硕士学位论文

Goodwillie Calculus 简介

A GENTLE INTRODUCTION TO GOODWILLIE CALCULUS

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摘　要

Goodwillie 于 20 世纪 90 年代提出了 Goodwillie 塔，这是同伦论中的一种基本工具，通过构造同伦函子的逼近，来研究拓扑空间的稳定同伦性质。

Goodwillie 塔的核心思想是用一系列的 n-切除逼近来分解同伦函子，从而使其更易于分析和计算。本文系统阐述了 Goodwillie 塔的构造，其中包括 n-切除逼近 P_nF，称为第 n 层，以及作为相邻两层自然诱导的同伦纤维 D_nF，它是一个 n-齐性函子。此外，本文还探讨了一些有关 Goodwillie 塔的基本理论，包括对称多重线性函子和 n-齐性函子之间的一一对应关系，这为研究 D_nF 提供了提供了可计算的新视角。

此外，本综述介绍了一些应用，例如 Snaith 分解。还综述了若干已知结果，包括 Goodwillie 塔关于恒等函子在一些特定空间下 ν_k-周期同伦的表现，以及 Goodwillie 塔的收敛性质，其对理解特定空间上同伦函子的行为具有重要意义。

关键词：Goodwillie 塔；Snaith 分解；恒等函子；ν_k-周期同伦

ABSTRACT

The Goodwillie tower, introduced by Thomas Goodwillie in the 1990s, is a fundamental tool in homotopy theory, designed to study the homotopy properties of topological spaces and approximate homotopy functors.

The core idea behind the Goodwillie tower is to decompose homotopy functors into a sequence of n-excisive approximations, making them easier for analysis and computation. This survey provides a detailed exploration of the construction of the Goodwillie tower, including the n-excisive approximation P_nF, called the n-th stage, and the homotopy fiber D_nF, an n-homogeneous functor. We also introduce its theoretical foundations, including the relationship between symmetric multilinear functors and n-homogeneous functors. This provides another viewpoint to calculate the D_nF.

Additionally, the survey highlights several key applications and examples, such as the Snaith splitting. The survey also discusses some results about the ν_k-periodic homotopy of the Goodwillie tower of the identity functor on some specific spaces, including the convergence properties of the Goodwillie tower, and its implications for understanding the behavior of homotopy functors on specific spaces.

Keywords: Goodwillie tower; Snaith splitting; identity funcor; ν_k-periodic homotopy

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CHAPTER 1 INTRODUCTION

The study of functorial approximation in homotopy theory has been profoundly shaped by the development of tower decompositions, which systematically break down complex homotopy invariants into more tractable components. One of the earliest and most influential examples is the *Postnikov tower*, which is a sequence of spaces

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots$$

such that $\pi_i(X) \cong \pi_i(X_n)$ for all $i \leq n$. The Postnikov tower was introduced by Mikhail Postnikov [1], see also [2, 3]. The Postnikov tower provides a sequential approximation of a topological space by truncating its higher homotopy groups, yielding a series of homotopy fibrations whose fibers are Eilenberg-MacLane spaces.

The homotopy theory for categories has grown in importance in modern mathematics. In the 1990s, Thomas Goodwillie [4–6] introduced the *Taylor tower* (also called Goodwillie tower) of homotopy functors from (topological) spaces to either spaces or spectra which takes the form

$$F \simeq P_0F \longleftarrow P_1F \longleftarrow \cdots \longleftarrow P_nF \longleftarrow \cdots,$$

where each P_nF is a homotopy functor from spaces to either spaces or spectra. Whereas the Postnikov tower decomposes spaces by Postnikov stages X_n, the Goodwillie tower applies to functors, approximating them through a sequence of polynomial stages P_nF. For each n, the *n-th layer* D_nF of the tower, which is the homotopy fiber of $P_nF \longrightarrow P_{n-1}F$, is an n-homogeneous functor (Proposition 2.3). As shown by the formula (3-1), D_nF can be treated as a kind of homotopy colimit (homotopy orbit) of a symmetric multilinear functor with n variables. Goodwillie found that the n-th layer can be determined by the *n-th derivative* ∂_nF, see the formula (3-2). Klein and Rognes [7] proved that the first derivative of the composition $F \circ G$ of functors F, G satisfies the chain rule:

$$\partial_1(F \circ G) \simeq \partial_1F \wedge \partial_1G.$$

Arone and Ching [8] showed that the higher order derivatives can be regarded as bimodules over operads. For details of operad, see [9]. For a well-established survey of Taylor towers, one may consult [10].

Given a homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ from category $\mathcal{C}_{/Y}$ of spaces over Y to the category $\mathcal{D}$ of spaces or spectra, we say that its Goodwillie tower converges on $X \in \mathcal{C}_{/Y}$ if there is a weak equivalence

$$F(X) \simeq holim\ P_nF(X).$$

In the case of Posnikov tower, there is always a homotopy equivalence $X \simeq lim\ X_n$ for any CW complex X. A natural question for Goodwillie towers is which classes of homotopy functors admit convergent Goodwillie towers. When the Goodwillie tower of a functor converges at a given space, it provides a complete homotopy decomposition of the functor at that object, effectively breaking it down into its constituent polynomial approximations of increasing degrees of excision. There are some remarkable results about the convergence of Goodwillie towers. In 2003, Goodwillie [6] proved that if F is ρ-analytic and the structural map $X \longrightarrow Y$ is $(\rho + 1)$-connected, then there is a weak equivalence

$$F(X) \simeq holim\ P_nF(X).$$

In Arone and Mahowald's work [11], they proved that the Goodwillie tower of the identity functor I on odd-dimensional spheres has only non-trivial layers $D_{p^k}I$. In works associated with Anderson and Davis [12], as well as Miller and Wilkerson [13], they proved that the v_{k-1}-periodic homotopy of $D_{p^j}I(S^{2s+1})$ is trivial when $j \geq k$. That is, the Goodwillie tower of I on an odd-dimensional sphere S^{2s+1} converges at an exponential speed, the unstable v_k-periodic homotopy of an odd-dimensional sphere can be resolved into a tower of fibrations with $k+1$ stages, where the fibers are infinite loop spaces. This shows that the Goodwillie tower is powerful to study the unstable v_k-periodic homotopy theory. In Behrens and Rezk's work [14], they characterized a class of spaces called $\Phi_{K(n)}$-good spaces, on which the Goodwillie tower of I is convergent under v_k-periodic homotopy. As an example, the sphere is a $\Phi_{K(n)}$-good space, which coincides with Arone and Mahowald's work [11] on spheres. On the other hand, Brantner and Heuts [15] found that the Goodwillie tower on the wedge sum of spheres, or mod p Moore spaces, is divergent under v_k-periodic homotopy.

Goodwillie towers have rich applications in topology and homotopy theory. For example, the Snaith splitting [16], which can be used to calculate the stable homotopy groups of delooping spheres ΩS^n. Bödigheimer [17] characterized the Goodwillie tower of stable homotopy functor $Q = \Sigma^\infty\Omega^\infty$, on unbased mapping space from an n-dimensional manifold M to unreduced suspension space $S^m X$ by

$$Q\mathrm{Map}(M, S^m X) \simeq \prod_{n\geq 1} Q(C(M, \partial M; n) \wedge_{\Sigma_n} X^{\wedge n}).$$

The functor $X \longmapsto Q(C(M, \partial M; n) \wedge_{\Sigma_n} X^{\wedge n})$ is n-homogeneous. Thus, this equivalence also splits. For the based version, Arone [18] described the n-excisive approximation of this stable homotopy functor on mapping space from a CW complex K to any X. In the same example, Goodwillie [6] described the n-th derivative of functor $X \longmapsto \Sigma^\infty \mathrm{Map}_*(K, X)$, thus giving the n-th layer. On smooth manifolds, Goodwillie tower can be used to decompose the embeddings between manifolds [19]. Bauer, Burke, and Ching's work [20] shows that the n-excisive functors are the direct analogues of n-jets of smooth maps between manifolds. This makes a more useful connection between stable homotopy and smooth manifolds. On the other hand, Goodwillie towers provide a framework for integrations between localization techniques and chromatic homotopy theory [10, 21], significantly advancing the study of height decompositions in homotopy theory. Also, it provides a tool to study the unstable homotopy theory by the stable homotopy theory [22, 23]. This integration enables deeper insights into complex problems in algebraic topology and related fields.

In this thesis, we adopt the following global notation and convention. All spaces are supposed to be compactly generated Hausdorff spaces. There are three homotopy categories frequently used in this thesis: **Top** is the model category of unbased spaces, $\mathbf{Top}_*$ is the based spaces, and **Sp** is the model category of spectra. Unless otherwise specified, the category $\mathcal{C}$ is usually **Top** or $\mathbf{Top}_*$, the category $\mathcal{D}$ is usually $\mathbf{Top}_*$ or **Sp**. The homotopy category of homotopy functors between $\mathcal{C}$ and $\mathcal{D}$ is denoted by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. The slice category $\mathcal{C}_Y$ is the category of spaces over Y. The constant functor $*_{F(Y)} : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ satisfies $*_{F(Y)}(X) \simeq F(Y) \in \mathcal{D}$ for all object $X \in \mathcal{C}_{/Y}$. The notation $hocolim$ denotes the homotopy colimit, and $holim$ denotes the homotopy limit.

This thesis is arranged as follows. In Chapter 2, we introduce the construction of the Goodwillie tower, including the universal n-excisive approximation and the n-th layer. There is an analogy between ordinary calculus and Goodwillie's functor calculus, see Section 2.1 for details. In Chapter 3, we introduce the one-to-one correspondence between n-homogeneous functors and multilinear symmetric multivariab functors, which is the framework developed by Goodwillie [6]. Chapter 4 is devoted to presenting some examples and applications of Goodwillie towers. Concretely, in Section 4.1, we introduced the Snaith splitting. In Section 4.2, we introduced the Goodwillie tower of the identity functor between spaces, and some results on spheres.

CHAPTER 2 CONSTRUCTION AND BASIC PROPERTIES

This chapter constructs the Goodwillie tower and studies its convergence. Analogous to the Postnikov tower for spaces, the Goodwillie tower decomposes homotopy functors into a sequence of n-excisive approximations. The n-excisive functor satisfies a universal homotopy property, yet directly computing its n-th stage (P_nF) is often difficult. Instead, we analyze the homotopy fiber D_nF (the n-th layer) to understand the tower's behavior on specific spaces. Primary references include [5, 6].

2.1 Goodwillie Calculus

There is an analogy between ordinary calculus and Goodwillie calculus. We can approximate homotopy functors between certain homotopy categories by the Goodwillie tower, as well as use the Taylor series to approximate smooth functions in the ordinary calculus.

Calculus	Goodwillie calculus
Smooth function $f : (y-\epsilon, y+\epsilon) \longrightarrow \mathbb{R}$	Homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$
n-degree polynomial approximation: $f(x) \longmapsto \dfrac{f^{(n)}(x)}{n!}(x-y)^n$	n-excisive approximation: $F(X) \longmapsto P_nF(X)$
Approximation of n-homogeneous function: $\dfrac{f^{(k)}(x)}{k!}(x-y)^k = 0$ for all $k > n$	Approximation of n-homogeneous functor: $P_kF(X) \simeq *$ for all $k < n$
The n-th derivative of function: $f^{(n)}(x) = \lim\limits_{x\longrightarrow y} \dfrac{f^{(n-1)}(x) - f^{(n-1)}(y)}{x-y}$	The n-th layer of functor: $D_nF = hofib\ (P_nF \longrightarrow P_{n-1}F)$

In the Taylor series, we approximate a smooth function by adding polynomial functions of different degrees. In the Goodwillie tower, we approximate a homotopy functor by taking the n-excisive functors as n goes to infinity. Besides, the Taylor series are determined by the n-th derivatives $f^{(n)}$ of functions f, as well as the Goodwillie towers are determined by the n-th layers D_nF of homotopy functors F.

The Goodwillie calculus is also viewed as the calculus in the infinity category. See [24] for more details.

2.2 Polynomial Approximation of Functors

Now we recall some related concepts. Consider the small category $\mathcal{P}(\underline{n})$ whose objects are exactly subsets of $\{1, 2, \cdots, n\}$, morphisms are induced by partially ordered relation "⊆". We denote

$$\underline{n} = \{1, \cdots, n\}$$

$$\mathcal{P}_0(\underline{n}) = \mathcal{P}(\underline{n}) - \emptyset,$$

$$\mathcal{P}_1(\underline{n}) = \mathcal{P}(\underline{n}) - \{1, \cdots, n\}.$$

Definition 2.1: An *n-cube* $\mathcal{X}$ in category $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}(\underline{n}) \longrightarrow \mathcal{C}$.

（1）An n-cube $\mathcal{X}$ is *cocartesian* if the map

$$\mathcal{X}(\{1, \cdots, n\}) \longrightarrow \underset{S \in \mathcal{P}_1(\underline{n})}{hocolim} \mathcal{X}(S)$$

induced by universal property of homotopy colimits is a weak equivalence.

（2）An n-cube $\mathcal{X}$ is strongly cocartesian is its every 2-cube surface is cocartesian.

（3）An n-cube $\mathcal{X}$ is *cartesian* if the map

$$\underset{S \in \mathcal{P}_0(\underline{n})}{holim} \mathcal{X}(S) \longrightarrow \mathcal{X}(\emptyset)$$

induced by universal property of homotopy limits is a weak equivalence.

Notice that if $n = 2$, the cocartesian 2-cube is exactly the homotopy pushout, the cartesian 2-cube is exactly the homotopy pullback. See [25] for more details.

To describe the n-excisive approximation P_nF of functor F, we need to focus on the slice category $\mathcal{C}_{/Y}$ where Y is the terminal object instead of $*$ on $\mathcal{C}$.

Definition 2.2: Let U be a finite set with the discrete topology. Given any object $X \in \mathcal{C}_{/Y}$, the *fiberwise join* $X *_Y U$ is defined by the homotopy pushout

$$\begin{tikzcd} X \times U \arrow[r, "f \times \mathbb{1}_U"] \arrow[d] & Y \times U \arrow[d] \\ X \arrow[r] & X *_Y U \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}.$$

Notice that this diagram

$$
\begin{tikzcd}
X \times U \arrow[r, "f \times \mathbb{1}_U"] \arrow[d, "p_X"'] & Y \times U \arrow[d, "p_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

is homotopy commutative where $p_X : (x,u) \longmapsto x$, $p_Y : (y,u) \longmapsto y$, thus there is an induced map $X *_Y U \longrightarrow Y$ by the universal property of homotopy colimits. Thus $X *_Y U$ is also an object in $\mathcal{C}_{/Y}$.

Example 2.1:

（1）If $U = \{1\}$, then $X *_Y \{1\} = I_f = Y \cup_f (X \times I)$ is the *fiberwise cone*.

（2）If $U = \{1,2\}$, then $X *_Y \{1,2\} = \Sigma_Y X$ is the *fiberwise suspension*.

（3）If $Y = *$, the fiberwise suspension is the (reduced) suspension ΣX.

（4）If $Y = *$, then $X *_Y U$ is an object gluing $|U|$ mapping cones.

（5）Note that there is a natural isomorphism of spaces over Y

$$(X *_Y U) *_Y V \cong X *_Y (U * V)$$

where $U * V$ is the where $U * V$ is the ordinary join of two spaces.

Lemma 2.1: The functor

$$\mathcal{X} : \mathcal{P}(\underline{n+1}) \longrightarrow \mathcal{C}_{/Y},\ U \longmapsto X *_Y U$$

is a strongly cocartesian n-cube in $\mathcal{C}_{/Y}$.

Proof: Every 2-cube in the $(n+1)$-cube $\mathcal{P}(\underline{n+1})$ is cocartesian, thus it is a strongly cocartesian $(n+1)$-cube. Since the functor $X *_Y -$ is a homotopy colimit, it preserves the cocartesian squares. Then the $(n+1)$-cube $\mathcal{X}$ strongly cocartesian. ■

The action of the functor $\mathcal{X}$ for $n = 2$ is given by the following.

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & \emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & \{1\} \arrow[dl] \arrow[dd] \\
\{2\} \arrow[rr, crossing over] \arrow[dd] & & \{1,2\} & \\
 & \{3\} \arrow[rr] \arrow[dl] & & \{1,3\} \arrow[dl] \\
\{2,3\} \arrow[rr] & & \{1,2,3\} \arrow[from=uu, crossing over] &
\end{tikzcd}
\xrightarrow{\ \mathcal{X}\ }
\begin{tikzcd}[row sep=small, column sep=small]
 & X *_Y \emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & X *_Y \{1\} \arrow[dl] \arrow[dd] \\
X *_Y \{2\} \arrow[rr, crossing over] \arrow[dd] & & X *_Y \{1,2\} & \\
 & X *_Y \{3\} \arrow[rr] \arrow[dl] & & X *_Y \{1,3\} \arrow[dl] \\
X *_Y \{2,3\} \arrow[rr] & & X *_Y \{1,2,3\} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

Strongly cocartesian cube　　　　　　Strongly cocartesian cube

Definition 2.3: A homotopy functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is *n-excisive* if $F(\mathcal{X})$ is an cartesian

$(n+1)$-cube for any strongly cocartesian $(n+1)$-cube $\mathcal{X}$ in $\mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be homotopy categories, for each n, $\mathbf{Exc}_n(\mathcal{C},\mathcal{D})$ denotes the homotopy category of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$.

Definition 2.4: Given a homotopy functor $F \in \mathbf{Fun}(\mathcal{C}_{/Y},\mathcal{D})$, the homotopy functor $T_nF \in \mathbf{Fun}(\mathcal{C}_{/Y},\mathcal{D})$ is defined by

$$T_nF(X) = \underset{U\in\mathcal{P}_0(\underline{n+1})}{holim} F(X *_Y U)$$

where $\mathcal{P}_0(\underline{n+1})$ is the nonempty set in $\mathcal{P}(\underline{n+1})$.

Since $X *_Y \emptyset = X$, there is an induced map $u : F(X) \longrightarrow T_nF(X)$ by the universal property of homotopy limit. Also, there is a natural transformation $t_nF : F \longrightarrow T_nF$.

If the homotopy functor F is n-excisive, then there is a weak equivalence

$$F(X) = F(X *_Y \emptyset) \simeq \underset{U\in\mathcal{P}_0(\underline{n+1})}{holim} F(X *_Y U).$$

Therefore we have the following lemma.

Lemma 2.2: If the homotopy functor F is n-excisive, then there is an equivalence

$$t_nF : F \longrightarrow T_nF.$$

Definition 2.5: For a homotopy functor $F \in \mathbf{Fun}(\mathcal{C}_{/Y},\mathcal{D})$, the *$n$-excisive approximation* (also called the *n-th polynomial stage)* $P_nF \in \mathbf{Fun}(\mathcal{C},\mathcal{D})$ is defined to be sequential homotopy colimit of

$$F(X) \xrightarrow{t_nF(X)} T_nF(X) \xrightarrow{t_nT_nF(X)} T_n^2F(X) \longrightarrow \cdots,$$

which is $P_nF(X) = hocolim\ T_n^k(X)$.

Given an n-cube, if every $(n-1)$-cube in it is cartesian, then it is a cartesian n-cube. We have inclusions

$$\mathbf{Exc}_0(\mathcal{C},\mathcal{D}) \subseteq \mathbf{Exc}_1(\mathcal{C},\mathcal{D}) \subseteq \mathbf{Exc}_2(\mathcal{C},\mathcal{D}) \subseteq \cdots \subseteq \mathbf{Exc}_n(\mathcal{C},\mathcal{D}) \subseteq \cdots.$$

Lemma 2.3: If the functor F is n-excisive, then there is a weak equivalence

$$p_nF : F(X) \longrightarrow P_nF(X).$$

Proof: If F is n-excisive, then $F(X) \simeq T_nF(X)$ by Lemma 2.2, thus $T_nF(X)$ is also n-excisive. By induction, we have $F(X) \simeq T_n^kF(X)$ for all k, then we take the homotopy colimit to get $F(X) \simeq P_nF(X)$. ∎

Definition 2.6: Let $\mathcal{J}$ be a small category, if:

（1）for any objects $a, b \in \mathcal{J}$, there is an object $c \in \mathcal{J}$ such that the set of morphisms from a to c and the set of morphisms from b to c are both nonempty.

（2）given objects $a, b, c \in \mathcal{J}$, for any morphisms $i, j : a \longrightarrow b$, there are morhpisms $i', j' : b \longrightarrow c$ such that $i' \circ i = j' \circ j$,

then $\mathcal{J}$ is a *filtered category* or a *filtered diagram*.

The homotopy colimit and homotopy limit on a filtered category are called *filtered homotopy colimit* and *filtered homotopy limit*, respectively. We denote the filtered homotopy colimit and filtered homotopy limit as $hocolim_{fil}$ and $holim_{fil}$, respectively.

Proposition 2.1:

(1) For $F_k \in \mathbf{Fun}(\mathcal{C}, \mathcal{D})$, we have equivalences between functors

$$T_n(holim\ F_k) \simeq holim\ T_n F_k,$$

$$P_n(\underset{k \in \text{ finite } K}{holim} F_k) \simeq \underset{k \in \text{ finite } K}{holim} P_n F_k.$$

(2) For $F_k \in \mathbf{Fun}(\mathcal{C}, \mathcal{D})$, we have equivalences between functors

$$T_n(hocolim_{fil} F_k) \simeq hocolim_{fil} T_n F_k,$$

$$P_n(hocolim_{fil} F_k) \simeq hocolim_{fil} P_n F_k.$$

(3) For $F_k \in \mathbf{Fun}(\mathcal{C}, \mathbf{Sp})$, we have equivalences between functors

$$T_n(hocolim\ F_k) \simeq hocolim\ T_n F_k,$$

$$P_n(hocolim\ F_k) \simeq hocolim\ P_n F_k.$$

Proof: See [6, Proposition 1.7]. ■

Lemma 2.4: For a strongly cocartesian $(n+1)$-cube $\mathcal{X}$ and homotopy functor F, $t_n F(\mathcal{X}) : F(\mathcal{X}) \longrightarrow T_n F(\mathcal{X})$ factors through some cartesian cubes.

Proof: See [6, Lemma 1.9]. ■

Notice that the following theorem by Goodwillie is called the **universal property** of the natural map $p_n F$.

Theorem 2.1 (Goodwillie [6], Theorem 1.8): For any homotopy functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, $P_n F$ is always n-excisive. Moreover, $p_n F : F \longrightarrow P_n F$ is the universal map from F to any n-excisive functor.

Proof: Suppose $\mathcal{X}$ is a strongly cocartesian n-cube, then $P_n F(\mathcal{X})$ can be treated as sequential homotopy colimit of cartesian cubes by Lemma 2.4, thus cartesian and $P_n F$ is n-exsicive.

$$P_n F(X) \xrightarrow{P_n(t_n F)} P_n T_n F(X) = P_n(\underset{S \in \mathcal{P}_0(\underline{n+1})}{holim} F(X *_Y S)) \simeq \underset{S \in \mathcal{P}_0(\underline{n+1})}{holim} P_n F(X *_Y S)$$

$$\simeq P_nF(X *_Y \emptyset) = P_nF(X),$$

thus $P_n(p_nF) : P_nF \longrightarrow P_nP_nF$ is an equivalence. Consider the diagram for another n-excisive functor P.

$$\begin{tikzcd} F \arrow[r, "u"] \arrow[d, "p_nF"'] & P \arrow[d, "p_nP"', "\simeq"] \\ P_nF \arrow[r, "P_nu"] & P_nP \end{tikzcd}$$

Thus, there is an induced map $v = (p_nP)^{-1} \circ P_nu$ which we will prove is unique up to homotopy. By Proposiyion 2.3, there is an equivalence $p_nP_nF : P_nF \longrightarrow P_nP_nF$ since P_nF is n-excisive. By the naturality, $P(p_nF) : P_nF \longrightarrow P_nP_nF$ is also an equivalence. Then consider the diagram

$$\begin{tikzcd} F \arrow[r, "p_nF"] \arrow[d, "p_nF"'] & P_nF \arrow[r, "v"] \arrow[d, "p_nP_nF", "\simeq"'] & P \arrow[d, "p_nP", "\simeq"'] \\ P_nF \arrow[r, "\simeq", "P_n(p_nF)"'] & P_nP_nF \arrow[r, "P_nv"'] & P_nP \end{tikzcd}$$

and v is determined by P_nv which is determined by $P_nv \circ P_n(p_nF) \simeq P_n(v \circ p_nF)$. ■

This theorem tells us the n-excisive approximation is unique up to homotopy. The uniqueness explains why we choose the n-excisive functors to approximate the homotopy functor but not others.

Corollary 2.1: For $0 \leq m \leq n$, $P_m(p_nF) : P_mF \longrightarrow P_mP_nF$ is an equivalence.

Proof: By the universal property and the fact that m-excisive functors are also n-excisive for $m \leq n$. ■

2.3 Homotopy Fiber of $P_nF \longrightarrow P_{n-1}F$

Lemma 2.5: The 0-th excisive approximation is $P_0F \simeq *_{F(Y)}$.

Proof: By Example 2.1 we have $T_0F(X) \simeq F(X *_Y \{1\}) = F(I_f) \simeq F(Y)$ for any $X \in \mathcal{C}_{/Y}$. ■

Since p_nF is universal, and $P_{n-1}F$ is also n-excisive as an $(n-1)$-excisive functor, there is an induced map $q_nF : P_nF \longrightarrow P_{n-1}F$, then we have the following theorem.

Theorem 2.2 (Goodwillie [6], Theorem 1.13): For a homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$, there is a tower of homotopy functors given by the following.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P_nF \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & P_1F \arrow[d, "q_1F"] & \\
F \arrow[uuur, "p_nF"] \arrow[ur, "p_1F"] \arrow[r, "p_0F"'] & P_0F \arrow[r, "\simeq"] & *_{F(Y)}
\end{tikzcd}$$

If there is a weak equivalence

$$F(X) \simeq holim\ P_nF(X),$$

then we say the Goodwillie tower of F *converges* at X.

To study the convergence of the tower, Goodwillie[5] characterizes the E_n condition and the O_n condition, to measure the approximation of functors.

Definition 2.7: A homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ satisfies the $E_n(c,\kappa)$-*condition* (or F is stably n-excisive) if for any strongly cocartesian n-cube $\mathcal{X}$ such that

（1）Maps $X *_Y \emptyset \longrightarrow X *_Y \{i\}$ are k_i-connected,

（2）$k_i \geq \kappa$ for all $1 \leq i \leq n$,

the cube $F(\mathcal{X})$ is $(\sum\limits_{i=1}^{n} k_i - c)$-cartesian.

A natursl transformation $u : F \longrightarrow G$ satisfies the $O_n(c,\kappa)$-*condition* if for any k-connected map $X \longrightarrow Y$ with $k \geq \kappa$, the induced map $F(X) \longrightarrow G(X)$ is $((n+1)k - c)$-connected.

Proposition 2.2: Let F be a homotopy functor satisfying the $E_n(c,\kappa)$-condition. Then the following hold:

(1) T_nF satisfies the $E_n(c-1,\kappa-1)$-condition,

(2) $t_nF : F \longrightarrow T_nF$ satisfies the $O_n(c,\kappa)$-condition,

(3) $p_nF : F \longrightarrow P_nF$ satisfies the $O_n(c,\kappa)$-condition,

(4) P_nF is n-excisive.

Proof: See [6, Proposition 1.4 and 1.5]. ■

A natural problem is what kind of functors converge for some certain $X \in C_{/Y}$, and what kind of conditions they should satisfy. Goodwillie described some conditions about this. This problem is still an open problem, and we will introduce it later.

Definition 2.8: If F satisfies $E_n(n\rho - q, \rho + 1)$-condition for all n and some q, then F is called *ρ-analytic*.

In the following theorem, Goodwillie described that such ρ-analytic functors have convergent Goodwillie towers.

Theorem 2.3 (Goodwillie [5], Theorem 1.13): If the homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ is ρ-analytic and $X \longrightarrow Y$ is $(\rho+1)$-connected, then $p_n F$ satisfies the $O_n(n\rho - q, \rho+1)$-condition. Thus, as $n \longrightarrow \infty$, we have a weak equivalence $F(X) \simeq holim\ P_n F(X)$.

The polynomial approximation functor $P_n F$, defined as the sequential homotopy colimit of finite homotopy limits, is often computationally intractable. To address this, we define the homotopy fiber of $P_n F \longrightarrow P_{n-1} F$, thereby decomposing the n-excisive approximation problem into a layer-wise homotopy computation.

Definition 2.9: Given a homotopy functor F, its *n-th layer* $D_n F$ is defined to be the homotopy functor

$$D_n F = hofib\ (P_n F \xrightarrow{q_n F} P_{n-1} F) = holim\ (P_n F \longrightarrow P_{n-1} F \longleftarrow *).$$

Definition 2.10: A homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ is *n-reduced* if $P_{n-1} F \simeq *$, is *n-homogeneous* if it is both n-excisive and n-reduced.

We denote by $\mathcal{H}_n(\mathcal{C}, \mathcal{D})$ the homotopy category of n-homogeneous functors.

Example 2.2:

（1）A 1-homogeneous functor is alternatively termed *linear*.

（2）A 1-reduced functor is simply called *reduced*.

By Lemma 2.5, we have the following corollary.

Corollary 2.2: Let $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ be a homotopy functor. The following are equivalent:

（1）F is 1-reduced,

（2）$F(Y) \simeq *$.

Lemma 2.6: Given integers $m \leq n$ and a homotopy functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$,

Let $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$ be a homotopy functor. If F is n-reduced for some $n \geq 1$, then for all integers $m \leq n$, F is automatically m-reduced.

Proof: By Corollary 2.1, we have $P_{m-1} F \simeq P_{m-1} P_{n-1} F \simeq *$. ■

Thus for any n-reduced functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$, we have an equivalence $F(Y) \simeq *$.

Example 2.3: Let $f : F_a \longrightarrow F_b$ be a natural transformation from an a-homogeneous functor F_a to a b-homogeneous functor F_b where $a < b$. Then the homotopy fiber $F = hofib\ (F_a \longrightarrow F_b)$ of f is an a-homogeneous functor.

Proof: By Corollary 2.1, we have

$$P_nF_a(X) \simeq \begin{cases} P_nP_{a-1}F_a(X) \simeq * & n \leq a-1 \\ F_a(X) & n \geq a \end{cases},$$

$$P_nF_b(X) \simeq \begin{cases} P_nP_{b-1}F_b(X) \simeq * & n \leq b-1 \\ F_b(X) & n \geq b \end{cases}.$$

Thus we have

$$P_nF \simeq \begin{cases} * & n \leq a-1 \\ F_a & a \leq n \leq b-1 \, . \\ F & n \leq b \end{cases}$$

■

Lemma 2.7: For an n-homogeneous functor $F : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$, we have

$$P_0F \simeq P_1F \simeq \cdots \simeq P_{n-1}F \simeq *,\ P_nF \simeq F, D_nF \simeq F.$$

Proof: Since F is n-reduced, $P_{n-1}F \simeq *$. By Corollary 2.1, $P_kF \simeq P_kP_{n-1}F \simeq *$ for all $0 \leq k \leq n-1$. F is n-excisive, by 2.3, we have $P_nF \simeq F$, $D_nF = hofib\ (F \longrightarrow *) \simeq F$. ■

Proposition 2.3: The n-th layer D_nF is always n-homogeneous.

Proof: By definition, we have

$$\begin{aligned} P_{n-1}D_nF &= P_{n-1}holim\ (P_nF \longrightarrow P_{n-1}F \longleftarrow *) \\ &\simeq holim\ (P_{n-1}P_nF \longrightarrow P_{n-1}P_{n-1}F \longleftarrow *). \end{aligned}$$

By Corollary Lemma 2.1, $P_{n-1}P_{n-1}F \simeq P_{n-1}F \simeq P_{n-1}P_nF$, thus $P_{n-1}D_nF \simeq *$, D_nF is n-reduced. Since P_nF, $P_{n-1}F$ and $*$ are all n-excisive, as a homotopy limit $D_nF = holim\ (P_nF \longrightarrow P_{n-1}F \longleftarrow *)$ is also n-excisive. Then we conclude that D_nF is n-homogeneous. ■

The n-th layer D_nF shares key properties with the n-excisive approximation P_nF since D_nF is a homotopy colimit involving $P_nF, P_{n-1}F$.

Proposition 2.4:

(1) There are natural isomorphisms

$$P_n(F \circ \Sigma_Y) \cong (P_nF) \circ \Sigma_Y,$$

$$D_n(F \circ \Sigma_Y) \cong (D_nF) \circ \Sigma_Y.$$

(2) For $F_k \in \mathbf{Fun}(\mathcal{C}, \mathcal{D})$, we have equivalences between functors

$$D_n(\underset{k\in \text{ finite } K}{holim} F_k) \simeq \underset{k\in \text{ finite } K}{holim} D_n F_k,$$

$$D_n(hocolim_{fil} F_k) \simeq filhocolim\ D_n F_k.$$

(3) For $F_k \in \mathbf{Fun}(\mathcal{C}, (\mathbf{Sp}))$, we have an equivalence between functors

$$D_n(hocolim\ F_k) \simeq hocolim\ D_n F_k.$$

Proof: See [6, Proposition 1.18]. ■

Lemma 2.8: Let F be an $(n-1)$-excisive functor and H be an n-homogeneous functor, then the fiber sequence $F \longrightarrow G \longrightarrow H$ always splits.

Proof: By Proposition 2.1, $P_n F \longrightarrow P_n G \longrightarrow P_n H$ is also a fiber sequence. Since $P_{n-1} H \simeq *$, we have $P_{n-1} F \simeq P_{n-1} G$. Since F is $(n-1)$-excisive, we have $F \simeq P_{n-1} F$ by Lemma 2.3. Consider the following diagrams

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\simeq"'] & G \arrow[d, "p_n G"] \\ P_{n-1}F \arrow[r, "\simeq"'] & P_{n-1}G \end{tikzcd}.$$

It is commutative by the naturality of P_n. Thus, the upper triangle in the diagram

$$\begin{tikzcd} F \arrow[rr, "\simeq"] \arrow[dr, "f"'] & & P_{n-1}G & \\ & G \arrow[ur, "p_n G"] \arrow[rr, dashed, "g"'] & & F \arrow[ul, "\simeq"'] \end{tikzcd}$$

is commutative. Then by the universal property of $P_{n-1} G$, there is an induced map $g : G \longrightarrow F$ such that $g \circ f \simeq \mathbb{1}_F$. Thus, it is splitting. ■

CHAPTER 3 EQUIVALENCE OF NATURAL TRANSFORMATIONS

To compute the n-th layer, Goodwillie [6] showed that n-homogeneous functors correspond bijectively to symmetric multilinear functors. This identifies the n-th layer with a spectrum (the n-th differential), carrying a Σ_n-action. The result offers a powerful computational tool for the Goodwillie tower. (See [5, 6].)

3.1 Delooping of the n-th Layer

Now we introduce how to deloop the n-th layer. Before proceeding, we recall a fundamental result by Goodwillie.

Lemma 3.1 (Goodwillie [6], Lemma 2.2): Given a reduced homotopy functor F : $\mathcal{C}_{/Y} \longrightarrow \mathbf{Top}_*$, up to natural equivalence there is a fiber sequence

$$P_nF \xrightarrow{q_nF} P_{n-1}F \longrightarrow R_nF$$

where R_nF is n-homogeneous.

Proof: When we define the T_nF, we take the $(n+1)$-cube moving out a corner. Then take the homotopy limit to obtain T_nF. This time we are moving out more to get more construction.

$n = 2$

$$B_n = \mathcal{P}_0(\underline{n+1}) - \{n+1\}$$

$$\begin{tikzcd}
 & & X_1 \\
X_2 & X_{12} \arrow[l] \arrow[ur] & \\
 & & X_{13} \arrow[uu] \\
X_{23} \arrow[uu] & X_{123} \arrow[l] \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

$$A_n = \mathcal{P}_0(\underline{n+1}) - \mathcal{P}_0(\underline{n})$$

$$\begin{tikzcd}
 & & X_{13} \\
X_{23} & X_{123} \arrow[l] \arrow[ur] &
\end{tikzcd}$$

As the same way as we get T_n^iF by $\mathcal{P}_0(\underline{n+1})^i$,

$$T_n^iF(X) = \underset{(U_1,\cdots,U_i)\in\mathcal{P}_0(\underline{n+1})^i}{holim} F(X *_Y (U_1 * \cdots * U_i)),$$

we also define

$$B_n^i = \mathcal{P}_0(\underline{n+1})^i - \{n+1\}^i,$$

$$A_{n,i} = \mathcal{P}_0(\underline{n+1})^i - \mathcal{P}_0(\underline{n})^i,$$

to get

$$S_{n-1}^i F(X) = \underset{(U_1,\cdots,U_i)\in B_n^i}{holim} F(X *_Y (U_1 * \cdots * U_i)),$$

$$K_{n,i} F(X) = \underset{(U_1,\cdots,U_i)\in A_{n,i}}{holim} F(X *_Y (U_1 * \cdots * U_i)).$$

Notice that $\mathcal{P}_0(\underline{n})$ is *left cofinal* (see [26] for more details) in B_n, we take the homotopy limit under these index categories

$$\mathcal{P}_0(\underline{n+1}) \supseteq B_n \supseteq \mathcal{P}_0(\underline{n}),$$

we get

$$q_{n,1} : T_n F \longrightarrow S_{n-1} F \xrightarrow{\simeq} T_{n-1} F.$$

Since $\mathcal{P}_0^i(\underline{n})$ is left cofinal in B_n^i, we have an equivalence $S_{n-1}^i F \longrightarrow T_{n-1}^i F$ for all i.

For a sequence of such maps $q_{n,1}, q_{n,2}, \cdots, q_{n,i}, \cdots$, if we take i to infinity, we exactly get the natural transformation $q_n F : P_n F \longrightarrow P_{n-1} F$.

If we take the homotopy limit

$$R_{n,i} F(X) = \underset{(U_1,\cdots,U_i)\in A_{n,i}\cap B_n^i}{holim} F(X *_Y (U_1 * \cdots * U_i)),$$

we get a cartesian square by [5, Lemma 1.9].

$$\begin{tikzcd}
\mathcal{P}_0(\underline{n+1})^i & B_n^i \arrow[l] & & T_n^i F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_{n-1}^i F \arrow[d] \\
A_{n,i} \arrow[u] & A_{n,i} \cap B_n^i \arrow[l] \arrow[u] \arrow[r, phantom, ""] & \arrow[r, "\text{Homotopy limit}"] & K_{n,i} F \arrow[r] & R_{n,i} F
\end{tikzcd}$$

Cartesian cube

We next prove that $K_{n,i} F \simeq *$. The diagram $A_{n,i}^* = \mathcal{P}_0(\underline{n+1}) - B_n^i$ is left cofinal in $A_{n,i}$. Thus

$$K_{n,i} F(X) = \underset{(U_1,\cdots,U_i)\in A_{n,i}}{holim} F(X *_Y (U_1 * \cdots * U_i)) = \underset{(U_1,\cdots,U_i)\in A_{n,i}^*}{holim} F(X *_Y (U_1 * \cdots * U_i))$$

is contractible since there always exists a $|U_j| = 1$ for some j and then $\{1\} * U \simeq *$.

Finally if F is reduced, $F(Y) \simeq *$, we have

$$holim\ F(X *_Y (U_1 * \cdots U_i) \simeq holim\ F(X *_Y *) \simeq holim\ F(Y) \simeq *.$$

We assume that F is not $(n-1)$-excisive, then by Proposition 2.1 and Corollary 2.1 we have

$$P_{n-1}R_{n,i}P_{n-1}F \simeq P_{n-1}P_{n-1}R_{n,i}F \simeq P_{n-1}R_{n,i}F.$$

Thus functor $P_{n-1}R_{n,i}$ coincides on $P_{n-1}F$ and F, without loss of generality, we assume that F is $(n-1)$-excisive. To prove that $R_{n,i}F$ is n-homogeneous, we need to prove $P_nR_{n,i}F \simeq R_{n,i}F$ and $P_{n-1}R_{n,i}F \simeq *$. One way to show this is to prove that, if F is $(n-1)$-excisive, then $R_{n,i}F \simeq *$. There is an isomorphism given by

$$A_{n,i} \cap B_n^i \cong \mathcal{P}_0(\underline{n})^i \times \mathcal{P}_0(\underline{i}),\ (U_1, \cdots, U_i) \longmapsto (V_1, \cdots, V_i, W),$$

where

$$V_i = U_i - \{n+1\},\ W = \{i \mid n+1 \in U_i\}.$$

If $1 \in W$, take $e_1 = \{n+1\}$, if not, take $e_1 = \emptyset$. And do for $2, 3, \cdots$ by induction. Then

$$\begin{aligned} R_{n,i}F(X) &= \underset{(V_1,\cdots,V_i,W)\in\mathcal{P}_0(\underline{n})^i\times\mathcal{P}_0(\underline{i})}{holim} F(X *_Y (U_1 * \cdots * U_i)) \\ &\simeq \underset{W\in\mathcal{P}_0(\underline{i})}{holim} F(X *_Y (e_1 * \cdots * e_i)) \\ &\simeq holim\ F(X *_Y D^k) \end{aligned}$$

for some $k \leq i-1$ since the join of i 1-cells is D^{i-1}.

At last, take the i to infinity in the cartesian cube above, we get a fiber sequence $P_nF \longrightarrow P_{n-1}F \longrightarrow R_nF$. ∎

What the map explicitly looks like before $i \longrightarrow \infty$ has been given, but it is complex to describe what the map looks like while $i \longrightarrow \infty$. For details, see Goodwillie's original article [6]. This theorem means that for the n-homogeneous functor D_nF, there is a delooping $\Omega R_nF \simeq D_nF$. Let $B = R_nD_n^{-1}$, then we have

$$\Omega R_nF = \Omega BD_nF \simeq D_nF.$$

Now we recall some concepts. For any spectrum E with structure map $\Sigma E_n \longrightarrow E_{n+1}$, there is a corresponding map $E_n \longrightarrow \Omega E_{n+1}$ given by

$$[\Sigma E_n, E_{n+1}]_* \cong [E_n, \Omega E_{n+1}]_*.$$

Replace the space E_n by $\Omega^k E_{n+k}$ for a $k \in \mathbb{N}^+$, by induction, there is a sequence

$$\cdots \longrightarrow \Omega^k E_{n+k} \longrightarrow \Omega^{k+1} E_{n+k+1} \longrightarrow \cdots.$$

We define the *omega spectrification* to be the Ω-spectrum E^Ω given by

$$E_n^\Omega = colim\ \Omega^k E_{n+k}.$$

It has the same homotopy groups as spectrum E. We define $\Omega^\infty E = \Omega^\infty E^\Omega \simeq E_0^\Omega$, the 0-th space of spectrum E^Ω, to be the *infinite loop space*, see [27] for more details.

Theorem 3.1 (Goodwillie [6], Theorem 2.1): The homotopy functor

$$\Omega^\infty : \mathcal{H}_n(\mathcal{C}, \mathbf{Sp}) \longrightarrow \mathcal{H}_n(\mathcal{C}, \mathbf{Top}_*)$$

has an inverse up to homotopy, we denote by this inverse B^∞.

Proof: By Lemma 3.1, there is a functor B such that $\Omega BF \simeq F$ for $F \in \mathcal{H}_n(\mathcal{C}, \mathbf{Top}_*)$. Notice that $F \simeq G \Leftrightarrow \Omega F \simeq \Omega G$, then $F \simeq G \Leftrightarrow BF \simeq BG$ under weak equivalence since they have the same homotopy groups for any X. Thus $\Omega^\infty B^\infty F(X) \simeq F(X)$. For $B^\infty \Omega^\infty F(X) \simeq F(X)$ we construct a bispectrum for $F \in \mathcal{H}_n(\mathcal{C}, \mathbf{Sp})$ and spectrum $F(X)$.

$$\begin{tikzcd}
B^0F_0(X) \arrow[r] \arrow[d] & B^1F_0(X) \arrow[r] \arrow[d] & B^2F_0(X) \arrow[r] \arrow[d] & \cdots \\
B^0F_1(X) \arrow[r] \arrow[d] & B^1F_1(X) \arrow[r] \arrow[d] & B^2F_1(X) \arrow[d] & \\
B^0F_2(X) \arrow[r] \arrow[d] & B^1F_2(X) \arrow[r] & B^2F_2(X) & \\
\vdots & & &
\end{tikzcd}$$

Then there are two equivalent spectra $B^0F(X) = F(X)$ and $B^\infty F_0(X) = B^\infty \Omega^\infty F(X)$ from this bispectrum. ■

This will be a huge help in constructing the one-to-one correspondence from n-homogeneous functors to symmetric multilinear functors.

3.2 Multivariable Functors

Now we introduce multivariable functors.

Definition 3.1: Let $F : \mathcal{C}^n \longrightarrow \mathcal{D}$ be a multivariable functor.

（1）F is $(d_1, \cdots, d_n)$-*excisive* if it is d_k-excisive in the k-th variable for all $1 \leq k \leq n$.

（2）F is $(d_1, \cdots, d_n)$-*reduced* if it is d_k-reduced in the k-th variable for all $1 \leq k \leq n$.

（3）F is $(d_1, \cdots, d_n)$-*homogeneous* if it is both $(d_1, \cdots, d_n)$-excisive and $(d_1, \cdots, d_n)$-reduced.

Proposition 3.1: Given a homotopy functor $\Delta : X \longmapsto (X, \cdots, X)$, the compositon $F \circ \Delta : \mathcal{C} \longrightarrow \mathcal{C}^n \longrightarrow \mathcal{D}$ is $(d_1 + \cdots + d_n)$-excisive if $F : \mathcal{C}^n \longrightarrow \mathcal{D}$ is $(d_1, \cdots, d_n)$-excisive.

Proof: See [5, Theorem 4.3]. ■

Let homotopy functor $F : \mathcal{C}^n \longrightarrow \mathcal{D}$ be $(1, \cdots, 1)$-excisive. By Proposition 3.1, the homotopy functor $F \circ \Delta : \mathcal{C} \longrightarrow \mathcal{D}$ is n-excisive.

Proposition 3.2: For any $(1, \cdots, 1)$-reduced homotopy functor F, the map

$$t_{n-1}(F \circ \Delta)(X) : F \circ \Delta(X) \longrightarrow T_{n-1}(F \circ \Delta)$$

factors through a weakly contractible object for any $X \in \mathcal{C}$.

Proof: We define

$$\xi = \{(U_1, \cdots, U_n) \mid U_i \in \mathcal{P}_0(\underline{n})^n, \text{ at least one } s \in \underline{n} \text{ such that } s \in U_s\},$$

$$\xi^* = \{(U_1, \cdots, U_n) \mid U_i \in \mathcal{P}_0(\underline{n})^n, \text{ at least one } s \in \underline{n} \text{ such that } \{s\} = U_s\}$$

where ξ is left cofinal in ξ^*.

Then the map $t_{n-1}(F \circ \Delta)(X) : F \circ \Delta(X) \longrightarrow T_{n-1} F \circ \Delta(X)$ factors as the composition

$$\begin{aligned} F \circ \Delta(X) = F(X, \cdots, X) &\xrightarrow{\cong} F(X *_Y \emptyset, \cdots, X *_Y \emptyset) \\ &\longrightarrow \underset{(U_1, \cdots, U_n) \in \xi}{holim} F(X *_Y U_1, \cdots, X *_Y U_n) \\ &\xrightarrow{\cong} \underset{(U_1, \cdots, U_n) \in \xi^*}{holim} F(X *_Y U_1, \cdots, X *_Y U_n) \\ &\longrightarrow \underset{U \in \mathcal{P}_0(\underline{n})}{holim} F(X *_Y U, \cdots, X *_Y U) = T_{n-1} F \circ \Delta(X). \end{aligned}$$

Since $U_s = \{s\}$, F is $(1, \cdots, 1)$-reduced, we have

$$F(X *_Y U_1, \cdots, X *_Y U_n) = F(X *_Y U_1, \cdots, Y, \cdots, X *_Y U_n) = *.$$

Then we have

$$\underset{(U_1, \cdots, U_n) \in \xi^*}{holim} F(X *_Y U_1, \cdots, X *_Y U_n) = *.$$

■

Proposition 3.3: Given a $(1, \cdots, 1)$-reduced homotopy functor $F : \mathcal{C}^n \longrightarrow \mathcal{D}$, we have an n-reduced functor $F \circ \Delta : \mathcal{C} \longrightarrow \mathcal{D}$.

Proof: By Proposition 3.2, factor all these maps through weakly contractible objects

$$F \circ \Delta(X) \longrightarrow \cdots \longrightarrow T_{n-1}^{i}(F \circ \Delta)(X) \xrightarrow{t_{n-1}T^{i}(F\circ\Delta)} T_{n-1}^{i+1}(F \circ \Delta)(X) \longrightarrow \cdots .$$

Thus, as the homotopy colimit of this, we have $P_{n-1}F \circ \Delta \simeq *$. ■

Definition 3.2: A $(1, \cdots, 1)$-homogeneous functor $F : \mathcal{C}^n \longrightarrow \mathcal{D}$ is called *multilinear*.

By Proposition 3.1 and 3.3, $F \circ \Delta$ is n-homogeneous for any multilinear functor F. This implies that the n-th layer can be treated as a composition of the diagonal Δ and a multilinear functor.

Definition 3.3: The multivariable functor $L : \mathcal{C}^n \longrightarrow \mathcal{D}$ is *symmetric* if it has additional structure consisting of isomorphisms $L(\sigma) : L(X_1, \cdots, X_n) \longrightarrow L(X_{\sigma(1)}, \cdots, X_{\sigma(n)})$ with $L(\sigma \circ \pi) = L(\pi) \circ L(\sigma)$ for $\sigma, \pi \in \Sigma_n$.

If symmetric functor $L : \mathcal{C}^n \longrightarrow \mathcal{D}$ is $(1, \cdots, 1)$-homogeneous (multilinear), then

$$L \circ \Delta : \mathcal{C} \longrightarrow \mathcal{C}^n \longrightarrow \mathcal{D},\ X \longmapsto (X, \cdots, X) \longmapsto L(X, \cdots, X)$$

is n-homogeneous, and it has a compactible Σ_n-action. We denote the homotopy category of symmetric multilinear functors as $\mathcal{L}_n(\mathcal{C}, \mathcal{D})$.

Definition 3.4: The universal covering of topological group G is defined as $EG = *_\infty G$ with free action $G \times EG \longrightarrow EG$, by Milnor [28]. Then we define the classifying space to be

$$BG = EG/G = \{\mathrm{Orb}(c) \mid x \in EG\}.$$

Given a space X with a G-action, the *homotopy orbit* of X is

$$X_{hG} = (EG \times X)/G,$$

or the homotopy colimit

$$X_{hG} = hocolim\ (BG \longleftarrow EG \times X \longrightarrow *).$$

Proposition 3.4: Given a spectrum-valued symmetric multilinear functor $L : \mathcal{C}^n \longrightarrow$ **Sp**, there is an n-homogeneous functor $\Delta_n L$ given by

$$\Delta_n L(X) = L(X, \cdots, X)_{h\Sigma_n}.$$

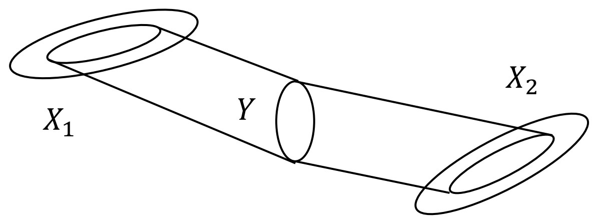
Proof: By Proposition 3.1 and 3.3, the functor $L \circ \Delta : X \longmapsto (X, \cdots, X) \longmapsto L(X, \cdots, X)$ is n-homogeneous. By Proposition 2.1, we know that the homotopy orbit as a homotopy colimit preserves the property of n-homogeneous. Then $\Delta_n L$ is also n-homogeneous. ■

Given a spectrum-valued symmetric multilinear functor, the homotopy orbit construction naturally defines an n-homogeneous functor. This shows how explicitly the spectrum-valued symmetric multilinear functor becomes an n-homogeneous functor. Conversely, Goodwillie showed how n-homogeneous functors become symmetric multilinear functors, by cross effect. That is exactly the one-to-one correspondence between n-homogeneous and symmetric multilinear functors.

In the category $\mathbf{Top}_{*/Y}$, the zero object is exactly Y. For any object $f : X \longrightarrow Y$, there is always a unique map $f' : Y \longrightarrow X$.

Definition 3.5: Given objects $f_1 : X_1 \longrightarrow Y$, $f_2 : X_2 \longrightarrow Y$ of the category $\mathbf{Top}_{*/Y}$, the *fiberwise whisker* is $X_1 + X_2 = I_{f_1'} \cup I_{f_2'}$.

$X_1 + X_2 :$



Notice that the whisker $X_1 + X_2$ is also an object in $\mathbf{Top}_{*/Y}$. With more objects of $\mathbf{Top}_{*/Y}$, we can also define

$$\sum_{i=1}^{k} X_i = X_1 + \cdots + X_k.$$

Example 3.1: If $Y = *$, then $X_1 + X_2 \simeq X_1 \vee X_2$.

Definition 3.6: Given a homotopy functor $F : \mathbf{Top}_{*/Y} \longrightarrow \mathcal{D}$, we define the *$n$-th cross effect* to be the multivariable functor $cr_n F : \mathbf{Top}_{*/Y}^{n} \longrightarrow \mathcal{D}$ given by

$$cr_n F(X_1, \cdots, X_n) = hofib\ (holim\ (F(S(X_1, \cdots, X_n)) - F(\sum_{i=1}^{n} X_i)) \longrightarrow F(\sum_{i=1}^{n} X_i))$$

where $\mathcal{D} = \mathbf{Sp}$ or $\mathbf{Top}_*$, and $S(X_1, \cdots, X_n)$ is a strongly cocartesian cube

$$\mathcal{P}(\underline{n}) \longrightarrow \mathbf{Top}_{*/Y},\ U \longmapsto \sum_{i \in \{1,\cdots,n\} - U} X_i.$$

Acturally, $hofib\ (holim\ (F(S(X_1, \cdots, X_n)) - F(\sum_{i=1}^{n} X_i)) \longrightarrow F(\sum_{i=1}^{n} X_i))$ is called the *total fiber*, see [25] for more details.

Lemma 3.2: If F is $(n-1)$-excisive, then $cr_n F \simeq *$.

Proof: Since $S(X_1, \cdots, X_n)$ is a strongly n-cube, then $F(S(X_1, \cdots, X_n))$ is cartesian, thus the total fiber $cr_n F(X_1, \cdots, X_n) \simeq *$. ■

The following proposition means that $cr_n F \simeq cr_n D_n F$ for any n-excisive functor F. That is, the n-th cross effect only captures the n-th layers of homotopy functors.

Proposition 3.5: If F is n-excisive, then $cr_n F \simeq cr_n D_n F$.

Proof: Since cr_n is a homotopy limit as the total fiber, by Proposition 2.4, we have

$$cr_n D_n F = cr_n hofib\ (P_n F \longrightarrow P_{n-1} F) \simeq hofib\ (cr_n P_n F \longrightarrow cr_n P_{n-1} F).$$

By Lemma 3.6, we have $cr_n P_{n-1} F \simeq *$. Thus we have $cr_n D_n F \simeq cr_n P_n F \simeq cr_n F$ since F is n-excisive. ■

Obviously, there is an isomorphism $S(X_1, \cdots, X_n) \cong S(X_{\sigma(1)}, \cdots, X_{\sigma(n)})$ for any $\sigma \in \Sigma_n$. Then we have the following lemma.

Lemma 3.3: The n-th cross effect $cr_n F$ is symmetric.

The following lemma tells us the 1-st cross effect is always 1-reduced.

Lemma 3.4: The 1-st cross effect $cr_1 F(X) = hofib\ (F(X) \longrightarrow F(Y))$ is 1-reduced.

Proof: By Lemma 2.5, we have an equivalence $P_0 cr_1 F \simeq *_{cr_1 F(Y)} \simeq *$. ■

By this lemma, we know that the n-th cross effect $cr_n F$ is $(1, \cdots, 1)$-reduced.

Proposition 3.6: Give an n-excisive functor $F : \mathbf{Top}_{*/Y} \longrightarrow \mathcal{D}$, the m-th cross effect $cr_m F : \mathbf{Top}_{*/Y}^{\ m} \longrightarrow \mathcal{D}$ is $(n-m+1, \cdots, n-m+1)$-excisive for $0 \leq m \leq n$.

Proof: When $m = 0$, this is trivial. We do the induction, assume that it holds for $m = k$, to prove this holds for $m = k+1$. Notice that

$$cr_{k+1} F(X_1, \cdots, X_k, A) \cong cr_k F_{+A}(X_1, \cdots, X_k)$$

where $F_{+A} = hofib\ (F(X+A) \longrightarrow F(X))$. And F is n-excisive implies F_{+A} is $(n-1)$-excisive. Then we prove this holds for $m = k+1$. ■

By this proposition, we know that $cr_n F$ is $(1, \cdots, 1)$-excisive, since it is $(1, \cdots, 1)$-reduced, we have the following proposition.

Proposition 3.7: Given a homotopy functor F, the n-th cross effect $cr_n F$ is symmetric multilinear.

Theorem 3.2 (Goodwillie [6], Proposition 3.4): Given n-homogeneous functors $F, G : \mathcal{C}_{/Y} \longrightarrow \mathcal{D}$, $F \simeq G$ if and only if $cr_n F \simeq cr_n G$.

By this theorem, with Proposition 3.4 and 3.7 we have this following corollary.

Corollary 3.1:

(1) For $\mathcal{D} = \mathbf{Sp}$ or $\mathbf{Top}_*$, there is a homotopy functor

$$cr_n : \mathcal{H}_n(\mathbf{Top}_{*/Y}, \mathcal{D}) \longrightarrow \mathcal{L}_n(\mathbf{Top}_{*/Y}, \mathcal{D}).$$

(2) For $\mathcal{C} = \mathbf{Top}_{*/Y}$ or $\mathbf{Top}_{/Y}$, there is a homotopy functor

$$\Delta_n : \mathcal{L}_n(\mathcal{C}, \mathbf{Sp}) \longrightarrow \mathcal{H}_n(\mathcal{C}, \mathbf{Sp}).$$

3.3 The Correspondence

Now we establish the one-to-one correspondence framework through a series of following theorems by Goodwillie. The following theorem tells us that functors cr_n and Δ_n are inverses of each other.

Theorem 3.3 (Goodwillie [6], Theorem 3.5): There are homotopy functors mutual inverses up to natural equivalence

$$cr_n : \mathcal{H}_n(\mathbf{Top}_{*/Y}) \longrightarrow \mathcal{L}_n(\mathbf{Top}_{*/Y}),$$

$$\Delta_n : \mathcal{L}_n(\mathbf{Top}_{*/Y}) \longrightarrow \mathcal{H}_n(\mathbf{Top}_{*/Y}).$$

We now introduce some definitions coming from the book [29]. Let V_n be a linear space, define

$$S(V_n) = \{(x_1, \cdots, x_n) \mid \sum_i x_i^2 = 1\},$$

$$D(V_n) = \{(x_1, \cdots, x_n) \mid \sum_i x_i^2 \leq 1\}.$$

Then define the sphere over V_n as $S^{V_n} \cong D(V_n)/S(V_n)$. Given a based space X, define the space $\Omega^{V_n}X = \mathrm{Map}_*(S^{V_n}, X)$. It is a generalization of the loop space $\Omega^n X$, corresponding to the extension of the definition of S^n to S^{V_n}.

In the equivariant homotopy theory, Ω^{V_n} serves as an important homotopy functor of study. Similarly, S^{V_n} generalizes the classical S^n by embedding it within the framework of representation theory, where the group action induces additional structure on the sphere. These generalized constructions are fundamental in equivariant homotopy theory, see [29, 30] for more details.

There is a delooping for the symmetric multilinear functors. Like we make a delooping construction for n-homogeneous functors. This time we treat n as $1 + \cdots + 1$.

Theorem 3.4 (Goodwillie [6], Proposition 3.7): There exists a homotopy functor $\Omega_*^\infty : \mathcal{L}_n(\mathcal{C}, \mathbf{Sp}) \longrightarrow \mathcal{L}_n(\mathcal{C}, \mathbf{Top}_*)$. It has an inverse up to homotopy.

Proof: For functor $L : \mathcal{C}_{/Y_1} \times \cdots \times \mathcal{C}_{/Y_n} \longrightarrow \mathcal{D}$, we can define

$$T_{d_1,\cdots,d_n} L(X_1, \cdots, X_n) = \underset{(U_1,\cdots,U_n) \in \mathcal{P}_0(\underline{d_1+1}) \times \cdots \times \mathcal{P}_0(\underline{d_n+1})}{holim} F(X_1 *_{Y_1} U_1, \cdots, X_n *_{Y_n} U_n).$$

Thus if L is symmetric multilinear, then

$$L(X_1, \cdots, X_n) \simeq T_{1,\cdots,1}(X_1, \cdots, X_n).$$

By definition, $T_1F \simeq \Omega F\Sigma_Y$ for homotopy functor F. Thus

$$L(X_1,\cdots,X_n) \simeq T_{1,\cdots,1}L(X_1,\cdots,X_n) \simeq \Omega^n L(\Sigma_Y X_1,\cdots,\Sigma_Y X_n).$$

But we need Ω^n to be compatible with symmetric property, thus rewrite Ω^n as Ω^{V_n} where $V_n = \{(x_1,\cdots,x_n) \mid x_1+\cdots+x_n = 0\}$ is the standard representation of Σ_n. Thus there is an equivalence $L \simeq \Omega BL$ where

$$BL(X_1,\cdots,X_n) = \Omega^{\overline{V_n}} L(\Sigma_Y X_1,\cdots,\Sigma_Y X_n)$$

and $V_n \cong \mathbb{R} \oplus \overline{V_n}$. Then we consider the same thing with Theorem 3.1 to get the inverse B_*^∞. ■

Proposition 3.8: The n-homogeneous homotopy functor $F : \mathbf{Top}_{*/Y} \longrightarrow \mathcal{D}$ is determined by $F \circ \Sigma_Y$.

Proof: By 3.2, F is determined by cr_nF, and we have

$$cr_nF(X_1,\cdots,X_n) \simeq \Omega^{V_n} cr_nF(\Sigma_Y X_1,\cdots,\Sigma_Y X_n) \cong \Omega^{V_n} cr_n(F\circ\Sigma_Y)(X_1,\cdots,X_n).$$

■

Theorem 3.5 (Goodwillie [6], Theorem 4.1): The following four functors all have inverses up to weak equivalence :

$$\phi_* : \mathcal{H}_n(\mathbf{Top}_{/Y},\mathbf{Sp}) \longrightarrow \mathcal{H}_n(\mathbf{Top}_{*/Y},\mathbf{Sp}),$$

$$\phi_* : \mathcal{H}_n(\mathbf{Top}_{/Y},\mathbf{Top}_*) \longrightarrow \mathcal{H}_n(\mathbf{Top}_{*/Y},\mathbf{Top}_*),$$

$$\phi_* : \mathcal{L}_n(\mathbf{Top}_{/Y},\mathbf{Sp}) \longrightarrow \mathcal{L}_n(\mathbf{Top}_{*/Y},\mathbf{Sp}),$$

$$\phi_* : \mathcal{L}_n(\mathbf{Top}_{/Y},\mathbf{Top}_*) \longrightarrow \mathcal{L}_n(\mathbf{Top}_{*/Y},\mathbf{Top}_*)$$

where $\phi : \mathbf{Top}_{*/Y} \longrightarrow \mathbf{Top}_{/Y}$ is the forgetful functor.

With these theorems established, we can ultimately construct a comprehensive framework that establishes a one-to-one correspondence between symmetric multilinear functors and n-homogeneous functors across different categories.

The one-to-one correspondence is given by this commutative diagram 3-1.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{H}_n\left(\mathbf{Top}_{/Y}, \mathbf{Sp}\right) \arrow[r, dashed, bend left, "\text{Induced}"] \arrow[d, bend left, "\phi_*"] \arrow[ddd, bend right=60, "\Omega_*^{\infty}"'] & \mathcal{L}_n\left(\mathbf{Top}_{/Y}, \mathbf{Sp}\right) \arrow[l, bend left, "\Delta_n"] \arrow[d, bend left, "\phi_*"] \arrow[ddd, bend left=60, "\Omega_*^{\infty}"] \\
\mathcal{H}_n\left(\mathbf{Top}_{*/Y}, \mathbf{Sp}\right) \arrow[u, dashed, bend left] \arrow[r, bend left, "cr_n"] \arrow[d, bend left, "\Omega_*^{\infty}"] & \mathcal{L}_n\left(\mathbf{Top}_{*/Y}, \mathbf{Sp}\right) \arrow[u, dashed, bend left] \arrow[l, bend left, "\Delta_n"] \arrow[d, bend left, "\Omega_*^{\infty}"] \\
\mathcal{H}_n\left(\mathbf{Top}_{*/Y}, \mathbf{Top}_*\right) \arrow[u, bend left, "B_*^{\infty}"] \arrow[r, bend left, "cr_n"] \arrow[d, dashed, bend left] & \mathcal{L}_n\left(\mathbf{Top}_{*/Y}, \mathbf{Top}_*\right) \arrow[u, bend left, "B_*^{\infty}"] \arrow[l, dashed, bend left, "\text{Induced}"] \arrow[d, dashed, bend left] \\
\mathcal{H}_n\left(\mathbf{Top}_{/Y}, \mathbf{Top}_*\right) \arrow[u, bend left, "\phi_*"] \arrow[uuu, bend left=20, "B_*^{\infty}"] & \mathcal{L}_n\left(\mathbf{Top}_{/Y}, \mathbf{Top}_*\right) \arrow[u, bend left, "\phi_*"] \arrow[uuu, bend right=20, "B_*^{\infty}"']
\end{tikzcd}
$$

Figure 3-1

In this diagram, Ω_*^{∞} between n-homogeneous functors comes from definition of infinity loop space, Ω_*^{∞} between symmetric multilinear functors comes from Theorem 3.1. As the inverse, B_*^{∞} between n-homogeneous functors comes from Theorem 3.4, B_*^{∞} between symmetric multilinear functors comes from Theorem 3.1. At last, the inverse of ϕ_* is from Theorem 3.5.

3.4 Differentials and Derivatives

Next, we can investigate how to associate the n-th layer D_nF to a symmetric multilinear functor explicitly. This approach clarifies the behavior of D_nF.

Definition 3.7: For the n-homogeneous functor $D_nF \in \mathcal{H}_n(\mathcal{C}_{/Y}, \mathcal{D})$, its correspondence in $\mathcal{L}_n(\mathcal{C}_{/Y}, \mathcal{D})$ is called *n-fold differential*, denoted by $D^{(n)}F$. The n-fold differential $D^{(n)}F \in \mathcal{L}_n(\mathcal{C}_{/Y}, \mathcal{D})$ determines D_nF by

$$D_nF(X) \simeq \Delta_n(D^{(n)}F)(X) = (D^{(n)}F)(X, \cdots, X)_{h\Sigma_n} \text{ for } \mathcal{D} = \mathbf{Sp}. \tag{3-1}$$

Then we consider the correspondence framework 3-1, to formularize the n-fold differential when $\mathcal{D} = \mathbf{Top}_*$.

$$\begin{tikzcd}
\Delta_n D^{(n)}F \arrow[rr, bend left, "cr_n"] \arrow[d, bend left, "\Omega^\infty_*"] & & D^{(n)}F \arrow[ll, bend left, "\Delta_n"] \arrow[d, bend left, "\Omega^\infty_*"] \\
\Omega^\infty \Delta_n D^{(n)}F \arrow[u, bend left, "B^\infty_*"] \arrow[rr, bend left, "cr_n"] & & \Omega^\infty D^{(n)}F \arrow[u, bend left, "B^\infty_*"] \arrow[ll, bend left, dashed, "\text{Induced}"]
\end{tikzcd}$$

There is a weak equivalence

$$(B^\infty D_nF)(X) \simeq (B^\infty D^{(n)}F)(X, \cdots, X)_{h\Sigma_n}.$$

Thus, we have

$$D_nF(X) \simeq \Omega^\infty (B^\infty D^{(n)}F)(X, \cdots, X)_{h\Sigma_n}.$$

Given a homotopy functor F, the n-fold differential $D^{(n)}F$ determines the n-th layer D_nF. On the other hand, the n-th layer D_nF also determines the n-fold differential by

$$D^{(n)}F \simeq cr_n D_nF.$$

Definition 3.8: A homotopy functor is *finitary* if it preserves filtered homotopy colimit.

Lemma 3.5: Given a spectrum C, the homotopy functor

$$F : \mathbf{Top}_*^n \longrightarrow \mathbf{Sp}, \ (X_1, \cdots, X_n) \longmapsto C \wedge (X_1 \wedge \cdots \wedge X_n)$$

is multilinear. If C has a Σ_n-action, then F is symmetric multilinear.

Conversely, given a multilinear functor $L : \mathbf{Top}_*^n \longrightarrow \mathbf{Sp}$ and a spectrum $C =$

$L(S^0, \cdots, S^0)$, there is an equivalence

$$L(S^0, \cdots, S^0) \wedge (X_1 \wedge \cdots \wedge X_n) \longrightarrow L(X_1, \cdots, X_n)$$

for finite CW complexes $X_1, \cdots, X_n$. If L is finitary, then it is an equivalence for any $X_1, \cdots, X_n$. If L is symmetric, then the spectrum $L(S^0, \cdots, S^0)$ has a Σ_n-action.

By this lemma, if we take $L = D^{(n)}F$ to be the n-fold differential, we get a spectrum $D^{(n)}F(S^0, \cdots, S^0)$ with a Σ_n-action. Thus for any homotopy functor $F : \mathbf{Top}_* \longrightarrow \mathbf{Sp}$, we have an equivelence

$$D_nF(X) \simeq D^{(n)}F(X, \cdots, X)_{h\Sigma_n} \simeq (\partial^{(n)}F(*) \wedge X^{\wedge n})_{h\Sigma_n} \tag{3-2}$$

where $\partial^{(n)}F(*) \simeq D^{(n)}F(S^0, \cdots, S^0)$.

Morever, for any homotopy functor $F : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$, we have an equivalence

$$D_nF(X) \simeq \Omega^\infty(B^\infty D^{(n)}F)(X, \cdots, X)_{h\Sigma_n} \simeq \Omega^\infty(\partial^{(n)}F(*) \wedge X^{\wedge n})_{h\Sigma_n}$$

where $\partial^{(n)}F(*) \simeq (B^\infty D^{(n)}F)(S^0, \cdots, S^0)$.

Definition 3.9: Let F be a homotopy functor from $\mathbf{Top}_*$ to $\mathbf{Top}_*$ or $\mathbf{Sp}$. The n-th layer D_nF is governed by a spectrum $\partial^{(n)}F(*)$ with a Σ_n-action, called the *n-th derivative* of F at $*$.

Define the homotopy functor F_Y to be the composition

$$F_Y = F \circ \phi : \mathbf{Top}_{/Y} \longrightarrow \mathbf{Top} \longrightarrow \mathcal{D}.$$

For the homotopy functor $F_Y : \mathbf{Top}_{/Y} \longrightarrow \mathbf{Sp}$, we have an equivalence

$$D_nF_Y(X) \simeq D_Y^{(n)}F(X, \cdots, X)_{h\Sigma_n} \simeq (\partial^{(n)}_{y_1,\cdots,y_n}(Y) \wedge X^{\wedge n})_{h\Sigma_n}$$

where $\partial^{(n)}_{y_1,\cdots,y_n}F(Y) \simeq D_Y^{(n)}F(S^0 \vee_{y_1} Y, \cdots, S^0 \vee_{y_n} Y)$.

Definition 3.10: Let F be a homotopy functor from $\mathbf{Top}$ to $\mathbf{Top}_*$ or $\mathbf{Sp}$. The n-th layer D_nF is governed by a spectrum $\partial^{(n)}_{y_1,\cdots,y_n}F(Y)$ with a Σ_n-action, called the *n-th derivative of F at $(Y, y_1, \cdots, y_n)$*.

CHAPTER 4 APPLICATIONS

In this chapter, we introduce some applications of Goodwillie towers. Notice that all homotopy functors in this chapter are reduced and finitary. We denote the suspension spectrum of based space X as $\Sigma^\infty X$.

4.1 Snaith Splitting

Snaith [16] gave the decomposition of space $\Omega^n\Sigma^n X$ for a connected based space X. Goodwillie described the Snaith splitting of spectrum $\Sigma^\infty\Omega\Sigma X$ as the following.

Theorem 4.1 (Goodwillie [6], Example 1.20): Given a connected based sapce X, there is a weak equivalence

$$\Sigma^\infty\Omega\Sigma X \simeq \prod_{n\geq 1}\Sigma^\infty X^{\wedge n}.$$

Notice that the functors $(X,\cdots,X)\longmapsto \Sigma^\infty X^{\wedge n}$ are multilinear for each n. Thus by Proposition 3.1 and 3.3, the functors $X\longmapsto \Sigma^\infty X^{\wedge n}$ are n-homogeneous for each n.

Proposition 4.1: There are weak equivalences for each n

$$P_n\Sigma^\infty\Omega\Sigma X \simeq \prod_{i=1}^{n}\Sigma^\infty X^{\wedge i},$$

$$D_n\Sigma^\infty\Omega\Sigma X \simeq \Sigma^\infty X^{\wedge n}.$$

Proof: For the m-homogeneous functor $\Sigma^\infty X^{\wedge m}$, we have $P_{m-1}\Sigma^\infty X^{\wedge m}\simeq *$. By Corollary 2.1, we have

$$*\simeq P_nP_{m-1}\Sigma^\infty X^{\wedge m}\simeq P_n\Sigma^\infty X^{\wedge m}$$

for $n\leq m-1$. Thus $P_1\Sigma^\infty X^{\wedge m}\simeq\cdots\simeq P_{m-1}\Sigma^\infty X^{\wedge m}\simeq *$. Thus we have a weak equivalence

$$P_n\Sigma^\infty\Omega\Sigma X \simeq \prod_{i=1}^{n}\Sigma^\infty X^{\wedge i}$$

according to Theorem 4.1. By definitions, we have the weak equivalence

$$D_n\Sigma^\infty\Omega\Sigma X \simeq hofib\,(\prod_{i=1}^{n}\Sigma^\infty X^{\wedge i}\longrightarrow\prod_{i=1}^{n-1}\Sigma^\infty X^{\wedge i})\simeq\Sigma^\infty X^{\wedge n}.$$

■

The Snaith splitting is an important result in algebraic topology that describes the decomposition properties of spectra. Specifically, certain spectra can be decomposed into direct sums of simpler spectra, thereby simplifying the analysis of their homotopical structures.

Example 4.1: Take $X = S^1$, by Theorem 4.1, we have

$$P_n\Sigma^\infty\Omega S^2 \simeq \prod_{i=1}^{n}\Sigma^\infty S^i = \prod_{i=1}^{n}\mathbb{S}^{(i)}, \quad holim\, P_n\Sigma^\infty\Omega S^2 \simeq \prod_{n\geq 1}\mathbb{S}^{(n)} \simeq \Sigma^\infty\Omega S^2,$$

where $\mathbb{S}^{(i)}$ is the suspension spectrum starts with i dimensional sphere S^i.

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& P_n\Sigma^\infty\Omega S^2 \simeq \prod_{i=1}^{n}\mathbb{S}^{(i)} \arrow[d] \\
& \vdots \arrow[d] \\
& P_2\Sigma^\infty\Omega S^2 \simeq \mathbb{S}\times\mathbb{S}^{(2)} \arrow[d] \\
\Sigma^\infty\Omega S^2 \arrow[uuur] \arrow[ur] \arrow[r] \arrow[dr] & P_1\Sigma^\infty\Omega S^2 \simeq \mathbb{S} \arrow[d] \\
& *
\end{tikzcd}$$

The core idea of the Snaith splitting is to construct the decomposition of spectra. Through this decomposition, we can break down the homotopical information of complex infinite loop space spectra into simpler parts, making it easier to compute their homotopy groups and homology groups.

The Snaith splitting has wide applications in stable homotopy theory. For example, McCarthy [22] studied the decomposition of relative K-theory, which has a convergent Goodwillie tower. Additionally, this splitting offers important tools for understanding the relationships between objects in generalized homology theories and the stable homotopy category.

Proposition 4.2: The n-th layer of functor $F : \mathbf{Top}_* \longrightarrow \mathbf{Sp}$, $X \longmapsto \Sigma^\infty \Omega X$ is

$$D_n F : X \longmapsto \Omega^n \Sigma^\infty X^{\wedge n}.$$

Proof: By Proposition 2.4, there is an equivalence

$$D_n F(\Sigma X) \simeq D_n(F \circ \Sigma)(X).$$

By the last example we have $\Sigma^\infty \Omega \Sigma X \simeq \prod_{n \geq 1} \Sigma^\infty X^{\wedge n}$. Since $\Sigma^\infty X^{\wedge n}$ is n-homogeneous, the n-th layer of $F \circ \Sigma$ is $\Sigma^\infty X^{\wedge n}$.

Thus we have a weak equivalence

$$D_n F(\Sigma X) \simeq D_n(F \circ \Sigma)(X) \simeq \Sigma^\infty X^{\wedge n} \simeq \Omega^n \Sigma^\infty (\Sigma X)^{\wedge n}$$

by $\Sigma(X \wedge X) \simeq \Sigma X \wedge X \simeq X \wedge \Sigma X$. Then by Proposition 3.8, there is an equivalence

$$D_n F(X) \simeq \Omega^n \Sigma^\infty X^{\wedge n},$$

which completes the proof. ■

4.2 The Identity Functor

We consider some results for the identity functor $I : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$. We start with a famous theorem.

Theorem 4.2 (Johnson [31]): The n-th derivative $\partial_n I(*)$ of the identity functor $I : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$ is

$$\partial_n I(*) \simeq (\mathbb{S}^{(1-n)})^{\wedge (n-1)!}.$$

By this theorem, we can calculate the layers of the identity functor I. By definitions, we have

$$T_1 I(X) \simeq \Omega \Sigma X,\ T_1^n I(X) \simeq \Omega^n \Sigma^n X,$$

$$P_1 I(X) \simeq hocolim_{fil} \Omega^n \Sigma^n X \simeq \Omega^\infty \Sigma^\infty X.$$

Since $P_0 I(X) \simeq F(*) = *$, we have

$$D_1 I(X) \simeq P_1 I(X) \simeq \Omega^\infty \Sigma^\infty X.$$

By $\partial_2 I(*) \simeq \mathbb{S}^{(-1)}$, we have

$$D_2 I(X) \simeq \Omega^\infty ((\mathbb{S}^{(-1)} \wedge (X \wedge X))_{h\Sigma_2}) \simeq \Omega^\infty (\Sigma^{-1} \Sigma^\infty (X \wedge X)_{h\Sigma_2}).$$

By $\partial_3 I(*) \simeq (\mathbb{S}^{(-2)})^{\wedge 2}$, we have

$$D_3 I(X) \simeq \Omega^\infty((\mathbb{S}^{(-2)} \wedge X^{\wedge 3})_{h\Sigma_3}) \simeq \Omega^\infty(\Sigma^{-2}\Sigma^\infty X^{\wedge 3}_{h\Sigma_3}).$$

By Lemma 3.1, there is a fiber sequence $P_n I \longrightarrow P_{n-1} I \longrightarrow R_n I$. Then the n-excisive approximation $P_n I \simeq hofib\,(P_{n-1} I \longrightarrow B\Omega^\infty(\Sigma^{-1}\Sigma^\infty X^{\wedge n}_{h\Sigma_n}))$. Then by induction, we have the following Goodwillie tower of the identity functor.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P_n I(X) \arrow[d, dotted] & D_n I(X) \simeq \Omega^\infty\left(\Sigma^{1-n}\Sigma^\infty X^{\wedge n}_{h\Sigma_n}\right) \arrow[l] \\
 & P_2 I(X) \arrow[d] & D_2 I(X) \simeq \Omega^\infty\left(\Sigma^{-1}\Sigma^\infty X^{\wedge 2}_{h\Sigma_2}\right) \arrow[l] \\
I(X) \arrow[r] \arrow[ur] \arrow[uur] & P_1 I(X) & D_1 I(X) \simeq \Omega^\infty\Sigma^\infty X \arrow[l]
\end{tikzcd}$$

4.2.1 v_k-Periodic Homotopy Theory

We recall some concepts. References are [15, 32–34].

Definition 4.1: For each $n \in \mathbb{N}$, the *Morava K-theory* $K(n)_\bullet$ at prime p is a generalized cohomology theory whose coefficient ring is

$$K(n)_\bullet(*) = \mathbb{F}_p[v_n, v_n^{-1}]$$

where $|v_n| = 2(p^n - 1)$. A based space V is *of type k* if $K(n)_\bullet(V) = 0$ for $n < k$, and $K(k)_\bullet(V) \neq 0$.

Mitchell [35] established that finite type k spaces exist for every $k \geq 0$.

Definition 4.2: A *v_k-self map* of a finite based space V is a map $v : \Sigma^d V \longrightarrow V$ with a given integer d such that

$$K(n)_\bullet(v) : K(n)_\bullet(V) \longrightarrow K(n)_\bullet(\Sigma^d V)$$

is an isomorphism for $n = k$ and nilpotent for all $n \neq k$.

Hopkins and Smith [36] showed that every type k space which admits a v_k self-map after suspending it sufficiently many times.

Since $K(n)_\bullet(V) \cong K(n)_{\bullet+1}(\Sigma V)$, we have the following lemma.

Lemma 4.1: For a given space V of type k, the t-times suspension $\Sigma^t V$ is of type k.

That is, for a given k, there always exists a space V of type k admits a v_k-self map $v_k : V \longrightarrow \Sigma^d V$. Then for any other based space X, there is a sequence

$$\mathrm{Map}_*(V,X) \xrightarrow{v_k^*} \mathrm{Map}_*(\Sigma^d V,X) \xrightarrow{v_k^*} \mathrm{Map}_*(\Sigma^{2d} V,X) \longrightarrow \cdots.$$

By the exponential law

$$\mathrm{Map}_*(A \wedge B, X) \simeq \mathrm{Map}_*(B, \mathrm{Map}_*(A,X)),$$

there is a map $v_k^* : \mathrm{Map}_*(V,X) \longrightarrow \Omega^d \mathrm{Map}_*(V,X)$.

Definition 4.3: The spectrum $\Phi_v X$ is defined as $(\Phi_v X)_{nd} = \mathrm{Map}_*(V,X)$ for all $n \geq 0$ with structure map $v_k^* : \mathrm{Map}_*(V,X) \longrightarrow \Omega^d \mathrm{Map}_*(V,X)$. The functor $\Phi_v : \mathbf{Top}_* \longrightarrow \mathbf{Sp}$ is called the *telescopic functor* associated to v_k.

Proposition 4.3: Given a based space X, the homotopy groups of spectrum $\Phi_v X$ are periodic with period d.

Proof: By definition, we have

$$\begin{aligned}
\pi_n(\Phi_v X) &\cong [\mathbb{S}^n, \Phi_v X]_* \\
&\cong colim\ \pi_{n+m}((\Phi_v X)_m) \\
&\cong colim\ \pi_{n+d+m}((\Phi_v X)_{m+d}) \\
&\cong colim\ \pi_{n+d+m}((\Phi_v X)_m) \\
&\cong [\mathbb{S}^{n+d}, \Phi_v X]_* \\
&\cong \pi_{n+d}(\Phi_v X).
\end{aligned}$$

■

By this proposition, we can define the v_k-periodic homotopy group of the based space X.

Definition 4.4: The v_k-periodic homotopy group of a based space X is defined as

$$\pi_\bullet(X;V) = \pi_\bullet(\Phi_v X).$$

4.2.2 On Spheres

Arone and Mahowald [11] found that the Goodwillie tower of the identity functor on odd dimensional sphere $S^{2s+1}_{(p)}$ localized at any prime p (see [37] for more details about localization of spaces), has non-contractible n-th layer $D_n I(S^{2s+1}_{(p)})$ only for $n = p^k$. This implies the tower converges at an exponential speed. Besides, they proved that all the layers after p^k are trivial in v_k-periodic homotopy. Thus, the only non-trivial layer appears at $n = p^0, \cdots, p^k$. Thus the unstable v_k-periodic homotopy of an odd dimensional sphere can be resolved into a tower of fibrations with $k+1$ stages.

Theorem 4.3 (Arone-Mahowald [11], Theorem 3.13): For an odd dimensional sphere $X = S^{2s+1}_{(p)}$ localized at a prime p, the n-th layer $D_nI(X)$ is weakly contractible for $n \neq p^k$, has only p-primary torsion for $n = p^k$.

The v_0-periodic homotopy is the same as the rational homotopy. Thus, the n-th layer $D_nI(S^{2s+1})$ is rationally contractible for all $n \geq 2$. Moreover, Arone and Mahowald proved that, for an odd shpere $X = S^{2s+1}_{(p)}$ localized at a prime p, the layer $D_{p^n}I(X)$ is trivial in v_k-periodic homotopy for all $n \geq k+1$, by using vanishing line theorems (see [38] and [39] for more details). Thus we have the Goodwillie tower of the identity functor I on $X = S^{2s+1}_{(p)}$, where non-trivial layer appear only for $n = 1, \cdots, p^k$.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P_{p^k}I(X) \arrow[d] & D_{p^k}I(X) \arrow[l] \\
 & \vdots \arrow[d] & \\
 & P_{p^2}I(X) \arrow[d] & D_{p^2}I(X) \arrow[l] \\
 & P_pI(X) \arrow[d] & D_pI(X) \arrow[l] \\
X = S^{2s+1}_{(p)} \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuuur] & P_1I(X) & D_1I(X) \arrow[l]
\end{tikzcd}
$$

Theorem 4.4 (Arone-Mahowald [11], Theorem 4.1): For $X = S^{2s+1}_{(p)}$, the map $X \longrightarrow P_{p^k}I(X)$ is v_j-periodic equivalence for all $0 \leq j \leq k$.

Thus the Goodwillie tower of the identity functor I on $X = S^{2s+1}_{(p)}$ converges in v_k-periodic homotopy. Then Arone and Mahowald made this into a fiber sequence (see [11] Proposition 4.7 for details)

$$P_nI(S^{2s-1}_{(p)}) \longrightarrow \Omega P_nI(S^{2s}_{(p)}) \longrightarrow \Omega P_nI(S^{4s-1}_{(p)})$$

to deduce that the Goodwillie tower of the identity functor I on $X = S^{2s}_{(p)}$ converges in v_k-periodic homotopy. Thus, the Goodwillie towers of the identity functor on all dimensional spheres converge in v_k-periodic homotopy.

Theorem 4.5 (Arone-Mahowald [11], Theorem 4.4): For an even dimensional shpere $X = S^{2s}_{(p)}$ localized at prime p, the n-th layer $D_nI(X)$ is weakly contractible for $n \neq p^k$ or $2p^k$, has only p-primary torsion for $n = p^k$ or $2p^k$.

We have the following Goodwillie tower of the identity functor on $X = S^{2s}_{(p)}$.

$$
\begin{tikzcd}
& \vdots \arrow[d] & & \vdots \arrow[d] & \\
D_{2p^k}I(X) \arrow[r] & P_{2p^k}I(X) \arrow[d] & & P_{p^k}I(X) \arrow[d] & D_{p^k}I(X) \arrow[l] \\
& \vdots \arrow[d] & & \vdots \arrow[d] & \\
D_{2p^2}I(X) \arrow[r] & P_{2p^2}I(X) \arrow[d] & & P_{p^2}I(X) \arrow[d] & D_{p^2}I(X) \arrow[l] \\
D_{2p}I(X) \arrow[r] & P_{2p}I(X) \arrow[d] & & P_pI(X) \arrow[d] & D_pI(X) \arrow[l] \\
D_2I(X) \arrow[r] & P_2I(X) & X = S^{2s}_{(p)} \arrow[l] \arrow[r] \arrow[ul] \arrow[uul] \arrow[uuuul] \arrow[ur] \arrow[uur] \arrow[uuuur] & P_1I(X) & D_1I(X) \arrow[l]
\end{tikzcd}
$$

Theorem 4.6 (Arone-Mahowald [11], Theorem 4.5): For $p > 2$, $X = S^{2s}_{(p)}$, the map $X \longrightarrow P_{2p^k}I(X)$ is v_k-periodic equivalence. If $p = 2$, then the map $X \longrightarrow P_{p^{k+1}}I(X)$ is v_k-periodic equivalence.

4.2.3 Examples of Divergent Towers

We now introduce some results about divergent Goodwillie towers.

Definition 4.5: Given k generators $x_1, \cdots, x_k$ corresponding based connected spaces $X_1, \cdots, X_k$, we say that x_i is a Lie word. If u, v are Lie words, then the Lie bracket $[u, v]$ is also a Lie word. See [40] for more details.

We denote the ordered set of these Lie words with k generators as L_k.

Example 4.2: We evaluate these Lie words on a k-tuple of based connected spaces $X_1, \cdots, X_k$ by letting the Lie bracket act as the smash product. For example:

（1）$[x_1, [x_2, x_2]](X_1, X_2, X_3) = X_1 \wedge X_2 \wedge X_2$,

（2）$[x_2, [x_3, x_3]](X_1, X_2, X_3) = X_2 \wedge X_3 \wedge X_3$.

Theorem 4.7 (Brantner-Heuts [15], Theorem 2.3): Given based connected spaces $X_1, \cdots, X_k$, there is a weak equivalence

$$\prod_{w \in L_k}{}' \Omega\Sigma(w(X_1, \cdots, X_k)) \longrightarrow \Omega\Sigma(X_1 \vee \cdots \vee X_k)$$

where $\prod'$ is the weak infinite product.

Then we need the following two lemmas.

Lemma 4.2 (Brantner-Heuts [15], Lemma 2.1): Let $G : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$ be a reduced finitary homotopy functor. We denote the iterated wedge sum functor of k terms as $\bigvee :$ $\mathbf{Top}_*^k \longrightarrow \mathbf{Top}$, let $F = G \circ \bigvee : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$. Then there are canonical equivalences of functors from $\mathbf{Top}_*^k$ to $\mathbf{Top}$:

$$(P_n G) \circ \bigvee \longrightarrow P_n F \longrightarrow \underset{n_1+\cdots+n_k \leq n}{holim} P_{n_1,\cdots,n_k} F.$$

This lemma give an equivalence between homotopy functor $(P_n G) \circ \bigvee$, $P_n F$ and multivariable functor $P_{n_1,\cdots,n_k} F$.

Lemma 4.3 (Brantner-Heuts [15], Lemma 2.4): Let $G : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$ be a reduced finitary homotopy functor. Given a k-tuple of natural numbers $(a_1, \cdots, a_k)$, define a homotopy functor $F : \mathbf{Top}_*^k \longrightarrow \mathbf{Top}_*$ as

$$F(X_1, \cdots, X_k) = G(X_1^{\wedge a_1} \wedge \cdots \wedge X_k^{\wedge a_k}).$$

Then for any $(n_1, \cdots, n_k)$ there is a weak equivalence

$$P_{n_1,\cdots,n_k} F(X_1, \cdots, X_k) \longrightarrow P_{min\{[\frac{n_1}{a_1}],\cdots,[\frac{n_k}{a_k}]\}} G(X_1^{\wedge a_1} \wedge \cdots \wedge X_k^{\wedge a_k})$$

where $[\frac{n_i}{a_i}]$ is the largest integer no greater than $\frac{n_i}{a_i}$.

Then associated with the P_n functor, we have the following.

Theorem 4.8 (Brantner-Heuts [15], Theorem 2.5): Given based connected spaces $X_1, \cdots, X_k$, there is a weak equivalence

$$\prod_{w \in L_k}{}' \Omega P_{[\frac{n}{|w|}]}(\Sigma w)(X_1, \cdots, X_k)) \longrightarrow \Omega P_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$$

where $[\frac{n}{|w|}]$ is the largest integer no greater than $\frac{n}{|w|}$.

Proof: By Theorem 4.7, we have a weak equivalence

$$\prod_{w \in L_k}{}' \Omega P_{n_1,\cdots,n_k}(\Sigma w)(X_1, \cdots, X_k)) \longrightarrow \Omega P_{n_1,\cdots,n_k} \Sigma(X_1 \vee \cdots \vee X_k).$$

By Lemma 4.2, we have a weak equivalence

$$\Omega P_n I(\Sigma X_1 \vee \cdots \vee \Sigma X_k) \longrightarrow \underset{n_1+\cdots+n_k \leq n}{holim} \Omega P_{n_1,\cdots,n_k} \Sigma(X_1 \vee \cdots \vee X_k).$$

By Lemma 4.3, we have a weak equivalence

$$\underset{n_1+\cdots+n_k \leq n}{holim} \prod_{w \in L_k}{}' \Omega P_{n_1,\cdots,n_k}(\Sigma w)(X_1, \cdots, X_k)) \longrightarrow \underset{n_1+\cdots+n_k \leq n}{holim} \prod_{w \in L_k}{}' \Omega P_{[\frac{n}{|w|}]}(\Sigma w)(X_1, \cdots, X_k),$$

which completes the proof. ■

Theorem 4.9 (Brantner-Heuts [15], Corollary 3.3): Given based connected spaces $X_1, \cdots, X_k$, If $\Phi_\nu\Sigma(w(X_1, \cdots, X_k))$ is not contractible for infinitely many $w \in L_k$, then the canonical map

$$\Phi_\nu(\Sigma X_1 \vee \cdots \vee \Sigma X_k) \longrightarrow holim\ \Phi_\nu P_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$$

is not an equivalence.

Brantner and Heuts [15] found that the Goodwillie tower on the wedge sum of spheres is divergent in v_k-periodic homotopy.

Theorem 4.10 (Brantner-Heuts [15], Theorem 3.4): For given $m, n \geq 2$, the Goodwillie tower of the identity functor on $S^m \vee S^n$ is infinite and fails to converge under v_k-periodic homotopy.

Proof: We consider the case where each X_i is a sphere of dimension at least 1. By Theorem 4.3 and 4.5, the stages become constant at stage p^k or $2p^k$ (meaning the sequence becomes constant after p^k or $2p^k$) in v_k-periodic homotopy. Then by Theorem 4.8, $\Omega P_{[\frac{n}{|w|}]}\Sigma(w(X_1, \cdots, X_k))$ only becomes constant for $|w|p^k$ or $2|w|p^k$ but they has no bound. Thus, the Goodwillie tower in this case is infinite. Then by Theorem 4.9, for each X_i is a sphere of dimension at least 1, this Goodwillie tower fails to converge. ■

Moreover, they also proved, for given $n \geq 5$ and an odd prime p, that the Goodwillie tower of the identity functor I is divergent on mod p Moore space S^n/p in v_1-periodic homotopy. See [15] Theorem 5.4 for details.

CONCLUSION

The contributions include the following.

(1) Explaining how to construct the Goodwillie tower of a homotopy functor F.

(2) Explaining how to construct the one-to-one correspondence between n-homogeneous functors and symmetric multilinear functors.

(3) Introducing how to calculate the stable homotopy group of ΩS^n by the Snaith Splitting.

(4) Introducing some applications on the unstable v_k-periodic homotopy.

Further researches include the following.

(1) Construct another kind of approximation for a homotopy functor F, which is better to calculate than n-excisive approximations.

(2) Extend the Goodwillie tower in the negative range, with some new definitions, where some divergent results become convergent in the negative range.

(3) Explicit calculations with v_k-periodic Goodwillie towers for $k = 2$, analogous to the $k = 1$ case studied by Arone and Mahowald.

(4) Explore new examples of convergent and divergent Goodwillie towers after Berehus-Rezk and Brantner-Heuts.

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

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