n \arrow[dd] \\
\mathrm{Hom}_\Gamma(N^n, C(\Gamma, M)^n) \arrow[d, "\mathrm{Hom}(\Delta,\tilde{\phi})"] & \\
\mathrm{Hom}_\Gamma(N, C(\Gamma, M)) \arrow[r, equal] & C(N,A,M)
\end{tikzcd}$$

also characterizes $C(N,\Gamma,M)$ as a homotopy associative and commutative differential graded algebra in Comod_Γ.

Proposition 4.2: Let (N,Γ,M) be a triple over $\mathbb{F}_p$ and $G \subset \Sigma_p$, then there is a unique chain homotopy class of $\mathbb{F}_p[G]$-equivariant maps $\Phi\colon W \otimes_k C(\Gamma,M) \to C(N,\Gamma,M)$ such that $(C(\Gamma,M),\Phi)$ is a G-structured $\mathbb{F}_p$-algebra (recall Definition 4.2).

Corollary 4.2: Let (N,Γ,M) be a triple over $\mathbb{F}_p$, then $C(N,\Gamma,M)$ has a $(\pi,\mathbb{F}_p)$-pair structure.

$$\begin{tikzcd}
W \otimes \mathrm{Hom}_\Gamma(N, C(\Gamma, M))^p \arrow[r, equal] \arrow[d, "\otimes"] & W \otimes C(N,\Gamma,M)^p \arrow[dd] \\
\mathrm{Hom}_\Gamma(N^p, W \otimes_k C(\Gamma, M)^p) \arrow[d, "\mathrm{Hom}(\Delta,\Phi)"] & \\
\mathrm{Hom}_\Gamma(N, C(\Gamma, M)) \arrow[r, equal] & C(N,A,M)
\end{tikzcd}$$

Construction 4.10 (Steenrod operations in Ext**):** Note that $\mathrm{Ext}_\Gamma^{s,t}(N,M)$ is the homology of $C(N,\Gamma,M)$ and here we let $G = C_p$. Let $x \in \mathrm{Ext}_\Gamma^{s,t}(N,M)$.

If $p = 2$, we define

$$\mathrm{P}^i = \mathrm{Sq}^i(x) := \theta_*(e_{i-t+s} \otimes_k x^2), \text{ if } i \geq t-s$$

If p is an odd prime, we define

$$\begin{cases} \mathrm{P}^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)} \otimes_k x^p) & 2i \geq t-s \\ \beta\mathrm{P}^i(x) = (-1)^i \nu(t-s)\theta_*(e_{(2i-t+s)(p-1)-1} \otimes_k x^p) & 2i > t-s \end{cases}$$

where $\nu(n) = (-1)^j (m!)^\varepsilon$ for $n = 2j + \varepsilon$ and $m = (p-1)/2$.

Let (N,Γ,M) be a triple over Zb such that N, Γ and M are torsion free. Let $\bar{N} = N \otimes \mathbb{Z}/p$, $\bar{\Gamma} = \Gamma \otimes \mathbb{Z}/p$ and $\bar{M} = M \otimes \mathbb{Z}/p$. Then $(\bar{N},\bar{\Gamma},\bar{M})$ is a triple over $\mathbb{F}_p$. The exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

induces the Bockstein operation

$$\beta : \mathrm{Ext}_{\bar{\Gamma}}^{s,t}(\bar{N}, \bar{M}) \to \mathrm{Ext}_{\bar{\Gamma}}^{s+1,t}(\bar{N}, \bar{M})$$

Theorem 4.5: The Steenrod operations in previous definition has the following properties.

（1）$\beta^{\varepsilon}\mathrm{P}^i : \mathrm{Ext}_{\Gamma}^{s,t} \to \mathrm{Ext}_{\Gamma}^{s+(t-2i)(p-1)+\varepsilon,pt)}$ where $\varepsilon = 0$ if $p = 2$.

（2）When $p = 2$, $\mathrm{P}^i = 0$ unless $t - s \geq i \geq t$. When p is an odd prime, $\mathrm{P}^i = 0$ unless $t - s \geq 2i \geq t$, and $\beta\mathrm{P}^i = 0$ unless $t - s + 1 \geq 2i \geq t$.

（3）$\mathrm{P}^i(x) = x^p$ if $p = 2$ and $i = t - s$ or if p is an odd prime and $2i = t - s$.

（4）The Cartan formulas hold:

$$\mathrm{P}^n(xy) = \sum_i \mathrm{P}^i(x)\mathrm{P}^{n-i}(y)$$

$$\beta\mathrm{P}^n(xy) = \sum_i \beta\mathrm{P}^i(x)\mathrm{P}^{n-i}(y) + \sum_i (-1)^{|x|}\mathrm{P}^i(x)\beta\mathrm{P}^{n-i}(y)$$

（5）The Adem relations hold: if $p = 2$ and $0 < a < 2b$, then

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

If p is an odd prime and $a < pb$, then

$$\mathrm{P}^a\mathrm{P}^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} \mathrm{P}^{a+b-j}\mathrm{P}^j$$

and if $a \leq b$, then

$$\begin{aligned} \mathrm{P}^a\beta\mathrm{P}^b = &\sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} \beta\mathrm{P}^{a+b-j}\mathrm{P}^j \\ &+ \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1}\binom{(p-1)(b-j)-1}{a-pj-1} \beta\mathrm{P}^{a+b-j}\mathrm{P}^j \end{aligned}$$

（6）Let $f : (N, \Gamma, M) \to (N', \Gamma', M')$ and $g : (N', \Gamma', M') \to (N'', \Gamma'', M'')$ be two morphisms of triples such that the following sequence is exact

$$0 \longrightarrow C(N, \Gamma, M) \xrightarrow{C(f)} C(N', \Gamma', M') \xrightarrow{C(g)} C(N'', \Gamma'', M'') \longrightarrow 0$$

and let $\delta : \mathrm{Ext}_A^{s,t}(N, M) \to \mathrm{Ext}_{A''}^{s,t}(N'', M'')$ be the boundary map in the associated long exact sequence. Then $\delta \circ \mathrm{P}^i = \mathrm{P}^i \circ \delta$ and $\delta \circ \beta\mathrm{P}^i = -\beta\mathrm{P}^i \circ \delta$.

（7）If (N, Γ, M) is the mod-p reduction of torsion free triple over $\mathbb{Z}$, then $\beta \circ$

$\mathrm{Sq}^{i+1} = i\mathrm{Sq}^i$ and $\beta \circ \mathrm{P}^i = \beta\mathrm{P}^i$ if p is an odd prime.

Proof: See[1]Chapter IV, Theorem 2.5. ■

4.4 Power operations for structured ring spectra

Recall that the ∞-category $\mathcal{S}\mathrm{p}$ of spectra is a stable symmetric monoidal ∞-category. A commutative algebra in $\mathcal{S}\mathrm{p}$ is said to be a commutative ring spectrum or an $\mathbb{E}_\infty$-ring spectrum. More specifically, an $\mathbb{E}_\infty$-ring spectrum is a section $s : N(\mathcal{F}in_*) \to \mathcal{S}\mathrm{p}$ and we let $E = s(\langle 1\rangle)$ be the underlying spectrum. Due to the Segal condition, we have that $s(\langle n\rangle) \simeq E^{\wedge n}$. The coherence of the multiplication on E is totally encoded by $N(\mathcal{F}in_*)$, especially the morphisms and higher morphisms. For example, the multiplication $E^n \to E$ is Σ_n-equivariant, if we let Σ_n act on E^n by permuting factors and act on E trivially. Thus we have

$$\mathrm{Sym}_n E = E^n/\Sigma_n \to E.$$

Note that the notion of Σ_n-equivariant action is in the context of ∞-categories and the quotient is the homotopy quotient actually. Actually, a more suitable form to exhibit Sym_n is the notion of extended powers.

Construction 4.11: Given an spectra E, the j^{th} *extended power* of E is defined to be

$$D_j E = (E\Sigma_j)_+ \wedge E^j)/\Sigma_j$$

where $E\Sigma_j$ is the universal principal-Σ_j bundle.

There are natural maps associated to extended powers:

- $\iota_j : M^j \to D_j M$;
- $\alpha_{j,k} : D_j M \wedge D_k M \to D_{j+k} M$ induced by the inclusion $\Sigma_j \times \Sigma_k \hookrightarrow \Sigma_{j+k}$;
- $\beta_{j,k} : D_j D_k M \to D_{jk} M$ induced by the wreath product $\Sigma_j \wr \Sigma_k \to \Sigma_{jk}$;
- $\delta_j : D_j(M \wedge N) \to D_j M \wedge D_j N$.

The extended powers and these maps for pointed spaces are defined similarly and they compatible with the suspension functor. Given two pointed spaces X, Y, the following diagrams commutative up to homotopy:

$$\begin{tikzcd} & D_j\Sigma^\infty X \arrow[dd] \\ \Sigma^\infty(X^j) \arrow[ur, "\iota_j"] \arrow[dr, "\Sigma^\infty\iota_j"'] & \\ & \Sigma^\infty D_j X \end{tikzcd}$$

$$\begin{tikzcd} D_j(\Sigma^\infty X\wedge\Sigma^\infty Y) \arrow[r, "\delta_j"] \arrow[d] & D_j(\Sigma^\infty X)\wedge D_j(\Sigma^\infty Y) \arrow[d] \\ \Sigma^\infty D_j(X\wedge Y) \arrow[r, "\Sigma^\infty\delta_j"] & \Sigma^\infty(D_jX\wedge D_jY) \end{tikzcd}$$

$$\begin{tikzcd} D_jD_k\Sigma^\infty X \arrow[r, "\beta_{j,k}"] \arrow[d] & D_{jk}\Sigma^\infty X \arrow[d] \\ \Sigma^\infty D_jD_kX \arrow[r, "\Sigma^\infty\beta_{j,k}"] & \Sigma^\infty D_{jk}X \end{tikzcd}$$

$$\begin{tikzcd} D_j\Sigma^\infty X\wedge D_k\Sigma^\infty X \arrow[r, "\alpha_{j,k}"] \arrow[d] & D_{i+k}\Sigma^\infty X \arrow[d] \\ \Sigma^\infty(D_jX\wedge D_kX) \arrow[r, "\Sigma^\infty\alpha_{j,k}"] & \Sigma^\infty D_{j+k}X \end{tikzcd}$$

Let $\tau : E\wedge F\to F\wedge E$ denote the commutative isomorphism in $\mathcal{S}$p. The following assertions demonstrate the homotopy coherence data carried by extended powers.

Proposition 4.3: The extender powers are equipped with the following structure maps and coherence.

（1）$\{\alpha_{j,k}\}$ is a commutative and associative system up to homotopy, namely for any i,j,k, we have

•$\alpha_{j,k}\circ\tau\simeq\alpha_{k,j}$;

•$\alpha_{i+j,k}\circ(\alpha_{i,j}\wedge\mathrm{id})\simeq\alpha_{i,j+k}\circ(\mathrm{id}\wedge\alpha_{j,k})$;

（2）$\{\beta_{j,k}\}$ is an associative system up to homotopy, namely for any i,j,k, we have

$$\beta_{ij,k}\circ\beta_{i,j}\simeq\beta_{i,jk}\circ D_i\beta_{j,k}.$$

(3) Each δ_j is commutative and associative with respect to the smash product up to homotopy:

•$\tau \circ \delta_j \simeq \delta_j \circ D_j\tau$;

•$(\delta_j \wedge \mathrm{id}) \circ \delta_j \simeq \delta_j \circ (\mathrm{id} \wedge \delta_j)$

(4) The following diagrams commute up to homotopy:

$$\begin{tikzcd} M^j \wedge M^k \arrow[r, equal] \arrow[d, "\iota_j \wedge \iota_k"'] & M^{j+k} \arrow[d, "\iota_{j+k}"] \\ D_j M \wedge D_k M \arrow[r, "\alpha_{j,k}"] & D_{j+k} M \end{tikzcd} \qquad \begin{tikzcd} (D_k M)^j \arrow[dr, "\alpha_{k,\dots,k}"] \arrow[d, "\iota_j"'] & \\ D_j D_k M \arrow[r, "\beta_{j,k}"'] & D_{jk} M \end{tikzcd}$$

(5) Let ν_j be the evident shuffle isomorphism, then the following diagram commute up to homotopy

$$\begin{tikzcd} (E \wedge F)^j \arrow[r, "\nu_j"] \arrow[d, "\iota_j"'] & E^j \wedge F^j \arrow[d, "\iota_j \wedge \iota_j"] \\ D_j(E \wedge F) \arrow[r, "\delta_j"] & D_j E \wedge D_j F \end{tikzcd}$$

(6) The following diagram commutes up to homotopy:

$$\begin{tikzcd} D_i D_k M \wedge D_j D_k M \arrow[r, "\beta_{i,k} \wedge \beta_{j,k}"] \arrow[d, "\alpha_{i,j}"'] & D_{ik} M \wedge D_{jk} M \arrow[d, "\alpha_{ik,jk}"] \\ D_{i+j} D_k M \arrow[r, "\beta_{i+j,k}"] & D_{ik+jk} M \end{tikzcd}$$

(7) The following diagrams commute up to homotopy

$$\begin{tikzcd}
D_j(M\wedge N)\wedge D_k(M\wedge N)) \arrow[rr, "\alpha_{j,k}"] \arrow[d, "\delta_j\wedge\delta_k"] & & D_{j+k}(M\wedge N) \arrow[d, "\delta_{j+k}"] \\
D_jM\wedge D_jN\wedge D_kM\wedge D_kN \arrow[r, "\mathrm{id}\wedge\tau\wedge\mathrm{id}"] & D_jM\wedge D_kM\wedge D_jN\wedge D_kF \arrow[r, "\alpha_{j,k}\wedge\alpha_{j,k}"] & D_{j+k}M\wedge D_{j+k}N
\end{tikzcd}$$

$$\begin{tikzcd}
D_jD_k(M\wedge N) \arrow[rr, "\beta_{j,k}"] \arrow[d, "D_j\delta_k"] & & D_{jk}(M\wedge N) \arrow[d, "\delta_{jk}"] \\
D_j(D_kM\wedge D_kN) \arrow[r, "\delta_j"] & D_jD_kM\wedge D_jD_kN \arrow[r, "\beta_{j,k}\wedge\beta_{j,k}"] & D_{jk}M\wedge D_{jk}N
\end{tikzcd}$$

(8) The following diagram commutes up to homotopy:

$$\begin{tikzcd}
D_i(D_jM\wedge D_kM) \arrow[r, "\delta_i"] \arrow[d, "D_i\alpha_{j,k}"] & D_iD_jM\wedge D_iD_kM \arrow[r, "\beta_{i,j}\wedge\beta_{i,k}"] & D_{ij}M\wedge D_{ik}M \arrow[d, "\alpha_{ij,ik}"] \\
D_iD_{j+k}M \arrow[rr, "\beta_{i,j+k}"] & & D_{ij+ik}M
\end{tikzcd}$$

Proof: See[1]Lemma 2.8, Lemma 2.9, Lemma 2.10, Lemma 2.11, Lemma 2.12, Lemma 2.13, Lemma 2.14, Lemma 2.15

■

Definition 4.12: [1]Definition 3.1 An $\mathbb{H}_\infty$-ring spectrum is a spectra M together with $\xi_j : D_jM \to M$ for $j \geq 0$ such that ξ_1 is the identity map and the following diagrams commute for $j,k \geq 0$ up to homotopy

$$\begin{tikzcd}
D_jM\wedge D_kM \arrow[rr, "\alpha_{j,k}"] \arrow[d, "\xi_j\wedge\xi_k"] & & D_{j+k}M \arrow[d, "\xi_{j+k}"] \\
M\wedge M \arrow[r, "\iota_2"] & D_2M \arrow[r, "\xi_2"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D_jD_kM \arrow[r, "\beta_{j,k}"] \arrow[d, "D_j\xi_k"] & D_{jk}M \arrow[d, "\xi_{jk}"] \\
D_jM \arrow[r, "\xi_j"] & M
\end{tikzcd}$$

Remark 4.9: The structured diagrams for $\mathbb{H}_\infty$-ring spectra commute in h$\mathcal{S}$p, while the structured diagrams for $\mathbb{E}_\infty$-ring spectra commute in $\mathcal{S}$p. In other words, in the case of $\mathbb{H}_\infty$-ring spectra, diagrams commute up to homotopy, while in the case of $\mathbb{E}_\infty$-ring spectra, diagrams commute homotopy-coherently (it depends on the feature of ∞-categories).

Construction 4.13: Let E be an $\mathbb{H}_\infty$-*ring spectrum* and $\tilde{E}$ be the associated cohomology theory, the *j-total power operation* $\mathcal{P}_j$ is defined to be

$$\begin{tikzcd}
\tilde{E}^*(X) \arrow[rrr, "\mathcal{P}_j"] \arrow[d, equal] & & & \tilde{E}^*(B\Sigma_{j+} \wedge X) \arrow[d, equal] \\
{[\Sigma^\infty X, E]} \arrow[r, "D_j"] & {[D_j(\Sigma^\infty X), D_j E]} \arrow[r, "\circ \xi_j"] & {[\Sigma^\infty (D_j X), E]} \arrow[r, "\Delta^*"] & {[\Sigma^\infty (B\Sigma_{j+} \wedge X), E]}
\end{tikzcd}$$

where Δ^* is induced by the diagonal map $X \to X^j$.

If we have $\tilde{E}^*(B\Sigma_{j+} \wedge X) \cong \tilde{E}^*(B\Sigma_{j+}) \otimes_{\pi_* E} \tilde{E}^*(X)$, then given $\alpha \in E^* B\Sigma_j$ and $x \in \tilde{E}^r(X)$, we can define $\alpha^*(x)$ to be $D\alpha^* \circ \mathcal{P}(x)$, where $D\alpha^* : \tilde{E}^r(X) \to \pi_* E$ is the dual function of α.

CHAPTER 5 THE YOGA OF SPECTRAL SEQUENCES

Spectral sequences are the most important computational tools in homotopy theory and homological algebra. For example, they can be used to compute graded invariants of a space such as homotopy groups, homology groups, or cohomology groups by taking approximations. Since it is hard to compute the invariants of an object directly, we replace an object by taking its filtration or resolution (so-called sequential replacement) consisting of relatively simple objects and compute the invariants step by step along the filtration or resolution, where spectral sequences play a role as an effective algorithm for us to successively approach the information of the object. Roughly speaking, spectral sequences compute "the (co)homology of (co)homology groups" and iterate the process inductively. The concrete procedure is to use exact couples and their derived couples inductively[61]. In this chapter, we follow this big picture and focus on the Adams-type spectral sequences and their generalization.

• In Section 5.1, we show how to derive a spectral sequence from a sequence of spectra. We focus on how a specific topological setting or spectra-level setting on the sequences corresponds to a computable homological setting on spectral sequences. In particular, we discuss two types of sequences and examples of spectral sequences respectively.

types	sequences	resulting spectral sequences
inverse sequence	Adams resolution	Adams-type spectral sequence
direct sequence	skeletal filtration	Atiyah-Hirzebruch-type spectral sequence

• In Section 5.2, we introduce how to construct an Adams-type spectral sequence systematically. The key construction is the canonical Adams resolution (Construction 5.5). Specifically, given a spectrum X and Y, the associated complex of the canonical Adams resolution of Y with respect to E and X is exactly the canonical complex $C(E_*X, E_*E, E_*Y)$, so that we can apply the result in Section 4.3 on Adams spectral sequences. In particular, we have power operations in the E_2-page of an Adams-type spectral sequence.

• In Section 5.5, we introduce a generalized version of the Adams-type spectral sequences and their properties. The generalized Adams spectral sequences are designed for mixed filtration. More specifically, the input of a generalized Adams spectral sequence consists of two sequences, one of which is an Adams resolution and the other of which carries extra information so that the Adams-type spectral sequence can be manipulated further. We will then use this construction together with the filtration on the extended powers to show how power operations detect homotopy operations.

- In Section 5.3, we introduce how extended powers display at the level of filtration. Given an $\mathbb{H}_\infty$-ring spectrum Y, the unique (up to homotopy) algebraic extended power structures on $C(\pi_*E, E_*E, E_*Y)$ mentioned in Section 4.3 can be derived from the extended powers in the $\mathbb{H}_\infty$-structure on Y. In detail, we construct a "filtration" of $\xi : D_G Y \to Y$ (See Theorem 5.3) and show this "filtration" endow $C(\pi_*E, E_*E, E_*Y)$ a Σ_n-enhanced algebraic structure which is equivalent to the one in Proposition 4.2 (See Corollary 5.2).

- In Section 5.6, we use a generalized Adams spectral sequence in Section 5.3 together with the filtration of extended powers in Section 5.3 to show how the power operations detect the homotopy operations on $\pi_*(Y)$ associated to the extended power. More specifically, there is a spectral sequence $E^{*,*}(\mathbb{S}, \Xi)$ associated to a filtration Ξ of $D_G S^n$, and each $x \in \pi_n(Y)$ determines a morphism of spectral sequences from $E^{*,*}(\mathbb{S}, \Xi)$ to $E^{*,*}(\mathbb{S}, Y)$, which exhibits $E^{*,*}(\mathbb{S}, Y)$ is a module over $E^{*,*}(\mathbb{S}, \Xi)$. This module structure indicates how the power operations detect the homotopy operations.

In summary, the following diagram demonstrates the outline of this chapter.

$$\begin{tikzcd}
\text{filtration for powers} \arrow[rr, Rightarrow] \arrow[dd] \arrow[dr, "\text{input}"] & & \text{homotopy operations} \arrow[dd] \\
& \boxed{\text{generalized Adams S.S.}} \arrow[ur, "\text{detect}"] \arrow[dr, "\text{detect}"'] & \\
\text{Adams resolution} \arrow[rr, Rightarrow] \arrow[ur, "\text{input}"'] & & \text{homotopy classes}
\end{tikzcd}$$

5.1 Spectral sequences from sequences of spectra

Definition 5.1: Let $\mathcal{C}$ be a category. An *inverse sequence* in $\mathcal{C}$ is an object in $\mathcal{F}\mathrm{un}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C})$, where $\mathbb{Z}_{\geq 0}^{op}$ is the category of non-negative integers with $\geq$ as morphisms. We may write it as

$$Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

Dually, a *direct sequence* in $\mathcal{C}$ is an object in $\mathcal{F}\mathrm{un}(\mathbb{Z}_{\geq 0}, \mathcal{C})$ and is written as

$$X_0 \to X_1 \leftarrow X_2 \to \cdots$$

More generally, a *tower* in $\mathcal{C}$ is an object in $\mathcal{F}\mathrm{un}(\mathbb{Z}, \mathcal{C})$ and we may regard an inverse sequence (resp. direct sequence) as a bounded-above (resp. bounded-below) tower. All these items are called *sequential objects* in $\mathcal{C}$.

In this section, we focus on sequential objects in the category of spectra and study how to derive spectral sequences from sequential objects.

Construction 5.2: Suppose there is an inverse sequence of spectra

$$Y \simeq Y_0 \xleftarrow{i_0} Y_1 \xleftarrow{i_1} Y_2 \xleftarrow{i_2} \cdots$$

Let $i_{s,r} : Y_{s+r} \xrightarrow{i_{s+r-1}} \cdots \xrightarrow{i_r} Y_s$ and $Y_{s,r}$ be the cofiber $\mathrm{cofib}(i_{s,r})$ of $i_{s,r}$. Then we have a cofiber sequence in a form of

$$\begin{tikzcd} Y_{s+r} \arrow[rr, "i_{s,r}"] & & Y_s \arrow[dl, "p_{s,r}"] \\ & Y_{s,r} \arrow[ul, "\partial_{s,r}"] & \end{tikzcd}$$

where $\partial_{s,r}$ is of degree -1. Then we have the following derived diagram

$$\begin{tikzcd} Y_{0,1} \arrow[r, dashed] \arrow[dr, "\partial"] & \Sigma Y_{1,1} \arrow[r, dashed] \arrow[dr, "\partial"] & \Sigma^2 Y_{2,1} \arrow[r, dashed] \arrow[dr, "\partial"] & \cdots \\ Y_0 \arrow[u] & \Sigma Y_1 \arrow[u] & \Sigma^2 Y_2 \arrow[u] & \cdots \arrow[u] \\ Y_1 \arrow[u] & \Sigma Y_2 \arrow[u] & \Sigma^2 Y_3 \arrow[u] & \cdots \arrow[u] \end{tikzcd} \tag{5-1}$$

Each column with the coboundary ∂ is a cofiber sequence induced by i_k and the dashed arrows are induced by evident compositions. Let X be a spectrum. If we apply $[X, -]$ to the diagram, then we obtain an exact couple

$$\begin{tikzcd} \bigoplus_{s,t}[X, Y_{s+1}]_{t-s} \arrow[rr, "i_*"] & & \bigoplus_{s,t}[X, Y_s]_{t-s} \arrow[dl, "p_*"] \\ & \bigoplus_{s,t}[X, Y_{s,1}]_{t-s} \arrow[ul, "\partial_*"] & \end{tikzcd}$$

and the $E_r^{s,t}$-term of the associated spectral sequence is

$$E_r^{s,t} = \frac{\mathrm{im}([X, Y_{s,r}]_{t-s} \to [X, Y_{s,1}]_{t-s})}{\ker([X, Y_{s,1}]_{t-s} \to [X, Y_{s-r+1,r}]_{t-s})}$$

To make the spectral sequence computable, we need to add some assumptions to make it meaningful in the setting of homological algebra. The following example is about the idea of classical Adams spectral sequences.

Example 5.1 (Adams-type spectral sequence): In this example, we abbreviate the mod-p cohomology as H^* simply. Let Y be a CW-spectrum of finite type and let $\{\alpha_i\}_{i=1}^n$ be a set of cohomology classes in $H^*(Y)$ such that they generate $H^*(Y)$ as an $\mathcal{A}_p^*$-module. Then we let $q_1 : Y_1 \to \prod_{i=1}^n \Sigma^{n_i} H\mathbb{F}_p$ be the map representing these cohomology classes. Recall that the homotopy groups of a generalized Eilenberg-Mac Lane spectrum K consisting of $H\mathbb{F}_p$ is exactly the graded module $\mathrm{Hom}_{\mathcal{A}_p^*}(H^*(E), \mathbb{F}_p)$[61]Proposition 2.1.2(d). Therefore, by using the Whitehead theorem, the natural map

$$\bigvee_{i=1}^{n} \Sigma^{n_i} H\mathbb{F}_p \to \prod_{i=1}^{n} \Sigma^{n_i} H\mathbb{F}_p$$

is an equivalence. In this way, we lift q_1 to

$$p_1 : Y \to \bigvee_{i=1}^{n} \Sigma^{n_i} H\mathbb{F}_p$$

and let $Y_1 \to Y$ be the fiber of p_1. Inductively, we repeat this procedure by replacing Y by Y_s to produce $Y_{s+1} \to Y_s$. Eventually, we have an inverse sequence

$$Y = Y_0 \xleftarrow{i_0} Y_1 \xleftarrow{i_1} Y_2 \xleftarrow{i_2} \cdots$$

According to our construction, the $Y_{i,1}$ is a wedge of $H\mathbb{F}_p$ with some degree shifting. If we apply H^* to the sequence 5-1, then we have

$$0 \leftarrow H^*(Y_0) \leftarrow H^*(Y_{0,1}) \leftarrow H^*(\Sigma Y_{1,1}) \leftarrow \cdots \leftarrow H^*(\Sigma^n Y_{n,1}) \leftarrow \cdots$$

where each arrow is a surjection. Moreover, it turns out to be a resolution of $H^*(Y)$ consisting of free $\mathcal{A}_p^*$-modules. Since $\mathrm{cofib}(i_{s,1})$ is a wedge of $H\mathbb{F}_p$, we still have

$$[X, \mathrm{cofib}(i_{s,1})] \cong \mathrm{Hom}_{\mathcal{A}_p^*}(H^*(Y_{s,1}), H^*(X))$$

and

$$[X, \mathrm{cofib}(i_{s,1})]_t \cong \mathrm{Hom}^t_{\mathcal{A}_p^*}(H^*(Y_{s,1}), H^*(X)).$$

The E_2-page of the associated spectral sequences can be written as

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(H^*Y, H^*X)$$

where $\mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(M, N)$ is the s^{th} right derived functor of $\mathrm{Hom}_{\mathcal{A}_p^*}(M, N)$.

The convergence problem is more complicated. More details can be found in Ravenel's textbook[61].

Based on this example, we can see the desired assumption for constructing an Adams-type spectral sequence, see Definition 5.4.

Dually, a direct sequence can also derive a spectral sequence in a similar way. Specifically, the key homological feature appears in the sequence 5-1 by applying a suitable cohomology functor to it. Similarly, a direct sequence can also derive a complex.

Construction 5.3: Suppose we have a direct sequence of spectra

$$X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_1} X_2 \xrightarrow{j_2} \cdots$$

For convenience, we write $\mathrm{cofib}(j_i)$ as X_{i+1}/X_i and denote

$$\bar{X}_i = \Sigma^{-1}(X_i/X_{i+1}).$$

Then we have an associated sequence

$$\begin{tikzcd}
\cdots \arrow[r, dashed] \arrow[d] & \Sigma^{-2}\bar{X}_2 \arrow[r, dashed] \arrow[d] & \Sigma^{-1}\bar{X}_1 \arrow[r, dashed] \arrow[d] & \bar{X}_0 \arrow[d] \\
\cdots \arrow[d] & \Sigma^{-2}X_2 \arrow[ur, "\partial"] \arrow[d] & \Sigma^{-1}X_1 \arrow[ur, "\partial"] \arrow[d] & X_0 \arrow[d] \\
\cdots \arrow[d] & \Sigma^{-2}X_3 \arrow[d] & \Sigma^{-1}X_2 \arrow[d] & X_1 \arrow[d] \\
\cdots & \Sigma^{-2}(X_3/X_2) & \Sigma^{-1}(X_2/X_1) & X_1/X_0
\end{tikzcd} \tag{5-2}$$

Let E be a spectrum, $P = E_*X$, and $P_i = E_*(\Sigma^{-i}\bar{X}_i)$. The *associated complex* is

$$E_*X \leftarrow E_*(\Sigma^{-1}\bar{X}_1) \leftarrow E_*(\Sigma^{-2}\bar{X}_2) \leftarrow \cdots \leftarrow E_*(\Sigma^{-n}\bar{X}_n) \leftarrow \cdots$$

Then it will form an exact couple similarly and thus derive a spectral sequence. In particular, the E_1-page is actually given by the associated complex. Moreover, the spectral sequence will converge to $E_*(\mathrm{colim}_s X_s)$ [62]Chapter 5, Section 2.

Similarly, we define a cohomological-associated complex by replacing E_* by E^*.

A bounded-below filtration is a special direct sequence. By using cellular approximation and the telescope construction, any direct sequence is equivalent to a bounded-below filtration with subcomplex inclusion. In this case, $\mathrm{cofib}(j_s) \simeq X_s/X_{s+1}$ exactly. The most important bounded-below filtration in topology is skeletal filtration.

Example 5.2 (Atiyah-Hirzebruch-type spectral sequences): Let A be a spectrum and its associated cohomology ring of a single point is denoted by A^*. Let X be a connected CW-complex (since we always can take it as a spectrum by taking the associated suspension spectra, there is no harm to work on spaces) and we take the skeletal filtration of X

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$$

Then the E_1-page is given by

$$E_1^{s,t} = A^{s+t}(X_s/X_{s-1}) = C^s_{\mathrm{cell}}(X; A^t)$$

where $C^s_{\mathrm{cell}}(X; A^t)$ is the cochain cellular complex with coefficient A^t. This is true since X_s/X_{s-1} is the wedge of all the s-cells in X. Moreover, its E_2-page is given by

$$E_2^{s,t} = H^s(X; A^t)$$

and we can write

$$E_2^{s,t} = H^s(X; A^t) \Rightarrow A^{s+t}(X)$$

This spectral sequence is called an *Atiyah-Hirzebruch spectral sequences*.

5.2 The Adams spectral sequences

In this section, we will follow the approach in Example 5.1 to study Adams-type spectral sequences more systematically. In particular, the homological computation in the set-up of the Adams spectral sequence is based on some results on the cohomology of Hopf algebroid discussed in Section 4.3.

Suppose E is a (homotopy) commutative ring spectrum with unit $\eta : \mathbb{S} \to E$ and product $\mu : E \wedge E \to E$. We assume the induced map $\eta_* : \pi_* E \to E_* E$ is flat. Therefore $(\pi_* E, E_* E)$ forms a Hopf algebroid naturally.

Proposition 5.1: With the assumption that η_* is flat, the natural map

$$E_*E \otimes_{\pi_* E} E_*X \to \pi_*(E \wedge E \wedge X)$$

is an isomorphism.

In particular, we have $E_*(E \wedge E) \cong E_*E \otimes_{\pi_* E} E_*E$. Now we let $A = \pi_* E$ and $\Gamma = E_* E$. The coproduct $\Psi : \Gamma \to \Gamma \otimes_A \Gamma$ is

$$x \in [\mathbb{S}, E \wedge E]_* \mapsto \eta \wedge x \in [\mathbb{S}, E \wedge E \wedge E]_*$$

For any spectrum X, E_*X is a right A-comodule in a similar way. (we just replace the suitable E by X in previous formula.)

Definition 5.4: Let E be a homotopy commutative ring spectrum. An *Adams resolution of a spectrum X with respect to E* is an inverse sequence

$$Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow Y_3 \longleftarrow \cdots$$

such that for each s

(1) $Y_{s,1}$ is a wedge of (suspensions) of E or a retract of $I_s \wedge E$ for some spectrum I_s.

(2) $E_*Y_s \to E_*Y_{s,1}$ is an $\pi_*(E)$-split monomorphism.

If we splice an Adams resolution of Y, we obtain a π_*E-injective resolution of E_*Y, because $E_*Y_{s,1}$ is a direct summand of of $E_*(X_s \wedge E) \cong E_*(X) \otimes_{\pi_*E} E_*E$ and the cofiber sequences induce short exact π_*E-split sequences[61]A1.2.8. Lemma.

$$\begin{tikzcd} 0 \arrow[r] & E_*Y \arrow[r] & E_*Y_{0,1} \arrow[r] \arrow[d] & E_*\Sigma Y_{1,1} \arrow[r] \arrow[d] & E_*\Sigma^2 Y_{2,1} \arrow[r] \arrow[d] & \cdots \\ & & E_*\Sigma Y_1 \arrow[ur] & E_*\Sigma^2 Y_2 \arrow[ur] & \cdots & \end{tikzcd} \tag{5-3}$$

Construction 5.5 (Canonical Adams resolution): Let $i : \bar{E} \to \mathbb{S}$ be the fiber of the unit $\eta : \mathbb{S} \to E$. Since a cofiber sequence in the stable homotopy category is a fiber sequence, the cofiber of i is exactly the unit. We defined the *canonical Adams resolution* inductively by setting $Y_0 = Y$, $Y_{s+1} = Y_s \wedge \bar{E}$ and $i_s = \mathrm{id} \wedge i : Y_s \wedge \bar{E} \to Y_s \wedge \mathbb{S} \cong Y_s$. Note that the cofiber Ci_s is $Y_s \wedge E$ according to the definition.

The *Adams spectral sequence* for $[X, Y]_*$ with respect to E is the associated spectral sequence of the exact couple obtained by applying $[X, -]_*$ to an Adams resolution of Y. We denote it by $E_r^{*,*}(X, Y)$.

Remark 5.1: If we smash E on the cofiber sequence

$$\mathbb{S} \xrightarrow{\eta} E \longrightarrow \Sigma\bar{E}$$

then we have another cofiber sequence

$$\mathbb{S} \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E \longrightarrow \bar{E} \wedge E \tag{5-4}$$

Note that $\eta \wedge \mathrm{id}$ has a section $\mu : E \wedge E \to E \cong \mathbb{S} \wedge E$. Then we have a π_*E-split exact sequence by applying π_* on the cofiber sequence (5-4)

$$0 \longrightarrow \pi_*E \xrightarrow{\eta_*} E_*E \longrightarrow E_*\Sigma\bar{E} \longrightarrow 0.$$

Therefore $E_*\Sigma\bar{E}$ is isomorphic to the cokernel of $\eta : \pi_*E \to E_*E$, which means that $E_*\Sigma\bar{E}$ play a role as $\bar{\Gamma}$ in the normalized resolution resolution in Definition 4.8.

Lemma 5.1: The spliced resolution in the form of (5-3) obtained from the canonical Adams resolution is the normalized canonical resolution $C(E_*E, E_*Y)$ in Definition 4.8.

Condition 5.6: For any Y and π_*E-projective E_*X, we have the following isomorphisms.

$$[X, Y \wedge E]_* \cong \mathrm{Hom}_{E_*E}(E_*X, E_*(Y \wedge E)) \cong \mathrm{Hom}_{\pi_*E}(E_*X, E_*Y)$$

The Condition 5.6 holds for $E = \mathbb{S}, H\mathbb{Z}/p, K, KO, MU, MO, MSp$, and BP, see [1]Chapter IV, Lemma 3.7.

Remark 5.2: Given Condition 5.6, for any Adams resolution in Definition 5.4, we have $[X, Y_{s,1}] \cong \mathrm{Hom}_{E_*E}(E_*E, E_*Y_{s,1})$. Therefore, the E_2-term in the associated Adams spectral sequence is $\mathrm{Ext}_{E_*E}(E_*X, E_*Y)$.

Suppose the Condition 5.6 is satisfied, then we have the following results.

Lemma 5.2: If E_*X is π_*E-projective, then

$$E_1^{s,t}(X, Y) = C_{s,t}(E_*X, E_*E, E_*Y)$$

If E_*X is π_*E-projective, then

$$E_2^{s,t}(X, Y) = \mathrm{Ext}_{E_*E}^{s,t}(E_*X, E_*Y)$$

Theorem 5.1 (Adams): Given a commutative ring spectrum E and two spectra X, Z satisfying the condition in [63]Theorem 15.1, the Adams spectral sequence $E_r^{s,t}(X, Z)$ converges to $[X, Z]_*^E$, where $[X, Z]_*^E$ is the graded group of homotopy classes in the E-localized stable homotopy category.

Proof: See [63]Part III, Chapter 15. ■

Remark 5.3: Adams's conditions for the convergence of $E_r^{s,t}(X, Z) \Rightarrow [X, Z]_*^E$ are

(1) Z is bounded below,

(2) E is connective and $\mu_* : \pi_0(E) \otimes \pi_0(E) \to \pi_0(E)$ is an isomorphism,

(3) if $R \subset \mathbb{Q}$ is maximal such that the natural ring homomorphism $\mathbb{Z} \to \pi_0(E)$ extends to $R \to \pi_0 E$, then $H_r E$ is finitely generated as an R-module for all r;

Theorem 5.2: Given a pair of Adams spectral sequences $E_*^{*,*}(L, K)$ and $E_*^{*,*}(L', K')$, we have that

$$\begin{array}{ccccc}
E_r^{*,*}(L, K) & \otimes & E_r^{*,*}(L', K') & \to & E_r^{*,*}(L \wedge L', K \wedge K') \\
\Downarrow & & \Downarrow & & \Downarrow \\
[L, K]_*^E & \otimes & [L', K']_*^E & \to & [L \wedge L', Y \wedge K']_*^E
\end{array}$$

In the case where both E_*L and E_*L' are π_*E-projective, the product on the E_2-pages is exactly the external product

$$\mathrm{Ext}(E_*L, E_*K) \otimes \mathrm{Ext}(E_*L', E_*K') \to \mathrm{Ext}(E_*L \otimes E_*L', E_*L \otimes E_*K')$$

by using the Künneth isomorphism.

Corollary 5.1:

(1) $\{E_r^{*,*}(\mathbb{S}, \mathbb{S})\}$ is a spectral sequence of bigraded commutative algebras.

(2) $E_r^{*,*}(X, Y)$ is a differential $E_r^{*,*}(\mathbb{S}, \mathbb{S})$-module.

(3) If X is a suspension spectral and Y is a commutative ring spectrum, then $\{E_r^{/*,*}\}$ is a spectral sequence of bigraded commutative $\{E_r^{*,*}(\mathbb{S},\mathbb{S})\}$-algebras whose product converges to the smash product on $[X,Y]$ defined by the diagonal map $\Delta\colon X \to X \wedge X$ and the product $\mu\colon Y \wedge Y \to Y$.

5.3 Filtration for extended powers

Suppose Y is an $\mathbb{H}_\infty$-ring spectrum, we let

$$\xi\colon D_r Y \to Y$$

be its r^{th} extended power.

Let $G \subset \Sigma_r$ and let EG_n be the n-skeleton of a contractible G-free CW-complex EG. We assume $W_0 = G$. Then we let $D_G^i Y := ((EG_i)_+ \wedge Y^{\mathrm{r}})/G$, which is a subcomplex of $D_G Y$. This construction induces a filtration of $D_G Y$.

$$D_G^0 Y \subset D_G^1 Y \subset D_G^2 Y \subset \cdots \subset D_G Y$$

Now we let E be a ring spectrum satisfying Condition 5.6 and $(\pi_* E, E_* E)$ forms a Hopf algebroid. Let

$$Y \simeq Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \ldots$$

be an Adams resolution with respect to E. Then we let

$$F_s = (Y_s)^r$$

$$Z = D_G Y_0 = ((EG)_+ \wedge Y_0^r)/G$$

$$Z_{i,s} = ((EG_i)_+ \wedge F_s)/G$$

Lemma 5.3: Let $B_i = EG_i/G$.

(1) $Z_{i-1,s}$ and $Z_{i,s+1}$ are subcomplex of $Z_{i,s}$.

(2) $\frac{Z_{i,s}}{Z_{i-1,s}} \simeq \frac{B_i}{B_{i-1}} \wedge F_s$.

(3) $\frac{Z_{i,s}}{Z_{i-1,s} \cup Z_{i,s+1}} \simeq \frac{B_i}{B_{i-1}} \wedge \frac{F_s}{F_{s+1}}$

(4) The following diagram commutes.

$$\begin{tikzcd}
\frac{Z_{i,s}}{Z_{i-1,s}\cup Z_{i,s+1}} \arrow[r, "\sim"] \arrow[d, "\partial"] & \frac{B_i}{B_{i-1}} \wedge \frac{F_s}{F_{s-1}} \arrow[d, "\partial\wedge\mathrm{id}\vee\mathrm{id}\wedge\partial"] \\
\frac{Z_{i-1,s}}{Z_{i-2,s}\cup Z_{i-1,s+1}} \vee \frac{Z_{i,s+1}}{Z_{i-1,s+1}\cup Z_{i,s+2}} \arrow[r, "\sim"] & \frac{B_{i-1}}{B_{i-2}} \wedge \frac{F_s}{F_{s+1}} \wedge \frac{B_i}{B_{i-1}} \wedge \frac{F_{s+1}}{F_{s+2}}
\end{tikzcd}$$

Proof: See[1]Chapter IV, Lemma 5.1. ■

Theorem 5.3: If E_*Y_s is π_*E-projective for each s, then there exists maps $\xi_{i,s} : Z_{i,s} \to Y_{s-i}$ such that the following diagrams commute

$$\begin{tikzcd} D_GY & Z_{i,s} \arrow[l] \arrow[d, "\xi_{i,s}"] \\ Y \arrow[from=u, "\xi"'] & Y_{s-i} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} Z_{i,s-1} \arrow[d, "\xi_{i,s-1}"] & Z_{i,s} \arrow[l] \arrow[d, "\xi_{i,s}"] & Z_{i-1,s} \arrow[l] \arrow[d, "\xi_{i-1,s}"] \\ Y_{s-i-1} & Y_{s-i} \arrow[l] & Y_{s-i+1} \arrow[l] \end{tikzcd}$$

Proof: See[1]Chapter IV, Theorem 5.2. ■

Remark 5.4: The mix $\{Z_{i,s}\}$ of the skeleton filtration on $E\pi$ and the Adams resolution of Y together with $\{\xi_{i,s}\}$ is the "resolution" of $\xi : D_GY \to Y$. We will see how $\{\xi_{i,s}\}$ "converge" to ξ along a generalized Adams spectral sequence later (see Theorem 5.8).

Let $W_k = \pi_k(EG_k/EG_{k-1})$ and $d : W_k \to W_{k-1}$ be the map induced by the geometric boundary map. Then we have a $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ with $W_0 = \mathbb{Z}[G]$. Let $C_{s,t} = E_{t-s}Y_{s,1}$. Then

$$0 \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

is the resolution of E_*Y associated to the Adams resolution. (Here the index i of C_i is the total degree of elements in $\bigoplus_{s,t} C_{s,t}$.) Note that $C_{s,t}$ coincides with $E_1^{s,t}(\mathbb{S}, Y)$, and $C_{s,t} = C_{s,t}(\pi_*E, E_*E, E_*Y)$.

Note that if each E_*Y_s is π_*E-projective, the Künneth homomorphism is an isomorphism from C^{R} to the resolution associated to $\{F_s\}$. Let $h_E : \pi_* \to E_*$ be the Hurewicz homomorphism, κ the Künneth homomorphism.

Corollary 5.2: If $\pi_0E = \mathbb{Z}/p$ and the chain map $\Phi' : W \otimes_k C^r \to C$ is defined to make the following diagram commute

$$\begin{tikzcd} W_k \otimes (\otimes_i C_{s_i,t_i}) \arrow[r, "h_E\otimes\mathrm{id}"] \arrow[dd, dashed, "\Phi'"'] & E_k(EG_k/EG_{k-1}) \otimes (\otimes_i E_{t_i-s_i}Y_{s_i,1}) \arrow[d, "\kappa"] \\ & E_{t-s+k}(EG_k/EG_{k-1} \wedge F_{s,1}) \arrow[d, "\xi_{k,s}"] \\ C_{s-k,t} \arrow[r, equal] & E_{t-s+k}Y_{s-k,1} \end{tikzcd}$$

where $t = t_1+\cdots+t_r$ and $s = s_1+\cdots+s_r$, then Φ' is chain homotopic to Φ in Proposition

4.2 (Here we let $\Gamma = E_*E$, $M = E_*Y$, and $N = \pi_*E$).

Corollary 5.3: Suppose X is a spectrum with a coproduct $\Delta : X \to X \wedge X$ and E_*X is π_*E-projective. Let $e \in W_k$ and $f_j \in [X, Y_{s_j,1}]_{t_j - s_j}$, then $\Phi_*(e \otimes f_{1*} \otimes \cdots \otimes f_{r*})$ is represented by the composite

$$\begin{tikzcd}
\Sigma^{t-s+k}X \arrow[rr, dashed] \arrow[d, "\Sigma^{t-s+k}\Delta^r"] & & Y_{s-k,1} \\
\Sigma^{t-s+k}X^r \arrow[r] & \Sigma^k(\bigwedge_j \Sigma^{t_j - s_j}X) \arrow[r, "e\wedge(\bigwedge_j f_j)"] & EG_k/EG_{k-1} \wedge (\bigwedge_j Y_{s_j,1}) \arrow[u, "\xi_{k,s}"']
\end{tikzcd}$$

Remark 5.5: The total power operations can be written down explicitly with the extended powers internalized by $\mathbb{H}_\infty$-structures. Furthermore, we can use them to study differentials of the form $d_r\beta^\varepsilon \mathrm{P}^i x$ and related homotopy operations. We will see its application in the following section.

5.4 Application: on the Hopf invariant one problem

Let $x \in \pi_{2n-1}(\mathrm{S}^n)$ be an element represented by a map $f : \mathrm{S}^{2n-1} \to \mathrm{S}^n$. We attach a $2n$ cell along f to get $X = \mathrm{S}^n \cup_f e^{2n}$. Then we have

$$H^i(X; A) = \begin{cases} A, & i = 0, n, 2n \\ 0, & \text{otherwise} \end{cases}$$

Specifically, we focus on $A = \mathbb{Z}$ or $\mathbb{F}_p$. Let σ be a generator of $H^n(X; A)$ and τ be a generator of $H^{2n}(X; A)$ given by the orientation of $H^*(-; A)$. Let $H(f) \in A$ such that

$$\sigma^2 = H(f) \cdot \tau$$

and $H(f)$ is said to be the *Hopf invariant* of f. Since the homotopy type of X is independent of the choice of representing maps for x, we may let $H(x) = H(f)$ and thus define Hopf invariants for homotopy classes. Moreover, The assignment

$$\begin{array}{rcl} \pi_{2n-1}(\mathrm{S}^n) & \longrightarrow & \mathbb{Z} \\ x & \longmapsto & H(x) \end{array}$$

is a group homomorphism. The question is: *given a positive integer n, does $\pi_{2n-1}(\mathrm{S}^n)$ has an element of Hopf invariant one?* When $n = 2, 4, 8$, the answer is positive, since $\mathbb{R}^n$ can be a division algebra over $\mathbb{R}$. However, the further question is: *are $n = 2, 4, 8$ the only cases for the existence of Hopf invariant one elements*? We will show how to use Adams spectral sequences and power operations to solve the problem in the mod-2 case following[17,24,64].

First, we need some data about the mod-2 Adams spectral sequences for the sphere spectrum.

Theorem 5.4 (Adams[17]): $\mathrm{Ext}^{1,*}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated h_i for $i \geq 0$, where h_i is dual to $\mathrm{Sq}^{2^i} \in \mathcal{A}_2^*$ and is in $\mathrm{Ext}^{1,2^i}$.

These elements are strongly related to elements of Hopf invariant one and that may be why we use the letter "h" to denote them.

Proposition 5.2 (Adams[17]): If there is an element h'' of Hopf invariant one in $\pi_{2n-1}(\mathrm{S}^n)$, then $n = 2^m$. The class h' of h'' in $E_\infty^{1,n}$ is detected by $h_m \in E_2^{1,2^m}$.

Conversely, if h_m survives to the E_∞-page, then there is an element of Hopf invariant one in $\pi_{2n-1}(\mathrm{S}^n)$ for $n = 2^m$.

Therefore, the Hopf invariant one problem is essentially about the computation of the differentials supported by h_m (i.e. the values of differentials at the classes of h_m). We will compute these differentials to solve the Hopf invariant one problem using some multiplicative structure and power operations in the Adams spectral sequences.

Recall that there is a multiplication in the Adams spectral sequence

$$E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'}$$

which coincides with the cup product when ignoring the sign issue and $r = 2$. Besides, the multiplication satisfies the Leibniz rule

$$d_r(uv) = d_r(u)v + (-1)^{t-s} d_r(v).$$

Theorem 5.5 (Adams[17]): The multiplicative relations in $\mathrm{Ext}^{2,*}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)$ are only subject to

$$h_i h_{i+1} = 0$$

and the multiplicative relations in $\mathrm{Ext}^{3,*}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)$ are only subject to

$$h_i h_{i+1} h_j = 0,\ (h_i)^2 h_{i+2} = (h_{i+3})^2,\ h_i (h_{i+2})^2 = 0.$$

Second, we need to compute the action of power operations on the E_2-page of the Adams spectral sequences.

Theorem 5.6 (Milgram[64]): The action of Steenrod square on $\mathrm{Ext}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)$ is totally determined by

$$\mathrm{Sq}^0(h_i) = h_{i+1},$$
$$\mathrm{Sq}^1(h_i) = h_i^2.$$

Third, we need to figure out the relations between differentials and power operations on the E_2-page.

Theorem 5.7 (Milgram[64]): Let $\mathrm{Sq}^i : \mathrm{Ext}^{s,t}_{\mathcal{A}_2^*} \to \mathrm{Ext}^{s+i,2t}_{\mathcal{A}_2^*}$ be operations described in Section 4.3. Then for any $a \in \mathrm{Ext}^{r,s}_{\mathcal{A}_2^*}$, we have

$$d_2(\mathrm{Sq}^i(a)) = \begin{cases} h_0 \mathrm{Sq}^{i+1}(a), & i \equiv s \mod 2 \\ 0, & \text{otherwise.} \end{cases}$$

The proof of this theorem mainly relies on the analysis of the filtration for extended powers in Section 5.3[64-65]. Milgram wrote down the $\mathbb{H}_\infty$-ring structure of the sphere spectrum via some group representations and used the geometric interpretation in Section 5.3 of the power operation in the spectral sequences to deduce the theorem about the differential[64].

Combine these theorems, we can see that

$$d_2(h_i) = d_2(\mathrm{Sq}^0 h_{i-1}) = h_0 \mathrm{Sq}^1(h_{i-1}) = h_0 h_{i-1}^2.$$

According to the multiplicative relations in Theorem 5.5, we can see that $d_2(h_i) \neq 0$ when $i > 3$. Therefore, there is no element of Hopf invariant one in $\pi_{2n-1}(\mathrm{S}^n)$ unless $n = 1, 2, 4, 8$, according to Proposition 5.2.

5.5 The generalized Adams spectral sequences

Theorem 5.8: Suppose E is a commutative ring spectrum such that $(\pi_* E, E_* E)$ is a Hopf algebroid which satisfies Condition 5.6. Let

$$\mathcal{Z} := (Z = Z_0 \xleftarrow{f_0} Z_1 \xleftarrow{f_1} Z_2 \xleftarrow{f_2} \cdots)$$

be an inverse sequence such that $E_* Z_i$ is $\pi_* E$-projective and $E_* f_i$ is a $\pi_* E$-split monomorphism for each i. Then

(1) there exists a spectral sequence $E_*^{*,*}(X, \mathcal{Z})$, natural with respect to maps of such sequences, such that

$$E_2^{s,t}(X, \mathcal{Z}) = \bigoplus_i E_2^{s-i,t-i}(X, Cf_i)$$

where $E_*^{*,*}(X, Cf_i)$ is the Adams spectral sequence converging to $[X, Cf_i]$ (recall Theorem 5.1);

(2) if $E_* Y'$ is $\pi_* E$-projective and we let

$$\mathcal{Z} \wedge Y' := (Z \wedge Y' = Z_0 \wedge Y' \xleftarrow{f_0 \wedge \mathrm{id}} Z_1 \wedge Y' \xleftarrow{f_1 \wedge \mathrm{id}} Z_2 \wedge Y' \xleftarrow{f_2 \wedge \mathrm{id}} \cdots)$$

there is a pairing

$$\begin{tikzcd} E_r^{*,*}(X,\mathcal{Z}) \otimes E_r^{*,*}(X',Y') \arrow[r] \arrow[d, Rightarrow] & E_r^{*,*}(X \wedge X', \mathcal{Z} \wedge Y') \arrow[d, Rightarrow] \\ {[X,Z]_*^E} \otimes {[X',Y']_*^E} \arrow[r] & {[X \wedge X', Z \wedge Y']_*^E} \end{tikzcd}$$

(3) if

$$\begin{tikzcd} & Z_0 \arrow[d, "c_0"'] & Z_1 \arrow[l, "f_0"'] \arrow[d, "c_1"'] & \cdots \arrow[l, "f_1"'] \\ Y \arrow[r, equal] & Y_0 & Y_1 \arrow[l, "i_0"'] & \cdots \arrow[l, "i_1"'] \end{tikzcd}$$

is an inverse-sequence morphism from $\mathcal{Z}$ to an Adams resolution of Y, then there is a homomorphism c of spectral sequences

$$\begin{tikzcd} E_r^{*,*}(X,\mathcal{Z}) \arrow[r, Rightarrow] \arrow[d, "c"'] & {[X,Z]_*^E} \arrow[d, "c_{0*}"] \\ E_r^{*,*}(X,Y) \arrow[r, Rightarrow] & {[X,Y]_*^E} \end{tikzcd}$$

which maps the pairing in (2) to the smash product pairing

$$\begin{tikzcd} E_r^{*,*}(X,\mathcal{Z}) \otimes E_r^{*,*}(X',Y') \arrow[r] \arrow[d, "c\otimes \mathrm{id}"'] & E_r^{*,*}(X \wedge X', \mathcal{Z} \wedge Y') \arrow[d, "c"] \\ E_r^{*,*}(X,Y) \otimes E_r^{*,*}(X',Y') \arrow[r] & E_r^{*,*}(X \wedge X', Y \wedge Y') \end{tikzcd}$$

(4) the spectral sequence $E_r^{*,*}(X,\mathcal{Z})$ converges to $[X,Z]_*^E$ if E and Z satisfies the Adams condition in Remark 5.3 and $E_*(\mathrm{Mic}\,\mathcal{Z}) = 0$, where $\mathrm{Mic}\,\mathcal{Z}$ is the microscope or homotopy limit of the inverse sequence $\mathcal{Z}$.

Proof: See[1]Chapter IV, Section 6. ■

5.6 How power operations detect homotopy operations

Let $x \in \pi_n(Y)$ be detected by $\bar{x} \in E_2^{s,n+s}(\mathbb{S},Y)$ (this means that the element $x : \mathrm{S}^n \to Y$ is detected by $\bar{x} : \mathrm{S}^n \to Y_s$ and the later one is an element in some E_*-homology group), the Adams spectral sequence with respect to a commutative ring spectrum E satisfying the condition in Theorem 5.8. Let Ξ be the sequence

$$\Xi = (D_G^{ps}\mathrm{S}^n \longleftarrow D_G^{ps-1}\mathrm{S}^n \longleftarrow D_G^{ps-2}\mathrm{S}^n \longleftarrow \cdots \longleftarrow D_G^{1}\mathrm{S}^n \longleftarrow \mathrm{S}^{np})$$

where $G = C_p$ and $D_G^i \mathrm{S}^n = ((W_i)_+ \wedge \mathrm{S}^{np})/\pi$ is the extended power of S^n, where W_i is the i-skeleton of the standard free G-CW-complex, i.e. the universal cover of the mod-p lens space where $W_{2i-1} = \mathrm{S}^{2i-1}$. By Theorem 5.3, if $E_* Y_j$ is $\pi_* E$-projective, then we have a morphism from Ξ to the canonical Adams resolution of Y.

$$\begin{tikzcd}
D_G^{ps}\mathrm{S}^n \arrow[d, "D_G\bar{x}"] & D_G^{ps-1}\mathrm{S}^n \arrow[l] \arrow[d, "D_G\bar{x}"] & \cdots \arrow[l] & D_G^1\mathrm{S}^n \arrow[l] \arrow[d, "D_G\bar{x}"] & \mathrm{S}^{np} \arrow[l] \arrow[d, "D_G\bar{x}"] \\
D_G^{ps}Y_s \arrow[d, "\xi_{ps,ps}"] & D_G^{ps-1}Y_s \arrow[l] \arrow[d, "\xi_{ps-1,ps}"] & \cdots \arrow[l] & D_G^1 Y_s \arrow[l] \arrow[d, "\xi_{1,\xi}"] & Y_s^p \arrow[l] \arrow[d, "\xi_{0,ps}"] \\
Y_0 & Y_1 \arrow[l] & \cdots \arrow[l] & Y_{ps-1} \arrow[l] & Y_{ps} \arrow[l]
\end{tikzcd}$$

By Theorem 5.8 3, we have a homomorphism

$$\mathcal{P}(x) \colon E_r^{*,*}(\mathbb{S}, \Xi) \to E_r^{*,*}(\mathbb{S}, Y)$$

of spectral sequences (here we assume the domain spectral sequences exist). Similarly, we have compatible maps

$$D_G^1 \mathrm{S}^n \wedge Y \to Y_{ps-i}$$

and a homomorphism

$$\mathcal{P}(x) \colon E_r^{*,*}(\mathbb{S}, \Xi \wedge Y) \to E_r^{*,*}(\mathbb{S}, Y)$$

Proposition 5.3: If $E_* D_G^{i-1} \mathrm{S}^n \to E_* D_G^i \mathrm{S}^n$ is a $\pi_* E$-split monomorphism for each $i \geq ps$, then the spectral sequence $E_r^{*,*}(\mathbb{S}, \Xi)$ exists and $E_2(\mathbb{S}, \Xi)$ is free over $E_2(\mathbb{S}, \mathbb{S})$ on generators $e_i \in E_2^{ps-i, ps+pn}(D_\pi^{ps} \mathrm{S}^n, \Xi)$. Similarly, $E_2(\mathrm{S}^n, \Xi \wedge Y)$ is free over $E_2(\mathbb{S}, Y)$ on the image of the e_i under the map induced by the unit $\mathbb{S} \to Y$.

Proof: See[1]Chapter IV, Proposition 7.5. ■

Remark 5.6: We may take e_i as the $np + i$-cell of $D_G \mathrm{S}^n$ (the smash product among p copies of S^n and the unique i-cell of W).

Theorem 5.9: Suppose the hypothesis of Proposition 5.3 holds and $E_* Y$ is $\pi_* E$-projective. Then $\mathcal{P}(x)$ sends e_i to $\Phi_*(e_i \otimes \bar{x}^p)$.

Remark 5.7: If $p = 2$, then $\mathcal{P}(x)$ sends e_i to $\mathrm{Sq}^{i+n}(\bar{x})$. If p is an odd prime, then $\mathcal{P}(x)$ sends $(-1)\nu(n) e_i$ to $\beta^\varepsilon \mathrm{P}^j \bar{x}$ if $i = (2j - n)(p - 1) - \varepsilon$. $\mathcal{P}(x)$ sends elements to 0 if i is not of this form.

Definition 5.7 (Homotopy operations): Suppose Y is an $\mathbb{H}_\infty$-ring spectrum. Given $\alpha \in Y_m(D_{j_1} \mathrm{S}^{n_1} \wedge \cdots \wedge D_{j_k} \mathrm{S}^{n_k})$, the associated *homotopy operation*

$$\alpha^* : \pi_{n_1} Y \times \cdots \times \pi_{n_k} Y \to \pi_m(Y)$$

is defined by sending $f_1 \times \cdots \times f_k \in \pi_{n_1} Y \times \cdots \times \pi_{n_k} Y$ to the composite

$$\mathrm{S}^m \xrightarrow{\alpha} D_{j_1}\mathrm{S}^{n_1} \wedge \cdots \wedge D_{j_k}\mathrm{S}^{n_k} \wedge Y \xrightarrow{D_{j_1}f_1 \wedge \cdots \wedge D_{j_k}f_k \wedge \mathrm{id}} D_{j_1}Y \wedge \cdots \wedge D_{j_k}Y \wedge Y \xrightarrow{\xi} Y$$

Now we show how Steenrod operations on the E_2-page detect homotopy operations. If we assume

$$E_r^{*,*}(\mathrm{S}, \Xi) \Rightarrow \pi_* D^{ps} G \mathrm{S}^n$$

then any $\alpha \in \pi_* D_G^{ps}\mathrm{S}^n$ can be detected by an element $\sum a_k e_k \in E_2(\mathbb{S}, \Xi)$, where $a_k \in E_2(\mathbb{S}, \mathbb{S})$. Applying $\mathcal{P}(x)$, we see that $\alpha^*(x)$ is detected by $\sum a_k \Phi_*(e_k \otimes \bar{x}^p) \in E_2(\mathbb{S}, \mathbb{S})$. Similarly, if $E_r^{*,*}(\mathbb{S}, \Xi \wedge Y)$ converges to $Y_* D_G^{ps}\mathrm{S}^n$, any $\alpha \in Y_* D_G^{ps}\mathrm{S}^n$ is detected by $\sum a_k e_k \in E_2(\mathbb{S}, \Xi \wedge Y)$, where $a_k \in E_2(\mathbb{S}, Y)$, then $\alpha^*(x)$ is detected by $\mathcal{P}(x)$.

CHAPTER 6 MOTIVIC HOMOTOPY THEORY

In this chapter, we use the framework in Chapter 2 and Chapter 3 to construct unstable motivic homotopy theory and stable homotopy theory respectively.

6.1 Unstable motivic homotopy theory

Let S be a quasi-compact and quasi-separated scheme. Let Sm_S be the category of finitely presented smooth S-schemes. Note that this category is essentially small, so we may regard Sm_S as a small ∞-category category.

Construction 6.1: Given a small ∞-category $\mathcal{C}$ with finite coproducts. Let $\mathcal{P}_\Sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ be the full subcategory of presheaves that transform finite coproducts into finite products. By the Yoneda embedding, $\mathcal{P}_\Sigma(\mathcal{C})$ is generated by $\mathcal{C}$ under sifted colimits (roughly speaking, sifted colimits are the combination of filtered colimits and reflexive coequalizers, see[66]).

Lemma 6.1: Let $\mathcal{C}$ be a small ∞-category that admits a final object and finite coproducts. Then the Yoneda embedding $\mathcal{C}_+ \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C})_\bullet$ can be extended to an equivalence $\mathcal{P}_\Sigma(\mathcal{C}_+) \simeq \mathcal{P}_\Sigma(\mathcal{C})_\bullet$.

Definition 6.2 (Nisnevich topology): Given a quasi-compact and quasi-separated scheme X, *Nisnevich covering* of X is an étale covering $\{U_i \to X\}$ such that the étale maps are jointly surjective on k-points for every field k. The Grothendieck topology generated by Nisnevich coverings is said to be *Nisnevich topology*.

We let $\mathcal{S}\mathrm{hv}_{nis}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$ be the subcategory of Nisnevich sheaves i.e. the Nisnevich-local objects. Note that a presheaf $\mathcal{F}$ is Nisnevich local if for any Nisnevich covering $U_\bullet \to X$ in Sm_S, we have the following equivalence

$$\lim_{\check{C}_\bullet(U_\bullet)} \mathcal{F} \simeq \mathcal{F}(X)$$

where $\check{C}_\bullet(U_\bullet)$ is the $\check{C}$ech nerve associated to the Nisnevich covering $U_\bullet$ given by

$$\check{C}_n(U_\bullet) = U_\bullet \times_X \cdots \times_X U_\bullet$$

the iterated product of $n+1$ copies of $U_\bullet$.

Proposition 6.1: Nisnevich sieves on X are generated by the following cartesian square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where p is an étale morphism and j is an open immersion such that $(Y\backslash V)_{\mathrm{red}} \to (X\backslash U)_{\mathrm{red}}$ is an isomorphism. Such a square is so-called a *elementary distinguished square*. Therefore, a presheaf is a Nisnevich sheaf if and only if it send each elementary distinguished square to cartesian square.

Proof: See[37]Section 3 Proposition 1.4. ∎

Definition 6.3: A presheaf $F \in \mathcal{P}(\mathrm{Sm}_S)$ is $\mathbb{A}$-*invariant* if for any $X \in \mathrm{Sm}_S$, the natural map $F(X) \to F(X\times\mathbb{A}^1)$ is an equivalences. Let the subcategory of $\mathbb{A}$-invariant presheaves denoted by $\mathcal{P}_{\mathbb{A}^1}(\mathrm{Sm}_S)$.

Construction 6.4: The ∞-category $\mathcal{H}(S)$ of motivic spaces over S is defined by

$$\mathcal{H}(S) = \mathcal{S}\mathrm{hv}_{nis}(\mathrm{Sm}_S) \cap \mathcal{P}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$$

Note that $\mathcal{S}\mathrm{hv}_{nis}(\mathrm{Sm}_S), \mathcal{P}_{\mathbb{A}^1}(\mathrm{Sm}_S)$ and $\mathcal{H}(S)$ are reflective subcategories of $\mathcal{P}(\mathrm{Sm}_S)$, since they are closed under colimits (here we use Theorem 2.5 implicitly). Therefore we have localizations $L_{nis}, L_{\mathbb{A}^1}$ and L_{mot} respectively. A morphism in $\mathcal{P}(\mathrm{Sm}_S)$ is said to be a *Nisnevich equivalence* (resp. $\mathbb{A}^1$-*equivalence* or *motivic equivalence*) if $L_{nis}(f)$ (resp. $L_{\mathbb{A}^1}(f)$ or L_{mot}) is an equivalence.

Similarly, we can construct ∞-category of $\mathcal{H}_\bullet(S)$ pointed motivic space by working on $\mathcal{P}(\mathrm{Sm}_{S+})$.

6.2 Stable motivic homotopy theory

Construction 6.5: Note that $\mathcal{H}_\bullet(S)$ is a presentable symmetric monoidal ∞-category. Let $\mathcal{T}$ be the set of Thom spaces over S. The *motivic stable homotopy category* $\mathcal{SH}(S)$ is a presentable symmetric monoidal ∞-category obtained by inverting $\mathbb{P}^1_S$.

Recall that if V is a vector bundle over a scheme S, the Thom space $\mathrm{Th}(V) \in \mathcal{H}_\bullet(S)$ is the motivic space defined by

$$\mathrm{Th}(V) = V/(V \setminus 0) \simeq \mathbb{P}(V \oplus \mathbb{A}^1_S)/\mathbb{P}(V),$$

and we may also write it as S^V. The functors Σ^V and Ω^V are defined by

$$\Sigma^V(-) := \mathrm{S}^V \wedge (-),\ \Omega^V(-) = \mathrm{Hom}(\mathrm{S}^V, -)$$

Proposition 6.2: [67]2.4.9 If $\mathbb{P}^1_S$ is invertible, then any Thom space over S is invertible.

Let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and $\mathcal{J}$ be a set of objects in $\mathcal{C}$. For any finite subset $I = \{X_1, \cdots, X_n\} \subset \mathcal{J}$, we denote $\otimes I = X_1 \otimes \cdot X_n$. The *formal inversion with respect to* $\mathcal{J}$ is given by

$$\mathcal{C}[\mathcal{J}^{-1}] = \operatorname*{colim}_{\substack{I \subset \mathcal{J} \\ I \text{ finite}}} \mathcal{C}[(\bigotimes I)^{-1}]$$

Corollary 6.1: Let $\mathcal{T}$ be the set of Thom spaces over S, then

$$\mathcal{SH}(S) \simeq \mathcal{H}_{\bullet}(S)[\mathcal{T}^{-1}].$$

The unit object in $\mathcal{SH}(S)$ is denoted by $\mathbf{1}_S$ and we may also call it the motivic sphere spectrum over S.

CHAPTER 7 MULTIPLICATIVE COHERENCE IN MOTIVIC HOMOTOPY THEORY

In the study of field extensions (especially separable field extensions), norm maps usually encode massive arithmetic information. The most significant results are about the class field theory, which indicates how norm maps characterize the Galois groups. Actually, norm maps are kinds of "multiplicative averaging" with respect to the Galois group action. Recall that given a finite Galois extension L/K with Galois group G, the norm map is given by

$$N_{L/K} : L \to K, \quad x \mapsto \prod_{\sigma \in G} \sigma(x)$$

This is a multiplicative map i.e. $N_{L/K}(xy) = N_{L/K}(x)N_{L/K}(y)$ clearly and this is well-defined according to the fundamental theorem of Galois theory since $\prod_{\sigma \in G} \sigma(x)$ is invariant under the action of G.

This section intends to study this idea in the context of motivic homotopy theory, based on the joint work of Bachmann and Hoyois[2]. Specifically, the corresponding is given in the following table. By taking the residue field for each stalk of the right column

field theory	scheme theory
fields	quasi-projective smooth schemes
finite separable extensions	finite étale morphisms

of this table, we exactly have the left column. Moreover, every motivic space can be written as a sifted colimit of quasi-projective smooth schemes, so we believe the notion of norms in motivic homotopy should be a good generalization of the classical norms in field theory.

In Section 7.1, we study the notion of Weil restrictions when dealing with push-forwards. If we consider the Weil restriction for a finite separable extension, we will see how multiplicative averaging performs on schemes explictly (see Remark 7.1), which is very similar to the classical notion.

In Section 7.2, we introduce how Bachmann and Hoyois construct norm functors for motivic spaces and motivic spectra. Norms in motivic homotopy theory are symmetric monoidal functors that generalize the notion of norms for finite Galois extensions of fields.

They are constructed for any finite locally free morphism of schemes, and they relate the pointed unstable motivic homotopy category over the source scheme to that over the target scheme. When the morphism is finite étale , they also stabilize to functors between the stable motivic homotopy categories over the source and target schemes.

In Section 7.3, we briefly show how the norm functors are compatible with other operations. After that, we introduce the notion of normed motivic spectra in Section 7.4 and roughly speaking it turns out to be a motivic spectrum that can be parametrized by finite étale morphisms coherently. If we only take the trivial projections in these finite étale morphisms, then the notion of motivic normed spectra (resp. motivic incoherent normed spectra) is very similar to the notion of $\mathbb{E}_\infty$-ring spectra (resp. $\mathbb{H}_\infty$-ring spectra). From this viewpoint, we can see why a normed spectrum is an enhanced motivic $\mathbb{E}_\infty$-ring spectrum. Recall that May's $\mathbb{E}_\infty$-ring spectra have a monadic interpretation derived from $\mathbb{E}_\infty$-operads. We also have a monadic interpretation for normed spectra.

7.1 The Weil restrictions

Definition 7.1: Given a morphism $\varphi: S' \to S$ of schemes and an S'-scheme X', we have a presheaf on Sch_S given by

$$\mathrm{R}_\varphi X' : T \to X'(T \times_S S')$$

If the presheaf $\mathrm{R}_\varphi X'$ is representable by an S-scheme X, then X is defined to be the Weil restriction of X' along φ.

Example 7.1: Let L/K be a finite Galois extension with Galois group G, let V be a L-scheme and W be a K-scheme. Note that W can be viewed as a L-scheme naturally. Let $p: W \to V$ be a morphism between L-schemes. Then we have

$$\prod_{\sigma\in G} \sigma p : W \to \prod_{\sigma\in G} V_\Sigma$$

given by $w \mapsto (\sigma^* p(w))_{\sigma^*\in G}$, where V_Σ is the image of V under the functor $\sigma^* : \mathrm{Sch}_L \to \mathrm{Sch}_L$. If the morphism $\prod_{\sigma\in G} \sigma p$ is an isomorphism, then W is the Weil restriction of V along the field extension $\mathrm{Spec}L \to \mathrm{Spec}K$, see[68]Section 1.3.

Remark 7.1: These examples essentially come from the theory of Galois descent. Let K'/K be a finite separable extension, let X' be a quasi-projective K'-scheme, and let L/K

be a finite Galois extension that contains K'. Let

$$\overline{X} := \prod_{j:K'\hookrightarrow L} X' \times_{K',j} \mathrm{Spec}(L)$$

where the product runs over all embeddings $K' \to L$ over K. Then, for $\sigma \in \mathrm{Gal}(L/K)$, there exists an isomorphism $\varphi_\sigma : \overline{X} \simeq \overline{X}$ over $\mathrm{Spec}(L)$ given by $\mathrm{id} \times \sigma^*$, such that $(\overline{X}, \{\varphi_\sigma\}_{\sigma\in\mathrm{Gal}(L/K)})$ is an effective descent data, giving the k-scheme $\mathrm{R}_{K'/K}(X)$. Compared to norm maps in field and Galois theory, it is reasonable to regard Weil restrictions as a kind of norm map for schemes.

Example 7.2: Given a finite field extension L/K of order d. If we specify a K-basis $\{e_1, \cdots, e_d\}$ of L, then for an affine L-space $\mathbb{A}_L^n = \mathrm{Spec}L[x_1, \cdots, x_n]$, the Weil restriction $\mathrm{R}_{L/K}(\mathbb{A}_L^n)$ is given by

$$\mathrm{Spec}K[y_{ij}] = \mathbb{A}_K^{nd}$$

Similarly, we have an analogous result for projective spaces.

Proposition 7.1: If V admits has the Weil restriction $(W, \prod_{\sigma\in G} \sigma p)$ as above, then either a Zariski-open L-subscheme of V or a closed L-subscheme has its Weil restriction.

Proof: See[68]Section 1.3. ■

Theorem 7.1: Let $p: T \to S$ be a finite locally free morphism between schemes, and X a quasi-projective T-scheme. Then the Weil restriction $\mathrm{R}_p X$ exists and is quasi-projective over S.

Proof: See[69]Section 7.6, Theorem 4. ■

Remark 7.2: We need to require that the morphism should be finite locally free, because we need to basis affine-locally, as we do in Example 7.2.

Proposition 7.2: Let E be an arbitrary scheme and $X \in \mathrm{Sm}_B$. Suppose there is a finite locally free morphism $p : E \to B$. $\mathrm{R}_p X$ is smooth over B whenever the Weil restriction exists.

7.2 Norms for motivic spaces and spectra

We intend to construct the multiplicative norm functor

$$p_\otimes : \mathcal{H}_\bullet(T) \to \mathcal{H}_\bullet(S)$$

and its stable version for an integral and universally open morphism of schemes $p: T \to S$. and we expect it to satisfy the following properties:

（1）The smash product should be preserved by $p_\otimes$.

（2）Sifted colimits should be preserved by $p_\otimes$.

（3）p_* should be extended by $p_\otimes$ i.e. $p_\otimes(Y_+) \simeq (p_*Y)_+$ for $Y \in \mathrm{Sm}_S$.

（4）$p_\otimes$ should be the n-fold smash product if $p : S \times \{\underline{n}\} \to S$ is the trivial projection (we may also write it into a fold map $S^{\sqcup n} \to S$).

Since we have already had a good description of p_*, at least for quasi-projective smooth schemes. We just need to deal with two issues

（1）Extend p_* from non-pointed case to pointed case at the level of presheaves. More specifically, we need to extend it to a functor $\mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \xrightarrow{p_\otimes} \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$ such that the requirements are satisfied;

（2）Verify that the extension preserves motivic equivalence.

The second issue is relatively easier, because we have the following proposition.

Proposition 7.3: [2] Given an integral morphism $p : E \to B$ of schemes, Nisnevich and motivic equivalences are preserved by the functor $\mathcal{P}_\Sigma(\mathrm{Sm}_E) \xrightarrow{p_*} \mathcal{P}_\Sigma(\mathrm{Sm}_B)$.

Here we need to require the morphism to be integral because integral morphisms are direct limits of finitely presented morphisms. In this way, we can reduce the case to finite morphisms and further to finite field extension stalkwisely.

Now we just need to deal with the first issue. Note that Property 2 and Property 3 should determine $p_\otimes$, since $\mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet$ is generated under sifted colimits by objects of the form X_+. However, there is an obstruction on the way to the pointed cases: some maps in $\mathcal{P}_\Sigma(\mathrm{Sm})_T$ may not come from the functor $X \mapsto X_+$. For example, $f : (X \sqcup Y)_+ \to X_+$ that collapse Y to the base point cannot come from any map $X \sqcup Y \to X$. Therefore, the key point is how we define such $p_*(X \sqcup Y)_+ \to p_*(X)_+$.

Here we specialize the case to the case where $X, Y \in \mathrm{Sm}_T$. Then for any $U \in \mathrm{Sm}_S$, we decode the items:

（1）$p_*(X \sqcup Y)_+(U) = \mathrm{Sm}_T(U \times_S T, X \sqcup Y)_+$;

（2）$p_*(X)_+ = \mathrm{Sm}_T(U \times_S T, X)_+$;

For any $s : U \times_S T \to X \sqcup Y$, how should we define $p_\otimes(f)(s) \in p_*(X)_+$? Notice that we should collapse the part $s|_{s^{-1}(Y)} : s^{-1}(Y) \to Y$ according to the definition of f.

$$\begin{tikzcd} U \times_S T \arrow[rr, "s"] & & X \sqcup Y \arrow[d, "\text{collapse the ``cross terms''}"] \\ U \times_S T - s^{-1}(Y) \arrow[rr] & & X \end{tikzcd}$$

However, the bottom arrow is not an element in $p_*(X)(U)$ evidently, which is regarded as a "cross term" in $p_*(X)(U)$. To make it more clear, we need to separate $s|_{s^{-1}(Y)} : s^{-1}(Y) \to$

Y from $s\colon U\times_S T\to X\sqcup Y$ in $p_*(X\sqcup Y)_+(U)$ by decomposing the presheaf $p_*(X\sqcup Y)_+$.

Definition 7.2: A *relatively representable* morphism is a morphism $Y\to X$ in $\mathcal{P}(\mathrm{Sm}_T)$ is such that the presheaf $V\times_X Y$ is representable whenever $V\to X$ for some $V\in\mathrm{Sm}_T$.

Lemma 7.1: For any coproduct decomposition $X=X_1\sqcup X_2$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, the natural inclusion $X_1\hookrightarrow X$ is relative representable.

Proof: Let $j_i: X_i\to X$ be the natural inclusion for each $i=1,2$. For any $Y\in\mathrm{Sm}_S$, by the universality of colimits, we have $Y=Y_1\sqcup Y_2$, where

$$Y_i=Y\times_X X_i \text{ for } i=1,2$$

Recall that $\mathcal{O}(Y)=\mathrm{Hom}_{\mathrm{Sm}_S}(Y,\mathbb{A}^1)$. Then we can decompose the ring of functions of Y into

$$\mathrm{Hom}_{\mathrm{Sm}_S}(Y,\mathbb{A}^1)=\mathrm{Hom}_{\mathrm{Sm}_S}(Y_1\sqcup Y_2,\mathbb{A}^1)=\mathrm{Hom}_{\mathrm{Sm}_S}(Y_1,\mathbb{A}^1)\times\mathrm{Hom}_{\mathrm{Sm}_S}(Y_2,\mathbb{A}^1)$$

By reducing the case to affine cases, we can decompose Y into two clopen subsets that represents Y_1 and Y_2 respectively. ■

Construction 7.3: Let $Y_1,\dots,Y_k\to X$ be relatively representable morphisms. For $U\in\mathrm{Sm}_S$, let

$$p_*(X|Y_1,\dots,Y_k)(U):=\{s\colon U\times_S T\to X\mid s^{-1}(Y_i)\to U \text{ is surjective for all } i\}$$

where $s^{-1}(Y_i)\to U$ is given by the middle vertical composition of arrows in the following diagram.

$$\begin{tikzcd}
 & s^{-1}(Y_i) \arrow[r] \arrow[d] & Y_i \arrow[d] \\
T \arrow[d, "p"] & U\times_S T \arrow[l] \arrow[r, "s"] \arrow[d, "p_U"] & X \\
S & U \arrow[l] &
\end{tikzcd}$$

Note that $p_*(X|Y_1,\dots,Y_k)$ is a subpresheaf of $p^*(X)$. If $X\in\mathcal{P}_\Sigma(\mathrm{Sm}_T)$,represents $p_*(X|Y_1,\dots,Y_k)$.

Lemma 7.2: Given a universally clopen morphism $p: T\to S$, $Y\in\mathcal{P}_\Sigma(\mathrm{Sm}_T)$, with relatively representable morphisms $Z_1,\dots,Z_k\to Y$, for every coproduct decomposition $Y\simeq Y'\sqcup Y''$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_T)$, there is a decomposition

$$p_*(Y|Z_1,\dots,Z_k)\simeq p_*(Y'|Z_1',\dots,Z_k')\sqcup p_*(Y|Y'',Z_1,\dots,Z_k)$$

in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, where $Z_i'=Z_i\times_Y Y'$.

Proof: First, we reduce the case to $k=0$:

(1) Let $\phi : p_*(Y') \sqcup p_*(Y|Y'') \to p_*(Y)$ be the morphism induced by the inclusions.

(2) Note that $p_*(Y'|Z'_1, \dots, Z'_k) = p_*(X') \cap p_*(Y|Z_1, \dots, Z_k)$.

(3) Note that $p_*(Y|Y'', Z_1, \dots, Z_k) = p_*(Y|Y'') \cap p_*(Y|Z_1, \dots, Z_k)$.

(4) Consider the following cartesian square

$$\begin{tikzcd}
p_*(Y'|Z'_1, \dots, Z'_k) \sqcup p_*(Y|Y'', Z_1, \dots, Z_k) \arrow[r] \arrow[d, hook] & p_*(Y|Z_1, \dots, Z_k) \arrow[d, hook] \\
p_*(Y') \sqcup p_*(Y|Y'') \arrow[r, "\phi"] & p_*(Y)
\end{tikzcd}$$

(5) We just need to show ϕ is an equivalence.

Then we specialize to the case $k = 0$, and show ϕ is a monomorphism:

(1) $p_*(Y') \times_{p_*(Y)} p_*(Y|Y'')$ has no sections over nonempty schemes, because $Y' \cap Y'' = \emptyset$ in Y.

(2) Hence $p_*(Y') \times_{p_*(Y)} p_*(Y|Y'')$ is an initial object of $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, which means that ϕ is an equivalence by the universality of colimits. In particular, ϕ is a monomorphism.

It remains to show that ϕ is objectwisely an effective epimorphism:

(1) Let $p_U : U \times_S T \to U$ be the morphism parallel to $p : T \to S$ in the evident cartesian square.

(2) Given $U \in \mathrm{Sm}_S$ and $s \in p_*(Y)(U)$, we will decompose U according to these data.

(3) Let $U' = \{y \in U \mid p_U^{-1}(x) \subset s^{-1}(Y')\}$.

(4) Let U'' be the complement of U' in U, and $U'' = p_U(s^{-1}(Y''))$, which is a clopen subset of U.

(5) The image of the restriction $s|_{U'} : U' \to Y$ is in Y', according to the construction. Hence $s|_{U'} \in p_*(Y')(U')$ and we have $U' \to p_*(Y')$.

(6) $s^{-1}(Y'') \to U''$ is surjective, according to the construction. Hence $s|_{U'} \in p_*(Y')(U')$ and we have $U'' \to p_*(Y|Y'')$.

(7) Combine these coproducts together to define a section

$$U = U' \sqcup U'' \to p_*(Y') \sqcup p_*(Y|Y'')$$

which is a preimage of s by ϕ_U.

■

Remark 7.3: The proof of the surjectivity is the essential part, where we notice that the decomposition

$$p_*(Y') \sqcup p_*(Y|Y'') \simeq p_*(Y)$$

essentially encodes the decomposition of each section $s: U \to p_*(Y)$

$$\begin{tikzcd}
 & U \arrow[d, equal] & \\
U' \arrow[r] \arrow[d] & U' \sqcup U'' \arrow[d] & U'' \arrow[l] \arrow[d] \\
p_*(Y') \arrow[r] & p_*(Y') \sqcup p_*(Y|Y'') \arrow[d, "\phi"] & p_*(Y|Y'') \arrow[l] \\
 & p_*(Y) &
\end{tikzcd}$$

We may conclude that $s: U \to p_*(Y)$ is in $p_*(Y|Y'')$ if and only if the corresponding map $U \times_S T \to X$ can be lift to $U \times_S T \to Y \to Y''$ along the inclusion $Y'' \hookrightarrow Y$. If we let $f: Y_+ \to Y'_+$ collapse Y'' to the base point, the right vertical arrow $U'' \to p_*(Y|Y'')$ can be interpreted as the "cross terms" that should collapse. Therefore, we can see how $p(Y|Y'')$ packs the "cross terms".

Example 7.3: Given a universally clopen morphism $p: E \to B$ and let $Y, Z \in \mathcal{P}_\Sigma(\mathrm{Sm}_E)$, we have the decomposition

$$p_*(Y \sqcup Z) \simeq p_*(\emptyset) \sqcup p_*(Y|Y) \sqcup p_*(Z|Z) \sqcup p_*(Y \sqcup Z|Y, Z)$$

Theorem 7.2: Given a universally clopen morphism $p: T \to S$, there is a unique symmetric monoidal functor

$$p_\otimes : \mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$$

such that

(1) sifted colimits are preserved by $p_\otimes$;

(2) there is a natural equivalence $p_\otimes(X_+) \simeq p_*(X)_+$ between symmetric monoidal functors;

(3) for every $g: Z_+ \to Y_+$ with $Y, Z \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, the map $p_\otimes(g)$ is the composite

$$p_*(Z)_+ \to p_*(g^{-1}(Y))_+ \xrightarrow{f} p_*(Y)_+$$

by collapsing the part $p_*(Z|Z \setminus f^{-1}(Y))$ to the base point.

(4) Nisnevich and motivic equivalences are preserved by $p_\otimes$ whenever p is integral.

Proof: [2]Theorem 3.3. ■

Proposition 7.4: If $p\colon S^{\sqcup n} \to S$ is the fold map, then

$$p_{\otimes}\colon \mathcal{P}_{\Sigma}(\mathrm{Sm}_{S^{\sqcup n}})_{\bullet} \simeq (\mathcal{P}_{\Sigma}(\mathrm{Sm}_S)_{\bullet})^n \to \mathcal{P}_{\Sigma}(\mathrm{Sm}_S)_{\bullet}$$

Proof: Note that the n-fold smash product preserves sifted colimits, it remains to check that the smash product has the functoriality described Theorem 7.2 (3).

We may reduce the case to $n = 2$. Let $i_1, i_2\colon S \hookrightarrow S \sqcup S$ be the two natural inclusions. Then the equivalence $\mathcal{P}_{\Sigma}(\mathrm{Sm}_{S\sqcup S})_{\bullet} \to \mathcal{P}_{\Sigma}(\mathrm{Sm}_S)_{\bullet} \times \mathcal{P}_{\Sigma}(\mathrm{Sm}_S)_{\bullet}$ is given by

$$F \mapsto (i_1^*F, i_2^*F)$$

Our goal is check that $F \mapsto i_1^*F \wedge i_2^*F$ satisfies Theorem 7.2 (3). Let $f : Y_+ \to X_+$ be a morphism in $\mathcal{P}_{\Sigma}(\mathrm{Sm}_S)_{\bullet}$ with $X, Y \in \mathcal{P}_{\Sigma}(\mathrm{Sm}_S)$. Let $U \in \mathrm{Sm}_S$ and $U \times_S S^{\sqcup 2} = U \sqcup U$. We consider the diagram

$$\begin{tikzcd}
 & i_1^*Y(U)_+ \wedge i_2^*Y(U)_+ \arrow[d, equal] & \\
Y(U \sqcup U)_+ \arrow[r, "\sim"] \arrow[d, "p_*f_U"] & (i_1^*Y(U) \times i_2 1^*Y(U))_+ \arrow[d] & (\alpha_1, \alpha_2) \arrow[d, mapsto] \\
X(U \sqcup U)_+ \arrow[r, "\sim"] & (i_1^*X(U) \times i_2^*X(U))_+ \arrow[d, equal] & f(\alpha_1, \alpha_2) \\
 & i_1^*X(U)_+ \wedge i_2^*X(U)_+ &
\end{tikzcd}$$

We just need to check that $f(\alpha_1, \alpha_2) = *$ if and only if $(\alpha_1, \alpha_2) \in p_*(Y \mid f^{-1}(X))$. By taking the right adjoint images of $\alpha_1 : U \to i_1^*Y$ and $\alpha_2 : U \to i_2^*Y$, we have a map $i_{1*}U \sqcup i_{2*}U \cong U \sqcup U \to Y$ that represents (α_1, α_2). Then $f(\alpha_1, \alpha_2) = *$ if and only if there is a lifting

$$\begin{tikzcd}
 & Y \setminus f^{-1}(X) \arrow[d] \\
U \sqcup U \arrow[r] \arrow[ur, dashed] & Y_+
\end{tikzcd}$$

if and only if $(\alpha_1, \alpha_2) \in p_*(X|Y \setminus f^{-1}(X))$ by Remark 7.3. ■

Definition 7.4: Let $p : T \to S$ be a morphism of schemes, let $X \in \mathcal{P}(\mathrm{Sm}_T)$, and let $Y \subset X$ be a subsheaf. For $U \in \mathrm{Sm}_S$, let

$$p_*(X||Y)(U) = \{s\colon U \times_S T \to X \mid s \text{ sends a clopen subset covering } U \text{ to } Y\}.$$

Note that $p_*(X||Y) \subset p_*(X)$, and it is in $\mathcal{P}_{\Sigma}$ whenever X and Y are.

Proposition 7.5: Given a universally clopen morphism $p\colon T \to S$, $X \in \mathcal{P}_{\Sigma}(\mathrm{Sm}_T)$, and a subpresheaf $Y \subset X$ in $\mathcal{P}_\Sigma$, there is a natural equivalence

$$p_\otimes(X/Y) \simeq p_*(X)/p_*(X||Y)$$

in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$.

Proof: See[2]Proposition 3.7. ■

Proposition 7.6: Let $p : T \to S$ be an integral universally open morphism, let $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, and let $Y \subset$ be an open subsheaf. Then there is a natural equivalence

$$p_\otimes(X/Y) \simeq p_*(X)/p_*(X|Y)$$

in $\mathcal{S}\mathrm{hv}_{nis}(\mathrm{Sm}_S)_\bullet$.

Proof: See[2]Corollary 3.11. ■

Proposition 7.7: Given a finite étale morphism, $p : T \to S$, $X \in \mathrm{Sm}_T$, and a closed subscheme $Z \subset X$, if the Weil restriction $\mathrm{R}_p X$ exists, then

$$p_\otimes(\frac{X}{X \setminus Z}) \simeq \frac{\mathrm{R}_p X}{\mathrm{R}_p X \setminus \mathrm{R}_p Z}$$

If $p : T \to S$ is a finite étale and $V \to T$ is a vector bundle, its Weil restriction $\mathrm{R}_p V \to S$ has a canonical structure of vector bundle (stalkwisely, it is Example 7.2).

Proposition 7.8: Given a finite étale morphism $p : T \to S$, a vector bundle V over T, we have $p_\otimes(S^V) \simeq S^{\mathrm{R}_p V}$ in $\mathcal{H}_\bullet(S)$.

Proposition 7.9: Given a finite étale morphism $p : T \to S$, the functor $\Sigma_\infty p_\otimes : \mathcal{H}_\bullet(T) \to \mathcal{SH}(S)$ has a unique symmetric monoidal extension

$$p_\otimes : \mathcal{SH}(T) \to \mathcal{SH}(S)$$

preserving sifted colimits.

Proof: Since $p_\otimes$ does not necessarily preserve colimits, we cannot use the universal characterization in Construction 6.5 directly. We borrow[2]Lemma 4.1 to revise the universal property and the result follows. ■

Remark 7.4: Let $p : T \to S$ be finite étale morphism and let $E \in \mathcal{SH}(T)$. Then we have

$$p_\otimes(E) \simeq \operatorname*{colim}_n \Sigma^{-\mathrm{R}_p \mathbb{A}^n} \Sigma^\infty p_\otimes(E_n)$$

where E_n is the nth space of E and $E \simeq \mathrm{colim}_n \Sigma^{-\mathbb{A}^n} \Sigma^\infty E_n$.

7.3 Properties and coherence of norms

In this section, we mainly introduce how the multiplicative norm functors interact with other operations coherently.

Proposition 7.10 (Composition): Given two universally clopen morphisms $f : F \to E$ and $g : E \to B$, there is a symmetric monoidal natural equivalence

$$(gf)_\otimes \simeq g_\otimes f_\otimes : \mathcal{P}_\Sigma(\mathrm{Sm}_F)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_E)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_B)_\bullet.$$

Hence, the same result holds in $\mathcal{H}_\bullet$ (resp. in $\mathcal{SH}$) if f and g are integral and universally open (resp. are finite étale).

Proposition 7.11 (Base change): Given a pull-back square of schemes as follows

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

where p is universally clopen. Let $\mathcal{C} \subset \mathrm{Sm}_T$ be a full subcategory and let $X \in \mathcal{P}_\Sigma(\mathcal{C})_\bullet$. Suppose either of the following assertions is true

（1）f is smooth;

（2）the Weil restriction $\mathrm{R}_p U$ is a smooth S-scheme for every $U \in \mathcal{C}$,

Then there exists a natural equivalence $\mathrm{Ex}^*_\otimes : f^* p_\otimes(X) \to q_\otimes g^*(X)$. In particular, if p is finite étale (resp. finite locally free), then there is an equivalence $\mathrm{Ex}^*_\otimes : f^* p_\otimes \to q_\otimes g^*$ equivalence in $\mathcal{SH}$ (resp. in $\mathcal{H}_\bullet$).

Remark 7.5: By taking adjunction for $\mathrm{Ex}^*_\otimes : f^* p_\otimes \to q_\otimes g^*$, we have

$$\mathrm{Ex}_{\otimes *} : p_\otimes g_* \to f_* q_\otimes$$

If f is smooth, we also have

$$\mathrm{Ex}_{\#\otimes} : f_\# q_\otimes \to p_\otimes g_\#$$

Given a finite locally free morphism $p : T \to S$ and a quasi-projective morphism $h : Q \to T$, we have the diagram

$$\begin{tikzcd} Q & \mathrm{R}_p Q \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & \mathrm{R}_p Q \arrow[d, "f"] \\ & T \arrow[from=1-1, "h"'] \arrow[r, "p"] & S \end{tikzcd}$$

where e is the counit of the adjunction (p^*, p_*), q and g are the canonical projections, and $f = \mathrm{R}_p(h)$. Then we define

$$\mathrm{Dis}_{\#*} : f_\# q_* e^* \xrightarrow{\mathrm{Ex}_{\#*}} p_* g_\# e^* \xrightarrow{\epsilon} p_* h_\# : \mathrm{QP}_U \to \mathrm{QP}_S$$

Furthermore, we consider

$$\mathrm{Dis}_{\#\otimes} : f_\# q_\otimes e^* \xrightarrow{\mathrm{Ex}_{\#\otimes}} p_\otimes g_\# e^* \xrightarrow{p}_\otimes h_\#,$$

$$\mathrm{Dis}_{\otimes *} : p_{\otimes} h_* \xrightarrow{\eta} p_{\otimes} g_* e^* \xrightarrow{\mathrm{Ex}_{\otimes *}} f_* q_{\otimes} e^*$$

To organize these properties and their coherence more efficiently, we introduce the notion of spans.

Definition 7.5: Given a category $\mathcal{C}$ with two classes of morphisms L and R such that

- they all contains equivalences,
- the pull-back of any arrow in L (resp. in R) along any arrow in R (resp. in L) is still in L (resp. in R),
- they are closed under compositions,

we construct a new ∞-category $\mathrm{Span}(\mathcal{C}, \mathrm{L}, \mathrm{R})$ whose objects are objects in $\mathcal{C}$ and morphisms are of the form

$$\bullet \xleftarrow{f} \bullet \xrightarrow{g} \bullet$$

where $f \in \mathrm{L}$ and $g \in \mathrm{R}$. The composition is given by pull-back.

Let S be a scheme. We write $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$ if $\mathcal{C}$ is a full subcategory of Sch_S that contains S and is closed under finite coproducts and finite étale extensions. We denote fét the class of finite étale morphisms.

Definition 7.6 (Normed ∞-category): Let S be a scheme and $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$. A *normed ∞-category* over $\mathcal{C}$ is a functor

$$\mathcal{A} : \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét}) \to \mathcal{C}\mathrm{at}_\infty, \ (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_{\otimes} f^*,$$

preserving finite products. $\mathcal{N}$ is said to be *presentably normed* if:

(1) $\mathcal{N}(X)$ is presentable for every $X \in \mathcal{C}$;

(2) $h^* : \mathcal{N}(X) \to \mathcal{N}(Y)$ has a left adjoint $h_\#$ for every finite étale morphism $h : Y \to X$;

(3) $f^* : \mathcal{N}(X) \to \mathcal{N}(Y)$ preserves colimits for every morphism $f : Y \to X$;

(4) for every pull-back square

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[f]{} & X \end{array}$$

where h is a finite étale morphism, there an equivalence

$$\mathrm{Ex}^*_\# : h'_\# g^* \to f^* h_\# : \mathcal{N}(Y) \to \mathcal{N}(X')$$

as an exchange transformation;

(5) $p_{\otimes} : \mathcal{A}(Y) \to \mathcal{A}(Z)$ preserves sifted colimits for every finite étale morphism

$p : Y \to Z$;

（6）for every diagram

$$\begin{tikzcd} U & \mathrm{R}_p U \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & \mathrm{R}U \arrow[d, "f"] \\ & T \arrow[r, "p"] & S \arrow[from=1-1, to=2-2, "h"'] \end{tikzcd}$$

where p and h are finite étale morphisms, there exists an equivalence

$$\mathrm{Dis}_{\#\otimes} : f_\# q_\otimes e^* \to p_\otimes h_\#$$

as the distributivity transformation .

Example 7.4: In this example, we will construct the functor

$$\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty),\ S \mapsto \mathcal{SH}(S), (U \xleftarrow{f} T \xrightarrow{p} S) \mapsto p_\otimes f^*$$

which will form normed categories. Note that $\mathcal{H}(S)$ is generated by SmQP_S under sifted colimits. Our construction is decomposed into the steps

$$\mathrm{SmQP}_{S+} \rightsquigarrow \mathcal{P}_\Sigma(\mathrm{SmQP}_S)_\bullet \rightsquigarrow \mathcal{H}_\bullet(S) \rightsquigarrow \mathcal{SH}(S)$$

7.4 The category of normed motivic spectra

Recall that if $\mathcal{A} : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ is a functor classifying a cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$, a *section* of $\mathcal{A}$ is a section $s : \mathcal{C} \to \mathcal{E}$ of p. More specifically, for any $c \in \mathcal{C}$, $s(c)$ is an object in $\mathcal{A}(c)$. We write

$$\int \mathcal{A} = \mathcal{E} \text{ and } \mathrm{Sect}(\mathcal{A}) = \mathcal{F}\mathrm{un}_\mathcal{C}(\mathcal{C}, \mathcal{E})$$

Definition 7.7: Let $S \in \mathrm{Sch}$ and $\mathcal{C} \subset_{fet} \mathrm{Sch}_S$. A *normed spectrum over* $\mathcal{C}$ is a section of $\mathcal{SH}^\otimes$ over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{f\acute{e}t})$ that is cocartesian over $\mathcal{C}^{\mathrm{op}}$. An *incoherent normed spectrum over* $\mathcal{C}$ is a section of $h\mathcal{SH}^\otimes$ over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{f\acute{e}t})$ that is cocartesian over $\mathcal{C}^{\mathrm{op}}$.

The full subcategory of normed spectra over $\mathcal{C}$ is denoted by $\mathrm{NAlg}_\mathcal{C}(\mathcal{SH}) \subset \mathrm{Sect}(\mathcal{SH}^\otimes \mid \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{f\acute{e}t}))$. The frequent choices of $\mathcal{C}$ are Sm_S, Sch_S and FEt_S. For convenience, we write $\mathrm{NAlg}_{\mathrm{Sm}}(\mathcal{SH}(S))$ instead of $\mathrm{NAlg}_{\mathrm{Sm}_S}(\mathcal{SH})$.

Roughly speaking, a normed spectrum E over $\mathcal{C}$ is to assign $E_X \in \mathcal{SH}(X)$ for any $X\mathcal{C}$ and $p_\otimes f^* E_X \to E_Z$ in $\mathcal{SH}(Z)$ for any span $X \xleftarrow{f} Y \xrightarrow{p} Z$. Note that by full-back, we have that $f^* E_Y = E_X$ naturally. Therefore, the extra data for an (incoherent) normed structure is a spectrum $E \in \mathrm{Sch}(S)$ equipped with a parametrized multiplicative transfer $\mu_p : p_\otimes E_V \to E_U$ for any finite étale morphism $p : V \to U$ in $\mathcal{C}$ such that the following coherence conditions are satisfied.

Condition 7.8 (Coherence conditions for incoherent normed spectra): （1）μ_p is an equivalence when p is the identity;

（2）The square with two arbitrary composable finite étale morphisms $q : W \to V$ and $p : V \to U$ in $\mathcal{C}$

$$\begin{tikzcd} p_\otimes q_\otimes E_W \arrow[r, "p_\otimes \mu_q"] \arrow[d, "\simeq"'] & p_\otimes E_V \arrow[d, "\mu_p"] \\ (pq)_\otimes E_W \arrow[r, "\mu_{pq}"'] & E_U \end{tikzcd}$$

commutes up to homotopy.

（3）for every pull-back square

$$\begin{tikzcd} V' \arrow[r, "g"] \arrow[d, "q"'] & V \arrow[d, "p"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

in $\mathcal{C}$ where p is a finite étale morphism, the following diagram

$$\begin{tikzcd} f^* p_\otimes E_V \arrow[r, "f^*\mu_p"] \arrow[d, "\simeq"'] & f^* E_U \arrow[dd, "\simeq"] \\ q_\otimes g^* E_V \arrow[d, "\simeq"'] & \\ q_\otimes E_{V'} \arrow[r, "\mu_q"'] & E_{U'} \end{tikzcd}$$

commutes up to homotopy.

In particular, these coherence conditions imply that $\mu_p : p_\otimes E_V \to E_U$ is homotopically equivariant for the action of $\mathrm{Aut}(A/U)$ on $p_\otimes E_V$. Thus we have

$$\mu_p : (p_\otimes E_V)_{h\mathrm{Aut}(V/U)} \to E_U$$

Basically, the multiplicative coherence data for a normed spectrum over $\mathcal{C} \subset_{\text{fét}} \mathrm{Sm}_S$ is parametrized by $\mathcal{C} \cap \mathrm{FEt}_S$.

Proposition 7.12: Suppose S is a scheme and $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$.

（1）The ∞-category $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathcal{SH}(S)$ admits all finite limits and colimits. If $\mathcal{C}$ is a small ∞-category, then $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$ is presentable.

（2）The forgetful functor $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathcal{SH}(S)$ is conservative and preserves sifted colimits and finite limits. If $\mathcal{C} \subset \mathrm{Sm}_S$, it preserves limits and hence is both monadic.

Proof: See[2]Proposition 7.6. ■

Remark 7.6: The forgetful functor $\mathrm{NAlg}(\mathcal{SH}) \to \mathcal{SH}(S)$ has a left adjoint $\mathrm{NSym}_{\mathcal{C}} : \mathcal{SH}(S) \to \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$. When $\mathcal{C} = \mathrm{Sm}_S$ or $\mathcal{C} = \mathrm{FEt}_S$, we have that

$$\mathrm{NSym}_{\mathcal{C}}(E) = \underset{\substack{f:X\to S \\ p:Y\to X}}{\mathrm{colim}} f_{\#}p_{\otimes}(E_Y)$$

where the indexing ∞-category is the source of the cartesian fibration classified by $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, $X \mapsto \mathrm{FEt}_X^{\simeq}$. Therefore the motivic norm structure on a spectrum $E \in \mathcal{SH}(S)$ can be exhibited as

$$\mathrm{NAlg}_{\mathcal{C}}(E) \to E$$

The monadic argument can be found in[2]Section 7.1,16.4.

CHAPTER 8 MOTIVIC EXTENDED POWERS AND OPERATIONS

In this chapter, our main goal is to study how norms in motivic homotopy can derive motivic power operations, following Bachmann, Elamnto and Heller[3].

• The first section introduces the notion of motivic colimits to study the free norm monad further.

• The second section shows that the free norm functors are essentially extended power in the context of motivic homotopy theory via motivic colimits.

• The third section introduces some equivariant motivic homotopy theory, which we provides us with more computable tools.

• The fourth section explains the extended powers from the viewpoint of equivariant motivic homotopy theory. One may recall how power operations in ordinary cohomology are derived from this viewpoint in Remark 4.4.

• The fifth section shows how we use all the previous notions and constructions to construct motivic power operations.

8.1 Motivic colimits and the fundamental diagram

Given a functor

$$\mathcal{C} : \mathrm{Sm}_S^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

such that

• If $p_X : X \to S \in \mathrm{Sm}_S$ be the structure map, then $\mathcal{C}(p_X) : \mathcal{C}(S) \to \mathcal{C}(X)$ has a left adjoint $p_{X\#}$.

• The category $\mathcal{C}(S)$ is cocomplete.

Let $(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}} \to \mathrm{Sm}_S$ be the cartesian fibration classified by $\mathcal{C}$, i.e. objects in $(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$ are pairs $(p_X : X \to S \in \mathrm{Sm}_S, E \in \mathcal{C}(X))$. Then we construct a functor

$$\begin{array}{rccc} M_0 : & (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}} & \longrightarrow & \mathcal{C}(S) \\ & (X, E) & \longmapsto & p_{X\#}E \end{array}$$

From a more conceptual perspective, let

$$M_0^{\mathrm{R}} : \mathcal{C}(S) \to (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$$

be a standard inclusion, since the fiber of $(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}} \to \mathrm{Sm}_S$ over S can be identified with $\mathcal{C}(S)$, M_0 is a left adjoint of M_0^{R}.

Since $\mathcal{C}(S)$ is cocomplete, we may extend M_0 to a cocontinuous functor

$$M : \mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}) \to \mathcal{C}(S)$$

The category $\mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}})$ is too large, so we need to restrict it to some reasonable subcategories. Since an ∞-groupoid is a Kan complex, we may regard $\mathcal{C}^{\simeq}$ as a presheaf of space on Sm_S. Then we let $\mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}}$ be the slice category over $\mathcal{C}^{\simeq}$ and let $(\mathrm{Sm}_S)_{/\mathcal{C}}$ be the restriction of representable presheaves for $\mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}}$. Besides, $(\mathrm{Sm}_S)_{/\mathcal{C}} \to \mathrm{Sm}_S$ is a cartesian fibration classifying $X \mapsto \mathcal{C}(X)^{\simeq}$ and $\mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}} \simeq \mathcal{P}((\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}})$. The inclusion $(\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}} \to (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$ induces $\mathcal{P}((\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}}) \to \mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}})$.

Definition 8.1: The *motivic colimit functors* for $\mathcal{C}$ are the compositions

$$\mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}) \xrightarrow{M} \mathcal{C}(S)$$

$$\mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}} \simeq \mathcal{P}((\mathrm{Sm}_S)_{/\mathcal{C}^{\simeq}}) \to \mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}) \xrightarrow{M} \mathcal{C}(S)$$

Remark 8.1: We may view $\mathcal{C}$ as ∞-categories parametrized by $\mathrm{Sm}_S^{\mathrm{op}}$ and $\mathcal{P}\mathrm{Sm}_{S/\mathcal{C}}$ can be regarded as to assign a diagram $F_X \to \mathcal{C}(X)$ for each $X \in \mathrm{Sm}_S$. From this point of view, a motivic diagram is to parametrize diagrams by $\mathrm{Sm}_S^{\mathrm{op}}$ and a motivic colimit functor encodes the colimits of the parametrized diagrams with coherence in the initial category $\mathcal{C}(S)$.

Example 8.1: Recall that we have a pair of adjunction

$$c : \mathcal{S} \rightleftarrows \mathcal{P}(\mathrm{Sm}_S) : \Gamma$$

where c sends a space $\mathcal{X}$ to a constant presheaf on Sm_S with value $\mathcal{X}$ and Γ is the global section. Now given a space $\mathcal{X}$ with $c\mathcal{X} \to \mathcal{C}^{\simeq} \in \mathcal{P}(\mathrm{Sm}_S)_{/S^{\simeq}}$, which is equivalent to a functor

$$\bar{\alpha} : \mathcal{X} \to \mathcal{C}(S)$$

Then one may find

$$M(\alpha) \simeq \operatorname*{colim}_{\mathcal{X}} \bar{\alpha}$$

Example 8.2: Given $X \in \mathrm{Sm}_S$ with the structure map $p_X : X \to S$. By the Yoneda embedding, we may take X as a discrete presheaf h_X, and $E \in \mathcal{C}(X)$ is classified by $E : h_X \to \mathcal{C}$. According to the definition, we have

$$\operatorname*{colim}_{h_X} E \simeq p_{X\#} E$$

More generally, let $(\mathcal{X} \xrightarrow{\alpha} \mathcal{C}) \in \mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}}$. First, we may write

$$\mathcal{X} = \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} h_X$$

where $X \in \mathrm{Sm}_S$ and $x : h_X \to \mathcal{X}$ (or equivalently $x \in \mathcal{X}(X)$). Then we have

$$(\mathcal{X} \xrightarrow{\alpha} \mathcal{C}) = \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} (h_X \xrightarrow{\alpha\circ x} \mathcal{C})$$

and consequently

$$M(\mathcal{X} \xrightarrow{\alpha} \mathcal{C}) \simeq \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} (X \to X)_\# \alpha(x)$$

where $\alpha(x) \in \mathcal{C}(X)$ is represented by $h_X \xrightarrow{\alpha\circ x} \mathcal{C}$.

Example 8.3 (The motivic Thom spectrum functor): Let $(\mathrm{Sm}_S)_{//\mathcal{SH}} \to \mathrm{Sm}_S$ be the cartesian fibration classified by $\mathcal{SH} : \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. Note that the objects in $(\mathrm{Sm}_S)_{//\mathcal{SH}}$ are pairs $(f : X \to S \in \mathrm{Sm}_S, Y \in \mathcal{SH}(X))$. The *motivic Thom spectrum functor*

$$\mathrm{M}_S : \mathcal{P}((\mathrm{Sm}_S)_{//\mathcal{SH}}) \to \mathcal{SH}(S)$$

is the colimit-preserving extension of

$$\begin{array}{rcl} (\mathrm{Sm}_S)_{//\mathcal{SH}} & \longrightarrow & \mathcal{SH}(S) \\ (f : X \to S, p \in \mathcal{SH}(X)) & \longmapsto & f_\# P \end{array}$$

Here we take $\mathcal{SH}$ as $\mathcal{C}$ in the setting. Let $\mathrm{Vect} \in \mathcal{P}(\mathrm{Sm}_S)$ be the functor that assign to X the nerve of the groupoid of vector bundles over X, denoted by $\mathrm{Vect}(X)$. Then by taking the Thom spaces, we obtain a symmetric monoidal functor

$$\mathrm{Vect}(X) \to \mathcal{SH}(X),\ \xi \to S^\xi$$

Since S^ξ is invertible in $\mathcal{SH}(X)$ according to the definition of the motivic stable homotopy category, we may extend it to a natural transformation

$$j : \mathrm{K}^\oplus \to \mathcal{SH}$$

by taking group completion. Let Gr_∞ be the infinite Grassmannian and let $\gamma : \mathrm{Gr}_\infty \to \mathrm{K}^\circ$ be the map whose restriction to Gr_n classifies the tautological bundle minus the trivial bundle of rank n. Then we have

$$\mathrm{MGL}_S = M_S(j \circ \gamma : \mathrm{Gr}_\infty \to \mathcal{SH})$$

More details can be found in [2]Section 16.

Next, we study how motivic colimit functors interact with multiplicative transfers.

$$
\begin{tikzcd}
\mathrm{Sm}_S^{\mathrm{op}} \arrow[r] \arrow[d, "\mathcal{C}"] & \mathrm{Span}(\mathrm{Sm}_S, \mathrm{all}, \mathrm{f\acute{e}t}) \arrow[dl, "\mathcal{C}^{\otimes}"] \\
\widehat{\mathcal{C}\mathrm{at}}_{\infty} &
\end{tikzcd}
$$

Let $\mathrm{FEt}_X^{\simeq}$ be the groupoid whose objects are finite étale morphisms $f : X \to Y$ and morphisms are isomorphisms. Then $\mathrm{FEt}^{\simeq} : X \mapsto \mathrm{FEt}_X^{\simeq}$ will form a presheaf on Sm_S. We construct

$$N : \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \to \mathcal{P}(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$$

informally given by

$$(\mathcal{X} \to \mathrm{FEt}^{\simeq}, E \in \mathcal{C}(S)) \mapsto (\mathcal{X} \to \mathcal{C}) \in \mathcal{P}(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$$

where $\mathcal{X}(Y) \to \mathrm{FEt}_Y^{\simeq} \xrightarrow{N_E(Y)} \mathcal{C}(Y)$ and

$$
\begin{array}{rccc}
N_E(Y) : & \mathrm{FEt}_Y^{\simeq} & \longrightarrow & \mathcal{C}(Y) \\
& (U \xrightarrow{p} Y) & \longmapsto & p_{\otimes} E_U
\end{array}
$$

Recall that given a small ∞-category $\mathcal{D}$, the Grothendieck construction gives an equivalence

$$\int : \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \widehat{\mathcal{C}\mathrm{at}}_{\infty}) \simeq \mathrm{Cart}_{\mathcal{D}}$$

where $\mathrm{Cart}_{\mathcal{D}} = \widehat{\mathcal{C}\mathrm{at}}^{\mathrm{cart}}_{\infty/\mathcal{D}}$ is the category of cartesian fibrations over $\mathcal{D}$. We let

$$\mathcal{F}\mathrm{un}_{\mathcal{D}}^{\mathrm{cart}}(\mathcal{E}_1, \mathcal{E}_2) \subset \mathcal{F}\mathrm{un}_{\mathcal{D}}(\mathcal{E}_1, \mathcal{E}_2)$$

be the full subcategory on the functors over $\mathcal{D}$ that preserves cartesian edges.

Example 8.4: Let $h : \mathcal{D} \to \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \subset \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \widehat{\mathcal{C}\mathrm{at}}_{\infty})$ be the Yoneda embedding, i.e. $h_d(-) := \mathrm{Map}_{\mathcal{D}}(-, d)$ for $d \in \mathcal{D}$. Through the Grothendieck construction, the corresponding cartesian fibration

$$\int h_d = \mathcal{D}_{/d} \to \mathcal{D}$$

is given by the slice category $\mathcal{D}_{/d}$.

Lemma 8.1: For any $d \in \mathcal{D}$, evaluation at (d, id) defines an equivalences

$$\mathcal{F}\mathrm{un}^{\mathrm{cart}}(\int h_d, \mathcal{E}) \simeq \mathcal{E}_d$$

where $\mathcal{E}_d$ is the fiber over d.

Proof: See[70]Lemma 2.5.7. ■

Construction 8.2 (Fundamental diagram): First, we identify

$$\mathcal{C}(S) \simeq \mathcal{F}\mathrm{un}^{\mathrm{cart}}_{\mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})}(\int h_S, \mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})_{/\!\!/\mathcal{C}})$$

using Lemma 8.1. Then by restriction $\mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})$ to Sm_S, we have

$$\mathcal{F}\mathrm{un}^{\mathrm{cart}}_{\mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})}(\int_{\mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})} h_S, \mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})_{/\!\!/\mathcal{C}}) \to \mathcal{F}\mathrm{un}^{\mathrm{cart}}_{\mathrm{Sm}_S}((\mathrm{Sm}_S)_{/\!\!/\mathrm{FEt}^{\simeq}}, (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}})$$

The cartesian fibration

$$\int_{\mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})} h_S \to \mathrm{Span}(\mathrm{Sm}_S,\mathrm{f\acute{e}t},\mathrm{all})$$

classifies $X \mapsto \{Y \xleftarrow{p} X \xrightarrow{f} S\}$ where p is a finite étale morphism and $f: X \to S$ is in Sm_S. Then by restriction to Sm_S, the value of $\int h_S$ at $X \to S$ is exactly $\mathrm{FEt}^{\simeq}_X$. Combining all these morphisms together, we have

$$N_0^{\dagger}: \mathcal{C}(S) \to \mathcal{F}\mathrm{un}^{\mathrm{cart}}_{\mathrm{Sm}_S}((\mathrm{Sm}_S)_{/\!\!/\mathrm{FEt}^{\simeq}}, (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}) \subset \mathcal{F}\mathrm{un}((\mathrm{Sm}_S)_{/\!\!/\mathrm{FEt}^{\simeq}}, (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}})$$

By taking adjunction, we have

$$N_0: (\mathrm{Sm}_S)_{/\!\!/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \to (\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$$

The composite

$$N: \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \to \mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S)) \xrightarrow{\mathcal{P}(N_0)} \mathcal{P}((\mathrm{Sm}_S)_{/\!\!/\mathcal{C}})$$

takes values in $\mathcal{P}(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$. The functor

$$\mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \xrightarrow{N} \mathcal{P}(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}}$$

is defined to be the *fundamental diagram*.

Remark 8.2: By the construction, N is cocontinuous functor in the first variable. Then, in analogy to Example 8.2, we may write $\alpha: \mathcal{X} \to \mathrm{FEt}^{\simeq} = \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} (h_X \xrightarrow{\alpha\circ x} \mathrm{FEt}^{\simeq})$, where $\alpha \circ x$ can be identified as $(p_{\alpha\circ x}: U \to X) \in \mathrm{FEt}^{\simeq}_X$. Then we have

$$N(\alpha, E \in \mathcal{C}(S)) = \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} (X, p_{\alpha\circ x \otimes}(E_U)).$$

8.2 The generalized motivic extended powers

Lemma 8.2: For any scheme S, we have a canonical equivalence

$$\mathrm{FEt}^{\simeq,n}|_{\mathrm{Sm}_S} \simeq B_{\acute{e}t}\Sigma_n \in \mathcal{P}(\mathrm{Sm}_S).$$

Proof: Let $B\mathrm{Tors}_{\acute{e}t}(\Sigma_n)$ be the motivic space associated to the presheaf on Sm_S that assigns to $U \in \mathrm{Sm}_S$ the nerve of the groupoid of étale G-torsors on U and sends $f : U' \to U$ to the associated base change pull-back. According to[71]Proposition 5.6, there is a motivic equivalence $B_{\text{ét}}\Sigma_n \xrightarrow{\simeq} B\mathrm{Tors}_{\text{ét}}(\Sigma_n)$. Then the result follows from the standard correspondence between étale Σ_n-torsors and finite étale morphisms of degree n[72](6.1.3).

■

Definition 8.3: Suppose that $\mathcal{C}(S)$ admits small colimits. The *generalized motivic extended power* functor is defined as the composite

$$D_{gen}^{mot} : \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \xrightarrow{N} \mathcal{P}(\mathrm{Sm}_S)_{/\!\!/\mathcal{C}} \xrightarrow{M} \mathcal{C}(S)$$

We adopt the following notation:

(1) For $\mathcal{X} \in \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}}$ fixed, we denote the functor $D_{gen}^{mot}(\mathcal{X}, -)$ by

$$D_{\mathcal{X}}^{mot} : \mathcal{C}(S) \to \mathcal{C}(S);$$

(2) if $\mathcal{X} = \mathrm{FEt}^{\simeq,n}$, the stack of finite étale morphisms of rank n, we denote the functor $D_{\mathcal{X}}^{mot}$ by

$$D_{n}^{mot} : \mathcal{C}(S) \to \mathcal{C}(S);$$

(3) if $G \to \Sigma_n$ is a group homomorphism, and $\mathcal{X} \in \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}}$ is given by the composite $B_{\text{ét}}G \to B_{\text{ét}}\Sigma_n \simeq \mathrm{FEt}^{\simeq,n} \hookrightarrow \mathrm{FEt}^{\simeq}$, then we denote $D_{\mathcal{X}}^{mot}$ by

$$D_{G}^{mot} : \mathcal{C}(S) \to \mathcal{C}(S)$$

(4) if BG is the constant presheaf, then there is a canonical map $BG \to B\Sigma_n \to B_{\text{ét}}\Sigma_n$, and we put

$$D_{BG}^{mot} =: D_n^{naive} : \mathcal{C}(S) \to \mathcal{C}(S),$$

and in particular

$$D_{B\Sigma_n}^{mot} =: D_n^{naive} : \mathcal{C}(S) \to \mathcal{C}(S).$$

Remark 8.3: According to Remark 8.2, we have that

$$D_{\mathcal{X}}^{mot}(E) = \operatorname*{colim}_{(X,x)\in(\mathrm{Sm}_S)_{/\!\!/\mathcal{X}}} p_{\alpha\circ x\otimes}(E_U)$$

where $(p_{\alpha\circ x} : U \to X) \in \mathrm{FEt}_X^{\simeq}$ is induced by $h_X \xrightarrow{\alpha\circ x} \mathrm{FEt}^{\simeq}$ via the Yoneda lemma.

Example 8.5: If we let $\mathcal{C}^{\otimes}$ be $\mathcal{SH}^{\otimes}$ and let the motivic Thom functor fit in the motivic generalized extended power, then we have

$$D_{gen}^{mot}(\mathrm{FEt}^{\simeq}, E) = \mathrm{NSym}_{\mathrm{Sm}_S}(E)$$

by referring to Remark 7.6, Example 8.3 and Remark 8.2. We may split $\mathrm{FEt}^{\simeq} \simeq \sqcup \mathrm{FEt}^{\simeq,n}$ (a L_Σ-equivalence) in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, and we have

$$\mathrm{NSym}(E) = D^{\simeq}_{\mathrm{FEt}^{\simeq}}(E) \simeq \bigvee_{n\geq 0} D^{mot}_n(E).$$

More details can be found in [2]Remark 16.27 and [3]Example 5.13.

Proposition 8.1: The functor $D^{mot}_{gen} : \mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}} \times \mathcal{C}(S) \to \mathcal{C}(S)$ preserves colimits in the factor $\mathcal{P}(\mathrm{Sm}_S)_{/\mathrm{FEt}^{\simeq}}$. If each of the functors $f^*, p_\otimes$ preserves colimits of some shape K, then D^{mot}_{gen} preserves colimits of shape K in the factor $\mathcal{C}(S)$.

Proof: See [3]Proposition 5.11. ■

Proposition 8.2: [3] Let $G \to \Sigma_n$ and $H \to \Sigma_m$ be group homomorphisms of finite groups. Let $H \wr G \to \Sigma_{nm}$ correspond to the canonical of $H \wr G$ on $\{1, \cdots, n\} \times \{1, \cdots, m\}$ with lexicographic order. Assume that $\mathcal{C}$ is a Zariski sheaf. Then there is a canonical equivalence of functors $D^{mot}_G \circ D^{mot}_H \simeq D^{mot}_{H\wr G}$.

Proof: See [3]Proposition 5.28. ■

8.3 Equivariant motivic homotopy theory

Given a quasi-compact and quasi-separated scheme S and a finite group G, we denote QP^G_S the category of finitely presented and quasi-projective S-schemes with G-action. The elements in QP^G_S are called G-schemes simply. Let SmQP^G_S be the full subcategory of smooth schemes in QP^G_S.

Definition 8.4: Suppose X is a G-scheme and $x \in X$. The subgroup

$$\mathrm{Stab}(x)\{g \in G \mid g \cdot x = x \text{ and } \mathrm{id}_{k(x)} = g^* : k(x) \to k(x)\}$$

is defined to be the *stabilizer of* x.

Definition 8.5: A family $\mathcal{F}$ of subgroups of G is a collection of subgroups such that

(1) If $H \in \mathcal{F}$, then any subgroup of H is in $\mathcal{F}$;

(2) If $H \in \mathcal{F}$ and H' is conjugate to H, then $H' \in \mathcal{F}$.

Example 8.6: The following families will be frequently used

(1) The trivial family $\mathcal{F}_{triv} = \{e\}$.

(2) The family of all proper subgroups $\mathcal{F}_{prop}$.

(3) The family of all subgroups $\mathcal{F}_{\mathrm{all}}$

If $\mathcal{F}$ is a family, we denote $\mathrm{co}(\mathcal{F}) = \mathcal{F}_{\mathrm{all}} \setminus \mathcal{F}$.

Definition 8.6: Let $\mathcal{F}$ be a family of G. $\mathrm{SmQP}^G_S[\mathcal{F}]$ is the full subcategory of SmQP^G_S

defined as follows:

$$\mathrm{SmQP}_S^G[\mathcal{F}] = \{X \in \mathrm{SmQP}_S^G \mid \mathrm{Stab}(x) \in \mathcal{F} \text{ for any } x \in X\}$$

Definition 8.7: Let $f : U \to V$ be an equivariant map in SmQP_S^G. If $\mathrm{Stab}(v) = \mathrm{Stab}(f(v))$ for any $v \in U$, then f is a *fixed point reflecting map*.

Definition 8.8: Given a finite group G and a G-scheme $X \in \mathrm{SmQP}_S^G$, a *fixed point Nisnevich covering* is generated by the following cartesian square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where p is a fixed point reflecting étale map and j is an open immersion such that $(Y \setminus V)_{red} \to (X \setminus U)_{red}$ is an isomorphism. The generated topology is so-called the *fixed point Nisnevich topology*.

Proposition 8.3: Let $\mathcal{C}_S \subset \mathrm{SmQP}_S^G$ be a full subcategory such that for any $B \in \mathcal{C}_S$, if $E \to B$ is an equivariant étale map, then E is also in $\mathcal{C}_S$. With this assumption, the Nisnevich topology on $\mathcal{C}_S$ agrees with the fixed point Nisnevich topology on $\mathcal{C}_S$.

Proof: See[73]Proposition 2.13. ■

Remark 8.4: Given a family $\mathcal{F}$ of subgroups in G, the category $\mathrm{SmQP}_S^G[\mathcal{F}]$ satisfies the assumption in Proposition 8.3.

Definition 8.9: (1) The category $\mathcal{H}^G(S)$ of *motivic G-spaces* is the full subcategory of Nisnevich and $\mathbb{A}^1$-invariant sheaves in $\mathcal{P}_\Sigma(\mathrm{SmQP}_S^G)$. Note that we have the reflective localization functor $L_{mot} : \mathcal{P}_\Sigma(\mathrm{SmQP}_S^G) \to \mathcal{H}^G(S)$. Similarly, we can define the pointed version $\mathcal{H}_\bullet^G(S)$.

(2) Given a family $\mathcal{F}$ of subgroups in G, the category $\mathcal{H}^G(S)[\mathcal{F}]$ is the full subcategory of Nisnevich and $\mathbb{A}^1$-invariant sheaves in $\mathcal{P}_\Sigma(\mathrm{SmQP}_S^G)[\mathcal{F}]$. Similarly, we can define the pointed version $\mathcal{H}_\bullet^G(S)[\mathcal{F}]$.

Definition 8.10: Suppose $\mathcal{F}$ is a family of subgroups in G. The *universal $\mathcal{F}$-space* is the presheaf on SmQP_S^G

$$\mathbb{E}\mathcal{F}(X) = \begin{cases} \emptyset, & X \notin \mathrm{SmQP}_S^G[\mathcal{F}] \\ *, & X \in \mathrm{SmQP}_S^G[\mathcal{F}] \end{cases}$$

In particular, we denote $\mathbb{E}G = \mathbb{E}\mathcal{F}_{triv}$.

Proposition 8.4: Let $\mathcal{F}$ be a family for a finite group G. The presheaf $\mathbb{E}_\mathcal{F}$ is motivic local and is represented by an ind-smooth G-scheme.

Proof: See[73]Proposition 3.3, Proposition 3.7. ■

8.4 Motivic extended powers via enhanced smash powers

Definition 8.11: The functor

$$\mathbb{D}_n : \mathcal{SH}(S) \xrightarrow{(-)^{\wedge \underline{n}}} \mathcal{SH}^{\Sigma_n}(S) \xrightarrow{(-)_{h\Sigma_n}} \mathcal{SH}(S).$$

If $H \subset \Sigma_n$ is a subgroup, then we denote $\mathbb{D}_H$ the functor

$$\mathbb{D}_H : \mathcal{SH}(S) \xrightarrow{(-)^{\wedge \underline{n}}} \mathcal{SH}^{\Sigma_n}(S) \to \mathcal{SH}^H(S) \xrightarrow{(-)_{hH}} \mathcal{SH}(S).$$

Proposition 8.5: Let $p : T \to S$ be a finite étale morphism of degree n, and $q : R \to S$ the associated Σ_n-torsor. Then we have an equivalence

$$p_{\otimes} p^* \simeq R_+ \wedge_{\Sigma_n} (-)^{\wedge \underline{n}} : \mathcal{SH}(S) \to \mathcal{SH}(S)$$

Proof: Since these two functors are symmetric monoidal and preserves sifted, we just reduce the case to $\mathrm{SmQP}_{S+} \to \mathcal{SH}(S)$. Thus it remains to show that for $X_+ \in \mathrm{SmQP}_{S+}$, there is a natural equivalence

$$p_{\otimes} p^* X = X_+^T \simeq (R \times_{\Sigma_n} X^n)_+.$$

Since both of them are étale sheaves, we may assume $T = S^{\amalg n}$ and $R = S \times \Sigma$ by taking an étale cover, then both of the sides are X^n. ■

Lemma 8.3: There is a canonical equivalence $\mathbb{E}G \simeq \mathrm{colim}_{R \in B_{\text{ét}} G} R \in \mathcal{S}\mathrm{hv}_{\ni}(\mathrm{Sm}_S^G)$.

Proof: First, we may write a presheaf as a colimit of representable presheaves

$$\mathbb{E}G = \operatorname*{colim}_{R \to \mathbb{E}G} R$$

where $R \in \mathrm{SmQP}_S^G$ is a representable sheaf. According to Definition 8.10, we have that

$$\operatorname*{colim}_{R \to \mathbb{E}G} R = \operatorname*{colim}_{R \in \mathrm{SmQP}_S^G[\mathcal{F}_{triv}]} R$$

By specifying a faithful embedding $G \to \Sigma$, the construction of $\mathbb{E}\mathcal{F}_{triv}$ [73]Proposition 3.7 coincides with the geometric model of $B_{\text{ét}} G$ [37]Section 4, Proposition 2.6. ■

Proposition 8.6: There is a canonical equivalence between $\mathbb{D}_H$ and D_H^{mot}.

Proof: By decoding the definition of $D_H^{mot}(X)$, it is a motivic colimit parametrized by $B_{\text{ét}} H \hookrightarrow B_{\text{ét}} \Sigma_n \xrightarrow{\simeq} \mathrm{FEt}^{\simeq, n}$. According to Proposition 8.5, we have

$$D_n^{mot}(-) \simeq \operatorname*{colim}_{R \in B_{\text{ét}} \Sigma_n} R_+ \wedge_{\Sigma_n} (-)^{\wedge \underline{n}}$$

By reducing to H, we have

$$D_H^{mot}(-) \simeq \operatorname*{colim}_{R \in B_{\text{ét}}H} R_+ \wedge_H (-)^{\wedge \underline{n}} \simeq \mathbb{E}H_+ \wedge_H (-)^{\wedge \underline{n}} \simeq \mathbb{D}_H(-)$$

where the middle equivalence is given by Lemma 8.3. ∎

8.5 Motivic power operations via norms

Given a normed motivic spectrum (or an incoherent motivic spectrum) $E \in \mathcal{SH}(S)$ and $X \in \mathcal{H}(S)$, the E-cohomology space of X at degree (p, q) is given by

$$X \mapsto \mathrm{Map}_{\mathcal{SH}(S)}(\Sigma^\infty X_+, \Sigma^{p,q} \wedge E)$$

The associated E-cohomology group $E^{p,q}(X) := \pi_0 \mathrm{Map}_{\mathcal{SH}(S)}(\Sigma^\infty X_+, \Sigma^{p,q} \wedge E)$. We

Construction 8.12 (Power operations): Let $p: T \to S$ be a finite étale morphism of rank n in Sm_S. Let $E_0 = \Omega^\infty E$ and by adjunction, $E^{0,0}(X) = \pi_0 \mathrm{Map}_{\mathcal{H}_\bullet(S)}(X_+, E_0)$.The *multiplication parametrized by* p is given by

$$\mathrm{Map}(X_+, E_0) \xrightarrow{p_\otimes p^*} \mathrm{Map}(p_\otimes p^* X_+, p_\otimes p^* E_0) \xrightarrow{\mu_p} \mathrm{Map}(p_\otimes p^* X_+, E_0) \simeq \mathrm{Map}(W \wedge_{\Sigma_n} (X_+)^{\wedge \underline{n}}, E_0)$$

where $\mu_p : p_\otimes p^* E_0 \to E_0$ is given by the normed structure of E, $W \to S$ is a Σ_n-torsor corresponding to $p: T \to S$, and the last equivalence is promised by Proposition 8.5.

Recall that Morel and Voevodsky constructed a geometric model for $B_{\text{ét}}G$ by using admissible gadgets[37]Section 4.2. Specifically, fixed a group embedding $G \to \Sigma_n$, let $U_i \subset \mathbb{A}^{ni}$ be the free-action locus with respect to the diagonal action of $\mathbb{A}^{ni} \cong (\mathbb{A}^n)^i$, and we have an equivalence

$$U_\infty/G := \operatorname*{colim}_{i \to \infty} U_i/G \to B_{\text{ét}}G \tag{8-1}$$

where $U_i/G \to B_{\text{ét}}G \simeq B\mathrm{Tors}_{\text{ét}}(G)$ (recall[71]Proposition 5.6) is the classifying map for the natural G-torsor $U_i \xrightarrow{f_i} U_i/G$, and $U_i \to U_{i+1}$ is a pointed natural inclusion of the first i copies of $\mathbb{A}^n$. Let $p_i: T_i \to S$ be the finite étale map associated to $f_i: U_i \to U_i/G$, and we have diagram

$$\begin{tikzcd} \cdots \arrow[r] & T_{i-1} \arrow[r] \arrow[dr, "p_{i-1}"'] & T_i \arrow[r] \arrow[d, "p_i"] & T_{i+1} \arrow[r] \arrow[dl, "p_{i+1}"] & \cdots \\ & & S & & \end{tikzcd}$$

If we apply the procedure (8-1) for this diagram by taking $G = \Sigma_n$ exactly, then we will have

$$\mathrm{Map}(X_+, E_0) \to \mathrm{Map}(U_\infty \wedge_{\Sigma_n} (X_+)^{\wedge \underline{n}}, E_0)$$

If we let $X_+ = S$ and replace Map by the internal motivic mapping space, then we have the *total power operation*

$$\mathrm{P}_n : E_0 \to \mathrm{Hom}(B_{\text{ét}}\Sigma_n, E_0)$$

Remark 8.5: Given a finite étale morphisms $f : Q \to S$ with the structure étale group scheme G associated to f, we can define a total power operation twisted by Q as follows

$$\mathrm{P}_T : E_0 \to \mathrm{Hom}(B_{\text{ét}}G, E_0).$$

More details can be found in[2]Example. Roughly speaking, we just replace $U_i/G \to S$ by $(U_i \times_S T) \to S$ and transfer them via $(U_i \times_S T) \to U_i/G$.

Remark 8.6 (Comparison with Voevodsky's construction on motivic power operations): Bachmann and Hoyois show that the motivic cohomology spectra $H\mathbb{Z}_S$ is a normed motivic spectrum[2]Section 14. They decode the motivic spectrum in terms of equidimensional relative cycles of dimension 0[2]Lemma 14.17. Meanwhile, Voevodsky constructs the total motivic power operations for mod p motivic cohomology spectra by decoding and manipulating them in terms of equidimensional relative cycles, too[38]Theorem 3.1. It is natural to believe that power operations via norms on the motivic cohomology spectra coincide with Voevodsky's construction, by observing that both the constructions intend to perform the following rough procedure:

（1）Take simplicial resolutions for the motivic spaces of equidimensional relative cycles to make them accessible;

（2）Take a suitable auxiliary vectors;

（3）Mark the cycles on fibers so that we can parametrize by manipulating vector bundles;

（4）Use Thom spaces or Thom compactification to bring the results back to the context of motivic cohomology.

We will show the role of equidimensional cycles in the next chapter briefly.

CHAPTER 9 MOTIVIC COHOMOLOGY AND ADAMS SPECTRAL SEQUENCES

Motivic Adams spectral sequences are tools for computing stable homotopy groups of motivic spectra, which are generalizations of topological spectra to algebraic geometry. They are motivic analogs of the classical Adams spectral sequences in topology, which use the cohomology of the Steenrod algebra to approximate homotopy groups. Motivic Adams spectral sequences use the cohomology of the motivic Steenrod algebra, which depends on a base field and a weight parameter. They have similar construction and convergence properties as their topological counterparts but also exhibit new features and challenges due to the richer structure of motivic homotopy theory. Ultimately, we give rise to several questions on motivic cohomology and motivic spectral sequences.

9.1 Motivic cohomology and the associated spectra

Definition 9.1: Fix a field k, we define an additive category Cor_k whose objects are objects in Sm_k.

$$\mathrm{Cor}_k(X, Y) = \mathbb{Z}\langle \text{elementary correspondences from } X \text{ to } Y\rangle$$

An elementary correspondence from X to Y is an irreducible closed subscheme $W \subset X \times_k Y$ such that $W \to X$ is finite and surjective. The composition is given by pull-back. The assignment $X \otimes Y := X \times_k Y$ endows Cor_k a symmetric monoidal structure.

Remark 9.1: There is an embedding

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Sm}_k}(X, Y) & \longrightarrow & \mathrm{Cor}_k(X, Y) \\ f & \longmapsto & \Gamma_f \end{array}$$

where $\Gamma_k \subset X \times_k Y$ is the *graph* of f. For convenience, we let $\Gamma : \mathrm{Sm}_k \to \mathrm{Cor}_k$ denote this functor.

Definition 9.2: A *presheaf with transfers* is a contravariant additive functor $F : \mathrm{Cor}_k \to$ **Abel**. The category of presheaves with transfers is denoted by $\mathrm{PST}(k)$.

Theorem 9.1: [74]Theorem 2.3 $\mathrm{PST}(k)$ has enough injectives and projectives.

Example 9.1: The presheaf of global units $\mathcal{O}^* : X \mapsto \Gamma(X)^*$ is a presheaf with transfers, whose transfers are defined by norm maps for field extensions. Specifically, If X is normal and $W \to X$ is finite and surjective, then there is a norm map $N : \mathcal{O}^*(W) \to \mathcal{O}^*(X)$ induced by the usual norm on the function field $k(W)^* \to k(X)^*$.

Similarly, we can define additive transfers for the presheaf of global sections. Specifically, $\mathrm{Tr} \colon \mathcal{O}(W) \to \mathcal{O}(X)$ is the trace map induced from $k(W) \to k(X)$.

Construction 9.3: Given $X \in \mathrm{Sm}_k$, we define $\mathbb{Z}_{tr}(X)$ to be

$$\mathbb{Z}_{tr}(X)(U) = \mathrm{Cor}_k(U, X)$$

For any abelian group A, we let

$$A_{tr}(X)(U) = \mathrm{Cor}_k(U, X) \otimes A$$

Given some pointed schemes (X_i, x_i) in Sm_k, we define

$$\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) := \mathrm{coker}(\bigoplus_{i=1}^{n} \mathbb{Z}_{tr}(X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n))$$

Construction 9.4: Given $F \in \mathrm{PST}(k)$ and $X \in \mathrm{Sm}_k$, we define

$$F^X(U) := F(U \times_k X)$$

Then we define the associated simplicial object $C_\bullet(F)$ by setting

$$C_i(F) := F^{\Delta_k^i}$$

where $\Delta_k^i := \mathrm{Spec} k[x_0, \ldots, x_n]/(\sum_i x_i = 1)$. The *normalized chain complex* is denoted by $C_*^{DK}(F)$. The differentials of another chain complex $C_*(F)$ are given by an alternative sum of face maps.

Definition 9.5: For any non-negative integer q, the *motivic complex* $\mathbb{Z}(q)$ is defined to be

$$\mathbb{Z}(q) := C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$$

Specifically, $\mathbb{Z}(q)^i = C_{q-i}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$.

Similarly, for any abelian group A, we can define $A(q)$ to be

$$A(q)^i = C_{q-i}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \otimes_{\mathbb{Z}} A$$

Lemma 9.1: $\mathbb{Z}_{tr}(Y)$ is a Zariski sheaf and $C_*\mathbb{Z}_{tr}Y$ is a complex of Zariski sheaves.

Definition 9.6: The *motivic cohomology group* $\mathrm{H}^{p,q}(X)$ is defined to be

$$\mathrm{H}^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))$$

(Here $\mathbb{H}^*_{Zar}$ is Zariski hypercohomology i.e. the cohomology of the total complex of the bicomplex of the injective resolutions for the Zariski chain complex.)

Remark 9.2: Let C_* be a left bounded complex. Then for each C_i, there is an injective resolution $I_{i,*}$ and we may extend these injective resolutions by their lifting properties into

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
I_{0,1} \arrow[u] & I_{1,1} \arrow[l] \arrow[u] & I_{2,1} \arrow[l] \arrow[u] & I_{3,1} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
I_{0,0} \arrow[u] & I_{1,0} \arrow[l] \arrow[u] & I_{2,0} \arrow[l] \arrow[u] & I_{3,0} \arrow[l] \arrow[u] & \cdots \arrow[l] \\
C_0 \arrow[u] & C_1 \arrow[l] \arrow[u] & C_2 \arrow[l] \arrow[u] & C_3 \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

Construction 9.7: There is a *multiplicative structure on* $\mathbb{Z}(n)$. First, there is a morphism in $\mathrm{PST}(k)$

$$\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_i) \otimes \mathbb{Z}_{tr}(X_{i+1} \wedge \cdots \wedge \mathbb{Z}_j) \to \mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_j)$$

Then we have

$$\mathbb{Z}(m) \otimes \mathbb{Z}(n) \to \mathbb{Z}(m+n)$$

(reduce to simplicial case and use Eilenberg-Zilber theorem)

$$C_\bullet \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m}) \otimes C_\bullet \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) \to C_\bullet \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m+n})$$

Therefore, we have a pairing

$$\mathrm{H}^{p,q} \otimes \mathrm{H}^{p',q'} \to \mathrm{H}^{p+p',q+q'}$$

which makes $\mathrm{H}^{*,*}$ a bi-graded algebra.

One of the significant features of motivic cohomology is about the connection to the Milnor K-theory and the higher Chow groups.

Theorem 9.2: [74]Theorem 5.1,Theorem 19.1 The motivic cohomology groups are connected to the Milnor K-theory and the higher Chow groups in the following ways:

- Let k be a field. For any n and any Abelian group A, there is an isomorphism

$$\mathrm{H}^{n,n}(\mathrm{Spec} k, A) \cong K_n^M(k).$$

- For any separated scheme $X \in \mathrm{Sm}_k$ for a perfect field k, there is an isomorphism

$$\mathrm{H}^{p,q}(X, \mathbb{Z}) \cong CH^q(X, 2q-p)$$

for any p, q.

The first part of Theorem 9.2 is related to the computation of the cohomology of the points.

Theorem 9.3: [39-40] Given a prime p, let k be a field with a primitive p^{th} root of unity ζ_p. Let $\tau \in \mathrm{H}^{0,1}(k, \mathbb{F}_p) \cong \mathbb{Z}/p$. Then we have

$$\mathrm{H}^{*,*}(k, \mathbb{F}_p) \cong K_*^M(k)[\tau] \otimes_{\mathbb{Z}} \mathbb{Z}/p$$

where the elements of degree n in $K_*^M(k)$ are assigned to be degree (n, n) and τ is of degree $(0, 1)$. The result also holds for fields with characteristics coprime to p. For convenience, we denote

$$\mathbb{M}_p(k) := \mathrm{H}^{*,*}(k, \mathbb{F}_p).$$

The second part of Theorem 9.2 essentially indicates how to construct motivic cohomology spectra in terms of Bloch's cycle complexes. In particular, we need to use equidimensional cycles in its proof.

$$\mathbb{Z}(n) \leftrightsquigarrow \text{Equidimensional cycles} \leftrightsquigarrow \text{Bloch's cycle complex} \tag{9-1}$$

The following diagram illustrates the relationship between the motivic complexes and equidimensional cycles.

$$\begin{tikzcd}
\text{PST level} \arrow[d, squiggly] & \mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - 0) \arrow[r, "\simeq"] \arrow[d, squiggly] & z_{equi}(\mathbb{A}^n, 0) \arrow[d, squiggly] \\
\text{simplicial level} \arrow[d, squiggly] & C_\bullet(\mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - 0)) \arrow[r, "\simeq"] \arrow[d, squiggly] & C_\bullet(z_{equi}(\mathbb{A}^n, 0)) \arrow[d, squiggly] \\
\text{complex level} \arrow[d, squiggly] & C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - 0)) \arrow[r, "\simeq"] \arrow[d, squiggly] & C_*(z_{equi}(\mathbb{A}^n, 0)) \arrow[d, squiggly] \\
\text{motivic complexes} & \mathbb{Z}(n)[2n] \arrow[r, "\simeq"] & \mathbb{Z}^{SF}(n)[2n]
\end{tikzcd} \tag{9-2}$$

where $C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - 0)) \simeq \mathbb{Z}(n)[2n]$ [74]Theorem 15.2, $z_{equi}(T, m)(X)$ is the free abelian group generated by the irreducible closed subvarieties of $X \times T$ that are dominant and equidimensional of relative dimension m over S, and $\mathbb{Z}^{SF}(n)$ is the *Suslin-Friedlander complex* defined as

$$\mathbb{Z}^{SF}(n) := C_* z_{equi}(\mathbb{A}^n, 0)[-2n].$$

Let $z^i_{equi}(X, m) := z_{equi}(X, \dim X - i)(\Delta^m)$ for an equidimensional X and $i \leq \dim X$. Then we write $z^i_{equi}(X, \bullet)$ for the corresponding simplicial object and $z^i_{equi}(X, *)$ for the corresponding complexes. Let $z^i(X, n)$ be the free abelian group generated by all subvarieties of codimension i on $X \times \Delta^n$ which intersect all faces $X \times \Delta^j$ properly. Then $z^i(X, \bullet)$ is a simplicial group and the associated complex $z^i(X, *)$ is the *Bloch's cycle complex*. The relationship between Bloch's cycle complexes and equidimensional cycles is

$$z^i_{equi}(X, *) \xhookrightarrow{\sim} z^i(X, *)$$

$$CH^p(X, q) = \pi_q(z^p(X, \bullet)) = H_q(z^p(X, *))$$

Therefore, $K(\mathbb{Z}(n), 2n)$ is the motivic space z^n associated to the simplicial presheaf

$$X \mapsto z^n(X, \bullet),$$

since

$$H^{2n,n}(X; \mathbb{Z}) = CH^n(X, 0) = \pi_0(z^n(X, \bullet)) = [X, z^n]_{\mathbb{A}^1}.$$

It reveals that motivic Eilenberg-MacLane spectra are built from motivic complexes essentially. For schemes over a Noetherian scheme S, we may define $\mathrm{Cor}_S(X, Y)$ using relative cycles, see[74]Appendix 1A. Similarly, we can further define motivic Eilenberg-Mac Lane spectra in $\mathcal{SH}(S)$. In the rest of this chapter, the based scheme S is assumed to be Noetherian.

9.2 The motivic Steenrod algebra

In this section, we recall some results about the structure of the motivic Steenrod algebra in[38-40].

Definition 9.8: Given a prime p and a field k of characteristic 0, the mod p motivic Steenrod Algebra $\mathcal{A}^{**}(k)$ is defined to be

$$\mathcal{A}^{**}(k; \mathbb{F}_p) = [H\mathbb{F}_p(k), H\mathbb{F}_p(k)]_{**}$$

Similarly to the structure of the classical Steenrod algebra, Voevodsky shows that there exist motivic Steenrod operations that generate the motivic Steenrod algebra as a $\mathbb{M}_p$-algebra with the motivic Adem relations[38-40,75].

Theorem 9.4: [39] Let ℓ be a prime and let $\mathrm{H}^{*,*}$ be the motivic cohomology functor over $\mathbb{F}_\ell$. Let $X \in \mathrm{Sm}_k$. There exists $\mathrm{P}^i_\ell : \mathrm{H}^{*,*}(X; \mathbb{Z}/\ell) \to \mathrm{H}^{*+2i(\ell-1),*+i(\ell-1)}(X; \mathbb{Z}/\ell)$ and $\mathrm{B}^i_\ell : H^{*,*}(X; \mathbb{Z}/\ell) \to \mathrm{H}^{*+2i(\ell-1)+1,*+i(\ell-1)}(X; \mathbb{Z}/\ell)$ such that

(1) $\mathrm{P}^0_\ell = \mathrm{id}$ and $\mathrm{P}^n_\ell(u) = u^n$ if $u \in H^{2n,n}$;

(2) If $\ell = 2$, then we denote $\mathrm{Sq}^{2i} = \mathrm{P}$ and $\mathrm{Sq}^{2i+1} = \mathrm{B}^i$

(3) Cartan formula: if $\ell \neq 2$,

$$\mathrm{P}_\ell^i(uv) = \sum_{j=0}^{i} \mathrm{P}_\ell^j(u)\mathrm{P}_\ell^{i-j}(v)$$

$$\mathrm{B}_\ell^i(uv) = \sum_{j=0}^{i} \mathrm{B}_\ell^j(u)\mathrm{P}_\ell^{i-j}(v) + (-1)^{\deg(u)}\mathrm{P}_\ell^j(u)\mathrm{B}_\ell^{i-j}(v)$$

(4) If $\ell = 2$, let τ be the generator of $H^{0,1}(K;\mathbb{Z}/2)$, and $\rho \in H^{1,1}(k;\mathbb{Z}/2)$ be the class of -1, then

$$\mathrm{Sq}^{2i}(uv) = \sum_{j=0}^{i} \mathrm{Sq}^{2j}(u)\mathrm{Sq}^{2i-2j}(v) + \tau \sum_{s=0}^{i-1} Sq^{2s+1}(u)Sq^{2i-2s-1}(v)$$

$$\begin{aligned}\mathrm{Sq}^{2i+1}(uv) &= \sum_{j=0}^{i} (\mathrm{Sq}^{2j+1}(u)\mathrm{Sq}^{2i-2j}(v) + \mathrm{Sq}^{2j}(u)\mathrm{Sq}^{2i-2j-1}(v)) \\ &+ \rho \sum_{s=0}^{i-1} \mathrm{Sq}^{2s+1}(u)\mathrm{Sq}^{2i-2s-1}(v)\end{aligned}$$

Definition 9.9: The dual motivic Steenrod algebra is defined by

$$\mathcal{A}_{**} = \mathrm{Hom}_{\mathbb{M}_p}(\mathcal{A}^{**}, \mathbb{M}_p)$$

The motivic Adem relations are given in [39]Theorem 10.2, Theorem 10.3. Now we focus on the mod 2 case.

Theorem 9.5: The mod 2 dual motivic Steenrod algebra $\mathcal{A}_{**}$ is a commutative $\mathbb{M}_2$-algebra given by

$$\mathcal{A}_{**}(k,\mathbb{F}_2) = \mathbb{M}_2[\tau_i, \xi_j \mid i \geq 0, j \geq 1]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_)\xi_{i+1})$$

where $\tau_i \in \mathcal{A}_{2^{i+1}-1,2^i-1}(k;\mathbb{F}_2)$ and $\xi_i \in \mathcal{A}_{2^{i+1}-1,2^i-1}(k;\mathbb{F}_2)$. Furthermore, it is a coalgebra with the coproduct Δ given by

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \tau_k$$

$$\Delta(\xi_j) = \sum_{k=0}^{j} \xi_{j-k}^{2^k} \otimes \xi_k$$

Then $(\mathbb{M}_2, \mathcal{A}_{**})$ forms a Hopf algebroid with

- the left unit $\eta_L : \mathbb{M}_2 \to \mathcal{A}_{**}$ is the natural inclusion
- the right unit $\eta_r : \mathbb{M}_2 \to \mathcal{A}_{**}$ is given by $\rho \mapsto \rho$ and $\tau \mapsto \tau + \rho\tau_0$.

9.3 Motivic Adams spectral sequences

In this section, we focus on the construction of the motivic spectral sequences following Dugger and Isasken[45].

For convenience, we write $H(S)$ for the ring spectrum $H\mathbb{F}_\ell(S) \in \mathcal{SH}(S)$, where $S = \mathrm{Spec}k$ for some field of characteristic 0. If the base scheme is given obviously, we just write H simply.

Definition 9.10: Let X be a set with bigraded objects $X = \{x_{a_i,b_i}\}_{i\in I}$. We say X is *motivically finite* if for any $i \in I$, there are only finitely many $j \in I$ such that $a_i \geq a_j$ and $2b_i - a_i \geq 2b_j - a_j$.

Definition 9.11: Let X be a spectrum in $\mathcal{SH}(S)$. An H-Adams resolution of X is an inversion sequences

$$X \simeq X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

such that

（1）the induced map $H_{**}(X_i) \to H_{**}(\bar{X}_i)$ is an H_{**}-split monomorphism.

（2）the cofiber $\bar{X}_i$ of $X_{i+1} \to X_i$ is a of the form

$$\bar{X}_i \simeq \bigvee_{i\in I} \Sigma^{a_i,b_i} H$$

where the set of indices $\{(a_i, b_i)\}_{i\in I}$ is motivically finite.

This definition is very similar to Definition 5.4. Therefore, the exact couple is given by the induced diagram for T-spectra

$$\begin{tikzcd}
0 \arrow[r] & H_{**}X \arrow[r] & H_{**}\bar{X}_0 \arrow[r] \arrow[d] & H_{**}\Sigma\bar{X}_1 \arrow[r] \arrow[d] & H_{**}\Sigma^2\bar{X}_2 \arrow[r] \arrow[d] & \cdots \\
 & & H_{**}\Sigma X_1 \arrow[ur] & H_{**}\Sigma^2 X_2 \arrow[ur] & \cdots &
\end{tikzcd}$$

Compared to Construction 5.5, we also have the notion of canonical motivic Adams resolution.

Construction 9.12 (Canonical motivic Adams resolution): Let $\bar{H}$ be the fiber of the unit map $\eta\colon \mathbf{1} \to H$. The *canonical H-Adams resolution* is given by assigning

$$X_i := \bar{H}^{\wedge i} \wedge X$$

and

$$X_{i+1} = \bar{H}^{\wedge i+1} \wedge X \xrightarrow{\eta\wedge\mathrm{id}} \mathbf{1} \wedge \bar{H}^{\wedge i} \wedge X \cong \bar{H}^{\wedge i} \wedge X = X_i$$

In particular, we have

$$\bar{X}_i = H \wedge \bar{H}^{\wedge i} \wedge X.$$

Recall that the canonical Adams resolution induces the normalized canonical resolution for a graded comodule over the Hopf algebroid. We also expect that its motivic version has such good homological-algebraic properties. Therefore, we need some extra assumptions for X.

Definition 9.13: [76] If $E \in \mathcal{SH}(S)$ is a homotopy colimit in terms of $\Sigma^{a,b}\mathbf{1}_S$, then E is said to be a *motivic cellular spectrum*.

Example 9.2: [76] The motivic spectra H, KGL and MGL are (stably cellular).

One of the key features of motivic cellular spectra is the Künneth isomorphisms[76]. Then we have the following proposition.

Proposition 9.1: If X is a cellular spectrum. Then the canonical motivic Adams resolution of X induces the normalized canonical resolution $C(H_{**}, H_{**}(X))$ (recall Construction 4.8). Therefore, the E_2-page of the motivic Adams spectral sequences induced by the canonical motivic Adams resolution of X with respect to H is given by

$$E_2^{s,(t,w)} \cong \mathrm{Ext}_{\mathcal{A}_{**}}^{s,(t,w)}(\mathbb{M}_2, H_{**}X)$$

The convergence problem of the motivic Adams spectral sequences is a complicated problem. In the case of algebraically closed fields, we have that

$$\mathrm{Ext}_{\mathcal{A}_{**}}^{*,(*,*)}(\mathbb{M}_\ell, \mathbb{M}_\ell) \Rightarrow \pi_{t,t}(\mathbf{1})_\ell^\wedge$$

This result refers to the joint work of Hu, Kriz and Ormsby[77-78].

9.4 Further directions

In this section, we will discuss some observations and further questions on motivic cohomology and motivic Adams spectral sequences.

Observation 9.14: Recall that the motivic cohomology spectra are constructed from motivic complexes and the motivic complexes also have normed structures[2]. *Can we write down the normed structure on Bloch's cycle complexes as we did in Section 4.2 for simplicial complexes?* We observe that both of these two kinds of complexes are constructed by $\Delta^\bullet$ and their difference can be shown in the following table Moreover, the product in Chow groups relies a moving lemma or deformation to the normal cone, which can be regarded as deformation parametrized by $\mathbb{A}^1$ in some sense. If we lift the product

	simplicial or cellular complexes	motivic cycle complexes
parameter interval	$\mathbb{R}^1$	$\mathbb{A}^1$
fundamental elements	simplices or cells	algebraic cycles

structure to Bloch's cycle complexes, we may need an extra structure to help us encode the $\mathbb{A}^1$-homotopy coherence, as the $\mathbb{E}_\infty$-coalgebra structure is a lifting of cup product to the level of complexes, see Remark 4.3. A possible strategy is to consider the $\mathbb{A}^1$-Milnor construction or more generally, $\mathbb{A}^1$-two sided bar construction. We hope that this idea can stimulate the development of intersection theory from the viewpoint of $\mathbb{A}^1$-homotopy coherence.

Observation 9.15: An $\mathbb{E}_\infty$-ring spectra has a monadic interpretation[79-80], while a normed motivic spectra also has monadic interpretation given by the free norm functor. Then we may have a rough correspondence. Recall that $\mathrm{NSym}(E) \simeq \bigvee_{n\geq 0} D_n^{mot}(E)$ (see

classical stable homotopy theory	motivic stable homotopy theory
$\mathbb{E}_\infty$-ring spectra	normed motivic spectra
$\mathbb{H}_\infty$-ring spectra	incoherent normed spectra

Example). Based on this observation, *can we describe Condition 7.8 of coherence in terms of motivic extended powers in analogous to the conditions in Construction 4.11 (notice their diagrams respectively)?* Moreover, *can we say that Proposition 4.3 is essentially the same as the propositions in Section 7.3?*

Observation 9.16: If we regard an incoherent normed spectrum as an motivic version of $\mathbb{H}_\infty$-ring spectrum, *can we design a generalized motivic Adams spectral sequence for it as we did in Chapter 5?* The key feature of the generalized Adams spectral sequence is that it can manipulate two sequences of filtration or towers and the mixed tower in spectral sequences[81]. In topology, we usually consider the mix of a skeletal filtration and an Adams resolution. Bruner used this technique to compute invariants for manifolds[82]. However, in algebra geometry, *what is the right notion for skeletal filtration that respect arithmetic structure? Can we use this type of spectral sequences to compute some arithmetic invariants?*

CONCLUSION

Our main contributions are:

• Providing an accessible introduction to methods of homotopy theory in algebraic geometry using cohomology operations as a guiding thread.

• Explaining how norms generalize notions from classical Galois theory into a higher-categorical setting.

• Exploring some connections and applications of norms and motivic power operations in motivic homotopy theory.

We hope that this paper will serve as a useful introduction and reference for readers who are interested in learning more about methods of homotopy theory in algebraic geometry. We also hope that it will stimulate further research on this fascinating topic.

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APPENDIX A THE ACYCLIC CARRIER THEOREM

A geometric complex K means a simplicial complex or a CW complex. By taking simplicial chain complex or cellular complex, we may identify a geometric complex with a chain complex $C_\bullet(K)$. Sometimes I may abbreviate $C_\bullet(K)$ by K and there is no harm.

The group $C_q(K)$ of *q-chains* is the free abelian group generated by the q-cells and the *boundary operator* is denoted by

$$\partial : C_q(K) \to C_{q-1}(K)$$

and $\partial \circ \partial = 0$. Suppose σ is a cell in K and τ is its face, then we may write $\tau < \sigma$. *The Kronecker index* $\mathrm{In}(c)$ of a 0-chain $c = \sum a_i x_i$ is defined to be $\sum a_i$, where x_i are 0-cells. We denote $Z_q(K)$, $B_q(K)$ and $H_q(K)$ the group of *q-cycles*, the group of *q-boundaries* and *the group of q homology classes*, respectively. We say two elements in $C_q(K)$ are *homologic* if they are different from a boundary.

Definition A.1 (Carrier): A *carrier* from complexes pair (K, L) to (K', L') is a function which assigns to each cell σ of K a non-trivial subcomplex $C(\sigma)$ of K such that $\sigma \in L$ implies $C(\sigma) \subset L'$ and if $\tau < \sigma$, then $C(\tau) \subset C(\sigma)$. A carrier is acyclic, if $C(\sigma)$ is acyclic for each cell $\sigma \in K$.

We say *a carrier carries a chain homotopy* h if for each cell σ, $h(\sigma) \in C(\sigma)$. Similarly, *a carrier carries a chain map* ϕ if $\phi(\sigma) \in C(\sigma)$.

Lemma A.1 (Acyclic carrier lemma): If C is an acyclic carrier $K \to K'$, then C carries a chain map ϕ. Moreover, if ϕ, ψ are two chain maps carried by C, then ϕ is homotopic to ψ.

Proof: We construct such ϕ inductively on the dimension. First, for each 0-cell $\sigma \in K$, we just let $\phi(\sigma) \in C(\sigma)$ with index 1, for example, a 0-cell in K'. Then we can extend it to a homomorphism from $C_0(K)$ to $C_0(K')$. Suppose we have already define $\phi : C_n(K) \to C_n(K')$ for $n < q$, we need to construct a homomorphism $C_{n+1}(K) \to C_{n+1}(K')$. Let σ be a q-cell, then $\partial\sigma = \sum a_i c_i$, where c_i is a face of q-cell. Since $\sum a_i c_i$ is a cycle, $\partial\sigma$ is also a cycle by the inductive hypothesis and $\sum a_i \phi(c_i) \in C(\sigma)$. Then there is a chain $\phi(\sigma) \in C(\sigma)$ such that $\partial\phi(\sigma) = \phi(\partial\sigma)$, because $C(\sigma)$ is acyclic, namely each cycle is a boundary.

Next, we prove any two chain map ϕ, ψ carried by C are homotopic. We first write down the diagram

$$\begin{tikzcd} C_{q+1}(K) \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_q(K) \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_{q-1}(K) \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] \\ C_{q+1}(K') \arrow[r, "\partial"] & C_q(K') \arrow[r, "\partial"] & C_{q-1}(K') \end{tikzcd}$$

We construct the chain homotopy inductively on the dimension of cells. Since $\mathrm{In}(\phi\sigma) - \mathrm{In}(\psi\sigma) = 0$, we can find an 1-chain $h(\sigma) \in C(\sigma)$ such that $\partial h(\sigma) = \phi\sigma - \psi\sigma$, due to the acyclicness. Now we suppose for each n-cell τ, $n < q$, we have such $h(\tau)$ to exhibit the chain homotopy at lower dimension, then we need to find $h(\sigma)$ such that $\partial h(\sigma) + h(\partial\sigma) = \phi\sigma - \psi\sigma$. Note that $\phi\sigma - \psi\sigma - h(\partial\sigma)$ is a cycle, because $\partial(\phi\sigma - \psi\sigma) = \phi(\partial\sigma) - \psi(\partial\sigma)$ and by inductive hypothesis

$$\phi\sigma - \psi\sigma - h(\partial\sigma) = \phi(\partial\sigma) - \psi(\partial\sigma) - h(\partial\sigma) = \partial h(\partial\partial\sigma) = 0$$

Since $C(\sigma)$ is acyclic, we can find $h(\sigma) \in C(\sigma)$ such that

$$\partial h(\sigma) = \phi\sigma - \psi\sigma - h(\partial\sigma)$$

which is what we need. ■

Definition A.2: An operation of degree i from K to K' is defined to be a sequence of homomorphisms

$$D_i : C_q(K) \to C_{q+i}(K')$$

for all q and commutes with boundary maps. Let O_i be the set of all operations of degree i and it forms an additive group naturally. We define the boundary operator $\omega : O_i \to O_{i-1}$ by

$$(\omega D_i)c = \partial D_i c + (-1)^{i+1} D_i \partial c \tag{A-1}$$

Clearly, $\omega\omega = 0$ and the *operator complex* is defined by $(\{O_i\}, \omega)$. Specifically, the operator complex from K to K' is denoted by $O(K, K')$.

Proposition A.1: Let W be a complex, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Ch}}(W, O(K, K')) \cong \mathrm{Hom}_{\mathrm{Ch}}(W \otimes C_\bullet(K), C_\bullet(K'))$$

Sketch proof: The isomorphism is given by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ch}}(W, O(K, K')) &\longrightarrow \mathrm{Hom}_{\mathrm{Ch}}(W \otimes C_\bullet(K), C_\bullet(K')) \\ f &\longmapsto [w_q \otimes c_q \mapsto f(w_q) \cdot c_q] \end{aligned}$$

The pattern is similar to

$$\mathrm{Hom}_{\mathrm{Mod}}(M \otimes N, P) \cong \mathrm{Hom}_{\mathrm{Mod}}(M, \mathrm{Hom}_{\mathrm{Mod}}(N, P))$$

in the category of modules. ■

Definition A.3: Let C be a carrier from K to K', the operator complex $O(C)$ associated to C is defined by

$$O(C)_q := \{D_q \in O_q \mid D_q(\sigma) \in C(\sigma), \forall \sigma \in C_q(K)\}$$

Lemma A.2: Let C be an acyclic carrier from K to K', then the associated operator complex $O(C)$ contains 0-cycle of index 1, and $O(C)$ is acyclic.

Sketch proof: The proof of this lemma is similar to the proof of Theorem A.1. ■

Example A.1 (The construction of the cup-i product): Let σ be an n-cell in X, let $\bar{\sigma}$ be the subcomplex containing all the faces of σ and it is acyclic. Let $C(\sigma) = \bar{\sigma} \otimes \bar{\sigma}$. By the definition, this forms a carrier from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$. Moreover, C is an acyclic carrier and T-invariant, namely, $TC(\sigma) \subseteq C(\sigma)$.

Since both D_0 and TD_0 are carried by C, by Lemma A.1, they are homotopic. We let D_1 be a chain homotopy from D_0 to TD_0 carried by C. More specifically, for any n-cell σ in X, $D_1(\sigma)$ is in $C(\sigma)$ such that

$$\partial D_1(\sigma) + D_1(\partial\sigma) = TD_0(\sigma) - D_0(\sigma)$$

or

$$\partial D_1(\sigma) = TD_0(\sigma) - D_0(\sigma) - D_1(\partial\sigma) \tag{A-2}$$

Similarly,

$$\partial TD_1(\sigma) = D_0(\sigma) - TD_0(\sigma) - TD_1(\partial\sigma) \tag{A-3}$$

Notice that $D_1 + TD_1$ is a homotopy of D_0 around a circuit back to itself (the addition between chain homotopies in Ch means the join of homotopies) and $D_1(\sigma) + TD_1(\sigma) \in C(\sigma)$ for each cell σ. Since both D_1 and the constant homotopy of D_0 are carried by the acyclic carrier C, apply Lemma A.1 again, and there is a chain homotopy D_2 from D_1+TD_1 to the constant homotopy of D_0 carried by C. Specifically, for any n-cell σ, there is an $n+2$-cell $D_2(\sigma)$ such that

$$\partial D_2(\sigma) = D_1(\sigma) + TD_1(\sigma) + D_2(\partial\sigma)$$

Now observe that $D_2 - TD_2$ is a homotopy from $D_1 + TD_1$ to itself. Similarly, $D_2 - TD_2$

is homotopic to the constant homotopy of $D_1 + TD_1$, namely, there exists D_3 such that

$$\partial D_3(\sigma) = D_2(\sigma) - TD_2(\sigma) - D_3(\partial\sigma)$$

Repeat the procedure inductively, then we have $\{D_n\}_{n=0}^{\infty}$ to exhibit higher homotopies. Note that D_n is an operation of degree n from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$. Recall Definition A.2 and the operator boundary A-1,

$$\omega D_i = D_{i-1} + (-1)^{i+1} TD_{i-1}$$

We let W be the subcomplex of $O(C)$ from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$ and W_n is freely generated by D_i and TD_i (since C is T-invariant, TD_i is also in $O(C)$).

Then according to Proposition A.1, the inclusion map $W \hookrightarrow O(C)$ uniquely determined a chain map

$$\begin{array}{rccc} \phi: & W \otimes C_\bullet(X) & \longrightarrow & C_\bullet(X) \otimes C_\bullet(X) \\ & D_i \otimes \sigma & \longmapsto & D_i(\sigma) \end{array} \tag{A-4}$$

Note that the diagram

$$\begin{tikzcd} W \otimes C_\bullet(X) \arrow[r, "\phi"] \arrow[d, "T\otimes\mathrm{id}"'] & C_\bullet(X) \otimes C_\bullet(X) \arrow[d, "T"] \\ W \otimes C_\bullet(X) \arrow[r, "\phi"] & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd} \tag{A-5}$$

For each $i \geq 0$, we define a product called *cup-i product* as follows. for $u \in C^p(X)$ and $v \in C^q(X)$, the cup product is defined by

$$u \smile_i v \cdot c = u \otimes v \cdot \phi(D_i \otimes c)$$

for $c \in C_{p+q-i}(X)$. By taking $\mathbb{F}_2$-coefficients, the i^{rmth} Steenrod square is given by $[u] \mapsto [u \smile_{\dim(u-i)} u]$[7].

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

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