硕士学位论文

椭圆上同调理论与 σ-定向

ELLIPTIC COHOMOLOGY THEORIES AND THE σ-ORIENTATION

研　　究　　生：梁嘉诚

指　导　教　师：朱一飞

南方科技大学

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学 位 申 请 人：梁嘉诚
指 导 教 师：朱一飞
学 科 名 称：基础数学
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Candidate:	Liang Jiacheng
Supervisor:	Zhu Yifei
Discipline:	Fundamental Mathematics
Date of Defence:	May, 2024
Affiliation:	Department of Mathematics
Degree-Conferring-Institution:	Southern University of Science and Technology

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摘要

Ando-Hopkins-Strickland 发现了从 $MU\langle 6\rangle$ 到椭圆上同调理论的一个特殊定向，称为 σ-定向。在本文中，我们将给出 σ-定向的代数拓扑和代数几何前置内容。此外，我们将引入形式群和形式群上的线丛的精确定义。另外我们会特别提到如何得到 $\mathbb{E}_\infty$-空间的 n-连通覆盖上的 $\mathbb{E}_\infty$-结构，这在一般参考文献中没有很好地描述。

在本文中，我们首先将以 fppf 层为基础建立一个良好的代数几何框架，该框架能扩展概形的范畴，使之包括形式概形和 p-可除群。我们会在这个框架内提供形式李群和形式李簇的精确定义，从而可以考虑它们上的拟凝聚层。通过对 Thom 谱的构造进行函子化，它会成为一个左 Quillen 伴随函子，这样我们可以很方便的得到所需的 $\mathbb{E}_\infty$-Thom 谱的结构。我们也会证明 Thom 同构是由对角余模结构产生的，并证明了将 Thom 谱函子与无限回路空间机制相结合会在 $\mathbb{E}_\infty$-空间的 n-联通覆盖上得到典范的 $\mathbb{E}_\infty$-结构。最后，我们证明了立方结构与 $MU\langle 6\rangle$-定向之间的对应关系，最终得到了本文最重要的定理：椭圆上同调理论有唯一的 $MU\langle 6\rangle$-定向，称为 σ-定向。

关键词： 同伦论；代数拓扑；定向；代数几何；算术几何；椭圆曲线

Abstract

Ando–Hopkins–Strickland found a special orientation from $MU\langle 6\rangle$ to elliptic cohomology theories, called σ-orientation. In this note we will give both topological and algebro-geometric settings of σ-orientation. Furthermore, we will introduce the precise definitions of formal groups, line bundles on a formal group, and particularly the n-connective cover of an $\mathbb{E}_\infty$-space, which seems not well-described in ordinary references.

Besides, we will establish a comprehensive framework for fppf sheaves that extends beyond the category of schemes to include formal schemes and p-divisible groups. We provide a precise definition of formal Lie groups and formal Lie varieties within this framework, allowing for the consideration of quasi-coherent sheaves on them. By functorializing the construction of Thom spectra and establishing it as a left Quillen adjoint, we obtain the desired $\mathbb{E}_\infty$-structure of a Thom spectrum. We show that the Thom isomorphism arises from a diagonal comodule structure and demonstrate that combining the Thom spectrum functor with the infinite loop space machine results in a canonical $\mathbb{E}_\infty$-structure on the n-connective cover of an $\mathbb{E}_\infty$-space. Finally, we prove the correspondence between cubical structures and $MU\langle 6\rangle$-orientations, culminating in the most important theorem of this paper.

Keywords: Homotopy theory, Algebraic topology, Oriention, Algebraic geometry, Arithmetic geometry, Elliptic curve

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Introduction

Given a homotopy commutative ring spectrum E, Quillen [32, 33] discovered a deep connection between (homotopy) complex orientation set $\mathrm{Or}_h(MU, E) = \mathrm{Hom}_{\mathrm{CAlg}(hSp)}(MU, E)$ and formal group laws over E_*, that is we have a map of sets

$$\{\textbf{Complex orientations over } E\} \to \{\textbf{Formal group laws over } E_*\}$$

which sends an orientation $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ to a formal group law $F(x, y) \in E^*[[x, y]]$. The group law structure is induced by H-structure map

$$E^*[[x]] \simeq E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*[[x, y]]$$

with $x \mapsto F(x, y)$.
Also, Quillen found that the orientation element $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ uniquely determines a ring spectrum map $MU \to E$. Formally speaking, that is a one-to-one correspondence

$$\mathrm{Or}_h(MU, E) \xrightarrow{\simeq} C^1\left(P_E; \mathcal{I}(0)\right)$$

which became the cornerstone of the chromatic homotopy theory. After that, Ando–Hopkins–Strickland discovered a correspondence

$$\mathrm{Or}_h(MU\langle 2k\rangle, E) \xrightarrow{\simeq} C^k\left(P_E; \mathcal{I}(0)\right)$$

between $MU\langle 2k\rangle$-orientations and n-cocycles when $k \leq 3$, which give an elegant connection between algebraic geometry and algebraic topology. By uniqueness of cubical structure on any line bundle of any abelian variety, we can endow a unique $MU\langle 6\rangle$-orientation to any elliptic cohomology theory [16–18].

In this paper, we will be investigating how to get this elegant correspondence connecting algebraic geometry with algebraic topology and also give a rigorous proof. The article is divided into two parts, algebraic geometry (chapter 1 and 2) and algebraic topology (chapter 3 and 4). In the chapter 1, we will introduce basic definitions and propositions about fppf sheaves and how to take the formal completion for a pointed fppf sheaf along its basepoint.

In the chapter 2, we can see when the given fppf sheaf is given by the Yoneda presheaf of an elliptic curve, the formal completion is a formal group. And then we will investigate how to use fppf sheaves to unify schemes, formal schemes and p-divisible groups. By that, we can easily generate the definition of quasi-coherent sheaves to that on a formal scheme.

For the topology part, in the chapter 3 we firstly introduce how to functorialize the construction of Thom spectra and establish it as a left Quillen adjoint in chapter 3. And then we prove that the Thom functor is compatible with operads, thus we can naturally obtain desired $\mathbb{E}_\infty$-rings like $MU\langle 2k\rangle$ and $MO\langle 2k\rangle$ respectively from a given $\mathbb{E}_\infty$-structure on $BU\langle 2k\rangle$ and $BO\langle 2k\rangle$ respectively if there exists. After that, we prove that there indeed exists a canonical $\mathbb{E}_\infty$-structure on both $BU\langle 2k\rangle$ and $BO\langle 2k\rangle$ by using infinite loop space machine and the connective cover of a spectrum to produce a natural $\mathbb{E}_\infty$-structure on the connective cover of an $\mathbb{E}_\infty$-space.

In the chapter 4, we firstly introduce how to define n-cocycles in any category with finite products. And then we introduce a technical concept called the even space, which can be easily controlled by the algebra structure on it. The fact that $BU\langle 2k\rangle$ is not an even space when $k > 3$ partly explains why the AHS correspondence fails when $k > 3$. Finally, we prove the main theorem after proving an algebro-geometric proposition that any line bundle on any abelian variety admits just a unique cubical structure.

1. Sites, fppf sheaves and completion

Grothendieck topology and topoi are an important algebro-geometric machinery for homotopists since lots of algebro-geometric objects like schemes, algebraic spaces, formal groups and p-divisible groups all can fully faithfully embed into the category of fppf sheaves.

Now, let me give an introduction to Grothendieck topology and sheavs on sites. A good reference for them is the stacks project [35].

1.1 Grothendieck topology

Definition 1.1. *[35] A site is defined by a category $\mathcal{C}$ and a collection* $\mathrm{Cov}(\mathcal{C}) \subset 2^{Mor(C)}$ *consisting of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, where I is a small set, referred to as coverings of $\mathcal{C}$. These coverings adhere to the following axioms [20]:*
(1) Any single isomorphism is a covering.
(2) The composition of coverings is still a covering, by which it means if $\{U_i \to U\}_{i\in I}$ are coverings for each i and $\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering, then $\{V_{ij} \to U\}_{i\in I, j\in J_i}$ is a covering.
(3) If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

Remark 1.2. *In axiom (3) we require the existence of the fibre products $U_i \times_U V$ for $i \in I$. Actually almost all sites appear in algebraic geometry have all pullbacks.*

Example 1.3. *[Big τ site]*
Let Sch be the category of schemes, and $\tau \in \{Zar, et, Smooth, fppf, fpqc\}$. Let T be a scheme. An τ covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that $T = \bigcup f_i(T_i)$ and each f_i is respectively
(1)open immersion;
(2)étale;
(3)smooth;
(4)flat, locally of finite presentation;
(5)flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}, j = 1, \ldots, n$ with $\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$.

We denote the corresponding site to be Sch_τ. Appearently we have

$$\mathrm{Cov}(Zar) \subset \mathrm{Cov}(et) \subset \mathrm{Cov}(Smooth) \subset \mathrm{Cov}(fppf) \subset \mathrm{Cov}(fpqc)$$

Definition 1.4 (Presheaf)**.** *Let $\mathcal{C}$ be a site. The presheaf category of $\mathcal{C}$ is just the functor category $Fun(C^{op}, Set)$. (Note $\mathcal{C}$ is not necessarily essentially small, so* $\mathrm{PSh}(\mathcal{C})$ *is not necessarily locally small)*

Definition 1.5 (Sheaf and topos)**.** *A topos is defined to be a category of sheaves on a site.*

Definition 1.6 (Sheafification)**.** *Let $\mathcal{J}_U$ be the category whose objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and whose morphisms are the refinements. It is worth mentioning that* $\{\mathrm{id}_U\} \in \mathrm{Ob}(\mathcal{J}_U)$ *and hence $\mathcal{J}_U$ is not empty. We define*

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{op}} H^0(\mathcal{U}, \mathcal{F})$$

We call $s\mathcal{F} = \mathcal{F}^{++}$ by the sheafification.

Actually, this colimit is a direct colimit because we have the following lemma, which implies different refinements between 2 covers induce the same morphism of H^0.

Warning: $\mathcal{J}_U$ is not necessarily a (essentially) small catgory, so not any presheaf on any site can be sheafificated. **Actually, there exists a presheaf on Sch_{fpqc} which admits no fpqc sheafification!**
However if we remove $fpqc$ and consider $\tau \in \{Zar, et, Smooth, fppf\}$, then all $\mathcal{J}_U$ in Sch_τ are essentially small and any presheaf in it can be sheafificated.

In the following context, we only consider the site whose $\mathcal{J}_U$ are essentially small and in which all pullbacks exists. (Actually, that holds for almost all sites in algebraic geometry except for fpqc ones.)

Proposition 1.7 (Adjoint)**.** *$PSh(\mathcal{C}) \rightleftarrows Sh(\mathcal{C})$ is a pair of adjunction.*

Proposition 1.8. *The sheafification functor $s : PSh(\mathcal{C}) \to Sh(\mathcal{C})$ preserves any finite limit (because the sheafification can be witten as a filtered colimit of underlying sets).*

Proposition 1.9 (Adjoint)**.** *We denote $PAb(\mathcal{C})$ and $Ab(\mathcal{C})$ to be the categories of abelian presheaves and abelian sheaves on $\mathcal{C}$ respectively. Then $PAb(\mathcal{C}) \rightleftarrows Ab(\mathcal{C})$ is still a pair of adjunction.*

Proposition 1.10. *$PAbSh(\mathcal{C})$ and $AbSh(\mathcal{C})$ are abelian categories.*

Proof: First, the kernel and cokernel $PAb(\mathcal{C})$ are created objectwise, so it is abelian. For the $AbSh(\mathcal{C})$, we need the following lemma.

□

Lemma 1.11. *Consider an adjoint pair of functors* $\mathcal{C} \underset{a}{\overset{b}{\rightleftarrows}} \mathcal{D}$, *where:*
(1) $\mathcal{C}$ *and* $\mathcal{D}$ *are additive categories, and* b *and* a *are additive functors.*
(2) $\mathcal{C}$ *is abelian, and* b *preserves finite limits.*
(3) $b \circ a \cong id_{\mathcal{D}}$.
Under these conditions, $\mathcal{D}$ *is also abelian.*

Remark 1.12. *By the Yoneda lemma, if a presheaf of abelian groups is representable by an object* H, *then* H *admits a natural abelian group structure.*

1.2 Localization of topoi

In 1.2 we give some useful propositions about topoi. [8, 12, 28]

Proposition 1.13. *Let* $\mathcal{C}$ *denote a site with a Grothendieck topology in which any Yoneda presheaf is a sheaf, and consider* U *is an object in* $\mathcal{C}$. *By defining a covering of* $\mathcal{C}/U$ *if it is a covering in* $\mathcal{C}$, *therefore we can view* $\mathcal{C}/U$ *as a site.*
Then we can identify $Sh(\mathcal{C}/U)$ *with* $Sh(\mathcal{C})/U$ *where we consider the latter* U *as the Yoneda sheaf of* U.

Proof: Actually we can give a natural categorical equivalence

$$Sh(\mathcal{C}/S^{'}) \rightleftarrows Sh(\mathcal{C}/S)_{\downarrow S'}$$

for any morhism $S^{'} \to S$ in $\mathcal{C}$.

For a sheaf Y in $Sh(\mathcal{C}/S^{'})$ let Y_S denote the functor on $(\mathcal{C}/S)^{\mathrm{op}}$ sending an S-object T to the set of pairs (ϵ, y), where $\epsilon : T \to S'$ is an S-morphism and $y \in Y(\epsilon : T \to S')$ is an element. There is a natural morphism of functors $f_Y : Y_S \to S'$ sending (ϵ, y) to ϵ.

Conversely, for a sheaf X in $Sh(\mathcal{C}/S)_{\downarrow S'}$, let $X_{S'}$ be the functor on $(\mathcal{C}/S')^{\mathrm{op}}$ whose value on $T \to S^{'}$ is the set of morphisms $T \to X$ in $Sh(\mathcal{C}/S)_{\downarrow S'}$. It is easy to show these two functorial constructions give an equivalence of categories.

□

Remark 1.14. *(1) In algebraic geometry, this equivalence tells us* $Sh(Sch_{/S})_\tau$ *is exactly the overcategory* $Sh(Sch)_\tau \downarrow h_S$.

(2) This equivalence still holds even if we replace U *by any sheaf* $\mathcal{F}$.

$$Sh(\mathcal{C}/\mathcal{F}) \rightleftarrows Sh(\mathcal{C})_{\downarrow \mathcal{F}}$$

Now let us focus on the big fppf site Sch_{fppf}. [13,19,25] Actually any representable functor is an fppf sheaf.

Proposition 1.15. *[30] Let S be a base scheme, X be an S-scheme, then the representable functor $Hom_S(-,X)$ is an fppf sheaf on $Sch_{/S}$.*

Now we introduce a useful equivalence. The intuition is that a sheaf is a gluing result.

Lemma 1.16. *Consider a site denoted by C, with $C' \subset C$ being a full subcategory satisfying the following conditions:*
(i) For each $U \in C$, there exists a covering $\{U_i \to U\}_{i\in I}$ of U where $U_i \in C'$ for all i.
(ii) If $\{U_i \to U\}$ is a covering of an object $U \in C'$, with $U_i \in C'$ for all i, and for any morphism $V \to U$ in C', the fiber products $V \times_U U_i$ belong to C'.
Under these conditions, a Grothendieck topology can be defined on C' such that a collection of morphisms $\{U_i \to U\}$ in C' is a covering if and only if it qualifies as a covering in C. Furthermore, the topos resulting from C' with this topology is equivalent to the topos derived from C.

Proposition 1.17. *For any $\tau \in \{Zar, et, Smooth, fppf\}$ (remove fpqc), $Aff \to Sch$ induces a natural equivalence of topoi*

$$Sh(Sch)_\tau \xrightarrow{\sim} Sh(Aff)_\tau$$

A τ-sheaf is determined by its values on affine schemes!

Corollary 1.18. *Note that any object in Aff_τ is compact, so the sheaf condition in it is a finite limit!*
So we get: In $Sh(Aff)_\tau$ any filtered colimit can be created in presheaf level, which commutes with any finite limit.

1.3 Completion of an fppf sheaf along a subsheaf

The most following definitions are from [26].

Definition 1.19. *Consider a monomorphism $Y \subset X$ of fppf sheaves on $Sch_{/S}$. We introduce $Inf_Y^k(X) \subset X$, a subsheaf defined as follows: its value on an object $T \to S$ is given by the condition that for $t \in X(T)$, $t \in Inf_Y^k(X)(T)$ if there exists an fppf covering $\{T_i \to T\}$ where each T_i corresponds to a closed subscheme T_i' defined by an ideal whose $k+1$ power is (0), such that $t_{T_i'} \in X(T_i')$ is contained in $Y(T_i')$.*

This definition is somewhat general, in most cases we only involve the completion of a scheme along a subscheme.

Example 1.20. *(1) If X and Y are S-schemes and $Y \to U \subset X$ is an immersion, then $Inf_Y^k(X) = Inf_Y^k(U) \simeq \mathrm{Spec}(\mathcal{O}_U/\mathcal{I}^{k+1})$ where $\mathcal{I} \subset \mathcal{O}_U$ is the according quasi-coherent ideal.*

(2) Let $Z \subset X$ be a closed immersion of S-schemes with the according quasi-coherent ideal $\mathcal{I}$, then the value of the sheaf $\hat{X}_Z = \varinjlim_k Inf_Z^k(X) = \varinjlim_k \mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^{k+1})$ on a S-scheme T equals $\{t \in X(T) | t^*(\mathcal{I})$ is locally nilpotent$\}$.

We mostly consider the case when Y is a given base point, i.e. $Y(T) = \{*\} = h_S(T)$ for any S-scheme T. In this case we get an endfunctor $\widehat{(-)} : Sh(Sch_{/S})^* \to Sh(Sch_{/S})^*$ by $(X, e) \mapsto (\varinjlim_k Inf_e^k(X), e)$, where $Sh(Sch_{/S})^*$ is denoted as the category of fppf sheaves over S with a basepoint.

We say an $X \in Sh(Sch_{/S})^*$ is complete (ind-infinitesimal in [26]) iff $\hat{X} = X$. It is easy to check we have a natural inclusion $\hat{X} \subset X$, and that $\hat{\hat{X}} \subset \hat{X}$ is a natural isomorphism. So any completion of a pointed fppf sheaf is complete. [5, 29, 34]

Proposition 1.21. *(a) The endfunctor $\widehat{(-)} : Sh(Sch_{/S})^* \to Sh(Sch_{/S})^*$ preserves finite limits. Let $CSh(Sch_{/S})^*$ be the category of complete pointed fppf sheaves, so $CSh(Sch_{/S})^*$ has finite limits, which are created in $Sh(Sch_{/S})^*$.*

(b) $CSh(Sch_{/S})^* \underset{\widehat{(-)}}{\overset{Forget}{\rightleftarrows}} Sh(Sch_{/S})^*$ is an adjoint pair.

(c) $CAb(Sch_{/S}) \underset{\widehat{(-)}}{\overset{Forget}{\rightleftarrows}} Ab(Sch_{/S})$ is an adjoint pair.

Proof: (a) It suffices to verify that $\widehat{(-)}$ preserves the final object and pullbacks. The case of the final object is straightforward. For a pullback $X \times_Z Y$, we aim to demonstrate that $\widehat{X \times_Z Y} \to \hat{X} \times_{\hat{Z}} \hat{Y}$ is naturally isomorphic. Evidently, this is a monomorphism of sheaves, and to establish it as an epimorphism is adequate. Consider $(f, g) \in \Gamma\left(T, \hat{X} \times_{\hat{Z}} \hat{Y}\right)$ where T is affine. Then, there exists a (finite) covering family $\{T_i \to T\}$ and nilpotent immersions of order $k, \bar{T}_i \longrightarrow T_i$ such that $f \mid \bar{T}_{\mathrm{i}} = 0$. Similarly, with an fppf covering family $\left\{T_j' \to T\right\}$ and nilpotent immersions of order k: $\bar{T}_j' \hookrightarrow T_j'$ corresponding to g.

And (b),(c) are direct corollaries of (a).

□

2. Formal groups and p-divisible groups

All (big) sheaves involved in 2 will always mean fppf sheaves.

2.1 Linearly topological rings

Before the introduction of formal groups, we need some preliminary knowledge of linear topological rings. In the category of linear topological rings ([37] chap 4), we have an excellent framework to deal with the completion.

Definition 2.1. *A filtration of ideals $\mathfrak{I}$ in R is a non-empty collection of ideals of R such that for any pair of ideals $I, J \in \mathfrak{I}$, there always exists a $I' \in \mathfrak{I}$ satisfying $I' \subset I \cap J$.*

Lemma 2.2. *Given a filtration of ideals $\mathfrak{I}$ in R, then*

(i) $\{a + I | a \in R, I \in \mathfrak{I}\}$ forms a topological basis in R, and we call it the (linear) topology induced by $\mathfrak{I}$.

(ii) The (linear) topology induced by $\mathfrak{I}$ makes R become a topological ring.

Proof: *Omitted.*

□

Definition 2.3. *A linearly topological ring R is a topological ring such that the topology induced by the filtrtion of open ideals in R is the same as its topology.*

Proposition 2.4. *A topological ring induced by a filtration of ideals is a linearly topological ring (note this is not a completely trivial statement).*

Example 2.5. *The linear topology induced by $\{I^n | n \geq 1\}$ for an ideal $I \in R$ is called I-adic topology. Note if $I = 0$, then this topology is discrete.*

Let us denote LRings to be the category of linearly topological rings with continuous ring maps.

Proposition 2.6.

(i) Given following morphisms in LRings

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

then the subring $a = \{b \in B | f(b) = g(b)\}$ with the linear topology by filtration

$$\{J = I \cap B | I \text{ open in } B\}$$

is the equalizer in LRings.

(ii) So we conclude LRings *has any limit.*

Now we start to introduce the completion of linearly topological rings

Definition 2.7. *Let R be a linearly topological ring. The completion of R is defined as the ring $\widehat{R} = \lim_{\leftarrow I} R/I$, where I ranges over the set of open ideals in R. There exists a natural mapping $R \to \widehat{R}$, and the composition $R \to \widehat{R} \to R/I$ is surjective, implying the existence of an ideal $\bar{I} \subset \widehat{R}$ such that $R/I = \widehat{R}/\bar{I}$. These ideals form a filtered system, allowing us to endow $\widehat{R}$ with a linear topology where they serve as a base for the neighborhoods of zero. It can be readily verified that $\widehat{\widehat{R}} = \widehat{R}$. A ring R is considered complete, if $R = \widehat{R}$. Hence, $\widehat{R}$ always represents a complete ring. We denote the category of complete rings as* FRings.

Remark 2.8. *It is important to notice that the completion $\widehat{R}$ from an I-adic topology is not always the same as the $I\widehat{R}$-adic topology on $\widehat{R}$! But it is the case when I is finitely generated, see [35] Algebra 96.3.*

Proposition 2.9.

(i) A linearly topological ring with the discrete topology is always complete.

(ii) Consider R, S, and A in the category of formal rings, denoted as FRings. *Suppose there are continuous homomorphisms $R \to S$ and $R \to A$, then it is evident that $\widehat{S \otimes_R A}$ can be identified as the pushout of S and A with respect to R in the category of formal rings, denoted as* FLings. *This observation leads us to the conclusion that the category* FRings *possesses finite colimits, as it contains the initial object ($\mathbb{Z}$ with the discrete topology) and all pushouts within its structure.*

(iii) Any limit in FRings *exists and could be created in* LRings.

Definition 2.10. *Let $(R, \mathfrak{m})$ be a local ring, we have a natural linear topology in R by the $\mathfrak{m}$-adic topology. So we get a functor:* LocalRings $\longrightarrow$ LRings. *In fact this functor is fully faithful because of the following lemma, and base on that we will always treat local rings as linearly topological rings.*

Lemma 2.11. *Let $A, B \in$* LRings. *Suppose their linear topology is induced by filtrations $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Let $f : A \longrightarrow B$ be a ring homomorphism. Then f is continuous if and only if $\forall J \in \mathfrak{B}$ there exists $I \in \mathfrak{A}$ such that $f(I) \subset J$.*

Proposition 2.12 ([35] Algebra chap 96,97)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, then* *(i)* $(\widehat{R}, \mathfrak{m}\widehat{R})$ *is still Noetherian local, and* $\widehat{\mathfrak{m}} = \lim_{\leftarrow n} \mathfrak{m}/\mathfrak{m}^n \simeq \mathfrak{m}\widehat{R}$ *.*

(ii) $(R, \mathfrak{m})$ *is regular if and only if* $(\widehat{R}, \widehat{\mathfrak{m}})$ *is.*

(iii) The topology on the completion $\widehat{R}$ *is the same as the* $\widehat{\mathfrak{m}}$*-adic topology on it, by 2.8.*

Remark 2.13. *(i) If a local ring* $(R, \mathfrak{m})$ *is not Noetherian, then* $(\widehat{R}, \mathfrak{m}\widehat{R})$ *is not necessarily local.*
(ii) When we consider the opposite category FRings^{op} *we usually write an object to be* $\text{Spf}(R)$ *instead of* R*.*

2.2 Formal completion of pointed k-schemes

Definition 2.14. *For a* k*-scheme* X *with a rational point* $e \in X(k)$ *we call it a pointed* k*-scheme. The formal completion* $\widehat{X}$ *of* X *"along"* e *is defined to be the complete linearly topological ring* $\text{Spf}(\widehat{\mathcal{O}_{X,e}})$*, the completion of* $\mathcal{O}_{X,e}$ *by* $\mathfrak{m}$*-adic topology. This induces a functor* $Sch_k^* \xrightarrow{\widehat{(-)}} k\text{-FRings}^{op}$ *where the left one is the category of pointed* k*-schemes.*

Lemma 2.15. *For a pointed* k*-scheme* (X, e)*, if* $\text{Spec}(A) \subset X$ *is an affine neighborhood of* e*. Let* $\mathfrak{m} \subset A$ *be the maximal ideal according to the closed point* e*, then by* $A/\mathfrak{m}^n = A_{\mathfrak{m}}/\mathfrak{m}^n$ *we have* $\widehat{X} = \widehat{\mathcal{O}_{X,e}} \cong \widehat{A}$ *where the right one is the* $\mathfrak{m}$*-adic completion of* A*.*

Theorem 2.16. *The functor* $\widehat{(-)}$ *preserves finite limits. Particularly, it preserves finite products and hence preserves (commutative) Monoid objects, (commutative) Group objects. So it takes group* k*-schemes to formal group* k*-schemes.*

Proof: Because any finite limit is a combination of pullbacks and terminal object, we only need to show that $\widehat{(-)}$ preserves pullbacks and terminal object. The terminal object is easy to check. For the case of pullbacks, given a pullback diagram in Sch_k^* (note that the pullback in it is the same as the ordinary fiber product of schemes),

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow p & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

we take neighbouhoods of basepoints $\text{Spec}(R) \subset Z, \text{Spec}(A) \subset X, \text{Spec}(B) \subset Y, \text{Spec}(A \otimes_R B) \subset X \times_Z Y$. We write corresponding maximal ideals of basepoints $e_X, e_Y, e_{X\times_Z Y}$ to be $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}$ respectively. It is easy to see the basepoint $e_{X\times_Z Y}$ corresponds to $A \otimes_R B \to k \otimes_R k = k \otimes_k k$, so actually $\mathfrak{m} = \mathfrak{m}_1 \otimes_R B + A \otimes_R \mathfrak{m}_2$. By the lemma above and the description of pushout of formal rings, the natural

$$\widehat{A}\widehat{\otimes}_{\widehat{R}}\widehat{B} \to \widehat{A \otimes_R B}$$

is isomorphic, then so is

$$\widehat{\mathcal{O}_{Y,e}}\widehat{\otimes}_{\widehat{\mathcal{O}_{Z,e}}}\widehat{\mathcal{O}_{X,e}} \to \widehat{\mathcal{O}_{X\times_Z Y,e}}$$

□

It is easy to check following 2 useful propositions.

Proposition 2.17. *(i) If $k \to F$ is a field extension, then for any $(X, e) \in Sch_k^*$ we have natural isomorphism $\widehat{\mathcal{O}_{X,e}}\widehat{\otimes}_k F \to \widehat{\mathcal{O}_{X_F,e_F}}$.*
(ii) If k is a field of $char(k) = p > 0$ and $(X, e) \in Sch_k^*$, then $\widehat{\mathcal{O}_{X,e}} \to \widehat{\mathcal{O}_{X,e}}$ induced by absolute Frobenius $F : X \to X$ is absolute Frobenius on $\widehat{\mathcal{O}_{X,e}}$, and $\widehat{\mathcal{O}_{X,e}}\widehat{\otimes}_{k,Frob}k \to \widehat{\mathcal{O}_{X,e}}$ induced by relative Frobenius $F : X \to X^{(p/k)}$ is the formal relative Frobenius on $\widehat{\mathcal{O}_{X,e}}$.

By Cohen structure theorem, we will see that a smooth group k-scheme of dim n can induce a formal group over k of dim n.

Theorem 2.18. *If G is a smooth group k-scheme of dim n, then $\widehat{G}$ is a formal group over k of dim n.*

Proof: We know "smooth" implies "regular", so $\widehat{\mathcal{O}_{G,e}}$ is a complete regular local ring of dim n. Then by the theorem above we win.

□

2.3 Formal Lie varieties

We have known that the equivalence of topoi $Sh(Sch)_{fppf} \longrightarrow Sh(Aff)_{fppf}$, so we will be free to exchange things from each other.

It is obvious that $\hat{\chi} \subset Sh(Aff)_{fppf}$. Actually $\hat{\chi}$ is the category of "formal schemes" in Strickland's sense [37], which equals $(Pro - Ring)^{op}$ or $Ind - Aff$. And we have fully faithful embeddings

$$FRing \to \hat{\chi}$$

by sending R to $Spf(R) = \varinjlim_{I\ open} \operatorname{Spec} R/I$ and natural inclusion

$$\hat{\chi} \to Sh(Aff)_{fppf}$$

Definition 2.19. *Let $X \in CSh(Sch_{/S})^*$, we call it a pointed formal Lie variety iff zariski locally on S, the F is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_n]])$ as pointed fppf sheaves for some $n \geq 0$.*

Proposition 2.20. *[26] Let $X \in CSh(Sch_{/S})^*$, the following are equivalent*
(1) X is a pointed formal Lie variety.
(2) Zariski locally on S, the X is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_n]])$ as sheaves (not necessarily pointed) for some $n \geq 0$.
(3)
(a) The $\mathrm{Inf}^k(X)$ is representable for all $k \geq 0$.
(b) The $\omega_X = e^*(\Omega_{\mathrm{Inf}^1(X)/S}) = e^*(\Omega_{\mathrm{Inf}^k(X)/S})$ is a finite locally free sheaf on S.
(c) Denoting by $gr_*^{inf}(X)$ the graded $\mathcal{O}_S$-algebra $\bigoplus_{k\geq 0} \mathcal{I}_k^k$, such that $gr_i^{\mathrm{inf}}(X) = gr_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$. We have an isomorphism $Sym_S(\omega_X)_* \xrightarrow{\sim} gr_*^{inf}(X)$ induced by the canonical mapping $\omega_X \xrightarrow{\sim} gr_1^{\mathrm{inf}}(X)$.

Proposition 2.21. *Let $X \to S$ be a smooth S-scheme with a base point $e : S \to X \in X(S)$, then $\hat{X}$ is a formal Lie variety.*

Proof: Select an affine open set U containing s within S. Choose another affine open set V in $f^{-1}(U)$ that includes x. Subsequently, select an affine open set U' in $e^{-1}(V)$ that contains s. It is noteworthy that $V' = f^{-1}(U') \cap V$ is affine due to its representation as the fiber product $V' = U' \times_U V$. Consequently, the maps $f : U' \to V'$ and $e : V' \to U'$ are identified as separated, smooth, and a section (specifically, a closed immersion). This leads to the result that $\hat{X}_{V'} = \hat{U'}_{V'}$. The proposition can be readily derived from the subsequent lemma.

□

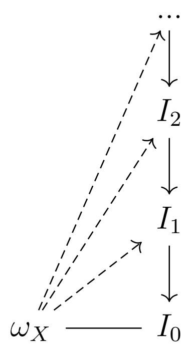
Lemma 2.22. *[35](Algebra 139.4) Consider a smooth ring morphism $\varphi : R \to S$ with a left inverse $\sigma : S \to R$ where $I = \mathrm{Ker}(\sigma)$. Then the following results hold:*
(1) The quotient module I/I^2 is a finitely generated projective R-module.
(2) If I/I^2 is a free R-module, then there is an isomorphism between the completion $S^\wedge$ with respect to the I-adic topology and $R[[t_1, \ldots, t_d]]$ as R-linear topological rings.
Proof: *Utilizing the exact sequence of Kahler differentials for $R \to S \to R$, we obtain $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since the module $\Omega_{S/R}$ is finitely generated projective over S due to the smoothness of the morphism, we establish the validity of (1).*
In the case where I/I^2 is free, consider the induced map $\Psi_n : P/J^n \to S/I^n$ for quotient rings. As $S/I^2 = \varphi(R) \oplus I/I^2$, the map Ψ_2 is an isomorphism. Let $\sigma_2 : S/I^2 \to P/J^2$ be the inverse of Ψ_2. By induction, we show the existence of an inverse $\sigma_n : S/I^n \to P/J^n$ for all $n > 2$ by the fact that S is formal smooth over R. This concludes the proof of the lemma.

□

Actually, any formal Lie variety on an affine base can be from the completion of a pointed smooth scheme, as the following.

Proposition 2.23. *Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety. If $S = \mathrm{Spec}(R)$ is affine, then we have a (non-canonical) isomorphism $X \to Spf(\widehat{Sym}_S(\omega_X))$ as pointed sheaves.*

Proof: Let $I_k \subset \mathcal{O}_X$ be the quasi coherent ideal according $S \to inf^k X$, and $I \to \omega_X \to 0$ be the projection of R-modules. Then we can lift following arrows one-by-one



Hence we get a sequence of isomorphisms

$$
\begin{array}{ccc}
\cdots & \xrightarrow{\simeq} & \cdots \\
\downarrow & & \downarrow \\
Sym(\omega_X)/(\omega_X^{k+1}) & \xrightarrow{\simeq} & \mathcal{O}_{inf^k X} \\
\downarrow & & \downarrow \\
Sym(\omega_X)/(\omega_X^{k}) & \xrightarrow{\simeq} & \mathcal{O}_{inf^{k-1} X} \\
\downarrow & & \downarrow \\
\cdots & \xrightarrow{\simeq} & \cdots
\end{array}
$$

which induces an isomorphism $X \to Spf(\widehat{Sym}_S(\omega_X))$.

□

Remark 2.24. *It is worth noting this theorem is based on the fact that a finite locally free sheaf on S is a projective object in $Qcoh(S)$ if S is affine.*

Corollary 2.25. *Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety (S here is not necessarily assumed to be affine), then X is a formally smooth fppf sheaf, which means $X(Spec(A)) \to X(Spec(A/I))$ is surjective for any $A \to A/I$ over S with a square-zero ideal I.*

Proof: To show that $X(Spec(A)) \to X(Spec(A/I))$ is surjective, we can assume $S = Spec(A)$ is affine. Then it is from the completion of a pointed smooth S-scheme $Y =$

$Spec(Sym_S(\omega_X))$ by the proposition above. So it suffices to show the following is a pullback diagram of sets.

$$\begin{array}{ccc} \hat{Y}(Spec(A)) & \longrightarrow & \hat{Y}(Spec(A/I)) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ Y(Spec(A)) & \longrightarrow & Y(Spec(A/I)) \end{array}$$

Let $u \in \hat{Y}(Spec(A/I))$, then $u \in Y(Spec(A/I))$ is from an element $v \in Y(Spec(A))$ by the formal smoothness of Y. Now we claim $v \in \hat{Y}(Spec(A))$.

There exists $n \geq 1$ such that $u : Spec(A/I) \to Y$ factors through $u : Spec(A/I) \to inf^k(Y)$ since $u \in \hat{Y}(Spec(A/I))$, then $u|Spec(A/I+J) = 0$ for some nilpotent ideal J. So $v \in \hat{Y}(Spec(A))$ by the fact $I + J$ is still nilpotent.

□

2.4 Formal Lie groups

Definition 2.26. *A formal Lie group is an abelian sheaf $X \in Ab(Sch_{/S})$ whose underlying pointed sheaf is a formal Lie variety.*

We more care about 1-dim formal Lie groups, which are called by "formal group" in most references. In 2.4 we will show that formal groups over an affine basis are equivalent to graded formal group laws on an even weakly periodic graded ring.

Definition 2.27 (EWP)**.** *A graded ring R_* is called EWP(even weakly periodic) iff it satisfies following conditions*
(a) $R_2 \otimes_{R_0} R_{-2} \to R_0$ is isomorphic;
(b) $R_1 = 0$.

Proposition 2.28. *From the definition, for an EWP ring R_* we immediately get*
(1) $R_2 \otimes_{R_0} R_n \to R_{n+2}$ is isomorphic for any $n \in \mathbb{Z}$.
(2) $R_{odd} = 0$.
(3) $R_2 \in Pic(R_0)$ with $(R_2)^{\otimes -1} = R_{-2}$.

Proof: We can directly check $R_* \simeq R[x^{\pm 1}], |x| = 2$ zariski locally on $Spec(R)$ and check these properties zariski locally.

□

Example 2.29. *Let R be a ring, and $L \in Pic(R)$. Then $Sym_R(L^{\pm 1})_* = \bigoplus_{i \in \mathbb{Z}} L^{\otimes i}$ is an EWP ring.*

Now let us calculate the data of a formal group.

Lemma 2.30. *For any $M, N \in Qcoh(S)$, we have*

$$Hom_{Sh(S)^*}(Spf(\widehat{Sym}_S(M)), Spf(\widehat{Sym}_S(N))) = \prod_{i=1}^{+\infty} Hom_{\mathcal{O}_S-Mod}(N, Sym_i(M))$$

Proof: Directly calculate by 1.21.

□

Corollary 2.31. *Let $X, Y \in CSh(Sch_{/S})^*$ be a pointed formal Lie variety of $dim = 1$ over an affine base $S = \mathrm{Spec}(R)$, then*
(1) $Hom_{Sh(S)^*}(X \times X, X) \simeq \prod_{(i,j)|i+j\geq 1} Hom_{\mathcal{O}_S-Mod}(\omega_X, \omega_X^{i+j}) = \prod_{(i,j)|i+j\geq 1} \omega_X^{i+j-1}$ where $Sh(S)^*$ denotes pointed fppf sheaves over S. So any $F \in Hom_{Sh(S)^*}(X \times X, X)$ corresponds an element $F(x,y) \in R_*[[x,y]], |x| = |y| = -2$ where $R_* = Sym_R(\omega_X^{\pm 1})_*$.
If it satisfies the associated (commutative) law then it coincides with a graded formal (commutative) group law on the EWP ring $Sym_R(\omega_X^{\pm 1})_*$ or on $Sym_R(\omega_X)_*$.
(2) We have $Hom_{Sh(S)^*}(X, Y) = \prod_{i=1}^{+\infty} Hom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X^i)$ and

$$Isom_{Sh(S)^*}(X, Y) = Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=2}^{+\infty} Hom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X^i) =$$

$$Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=2}^{+\infty} \omega_X^{i-1} = Isom_{\mathcal{O}_S-Mod}(\omega_Y, \omega_X) \times \prod_{i=1}^{+\infty} \omega_X^{i}$$

Theorem 2.32. *Let $p : \mathcal{M}_{FGL_s(EWP)} \to Aff$ be the moduli stack of formal group laws on EWP rings whose objects are pairs (E_*, F) with F a formal group law on E_*, whose morphisms are (oppositely) pairs (ϕ, f) with $\phi : E_{1*} \to E_{2*}$ a morphism of graded rings and $f : \phi^* F_1 \xrightarrow{\simeq} F_2$ an isomorphism of formal group laws on E_{2*}. And $p(E_*, F) = Spec(E_0)$.*

Then The construction in last corollary actually gives an equivalence of moduli stacks

$$\begin{array}{ccc} \mathcal{M}_{FG} & \xrightarrow{\sim} & \mathcal{M}_{FGL_s(EWP)} \\ & \searrow \quad \swarrow & \\ & Aff & \end{array}$$

Remark 2.33. *This theorem provides a natural **graded** structure to a 1-dim formal group over an affine base, which is important when we consider the Landweber exact theorem.*

2.5 Barsotti-Tate groups (p-divisible groups)

Definition 2.34. *A Barsotti-Tate group over a base scheme S is an fppf abelian sheaf G in $Ab(Sch/S)$ satisfying the following conditions:*
(1) $\varinjlim_n G[p^n] \to G$ is naturally isomorphic. (p-torsion)
(2) $G \xrightarrow{p} G$ is an epimorphism of abelian sheaves. (p-divisible).
(3) $G[p^n]$ is representable by a scheme finite locally free over S for any $n \geq 1$.

Lemma 2.35. *Let G be an abelian fppf sheaf over S satisfying (1) and (2). Then for any $m, n \geq 0$ we have a short exact sequence of abelian sheaf*

$$0 \to G[p^n] \to G[p^{m+n}] \xrightarrow{p^n} G[p^m] \to 0$$

So by fppf descent theory of finite group schemes [10], the (3) in the definition can be replaced by the following
(3)$'$ $G[p]$ is representable by a scheme finite locally free over S.

Proposition 2.36. *If $G_0 \to G_1 \to ... \to G_n \to ...$ be an sequence of morphisms of abelian sheaves over S satisfying the following conditions:*
(1) G_i is a scheme finite locally free of degree p^{hi} over S, where $h \geq 0$ is a number independent on i;
(2) $G_n \to G_{n+1}$ is a closed immersion for any $n \geq 0$;
(3) $0 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1}$ is exact for any $n \geq 0$,
then $G = \varinjlim G_n$ is a Barsotti-Tate group over S, and $G[p^n] = G_n$ for every $n \geq 0$.

Proof: The condition (3) implies $G_{n+1}[p^n] = G_n$, by induction we get $G_{n+m}[p^n] = G_n$, and hence $G[p^n] = G_n$ and $G = \varinjlim_n G[p^n]$.
On the other hand we get a new exact sequence $0 \to G_n \to G_{m+n} \xrightarrow{p^n} G_m$. We claim $G_{m+n} \xrightarrow{p^n} G_m$ is epimorphic. By fppf descent theory, we have a factorization

$$\begin{array}{ccc} G_{m+n} & \xrightarrow{\quad p^n \quad} & G_m \\ & \searrow \qquad \nearrow_{i} & \\ & G_{m+n}/G_n & \end{array}$$

where G_{m+n}/G_n is a finite locally free group of degree p^{mi} over S and i is a monomorphism. However, any proper monomorphism is a closed immersion. So i is a closed immersion between finite locally free schemes of the same degree over S, and hence an isomorphism.
Let $n = 1$, we get $G_{m+1} \xrightarrow{p} G_m \to 0$. Therefore take the direct colimit about m we get

$G \xrightarrow{p} G \to 0.$

□

Remark 2.37. *Actually, the proposition above is a local definition of the Barsotti-Tate group. Because for any BT group G and $s \in S$, $G[p]_s = G_s[p]$ is annihilated by p, which implies its rank must be p^{h_s} for some number h_s by the theory of algebraic groups.*

3. Thom spectrum functor and infinite loop space machine

Before getting into the σ-orientation we introduce two important topological settings which are infinite loop space machine and Thom spectrum functor respectively.
Here we only consider Thom spectra from a map into a classifying space of some topological **group**, from which Thom spectra admit more useful properties compared with those from a topological monoid.

Definition 3.1 ([11] Thom spectrum functor)**.** *Let $(f : X \to BO) \in Top_{\downarrow BO}$, then the standard filtration $X_V = f^{-1}(BO(V))$ gives a Thom prespectrum*

$$M_p(f)(V) = Th(E(X_V) \to X_V) = E(X_V)_+ \wedge_{O(V)_+} S^V$$

The spectrification $M(f)$ of $M_p(f)$ is called the Thom spectrum corresponding f.

Remark 3.2. *(i) Actually, any filtration $\varinjlim_{V\subset\mathbb{R}^\infty} F_V X = X$ where $F_V X$ is closed subset in X so that $F_V X \subset X_V$ gives the same [11] Thom spectrum (though not the same prespectra). (ii) For $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$, the construction above also applies.*

3.1 Properties of the Thom spectrum functor

For any spectrum $E \in Sp$ and any $V \subset \mathbb{R}^\infty$, $\Omega^\infty E$ admits a right $O(V)$-action since $\Omega^\infty E = E_0 = \Omega^V E_V = F(S^V, E_V)$. These actions are coherent between different V, so we actually get a right O-action on $\Omega^\infty E$. [10, 31, 39]

In the following content we always assume $G = Sp(\infty), U(\infty), SU(\infty), O(\infty)$ or $SO(\infty)$.

Theorem 3.3. *The Thom spectrum functor induces a continuous adjoint pair*

$$Top_{\downarrow BG} \underset{EG\times_G\Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$$

Given a map $(f : X \to BG) \in \mathcal{U}/BG$ and $E \in Sp$, then

$$\mathrm{Hom}_{Sp}(Mf, E) = \mathrm{Hom}_{\mathcal{U}[G]}(f^*EG, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

Proof: Let us denote $\mathcal{U}$ and $\mathcal{S}$ to be the categories of unbased Topological spaces, and spectra respectively. First we have

$$\mathrm{Hom}_{\mathcal{S}}(MX, E) = \mathrm{Hom}_{\mathcal{S}}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccc} EX_V & \longrightarrow & B(*,G(V),G(V)) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & B(*,G(V),*) \end{array} \qquad \begin{array}{ccc} Z_V & \longrightarrow & B(*,G(V),G) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & B(*,G(V),*) \end{array}$$

then

$\lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{G(V)} S^V, E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}_*[G_{V+}]}(EX_{V+}, \Omega^V E_V)$

$= \lim_V \mathrm{Hom}_{\mathcal{U}[G_V]}(EX_V, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(EX_V \times_{G_V} G, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_G(p^* X, \Omega^\infty E)$

Since equivariant maps from a principle G-bundle to a G-space are equivalent to the following sections, we can conclude

$$\mathrm{Hom}_G(p^* X, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, p^* X \times_G \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

□

Proposition 3.4. *This adjunction* $Top_{\downarrow BG} \underset{EG\times_G\Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$ *is actually a Quillen adjunction since* $M(S^{n-1} \to D^n)$ *is a cell pair of spectra and* $M(D^n \times 0 \to D^n \times I)$ *is a weak equivalent cell pair for those morphisms over* BG.

Proposition 3.5. *Let* $f : X \to BG$ *be a map and* A *a space. Let* g *be the composite* $X \times A \to X \to BG$*, where the first map is the projection away from* A*. Then* $T(g) = A_+ \wedge T(f)$*, which implies Thom spectrum functor preserves tensors, and hence is a topological Quillen functor.*

Proposition 3.6. *Thom spectrum functor* $T(-)$ *preserves weak equivalences. Any Thom spectrum* $T(f)$ *from a map* $F : X \to BG$ *is* (-1)*-connective.*

3.2 Monads and Thom spectrum functor

Proposition 3.7. *Let* $\mathcal{V}_1, \mathcal{V}_2$ *be two real universes.*
(i) Given maps $B \to \mathcal{L}(V_1, V_2)$ *and* $f : X \to BO(\mathcal{V}_1)$*, denote* g *to be the composition* $B \times X \to B \times BO(\mathcal{V}_1) \to BO(\mathcal{V}_2)$*. Then we have the natural isomorphism* $T(g) \cong B \ltimes T(f)$*.*
(ii) Given maps $f : X \to BO(\mathcal{V}_1)$ *and* $g : Y \to BO(\mathcal{V}_2)$*, denote* $f \times g$ *to be the composition* $X \times Y \to BO(\mathcal{V}_1) \times BO(\mathcal{V}_2) \to BO(\mathcal{V}_1 \oplus \mathcal{V}_2)$*. Then* $T(f \times g) \cong T(f) \barwedge T(g)$*.*

Proposition 3.8. *Let $\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{\infty\times n}, \mathbb{R}^\infty)$, then for any map $f : X \to BO$ we have*

$$T(g) = \bigvee_{n\geq 0} \mathcal{L}(n) \times_{\Sigma_n} T(f)^{\bar{\wedge} n}$$

where g is the composition $\coprod_{n\geq 0} \mathcal{L}(n) \times_{\Sigma_n} X^n \to \coprod_{n\geq 0} \mathcal{L}(n) \times_{\Sigma_n} BO^n \to BO$.

Now we introduce a quite useful lemma [6] which tells how to get the adjoint functor between monadic algebra categories.

Lemma 3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be topological powered and copowered categories, and $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ and $\mathbb{B} : \mathcal{D} \to \mathcal{D}$ be continuous monads. Given a continuous functor $F : \mathcal{C} \to \mathcal{D}$ which is coherent with the monad structure, therefore it yields a functor $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$.*

If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint functor preserving copowers, and the monads $\mathbb{A}$ and $\mathbb{B}$ preserve reflexive coequalizers, then $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$ is still a left adjoint functor preserving copowers.

Corollary 3.10. *Thom spectrum functor induces topological Quillen adjoint pairs*

$$Top[\mathcal{L}(1)]_{\downarrow BO} \rightleftharpoons Sp[\mathcal{L}(1)] \quad \text{and} \quad Top[E_\infty]_{\downarrow BO} \rightleftharpoons Sp[E_\infty]$$

where the $\mathcal{L}(1)$-spectrum is the $\mathbb{L}$-spectrum in EKMM [9] sense.

Remark 3.11. *This section 3.2 also applies to $G = U(\infty)$ or $G = Sp(\infty)$ if we replace real isometries operad by complex or symplectic isometries operads.*

3.3 Diagonal and Thom isomorphism

Definition 3.12 (coaction)**.** *For any map $f : X \to BG$, the diagonal induces a coaction $X \to X \times X$ in $Top_{\downarrow BG}$, where $X \times X \to BG$ is the projection of the second variable. It gives a natural coaction on Thom spectra: $Mf \to X_+ \wedge Mf$.*

Definition 3.13 (Thom morphism [11])**.** *With the same hypothesis above, given a homotopy commutative phantom ring spectrum (a commutative monoid in $Ho(Sp)/phantoms$) E and a morphism of spectra $Mf \to E$ we have a natural morphism $E \wedge Mf \to E \wedge X_+ \wedge Mf \to E \wedge X_+ \wedge E \to E \wedge X_+$ in $Ho(Sp)/phantoms$. It induces a natural homological morphism $\phi_f : E_*(Mf) \to E_*(X)$.*

Under certain condition ϕ_f will be an isomorphism, which is called Thom isomorphism.

Theorem 3.14 (Thom isomorphism)**.** *Let $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$ or $Spin(\infty)$. Let E be a homotopy commutative ring (phantom) spectrum.*

(i) Given a phantom ring spectrum morphism $MG \to E$, then for any map $X \to BG$ the Thom morphism $E_*(Mf) \to E_*(X)$ is an isomorphism.
Moreover, if X is E_∞ and f is an E_∞ map, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.
(ii) Given an E_∞ space X and an E_∞ map $f : X \to BG$. Let $Mf \to E$ be a phantom ring spectrum morphism. If X is 0-connected, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.

Example 3.15. *Let $MO \to H\mathbb{Z}/2$ and $MU \to H\mathbb{Z}$ be ring spectrum morphisms from the 0-th postnikov tower. Then we have natural Thom isomorphisms $H_*(MO;\mathbb{Z}/2) \to H_*(BO;\mathbb{Z}/2)$ and $H_*(MU) \to H_*(BU)$.*

3.4 Infinite loop space machine

Now we turn to the infinite loop space machine, which is an important technique in stable homotopy theory.

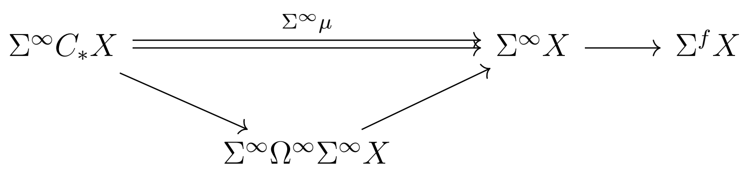
Definition 3.16. *(1). A commutative H-space space X i.e. a commutative monoid in $Ho(Top)$ is called group-like iff the monoid $\pi_0(X)$ is a group.*
(2). We define group-like E_∞-spaces as infinite loop spaces.
(3). Let $X \to Y$ be an H-map between commutative H-spaces, we call it the completion map of X iff $\pi_0(Y)$ is a group and $H_*(X)[(\pi_0 X)^{-1}] \to H_*(Y)$ is isomorphic.

Now let me introduce the existence and uniqueness of additive infinite loop space machine. [27, 30, 36]

Theorem 3.17 (Additive infinite loop space machine [1])**.** *Let C be a cofibrant unital E_∞ operad in Top and $f : C_* \to \Omega^\infty\Sigma^\infty$ be a morphism of monads on Top_*. Then the Quillen pair (Σ^f, Ω^f) induces a equivalence of categories if we restrict it to the following Top-enriched subcategories (so actually an equivalence of ∞-categories)*

$$\text{group-like } Ho(E_\infty\text{-spaces}) \rightleftharpoons (-1)\text{-connective } Ho(Sp)$$

where $\Sigma^f(-) = \Sigma^\infty \otimes_{C_*} (-)$ is the coequalizer of the following diagram in Sp



And $\Omega^f X = \Omega^\infty X$ is endowed with the C_*-action $C_*\Omega^\infty X \to \Omega^\infty\Sigma^\infty\Omega^\infty X \to \Omega^\infty X$.

Theorem 3.18 (Uniqueness of additive infinite loop space machine [23])**.** *We define an (additive) infinite loop space machine to be an adjoint pair* (F, G)

$$Ho(E_\infty\text{-spaces}) \underset{G}{\overset{F}{\rightleftarrows}} (-1)\text{-connective } Ho(Sp)$$

such that
(1) The composition (-1)*-connective* $Ho(Sp) \overset{G}{\to} Ho(E_\infty\text{-spaces}) \to CMon(Ho(Top_*))$ *is equivalent to* Ω^∞*;*
(2) For any $X \in Ho(E_\infty\text{-spaces})$*,* $X \to GF(X)$ *is a group completion, which means* $\pi_0 GF(X)$ *is a group and* $H_*(X)[(\pi_0 X)^{-1}] \to H_* GF(X)$ *is isomorphic.*

Now, if (F_1, G_1) *and* (F_2, G_2) *are two infinite loop space machines, then there exists a natural equivalence between* F_1 *and* F_2*.*

Remark 3.19. *The existence of an additive infinite loop space machine* (F, G) *implies that for any group-like* E_∞*-space* X*, the induced pointed* H*-space is actually an* H*-group because* $X \cong \Omega^\infty FX$ *in* $CMon(Ho(Top_*))$ *and* $\Omega^\infty FX$ *is a pointed* H*-group.*

Furthermore, beyond the additive, there exists multiplicative infinite loop space machine as the following constructed by May:

Theorem 3.20 ([24] Multiplicative infinite loop space machine)**.** *Let* K *be the Steiner* E_∞ *operad. We can construct a explicit morphism of monads* $f : K_* \to \Omega^\infty\Sigma^\infty$ *on* Top_**, which further induces a morphism of monads on* $Top_*[\mathcal{L}_+]$ *where* $\mathcal{L}$ *is the real linear isometries operad. Then the Quillen pair* (Σ^f_m, Ω^f_m) *induces a equivalence of categories if we restrict it to the following subcategories (enriched in* $Ho(Top)$*.)*

$$\text{ring-like } Ho(E_\infty\text{-ring spaces}) \rightleftharpoons (-1)\text{-connective } Ho(E_\infty\text{-}Sp)$$

where E_∞*-ring spaces means* $(Top_*[\mathcal{L}_+])[K_*]$ *and "ring like" means it is group-like after forgetting in* $Top_*[K_*]$*. The* $\Sigma^f_m(-) = \Sigma^\infty \otimes_{K_*} (-)$ *here should be the coequalizer of the following diagram in* $Sp[\mathcal{L}]$ *instead of in* Sp *in the additive case.*

$$\begin{array}{ccccc} \Sigma^\infty K_* X & \overset{\Sigma^\infty \mu}{\rightrightarrows} & \Sigma^\infty X & \longrightarrow & \Sigma^f_m X \\ & \searrow & \nearrow & & \\ & & \Sigma^\infty\Omega^\infty\Sigma^\infty X & & \end{array}$$

And $\Omega^f_m X = \Omega^\infty X$ *is endowed with the* K_**-action* $K_*\Omega^\infty X \to \Omega^\infty\Sigma^\infty\Omega^\infty X \to \Omega^\infty X$*.*

Remark 3.21. *(1) Note that for a unital operad* C *on* Top*, the* C_* *and* C_+ *are different* *constructions of operads on Top_*. The C_+ is added to an extra base point, while the $C_*(X)$ for an $X \in Top_*$ is defined as the following pushout diagram in $Top[C]$, which makes $C_*(X)$ become an object in Top_* by $C(\varnothing) = * \to C_*(X)$.*

$$\begin{array}{ccc} C(*) & \longrightarrow & C(\varnothing) = * \\ \downarrow & & \downarrow \\ C(X) & \longrightarrow & C_*(X) \end{array}$$

(2) An E_∞-ring space, i.e. an object in $(Top_*[\mathcal{L}_+])[K_*]$, can induce an additive monoid in $(Ho(Top_*), \times)$ and a multiplicative monoid in $(Ho(Top_*), \wedge)$, i.e. a semi-ring object in $(Ho(Top_*), \times, \wedge)$.

3.5 The E_∞-structures of $MString$ and $MU\langle 6\rangle$

Let bu the connective complex K-theory. By strategy of [22], $bu = L\Sigma^f_m(\bigsqcup_{i\geq 0} BU(i))$ 3.17 which means bu is a connective E_∞-ring and $bu^* = \mathbb{Z}[v], |v| = -2$.
We define $BU\langle 2k\rangle = R\Omega^f(\Sigma^{2k}bu)$, a group-like E_∞-space, then $bu^{2t}(X) = [X, BU\langle 2t\rangle]$. When $t = 0$, actually we have $BU\langle 0\rangle = \mathbb{Z} \times BU$ in $Ho(Top)$.

Multiplication by $v^t : \Sigma^{2t}bu \to bu$ gives the $(2t-1)$-connective cover of bu. Under this identification, we get a sequence of morphisms in $Ho(Top[E_\infty])$ by the infinite loop space machine

$$... \to BU\langle 2k\rangle \to ... \to BU\langle 6\rangle \to BSU \to BU \to BU\langle 0\rangle$$

derived from infinite loop space machine.

However, in order to get a Thom spectrum we need an actual over-map instead of a homotopy class of over-map which is what we only have now. The similar problem also appeared in [37]P87.

Lemma 3.22. *Let Sp denote the ∞-category of spectra, then the inclusions $Sp_{\geq n} \subset Sp_{\geq 0}, n \geq 0$ and $Sp_{\geq 0} \subset Sp$ are coreflective subcategories, which means the inclusion admits a left adjunction.*

Proof: It is a direct conclusion from the canonical t-structure on Sp.

□

Remark 3.23. *The 3.17 actually gives an equivalence between the ∞-category of connective* *spectra and the ∞-category of group-like E_∞-spaces.*

$$\mathcal{S}[E_\infty]^{gl} \xrightarrow{\sim} Sp_{\geq 0}$$

This equivalence can be produced easier in ∞-categories by the fact that $Sp_{\geq 0}$ is an additive ∞-category and the reflective adjunctions

$$\mathrm{Pr}^L \rightleftarrows \mathrm{Pr}^L_{Add} \rightleftarrows \mathrm{Pr}^L_{st}$$

make $\mathcal{S}[E_\infty]^{gl}$ universal among all additive ∞-categories. So we have the following unique induced functor which is an equivalence.

$$\begin{array}{ccc} & \mathcal{S} & \\ \swarrow & & \searrow^{\Sigma^\infty_+} \\ \mathcal{S}[E_\infty]^{gl} & \dashrightarrow & Sp_{\geq 0} \end{array}$$

Corollary 3.24. *(1) By the infinite loop space machine, for any $n \geq 0$ the ∞-category of $(n-1)$-connective group-like E_∞-spaces $\mathcal{S}[E_\infty]^{gl}_{\geq n} \subset \mathcal{S}[E_\infty]^{gl}$ is a coreflective subcategory.*
(2) Given an $(n-1)$-connective covering $X_n \to X$ of group-like E_∞-spaces, $Y \in \mathcal{S}[E_\infty]^{gl}_{\geq n}$ and an arrow $f : Y \to X$, then $Map_{\mathcal{S}[E_\infty]^{gl}_{/X}}(Y, X_n)$ is contractible.

proof of (2): It follows from the following homotopy pullback diagram of spaces.

$$\begin{array}{ccc} Map_{\mathcal{S}[E_\infty]^{gl}_{/X}}(Y, X_n) & \longrightarrow & Map_{\mathcal{S}[E_\infty]^{gl}}(Y, X_n) \\ \downarrow & & \downarrow \sim \\ * & \xrightarrow{\{f\}} & Map_{\mathcal{S}[E_\infty]^{gl}}(Y, X) \end{array}$$

The corollary illustrates the n-connective cover of a group like E_∞-space is up to contractible choices.

Proposition 3.25. *By the contractiblity above, we get for any group-like E_∞-space X the full sub ∞-category $Cov_n(X) \subset \mathcal{S}[E_\infty]^{gl}_{/X}$ is a contractible Kan complex.*

Theorem 3.26 (E_∞ structure of $MO\langle n\rangle$ and $MU\langle 2k\rangle$)**.**
By proposition above, we get the contractibility of choices for $BO\langle n\rangle$ and $BU\langle 2k\rangle$ when we take $X = BO$ and $X = BU$ respectively. Moreover, there is a following homotopy diagram

in $h(\mathcal{S}[E_\infty]^{gl}_{/BO})$ determined by the canonical E_∞ map $BU \to BO$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & BU\langle 6\rangle & \longrightarrow & BSU = BU\langle 4\rangle & \longrightarrow & BU & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & \\
\cdots & \longrightarrow & Bstring = BO\langle 6\rangle = BO\langle 8\rangle & \longrightarrow & BSpin = BO\langle 4\rangle & \longrightarrow & BSO & \longrightarrow & BO
\end{array}$$

Taking the E_∞ Thom spectrum functor 3.10 over BO, we get the following homotopy diagram in $h(Sp[E_\infty])$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & MU\langle 6\rangle & \longrightarrow & MSU = MU\langle 4\rangle & \longrightarrow & MU & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & \\
\cdots & \longrightarrow & Mstring = MO\langle 6\rangle = MO\langle 8\rangle & \longrightarrow & MSpin = MO\langle 4\rangle & \longrightarrow & MSO & \longrightarrow & MO
\end{array}$$

4. σ-orientation

We know that any commutative ring spectrum E with $E_{odd} = 0$ (actually $E_{2n+1} = 0$ for every $n \geq 1$ suffices) is complex orientable. So any elliptic cohomology theory is complex orientable. However we can not find a canonical complex orientation on an elliptic cohomology theory without extra data.

But this can be done when comes to $MU\langle 6\rangle$-orientation of an elliptic cohomology theory. The main result in [3] is that $MU\langle 6\rangle$-orientations of an EWP(2.27) ring spectrum E coincides with cubical structures of the bundle $\mathcal{I}(0)$ on Spf(E^0CP^∞).

Remark 4.1. *Throughout the whole section 4, E is denoted as an EWP commutative ring phantom-spectrum. Here we use ring phantom-spectrum because by localizing a ring (phantom-)spectrum we can only get a phantom spectrum: for any EWP commutative ring phantom-spectrum E and $f \in E_0$, the homology theory $E[f^{-1}]_*(-) = E_*[f^{-1}] \otimes_{E_*} E_*(-)$ induces a commutative ring phantom-spectrum $E[f^{-1}]$.*

4.1 n-cocycles

Definition 4.2. *Let C be a category admitting finite products. If A and T are commutative monoid objects in $CMon(C)$, we define $C^0(A,T)$ to be the set*

$$C^0(A,T) \overset{def}{=} \mathrm{Hom}_C(A,T)$$

and for $k \geq 1$ we let $C^k(A,T)$ be the submonoid of $f \in \mathrm{Hom}_C(A^k,T)$ such that
(a) $f(a_1,\ldots) = 0$ when one of $\{a_i\}$ is zero ;
(b) $f(a_1,\ldots)$ is a symmetric function ;
(c) f satisfies the cocycle condition, that is,

$$f(a_1,a_2,a_3,\ldots) + f(a_0,a_1+a_2,a_3,\ldots) = f(a_0+a_1,a_2,a_3,\ldots) + f(a_0,a_1,a_3,\ldots)$$

when $k \geq 2$.

Remark 4.3. *(1) The $C^n(A,T)$ is commutative monoid set induced by T.*
(2) If T is an abelian group object, then in definition (a) can be replaced by (a)': $f(0,0,...,0) = 0$.

Definition 4.4. *In the case where G and T are abelian group objects, and for $k \geq 0$ with*

$f \in C^k(G,T)$, the transformation $\delta(f) \in C^{k+1}(G,T)$ is defined as follows: for $k \geq 1$, the map is determined by $\delta(f)(a_0,\ldots) = f(a_0,a_2,\ldots) + f(a_1,a_2,\ldots) - f(a_0+a_1,a_2,\ldots)$.
In the special case where $k=0$, the map is specified as $\delta(f)(a) = f(0) - f(a)$.

Definition 4.5 (Sheafification)**.** *From definition we can make n-cocycles a sheaf as the following: let X,Y are commutative momoid fppf sheaves over S, we define $\underline{C}^k(X,Y)(T) = C^k(X_T,Y_T)$. It is actually a representable commutative monoid sheaf in $Sh(Sch/S)_{fppf}$ in certain case [21].*

4.2 Even spaces

Before into the topology cocycle, we introduce a useful concept.

Definition 4.6. *(1) We say a space X to be "even" iff $H_*(X)$ is concentrated in even degrees and $H_n(X)$ is free abelian for all n.*
(2) An H-space means a monoid object in $Ho(Top)$.

Lemma 4.7 ([14]4C.1)**.** *If X is even and simply-connected, then there exists a CW approximation $W \to X$ so that W only consists of cells of even dimension.*

Proposition 4.8. *Let E be an EWP commutative ring phantom-spectrum. Then for any even space X,*
(1) The A-T spectral sequence $H_*(X;E_*) \implies E_*(X)$ collapses . Therefore $E_*(X)$ is a free E_*-module and $E^*(X) \to Hom^*_{E_*}(E_*X, E_*)$ is bijective.
(2) The $E_0(X)$ is a cocommutative E_0-coalgebra by kunneth theorem. Furthermore, If X is an even H-space, we define $X_E = \operatorname{Spf} E^0X$, then the natural Cartier morphism $\operatorname{Spec} E_0X \to \underline{Hom}_{Grp/E}(X_E, \mathbb{G}_{m,E})$ is isomorphic, which is the Cartier duality.

Definition 4.9. *To begin, the map $\rho_0 : \mathbb{C}P^\infty \to 1 \times BU \subset \mathbb{Z} \times BU = BU\langle 0\rangle$ is initially defined as the mapping that classifies the tautological line bundle L [15].*
For $t > 0$, consider $L_1,\ldots,L_t$ as the evident line bundles over $(\mathbb{C}P^\infty)^t$. Introduce $x_i \in ku^2((\mathbb{C}P^\infty)^t)$ as defined by the expression

$$vx_i = 1 - L_i.$$

Subsequently, the following isomorphisms hold:

$$ku^*\left((\mathbb{C}P^\infty)^t\right) \cong \mathbb{Z}[v][[x_1,\ldots,x_t]]$$

The element $\prod_i x_i \in bu^{2t}(P^t)$ yields the map $\rho_t : (\mathbb{C}P^\infty)^t \to BU\langle 2t\rangle$.

Remark 4.10. *The composition* $(\mathbb{C}P^{\infty})^t \xrightarrow{\rho_t} BU\langle 2t\rangle \to BU\langle 0\rangle$ *happens to classify the bundle* $\prod_i (1 - L_i)$.

Proposition 4.11. *Let* X *be an even commutative H-space, we have the following diagram of commutative monoid sets for any* $k \geq 0$,

$$\begin{array}{ccccc} C^k(P,X) & \longrightarrow & C^k_{E_0-CcoAl}(E_0P, E_0X) & \dashrightarrow & Hom_{Mon/E}(X^E, \underline{C}^k(P_E, \mathbb{G}_{m,E})) \\ & & \downarrow & \searrow & \downarrow \\ & & \underline{C}^k(P_E, \mathbb{M}_{m,E})(\operatorname{Spec} E_0X) & \longleftarrow & \underline{C}^k(P_E, \mathbb{G}_{m,E})(\operatorname{Spec} E_0X) \end{array}$$

where $P = \mathbb{C}P^{\infty}$ *and* $P_E = \operatorname{Spf} E^0P$, $X^E = \operatorname{Spec} E_0X$. *The dashed liftings exist only when* $k \geq 1$ *or* X *is an* H*-group, and in those 2 cases all sets in the diagram are abelian groups.*

Definition 4.12. *For* $0 \leq t \leq 3$, $BU\langle 2t\rangle$ *is an even space [3]. Apply the above to* $\rho_t \in C^t(P, BU\langle 2t\rangle)$, *we get morphisms of commutative group schemes over* $\operatorname{Spec}(E_0)$

$$f_t : \operatorname{Spec} E_0BU\langle 2t\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E}).$$

Theorem 4.13 (Ando-Hopkins-Strickland [3])**.** *The morphism* $f_k : \operatorname{Spec} E_0BU\langle 2k\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E})$ *is an isomorphism of commuatative group schemes over* $\operatorname{Spec} E_0$ *when* $0 \leq k \leq 3$.

Proof: **Sketch:** First, we note that the formation of the map

$$f_k : \operatorname{Spec} E_0BU\langle 2k\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E})$$

is preserved under base change. Second, by 4.1, locally on $\operatorname{Spec} E_0$, we can assume E is MP-orientable. Thus, it suffices to show f_k is an isomorphism for $E = MP$.
In this case, we have a map of graded rings

$$\mathcal{O}_C \to MP_0BU\langle 2k\rangle = MU_*BU\langle 2k\rangle,$$

both of which are free of finite type over $\mathbb{Z}$. This map is a rational isomorphism by some easy calculation, so it must be injective, and the source and target must have the same Poincaré series. It will thus suffice to prove that it is surjective. Recall that I denotes the kernel of the map

$$MP_0 \to \mathbb{Z} = HP_0$$

that classifies the additive formal group law, or equivalently, the ideal generated by elements of strictly positive dimension in MU_*. By induction on degrees, it will suffice to prove that the map

$$\mathcal{O}_C/I \to MP_0BU\langle 2k\rangle/I$$

is surjective.
Base change and the Atiyah-Hirzebruch spectral sequence identify this map with the map

$$\mathcal{O}_{\underline{C}^3(\widehat{\mathbb{G}}_a,\mathbb{G}_m)} \to HP_0BU\langle 2k\rangle,$$

in other words, the case $E = HP$ of the proposition. This case was proved in Proposition 4.4 (for $k = 2$) or Corollary 4.14 (for $k = 3$) of [3].

□

4.3 The Line bundle on a formal group

Now we turn to the connection between n-cocycles for a line bundle and Thom spectrum orientation.

Firstly we need a well-behavior definition of the line bundle on a formal group.

Definition 4.14. *Consider $X \in Sh(Aff)_{Zar}$ as a large Zariski sheaf. The category $QCoh(X)$ is defined as follows:*
A quasi-coherent sheaf $\mathcal{F} \in QCoh(X)$ entails the following elements:
(a) For each (R, x) where R is a commutative ring and $x \in X(R)$, we assign an R-module M_x.
(b) For every map $f : (R, x) \to (S, y)$, there is an isomorphism $\phi(f, x) : S \otimes_R M_x \to M_y$ of S-modules. These isomorphisms $\phi(f, x)$ must adhere to the functoriality conditions:
(i) For $f = id : (R, x) \to (R, x)$, the requirement is $\phi(id, x) = id : M_x \to M_x$.
(ii) The morphism ϕ satisfies the associative law.

Remark 4.15. *(1) The category $QCoh(X)$ supports direct sums and tensor products which are defined pointwise.*
(2) A line bundle is defined to be a quasi-coherent sheaf on X such that all M_x is a projective module of rank 1 on R.
(3) It can be checked the definition agrees with the ordinary case when X is a scheme.

Proposition 4.16. *Let $X \in Sh(Aff)_{Zar}$ be a big Zariski sheaf, then the following statements hold:*

(1) There is a natural equivalence $p_X : \mathbb{G}_{m,X}$*-tor* $\to PIC(X)^{\simeq}$ *between the category of* $\mathbb{G}_{m,X}$*-torsors (on big Zariski site* $Aff_{/X}$*) and the maximal groupoid of the full category* $PIC(X) \subset QCoh(X)$ *of line bundles.*
(2) If $X = \varprojlim_{I^{op}} X_i$ *is an inverse limit of a filtered diagram* I*, then we have following equivalences by homotopy limit(or 2-limit) of categories*
(i) $QCoh(X) \simeq \varprojlim_{I^{op}} QCoh(X_i)$*;*
(ii) $\mathbb{G}_{m,X}$*-tor* $\simeq \varprojlim_{I^{op}} \mathbb{G}_{m,X_i}$*-tor ;*
(iii) $p_X = \varprojlim_{I^{op}} p_{X_i}$

Proof: (1)
Let $T \in \mathbb{G}_{m,X}$-tor, we define $p_X(T) \in PIC(X)^{\simeq}$ by $p_X(T)(R,x) = Hom_{\mathbb{G}_{m,R}}(T_R, \mathbb{A}^1_R)$, the $\mathbb{G}_{m,R}$-equivariant morphismsm, which is a R-module induced by $\mathbb{A}^1_R$.
Conversely, let $\mathcal{L} \in PIC(X)^{\simeq}$, we define the $\varphi_X(\mathcal{L}) \in \mathbb{G}_{m,X}$-tor by $\varphi_X(\mathcal{L})(R,x) = Iso_R(R, \mathcal{L}(R,x))$, the trivializations of $\mathcal{L}(R,x)$. It is not hard to verify p_X is the inverse of φ_X.

□

A Θ^3-structure on a line bundles is called by a cubical structure.

Definition 4.17. *In this thesis, we denote by* $C^k(G,\mathcal{L})$ *the collection of* Θ^k*-structures on* $\mathcal{L}$ *over* G*. It is important to note that* $C^0(G,\mathcal{L})$ *represents the trivializations of* $\mathcal{L}$*, while* $C^1(G,\mathcal{L})$ *corresponds to the rigid trivializations of* $\Theta^1(\mathcal{L})$*. Additionally, we introduce an fppf sheaf given by* $\underline{C}^k(G,\mathcal{L})(R) = C^k(G_R, \mathcal{L}_R)$*.*

Remark 4.18. *It is worth mentioning that when considering the trivial line bundle* $\mathcal{O}_G$*, we have that the set* $C^k(G;\mathcal{O}_G)$ *simplifies to the group* $\mathbb{C}^k(G,\mathbb{G}_m)$ *of previously introduced cocycles.*

For any pair of line bundles $\mathcal{L}_1, \mathcal{L}_2$, there exists a natural map $C^k(G;\mathcal{L}_1) \times C^k(G;\mathcal{L}_2) \to C^k(G;\mathcal{L}_1 \otimes \mathcal{L}_2)$ defined by $(s_1,s_2) \mapsto s_1 \otimes s_2$. Consequently, when $\mathcal{L}_1$ is trivial, a natural group action $C^k(G;\mathbb{G}_m) \times C^k(G;\mathcal{L}) \to C^k(G;\mathcal{L})$ can be obtained for any line bundle $\mathcal{L}$.

Proposition 4.19. *Furthermore, if* $\mathcal{L}$ *is a line bundle over* G *where* G *is a formal group over* S *and where* $\mathcal{L}$ *can be trivialized Zariski locally on* S *(warning: this is not equivalent to locally trivial on* G*), then the fppf sheaf* $\underline{C}^k(G,\mathcal{L})$ *forms a scheme, which is invariant under change of base. Moreover,* $\underline{C}^k(G,\mathcal{L})$ *acts as a torsor over* $\underline{C}^k(G,\mathbb{G}_m)$*.*

Now return to the topology.

Definition 4.20. *Let X be a finite even complex and V be a virtual complex vector bundle classified by a map $X \to Z \times BU$. We denote the Thom spectrum of X^V. The coaction of the Thom spectrum results in E^0X^V being an E^0X-module, which, by Thom isomorphism Zariski locally, can be further understood as a line bundle.*

Proposition 4.21. *In the master thesis, the proposition labeled as 4.21 discusses the scenario where X is a finite complex and V is a virtual bundle over X. The notation $\mathbb{L}(V)$ is used to denote the line bundle $\widetilde{E^0X^V}$, and $\mathbb{L}$ establishes a functor from vector bundles over X to line bundles on X_E. The proposition outlines the following key points:*
(i) The functor $\mathbb{L}(-)$ takes the direct sum into the tensor product of line bundles on X_E.
(ii) Additionally, let $f : Y \to X$ be a continuous map, then a natural isomorphism $f^*\mathbb{L}(-) \cong \mathbb{L}\left(f^*(-)\right)$ of line bundles over Y_E is established.

In the case where X is an (infinite) even complex and V is a virtual bundle classified by $f : X \to BU\langle 0\rangle$, $\mathbb{L}(V)$ is considered a quasi-coherent sheaf on $\operatorname{Spf} E^0X$ through (co)limits. It is emphasized that the proposition mentioned earlier is applicable even for infinite even complexes.

Lemma 4.22. *If $T(\rho_0) = \Sigma^\infty Th(\mathcal{L})$ denotes the Thom spectrum associated with $\rho_0 : P \to Z \times BU$ by the tautological bundle $\mathcal{L}$, then the Thom sheaf $E^0T(\rho_0)$ is naturally isomorphic to $\mathcal{I}(0) = ker(E^0P \to E^0)$ in $Qcoh(P_E)$. This isomorphism is induced by a homotopy equivalence of P_+-comodule pointed spaces $P \to Th(\mathcal{L})$.*

Proof: We can see the equivalence $P \to Th(\mathcal{L})$ preserved the P_+-comodule action by the following diagram.

$$\begin{array}{ccc}
P & \xrightarrow{\Delta} & P \times P \\
{\scriptstyle p}\big\uparrow\big\downarrow{\scriptstyle s} & & {\scriptstyle p\times id}\big\uparrow\big\downarrow{\scriptstyle s\times id} \\
D(EU_1) & \xrightarrow{(id,p)} & D(EU_1) \times P \\
\big\downarrow & & \big\downarrow \\
Th(\mathcal{L}) & \longrightarrow & Th(\mathcal{L}) \wedge P_+
\end{array}$$

□

Proposition 4.23. *The section s_k is a Θ^k-structure, and hence an element of*

$$\underline{C}^k\left(P_E; \mathcal{I}(0)\right)\left(MU\langle 2k\rangle^E\right)$$

Proof: Firstly we have an isomorphism $BU\langle 2k\rangle^E \cong \underline{C}^k\left(P_E, \mathbb{G}_m\right)$, which imparts the structure of a torsor over the group scheme $BU\langle 2k\rangle^E$ to $\underline{C}^k\left(P_E; \mathcal{I}(0)\right)$ when $k \leq 3$. The equivariant morphism between torsors is automatically deemed an isomorphism, as observed in the case of g_k.

□

Proposition 4.24. *The following diagram is commutative when $0 \le k \le 3$*

$$\begin{array}{ccc} BU\langle 2k\rangle^E \times MU\langle 2k\rangle^E & \longrightarrow & \underline{C}^k\left(P_E;\mathbb{G}_{m,E}\right) \times \underline{C}^k\left(P_E;\mathcal{I}(0)\right) \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle^E & \longrightarrow & \underline{C}^k\left(P_E;\mathcal{I}(0)\right) \end{array}$$

which is concluded by the following naturality of coactions on Thom spectra

$$\begin{array}{ccc} (P^k)^V & \longrightarrow & P^k_+ \wedge (P^k)^V \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle & \longrightarrow & BU\langle 2k\rangle_+ \wedge MU\langle 2k\rangle \end{array}$$

Theorem 4.25 (Ando-Hopkins-Strickland)**.** *The morphism $MU\langle 2k\rangle^E \xrightarrow{g_k} \underline{C}^k\left(P_E;\mathcal{I}(0)\right)$ is an isomorphism of $BU\langle 2k\rangle^E$-torsors when $0 \le k \le 3$.*

Proof: Since any morphism of torsors is an isomorphism, it follows from 4.24.

□

Since $MU\langle 2k\rangle$ is a bounded-below even spectrum when $k \le 3$, we have natural isomorphisms

$$[MU\langle 2k\rangle, E] = E^0(MU\langle 2k\rangle) \to Hom_{E_*}(E_*MU\langle 2k\rangle, E_*) = Hom_{E_0}(E_0MU\langle 2k\rangle, E_0)$$

and

$$[MU\langle 2k\rangle, E]_{ring} = Hom_{E_0-Al}(E_0MU\langle 2k\rangle, E_0) = MU\langle 2k\rangle^E(S^E).$$

Corollary 4.26 (Orientations correspond Θ^k-structures)**.** *When $k \le 3$, the isomorphism g_k induces a bijection*

$$[MU\langle 2k\rangle, E]_{ring} \to C^k\left(P_E;\mathcal{I}(0)\right)(S^E).$$

By the corollary above we can state our first main theorem as follows.

Theorem 4.27 (Main A)**.** *Let $E \to F$ be a ring (phantom-)morphism between EWP ring (phantom-)spectra, and $MU\langle 2k\rangle \to E$ and $MU\langle 2k\rangle \to F$ be two orientations. Then*

$$\begin{array}{ccc} & MU\langle 2k\rangle & \\ \swarrow & & \searrow \\ E & \longrightarrow & F \end{array}$$

commutes if and only if

$$\begin{array}{ccc} S^F & \longrightarrow & S^E \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle^F & \longrightarrow & MU\langle 2k\rangle^E \end{array}$$

commutes for the corresponding sections.

4.4 Cubical structure on elliptic curves

In 4.4, we will see any elliptic cohomology theory has a unique $MU\langle 6\rangle$-orientation.

Lemma 4.28 (Theorem of the cube [8])**.** *Let $X \to S$ be an abelian scheme over S. Then for any $\mathcal{L} \in Pic(X)$, the $\Theta^3(\mathcal{L}) \cong p^*\mathcal{M}$ for some $\mathcal{M} \in Pic(S)$ where p denote the projection $X_S \times X_S \times_S X \to S$.*
Furthermore, $\mathcal{O}_S \cong e^*\Theta^3(\mathcal{L})$ is naturally rigidificated, so $\mathcal{M} \cong e^*p^*\mathcal{M} \cong e^*\Theta^3(\mathcal{L}) \cong \mathcal{O}_S$ is trivial, and hence $\Theta^3(\mathcal{L})$ is also trivial.

Lemma 4.29. *Let $p : X \to S$ be a proper smooth morphism with geometrically connected fibers, then*
(i) [38]28.1H: The natural $\mathcal{O}_S \to p_*\mathcal{O}_X$ is isomorphic;
(ii) Let $e : S \to X$ be a section, and let $\mathcal{L}_1, \mathcal{L}_2$ be trivializable line bundles on X, then

$$Hom_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{L}_2) \to Hom_{\mathcal{O}_S}(e^*\mathcal{L}_1, e^*\mathcal{L}_2)$$

is bijective.

Theorem 4.30 (Unique cubical structure for abelian schemes)**.** *Let $p : X \to S$ be an abelian scheme over S. Then for any $\mathcal{L} \in Pic(X)$, there exists exactly one Θ^3-structure on $\mathcal{L}$.*

Proof: Since $Hom_{\mathcal{O}_{X^3}}(\mathcal{O}_{X^3}, \Theta^3(\mathcal{L})) \to Hom_{\mathcal{O}_S}(\mathcal{O}_S, e^*\Theta^3(\mathcal{L}))$ is bijective by lemma above. The natural rigidification $\mathcal{O}_S \xrightarrow{1} e^*\Theta^3(\mathcal{L})$ determines unique trivialization $u : \mathcal{O}_{X^3} \to \Theta^3(\mathcal{L})$.
Recall the axioms of cubical structures:
(i) $s(0) = 1$;
(ii) $s(a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}) = s(a_1, a_2, a_3)$ is symmetric for any $\sigma \in \Sigma_3$;
(iii) the section $s(a_1, a_2, a_3) \otimes s(a_0 + a_1, a_2, a_3)^{-1} \otimes s(a_0, a_1 + a_2, a_3) \otimes s(a_0, a_1, a_3)^{-1} = 1$.

However, all conditions automatically hold for u by $u(0) = 1$ when we pullback to S along e, which means u is exactly the unique cubical structure.

Now we can state the main theorem of this paper.

Theorem 4.31 (Main B)**.**
(i) For any elliptic cohomology theories E we have natural σ-orientation $MU\langle 6\rangle \to E$.
(ii) The σ-orientation commutes for any elliptic morphism of elliptic cohomology theories $E \to F$ induced by a morphism $C_1 \to C_2$ of elliptic curves.

$$\begin{array}{ccc} & MU\langle 2k\rangle & \\ \swarrow & & \searrow \\ E & \longrightarrow & F \end{array}$$

commutes by

$$\begin{array}{ccccccc} MU\langle 6\rangle^F & \xrightarrow{\simeq} & \underline{C}^3\left(P_F;\mathcal{I}(0)\right) & \longleftarrow & \underline{C}^3\left(C_1;\mathcal{I}(0)\right) & \xleftarrow{\simeq} & S^F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ MU\langle 6\rangle^E & \xrightarrow{\simeq} & \underline{C}^3\left(P_E;\mathcal{I}(0)\right) & \longleftarrow & \underline{C}^3\left(C_2;\mathcal{I}(0)\right) & \xleftarrow{\simeq} & S^E \end{array}$$

Proof: Combining 4.27 and 4.30 we obtain the result.

□

4.5 Further developments

When comes to $\mathbb{E}_\infty$-orientation space $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R) = \mathrm{Map}_{CAlg(Sp)}(Mf, R)$, combining the Thom adjunction $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})_{/Pic(Sp)} \overset{M(-)}{\rightleftarrows} CAlg(Sp)$ and the infinite loop space machine $\mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq Sp_{\geq 0}$ we can produce many interesting results.

$$Sp_{\geq 0} \xrightarrow{\sim} \mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \underset{GL_1}{\rightleftarrows} \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{CAlg}(Sp), \qquad gl_1 : \mathrm{CAlg}(Sp) \to Sp_{\geq 0}$$

By this adjunction we can get the following theorem.

Theorem 4.32 (Ando–Blumberg–Gepner–Hopkins–Rezk [1])**.** *Let Mf be the Thom $\mathbb{E}_\infty$-spectrum induced by a map $f : X \to \mathrm{pic}(Sp)$ in $Sp_{\geq 0}$ and let R be an $\mathbb{E}_\infty$-ring. Then $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R)$ is a torsor over the H-space $\mathrm{Map}_{Sp}(X, gl_1(R))$, meaning $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R)$ is either empty or homotopy equivalent to $\mathrm{Map}_{Sp}(X, gl_1(R))$.*

Example 4.33. *Particularly, combining with the Chromatic Nullstellensatz [7] and some further calculations, we can deduce that for any* height $= n > 0$, *the* $\mathrm{Or}_{\mathbb{E}_\infty}(MUP, E(\overline{\mathbb{F}}_p))$ *is non-empty and hence homotopy equivalent to* $\mathrm{Map}_{Sp}(ku, gl_1(E(\overline{\mathbb{F}}_p)))$.

Example 4.34. *The following genera for MSO_* are also examples which can be taken* *consideration by orientation theory.*

1. *L-genus*

$$\log_{\mathrm{Sign}}(x) = \sum_{n\geq 1} \frac{x^{2n+1}}{2n+1}$$

2. *$\widehat{A}$-genus*

$$\exp_{\widehat{A}}(u) = 2\sinh(u/2)$$

3. *Ochanine genus*

$$\log_{\mathrm{Och}}(x) = \int_0^x \frac{dt}{\sqrt{1-2\delta t^2+\epsilon t^4}}$$

4. *Witten genus*

$$\frac{u}{\exp_{\mathrm{Wit}}(u)} = \frac{u/2}{\sinh(u/2)} \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^u)(1-q^n e^{-u})}$$

5. *Witten signature*

$$\frac{u}{\exp_{\mathrm{WSig}}(u)} = \frac{u/2}{\tanh(u/2)} \prod_{n=1}^{\infty} \left(\frac{1+q^n e^u}{1-q^n e^u} \cdot \frac{1+q^n e^{-u}}{1-q^n e^{-u}}\right) / \left(\frac{1+q^n}{1-q^n}\right)^2$$

Proposition 4.35. *Cobordism spectra From the commutative diagram in $Ho(Top[E_\infty]_{/BO})$*

$$\begin{array}{ccccccc}
BU\langle 6\rangle & \longrightarrow & BSU & \longrightarrow & BU & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & \\
BString & \longrightarrow & BSpin & \longrightarrow & BSO & \longrightarrow & BO
\end{array}$$

we get a natural commutative diagram in $Ho(E_\infty\text{-}Sp)$ by applying Thom spectrum functor

$$\begin{array}{ccccccc}
MU\langle 6\rangle & \longrightarrow & MSU & \longrightarrow & MU & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & \\
MString & \longrightarrow & MSpin & \longrightarrow & MSO & \longrightarrow & MO
\end{array}$$

Theorem 4.36. *Ando–Hopkins–Strickland 2001 [3]*
The $[MU\langle 6\rangle, E]_{CAl(hSp)} \xrightarrow{\simeq} C^3\left(P_E; \mathcal{I}(0)\right)$ can induces a bijection of subsets

$$[MString, E]_{CAl(hSp)} \xrightarrow{\simeq} C^3_{is}\left(P_E; \mathcal{I}(0)\right) = \left\{f \in C^3\left(P_E; \mathcal{I}(0)\right) | f(a,b,-(a+b)) = 1\right\}$$

if $1/2 \in E_0$. So in the case 2 invertible, the σ-orientation of an elliptic cohomology theory has a (unique) factorization of homotopy ring spectra

$$\begin{array}{ccc}
MU\langle 6\rangle & \longrightarrow & MString \\
& \searrow \quad \swarrow & \\
& E &
\end{array}$$

which corresponds with the Witten genus.

Theorem 4.37 (Ando–Hopkins–Strickland 2004 [4]).
The σ-orientation $MU\langle 6\rangle \to E$ is an H_∞-map.

Theorem 4.38 (Ando–Hopkins–Rezk 2010 [2]).

1. *There exist, up to homotopy, unique E_∞-ring maps*

$$\sigma_L, \sigma_{\hat{A}} : \mathrm{MSpin} \longrightarrow KO$$

refining the L-genus and $\hat{A}$-genus.

2. *The σ-orientation $MU\langle 6\rangle \to E$ is an E_∞-map, which can be refined to be an E_∞ map $MString \to tmf$, a string orientation to global section of the E_∞-sheaf of moduli stack of elliptic curves.*

Theorem 4.39 (Dylan Wilson 2018 [40]).
There exist, up to homotopy, unique E_∞-ring maps

$$\sigma_{\mathrm{Och}}, \sigma_{\mathrm{WSig}} : \mathrm{MSpin} \longrightarrow tmf_0(2)$$

refining the Ochanine genus and Witten signature

The road ahead

(1) Could we generalize the σ-orientation to the PEL-type abelian varieties or even further refine it to be a $\mathbb{E}_\infty$ morphism $MU\langle 6\rangle \to TAF$?
(2) The geometric interpretation of higher Thom spectra $MU\langle 2k\rangle$, $MO\langle 2k\rangle$ and their $\mathbb{E}_\infty$-orientations?
(3) Are there higher viewpoints in Spectral Algebraic geometry?

Conclusion

Main contributions:

1. We provide a bigger category of fppf sheaves than the category of schemes. It can contain schemes, formal schemes and p-divisible groups.

2. We provide a precise definition of formal Lie groups and formal Lie varieties in the framework of fppf sheaves. So we can seriously consider the quasi-coherent sheaves on them.

3. We functorialize the construction of Thom spectra and make it become a left Quillen adjoint. In this point of view, we can easily gain the desired $\mathbb{E}_\infty$-structure of a Thom spectrum.

4. We give a statement that the Thom isomorphism actually comes from a diagonal comodule structure.

5. Combining the Thom spectrum functor with the infinite loop space machine, we can endow a canonical $\mathbb{E}_\infty$-structure to the n-connective cover of an $\mathbb{E}_\infty$-space.

6. Combining all of the above, we can get the most important theorem in this paper, that is, the correspondence between cubical structures and $MU\langle 6\rangle$-orientations.

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Resume and academic achievements

Liang Jiacheng was born in 2000, in Liuzhou, Guangxi, China.
Homepage: https://552jc.github.io/ljc552.github.io/

In September 2018, he was admitted to Sichuan University. In June 2022, he obtained a bachelor’s degree in science from the Department of Physics, Sichuan University.

From September 2022, he started to pursue his master degree of science in Mathematics in the Department of Mathematics, SUSTech.