博士学位论文

关于持续模结构的研究

TOPICS IN THE STRUCTURE OF PERSISTENCE MODULES

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TOPICS IN THE STRUCTURE OF PERSISTENCE MODULES

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摘　要

随着大数据和人工智能的快速发展，从复杂高维数据中提取有意义信息已成为当前研究热点之一。持续同调作为新兴的数据分析方法，以其独特的视角和强大的数据表征能力，在处理复杂非线性、非欧氏结构数据方面展现出巨大潜力。而持续模作为持续同调的理论基础，从持续同调兴起后不久就受到众多数学家的关注。当前研究者们在理论层面对持续模的关注主要集中于多参数持续模的离散不变量的寻找以及将持续模的思想与其他数学领域相结合。

在本文中，我们首先介绍了持续同调的基础知识与基本结果，特别是稳定性定理。之后我们介绍了持续模的相关结果，主要是分类定理和参数化定理。由于对持续模不能直接使用瓶颈距离，我们进一步介绍了瓶颈距离的推广，即交错距离，交错距离是持续模中的一个伪度量。

虽然单参数持续模的结构已经得到了完整地刻画，但是对于多参数持续模的研究还存在诸多挑战。因为多参数持续模可以被看作是多元多项式环 $\Bbbk[x_1,\cdots,x_n]$ 上的模，而这个模的分解是一个极其复杂的问题。因此研究者们便将目光转向了寻找多参数持续模的不完全离散不变量以及对一些特殊的持续模做完全分解。作为本文的一个主要结果，我们将 2 参数持续模的强正合性条件推广到 3 参数的情况，并且证明了这个条件是 3 参数持续模能够进行块分解的充分必要条件。

同时，研究者们关注的另一个问题就是持续模的稳定性。在这个问题中，科研工作者们关注的持续模不仅是通常所考虑的持续模，即函子 $(\mathbb{R},\leq)\rightarrow\mathbf{Vec}_{\Bbbk}$，其中 $\mathbf{Vec}_{\Bbbk}$ 是有限维向量空间的范畴，也会考虑更一般的持续模 $P\rightarrow C$，其中 P 是一个偏序集，C 是任意范畴。这里的目标范畴 C 可以是拓扑空间的范畴 **Top**，也可以是其他代数对象的范畴，例如微分分次 Lie 代数的范畴 **DGL**，交换微分分次代数的范畴 **CDGA** 等。在持续模的稳定性的研究中，我们的主要贡献是定义了有理 $\mathbb{R}$-空间 $\mathbb{X}:(\mathbb{R},\leq)\rightarrow\mathbf{Top}_{\mathbb{Q}}$ 的一种代数模型，称为持续自由李模型 $M_{Qui}(\mathbb{X}):(\mathbb{R},\leq)\rightarrow\mathbf{DGL}$，并且我们证明了这个模型的存在性与稳定性。

关键词：持续模；稳定性；强正合性；持续自由李模型

ABSTRACT

With the rapid development of big data and artificial intelligence, extracting meaningful information from complex high-dimensional data has become a current research hotspot. Persistent homology, as an emerging topological data analysis method, demonstrates immense potential in handling complex nonlinear and non-Euclidean structured data through its unique perspective and powerful data characterization capabilities. Persistence modules, serving as the theoretical foundation of persistent homology, have attracted significant attention from mathematicians since the emergence of persistent homology.

In this paper, we first introduce fundamental knowledge and key results of persistent homology, particularly the stability theorem. Subsequently, we present core results about persistence modules, focusing on classification theorems and parameterization theorems. Since the bottleneck distance cannot be directly applied to persistence modules, we introduce its generalization - the interleaving distance d_I, which serves as a pseudo-metric on persistence modules.

The structure of 1-parameter persistence modules has been fully characterized; however, research on multi-parameter persistence modules still presents significant challenges. This is because multi-parameter persistence modules can be viewed as modules over the multivariate polynomial ring $\Bbbk[x_1, \cdots, x_n]$, and the decomposition of such modules is an extremely complex problem. Therefore, researchers have shifted their focus to finding incomplete discrete invariants for multi-parameter persistence modules and to performing complete decompositions for some special class of multi-parameter persistence modules. One of this paper's main results of this paper is the extension of the strong exactness condition for 2-parameter persistence modules to the 3-parameter case. This condition is a necessary and sufficient condition for the block-decomposition of three-parameter persistence modules.

Meanwhile, another central question in the study of persistence modules is the stability of persistence modules. In this context, researchers focus not only on the commonly considered persistence modules, that are functors $(\mathbb{R}, \leq) \to \mathbf{Vec}_{\Bbbk}$ to the category of finitely dimensional vector spaces, but also on the more general persistence modules of the form $P \to C$, in which P is a poset and C is any category. The target category C

may be the category of topological spaces **Top**, or categories of other algebraic objects, such as the category of differential graded Lie algebras **DGL**, the category of commutative differential graded algebras **CDGA**, etc. In the study of the stability of persistent modules, our main contribution is the definition of an algebraic model for the rational $\mathbb{R}$-space $\mathbb{X} : \mathbb{R} \to \mathbf{Top}_{\mathbb{Q}}$, termed the persistence free Lie model $M_{Qui}(\mathbb{X}) : \mathbb{R} \to \mathbf{DGL}$, along with the proof of the existence and stability of this model.

Keywords: Persistence module; Stability; Strong exactness; Persistence free Lie model

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LIST OF SYMBOLS AND ACRONYMS

$\mathbb{k}$	the field
$\mathbb{Q}$	the field of rational numbers
$\mathbb{P}$	the projective space
$\mathbb{X}, \mathbb{Y}, \mathbb{V}, \mathbb{W} \cdots$	some persistence modules
$\mathcal{R}$	the Vietoris-Rip complexes
$\mathcal{C}$	the Cech complexes
$\mathcal{B}$	the barcode
A_n	the multivariate polynomial ring over $\mathbb{k}$, i.e. $\mathbb{k}[x_1, \cdots, x_n]$
d_I	the interleaving distance
$\mathbf{Vec}_{\mathbb{k}}$	the category of finite-dimensional vector spaces over $\mathbb{k}$
Top	the category of topological spaces
$\mathbf{Top}_{\mathrm{CGWH}}$	the category of compactly-generated weakly Hausdorff topological spaces
$\mathbf{Top}_{\mathbb{Q}}$	the category of simply connected rational spaces of finite type
SSet	the category of simplicial sets
CDGA	the category of 1-connected commutative differential graded algebras over $\mathbb{Q}$
CDGC	the category of 1-connected cocommutative differential graded coalgebras over $\mathbb{Q}$
DGL	the category of connected differential graded Lie algebras over $\mathbb{Q}$

CHAPTER 1 INTRODUCTION

1.1 Background

Persistence modules[28,84,106], as algebraic structures encoding the evolution of topological features across scales, have become central to the mathematical framework of topological data analysis(TDA)[22,32,41,106]. Their development, applications, and theoretical richness bridge pure mathematics, computational topology, and data science. This section traces the origins of persistence modules, their relationship to persistent homology[41,106], their significance in modern mathematics, and recent advances in the field.

The development of multi-parameter persistent homology exemplifies such progress. Where conventional TDA relies on radius-based filtrations, novel approaches in applied domains like molecular science[101] now incorporate additional parameters(e.g., curvature). The introduction of multi-parameter approaches creates fundamental challenges: the decomposition of multi-parameter persistence modules and the identification of their discrete invariants require new mathematical frameworks. Furthermore, while homology groups provide coarse topological characterizations, researchers are developing novel algebraic frameworks that preserve finer topological information without compromising computational traceability.

Within this context, our work makes two critical contributions: the block-decomposition of multi-parameter persistence modules and the development of a novel algebra model for the filtration of simplicial complexes. These advances address both theoretical and practical dimensions of contemporary persistent homology research.

Historical Development of Persistence Modules

The concept of persistence modules emerged in the early 2000s as a formalization of ideas in persistent homology, but its roots can be traced back to earlier mathematical frameworks. The early focus of persistent homology research was on what is now known as persistent Betti numbers, specifically rank $(H_i(\mathbb{X}_s) \to H_i(\mathbb{X}_t))$. This concept can be traced back to the work of Frosini[88] in 1992, although he used the term 'size function' at the time. Independently of Frosini's research, Robins[93] formally introduced the term 'persistent Betti numbers' in 1999, aiming to quantify rank $(H_i(\mathbb{X}_s) \to H_i(\mathbb{X}_t))$. It is worth noting that the stability discussed in Robins' paper was associated with the Hausdorff distance. In Morse theory[81] and spectral sequences[80], the idea of tracking topological features as varying parameters was implicit, laying the conceptual groundwork for persistence. Subsequently, the computational turn in the 1990s, driven by Edelsbrunner, Letscher, and Zomorodian's work[41] on alpha shapes and persistent homology, crystallized the need for a discrete, computable framework. This shift marked the transition from theoretical ideas to practical tools, setting the stage for the formalization of persistence modules.

The foundational work of Zomorodian and Carlsson[106] established persistence modules as functors $(\mathbb{N}, \leq) \to \mathbf{Vec}_{\Bbbk}$, effectively modeling them as graded modules over polynomial rings. This algebraic framework enabled the encoding of topological features over filtrations, with their structure theorem[106] asserting that persistence modules decompose into interval summands under mild assumptions. This result provided the theoretical foundation for persistence diagrams and barcodes, which have become ubiquitous tools in topological data analysis (TDA).

The introduction of the interleaving distance by Chazal et al.[26] further advanced the field by extending the bottleneck distance, used in persistent homology, to the broader context of persistence modules. The interleaving distance allowed persistence modules to be compared and ensured stability under small perturbations, a critical property for real-world applications.

After Zomorodian and Carlsson defined and studied the 1-parameter persistence module, they pioneered the study of multi-parameter persistence modules[23], extending the framework to higher-dimensional parameter spaces. This generalization introduced new algebraic and computational challenges, particularly in decomposing and comparing such persistence modules. Lesnick[75] later formalized the interleaving distance for multi-parameter persistence modules and proved the stability theorem for multi-parameter persistence modules, establishing theoretical guarantees for their use in data analysis.

In summary, the development of persistence modules reflects a rich interplay of ideas from algebraic topology, computational geometry, and data science. From their early roots in Morse theory to their formalization and extension to multi-parameter settings, persistence modules have become a cornerstone of modern topological data analysis.

Persistent Homology and Persistence Modules: A Symbiotic Relationship

Persistent homology, the computational engine of TDA, relies fundamentally on persistence modules. The process begins with a filtration of simplicial complexes $\{K_t\}_{t\in\mathbb{R}}$, where each inclusion $K_s \hookrightarrow K_t$ for $s \leq t$ induces homology maps $H_p(K_s;\Bbbk) \to H_p(K_t;\Bbbk)$. The collection of these homology groups and linear maps forms a persistence module that can be decomposed into intervals representing the birth and death of topological features (e.g., connected components, loops).

Such persistence modules have some key properties:

• **Stability**: The interleaving distance d_I, a pseudo-metric on persistence modules, ensures that small perturbations in point clouds yield only small changes in persistence diagrams[3,11].

• **Computability**: the decomposition theorems and the parameterization theorem[106] enable efficient algorithms for computing persistence diagrams and barcodes, implemented in libraries like Gudhi[78] and Ripser[4].

• **Interpretability**: Persistence diagrams summarize topological features of point clouds, bridging qualitative analysis with quantitative analysis[57,63,90,97].

For multi-parameter persistence modules $(\mathbb{N}^n, \leq) \to \mathbf{Vec}_{\Bbbk}$, however, decomposition fails in general, leading to active research into alternative invariants (e.g., rank functions[39], generalized persistence diagrams[68], and Hilbert functions[83]) and algebraic formulations (e.g., quiver representations[94]).

Applications of Persistence Modules

The versatility of persistence modules has driven their adoption across a wide range of disciplines, including but not limited to the applications discussed below.

In the field of materials science, persistent homology serves as a powerful tool for detecting and quantifying the microstructure of materials[87]. Specifically, it enables the identification of the number, size, distribution, and density of voids within material samples. Additionally, persistent homology can be applied to structural analysis of materials, including the study of crystallization in granular systems and the formation of crazes in polymers[19]. Beyond the applications described here, more extensive applications of persistent homology in materials science can be found in references[64,72,76,99].

In robotics, Adams and Carlsson[2] employed zigzag persistence to investigate the existence of evasion paths in sensor networks. Similarly, Silva and Ghrist[37] utilized persistent homology to address the coverage problem in sensor networks with minimal sensing capabilities, demonstrating its versatility in solving complex network-related challenges. In addition, there are also some other works on the application of persistence modules in robotics[10,56,86,100,104].

In the biomedical field, persistent homology has been effectively applied to a variety of problems. For instance, Chan et al.[25] used persistent homology to characterize clonal evolution, reassortment, and recombination in RNA viruses. Meanwhile, Y. Dabaghian et al.[35] developed a topological framework for hippocampal spatial maps using persistent homology. Additionally, Giusti et al.[53] leveraged persistent homology to identify meaningful structures in neural activity and connectivity data, showcasing its potential in advancing neuroscience research. Readers can also find more applications in the references[1,16,38].

In time series analysis, persistence modules also play a significant role. By employing Takens' embedding, time series data can be reconstructed into geometric spaces, allowing persistent homology to extract and analyze their topological features[43,67,85]. This approach provides valuable insights into the underlying structure and dynamics of time-dependent data. At the same time, many scholars have done a lot of research on this topic[29,58,91,98].

These applications underscore the dual role of persistence modules as mathematical objects of intrinsic interest and as tools for extracting meaning from complex data.

Recent Theoretical Advances

In recent years, the persistence module has made a lot of progress in theoretical research.

Multi-parameter persistence modules (e.g., indexed by $\mathbb{R}^n$) generalize the 1-parameter case but face algebraic complexity. First, Carlsson et al. showed us the algebraic complexity of multi-parameter persistence modules $(\mathbb{N}^n, \leq) \to \mathbf{Vec}_{\mathbb{k}}$. Secondly, many scholars used different methods to find the incomplete discrete invariants of multi-parameter persistence modules and the complete discrete invariants of special persistence modules. Oudot and Scoccola[83] used the Betti number and Hilbert function as the invariants and proved the stability of the invariants. Mémoli et al. used rank invariants as the invariant and designed pseudo-code to compute rank invariants[39]. Cochoy and Oudot proved the block-decomposition theorem[31] of special 2-parameter persistence modules satisfying the 2-parameter strong exactness. Additionally, some scholars try to reinterpret the persistence module with theories other than the quiver representation theory. Kashiwara and Schapira[66] interpret some results of persistent homology and barcodes (in any dimension) with the language of microlocal sheaf theory. Fersztand et al. used the Harder-Narasimhan filtration to study persistence modules and find invariants[48-49].

When considering the persistence modules $(\mathbb{N}, \leq) \to \mathbf{Vec}_{\mathbb{k}}$, the bottleneck distance d_B is equal to the interleaving distance d_I. However, when the target category we consider is not $\mathbf{Vec}_{\mathbb{k}}$, or when persistence modules cannot be decomposed into the sum of interval modules, we cannot use the bottleneck distance to describe the difference between the two persistence modules. Extending the bottleneck distance to the interleaving distance allows us to compare the differences between two persistence modules. Blumberg and Lesnick[12] defined the homotopy-interleaving distance d_{HI} and proved the stability and the universality of d_{HI} Lesnick[75] discussed the stability of multi-parameter persistence modules and further extended the definition of interleaving distance. Zhou[105] combined persistence modules with rational homotopy theory, defined persistence Sullivan models and proved the stability of persistence Sullivan models.

1.2 Statement of Results

This work establishes several results in the study of persistence modules, with the main contributions organized into two parts. On the one hand, we introduce the idea of rational homotopy theory into persistence modules $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$, define persistence minimal free Lie models $M_{Qui}(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{Ho}(\mathbf{DGL})$, and prove the existence of persistence minimal free Lie models. At the same time, we also discuss the stability of persistence minimal free Lie models. On the other hand, based on Cochoy and Oudot's results[31] on the block-decomposition of 2-parameter persistence modules, we generalize the 2-parameter strong exactness condition and prove the block-decomposition theorem of 3-parameter persistence modules $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$.

Note: If a persistence module is a functor to $\mathbf{Vec}_{\mathbb{k}}$, we call the persistence module pointwise finite-dimensional and abbreviate it as **pfd**.

A. Persistence Minimal Free Lie Models

Rational homotopy theory, pioneered by Quillen[89], associates to a rational space $X \in \text{ob}\, \mathbf{Top}_{\mathbb{Q}}$ a minimal free Lie algebra encoding its homotopy type. By integrating persistence modules into this framework, we define a persistence minimal free Lie model that is a functor $M_{Qui}(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{Ho}(\mathbf{DGL})$ for any rational $\mathbb{R}$-space $\mathbb{X}$, where a rational $\mathbb{R}$-space is a functor from $\mathbb{R}$ to the category of simply connected rational topological spaces of finite type. The persistence module $M_{Qui}(\mathbb{X}) : \mathbb{R} \to \mathbf{Ho}(\mathbf{DGL})$ defined by us is an algebraic model for the rational $\mathbb{R}$-space $\mathbb{X}$. Our results provide novel theoretical frameworks with implications for topological data analysis, which allows us to identify more topological information about point clouds.

Theorem 1.1: For any rational $\mathbb{R}$-space $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$, there exists a persistence minimal free Lie model $M_{Qui}(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{Ho}(\mathbf{DGL})$ such that $M_{Qui}(\mathbb{X})_t$ is a minimal free Lie model of $\mathbb{X}_r$ and $M_{Qui}(\mathbb{X})(s \leq t)$ is a Lie representative of $\mathbb{X}(s \leq t)$ up to weak equivalences.

What's more, we discuss the stability of persistence minimal free Lie models under the interleaving distance d_I. The following Theorem tells us that the persistence minimal free Lie model is a reasonable algebraic model for a rational $\mathbb{R}$-space $\mathbb{X} : \mathbb{R} \to \mathbf{Top}_{\mathbb{Q}}$, and is more refined than the persistence module generated by computing homology groups.

Theorem 1.2: For any rational $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$, we have

- $d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y})) \leq d_{HI}(\mathbb{X}, \mathbb{Y}) \leq d_I(\mathbb{X}, \mathbb{Y})$
- $$\begin{aligned} d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) &= d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_* \circ M_{Qui}(\mathbb{X}), H_* \circ M_{Qui}(\mathbb{Y})) \\ &\leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y})) \end{aligned}$$
- $d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) = d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\mathbb{V}, \mathbb{W}) \leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y}))$

This means that $M_{Qui}(\mathbb{X})$ retains more topological information than persistence modules $H_* \circ \mathbb{X} : \mathbb{R} \to \mathbf{Vec}_{\mathbb{Q}}$ for the rational $\mathbb{R}$-space $\mathbb{X}$. Meanwhile, $d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) = d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_* \circ M_{Qui}(\mathbb{X}), H_* \circ M_{Qui}(\mathbb{Y}))$ and $d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) = d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\mathbb{V}, \mathbb{W})$ indicate that $\pi_*(\mathbb{X}) \cong H_* \circ M_{Qui}(\mathbb{X})$ and $H_*(\mathbb{X}) \cong \mathbb{V}$ respectively. In fact, we also proved this conclusion in the proof of the theorem.

B. block-decomposition of 3-Parameter Persistence Modules

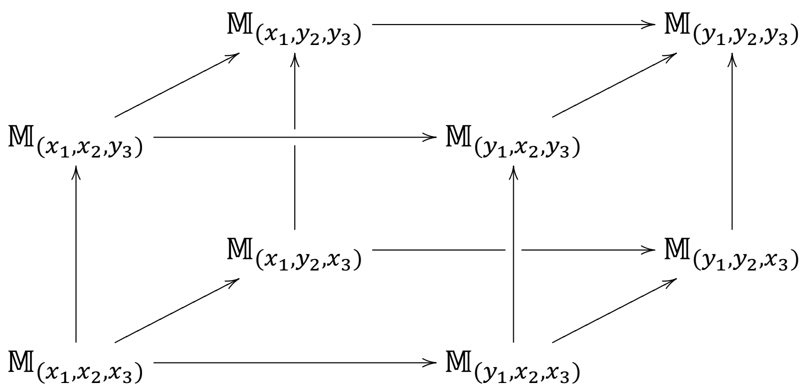
Multi-parameter persistent homology has long been a pivotal direction in the development of persistent homology. Multi-parameter filtrations can capture richer topological features of point clouds compared to 1-parameter approaches, yet extracting data features from multi-parameter persistent homology remains a significant challenge. Our proposed block-decomposition theorem for 3-parameter persistence modules provides a theoretical foundation for feature extraction of 3-parameter persistent homology.

Building on Cochoy and Oudot's block-decomposition theorems for 2-parameter strongly exact persistence modules[31], we generalize block-decomposition strategies to 3-parameter conditions.

In 2-parameter persistence modules $\mathbb{M} : (\mathbb{R}^2, \leq) \to \mathbf{Vec}_{\Bbbk}$, Cochoy and Oudot[31] call the 2-parameter persistence module $\mathbb{M}$ is 2-parameter strongly exact, if for all $(x_1, x_2) \leq (y_1, y_2) \in \mathbb{R}^2$, the following sequence is exact

$$\mathbb{M}_{(x_1,x_2)} \xrightarrow{(\rho_{(x_1,x_2)}^{(x_1,y_2)}, \rho_{(x_1,x_2)}^{(y_1,x_2)})} \mathbb{M}_{(x_1,y_2)} \bigoplus \mathbb{M}_{(y_1,x_2)} \xrightarrow{\rho_{(x_1,y_2)}^{(y_1,y_2)} - \rho_{(y_1,x_2)}^{(y_1,y_2)}} \mathbb{M}_{(y_1,y_2)}$$

When considering the 3-dimensional persistence modules $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\Bbbk}$, for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3) \in \mathbb{R}^3$, there is a cubical commutative diagram and the diagram induces the functor $\mathcal{X}(S) : \mathcal{P}(S) \to \mathbf{Vec}_{\Bbbk}$ with $|S| = 3$, resulting in two morphisms $\psi : \mathcal{X}(\emptyset) \to \lim_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ and $\varphi : \operatorname{colim}_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$, where $\mathcal{P}(S)$ is the power set of S, and $\mathcal{P}_0(S) := \mathcal{P}(S) \setminus \{\emptyset\}$ and $\mathcal{P}_0(S) := \mathcal{P}(S) \setminus \{S\}$.



We call $\mathbb{M}$ 3-parameter strongly exact, if $\mathbb{M}$ satisfies following conditions:

- for any $r \in \mathbb{R}$, $\mathbb{M}|_{\{r\}\times\mathbb{R}\times\mathbb{R}}$, $\mathbb{M}|_{\mathbb{R}\times\{r\}\times\mathbb{R}}$, $\mathbb{M}|_{\mathbb{R}\times\mathbb{R}\times\{r\}}$ are among 2-parameter strongly exact.
- for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3) \in \mathbb{R}^3$, the associated morphisms ψ and φ is surjective and injective respectively.

Thus, following Cochoy and Oudot's proof for the 2-parameter case, we prove the decomposition theorem for the 3-parameter case.

Theorem 1.3: Let $\mathbb{M}$ be a pointwise finite-dimensional 3-parameter persistence module satisfying the 3-parameter strong exactness. Then $\mathbb{M}$ may decompose uniquely (up to isomorphism and reordering of the terms) as a direct sum of block modules:

$$\mathbb{M} \cong \bigoplus_{B:\text{blocks}} \mathbb{M}_B$$

in which $\mathbb{M}_B \cong \bigoplus_{i=1}^{n_B} \mathbb{k}_B$ in which n_B are determined by the counting functor $\mathcal{CF}$.

Remark 1.1: Although Lerch et al.[74] also obtained this result around the same time independently, and a few months later, they extended the result to the case of any finite-dimensional persistence module that satisfies certain exact conditions, our research method has advantages. The main reason is that we use more general language to generalize the strong exactness, so our conditional generalization method has high mobility. Through the universal property of limits and colimits, we can understand the reason why there is no block-decomposition in general persistence modules to some extent.

1.3 Outline

In this chapter, we have introduced the history of persistence modules, the relationship between persistence modules and persistent homology, and the applications and theoretical advances of persistence modules. Additionally, we have presented some of our results.

In Chapter 2, we review the definitions and key results of persistent homology and persistence modules, including some methods for constructing simplicial complexes from point clouds. At the end of this chapter, we introduce a novel approach to identifying invariants of multi-parameter persistence modules: the Harder-Narasimhan filtration. By introducing persistent homology as a starting point, we aim to clarify the motivations for studying persistence modules and the significance of extracting invariants from them in topological data analysis. This chapter establishes both the practical foundation and theoretical basis for our presentation of one of our results in Chapter 5, which is the block-decomposition theorem for 3-parameter persistence modules.

In Chapter 3, we begin with the robustness of persistent homology, introducing the bottleneck distance and the stability theorem for persistent homology. We then present the interleaving distance, which strictly generalizes the bottleneck distance. Specifically, when considering 1-parameter interval-decomposable persistence modules, the bottleneck distance and interleaving distance coincide. Finally, we introduce the homotopy interleaving distance[12] on the persistence modules $\mathbb{R} \to \mathbf{Top}_{\mathrm{CGWH}}$, which can be viewed as a generalization of the interleaving distance under homotopy invariance. These concepts and results discussed in this chapter will be instrumental in Chapter 4, where we present another main result: the persistence minimal free Lie model.

Chapter 4 initiates with a foundational review of rational homotopy theory, followed by an exposition of Zhou's contributions[105]. Zhou's work defined the persistence Sullivan models, with proof of its stability. In this chapter, our results are organized into two key contributions: the definition and proof of the existence of persistence minimal free Lie models, which extend the classical minimal free Lie model framework to the context of persistence modules, and a discussion of the stability of persistence minimal free Lie models.

In Chapter 5, we began by revisiting Cochoy and Oudot's work[31] on the block-decomposition of 2-parameter persistence modules. Subsequently, we generalized the 2-parameter definition of strong exactness to 3-parameter settings and, adapting the proof strategy developed by Oudot and Cochoy, rigorously established the block-decomposition theorem for 3-parameter persistence modules.

CHAPTER 2 PERSISTENT HOMOLOGY AND PERSISTENCE MODULES

In this chapter, we introduce persistent homology and persistence modules. Specifically, we cover the foundational results of persistent homology, excluding the stability theorem, which will be discussed in Chapter 3. We also introduce persistence modules, focusing on the correspondence, classification, and parameterization for 1-parameter and multi-parameter persistence modules. Finally, we present a novel approach to identifying discrete invariants of persistence modules.

Through this chapter, we aim to demonstrate that studying the decomposition of persistence modules is a meaningful endeavor, as persistence modules serve as the mathematical abstraction of persistent homology. By introducing both 1-parameter and multi-parameter persistence modules, we provide readers with an intuitive understanding of the challenges in studying multi-parameter persistence modules compared to their 1-parameter counterparts.

2.1 Persistent Homology

The principle of persistent homology is to approximate topological spaces through filtrations of simplicial complexes. We assume that the underlying space of the point cloud is a topological space, and we can approximate the homology group of the topological space by constructing the filtration of simplicial complexes and computing the homology of simplicial complexes, and then inferring the topological properties of the space. Unless otherwise specified, the coefficients are any field denoted as $\Bbbk$.

In this section, we will first introduce the constructions and properties of filtrations of simplicial complexes such as Čech complexes, Vietoris-Rips complexes, and others. Second, we will introduce persistent homology, persistence diagrams[41], and barcodes[24]. More details can be found in the reference[40].

Constructions of Simplicial Complexes and Nerve Theorem

The common ways to construct a filtration of simplicial complexes include Čech complexes[51], Vietoris-Rips complexes[92], Alpha complexes[21], Witness complexes[36].

For the purpose of elucidating and showcasing the principle of persistent homology, we will only introduce the Čech complex and Vietoris-Rips complex in this section, and other constructions can be found in the reference[40].

To elaborate Čech complexes, we need to state the Nerve theorem.

Definition 2.1: Let F be the finite collection of sets. The nerve consists of all non-empty subcollections whose sets have a non-empty common intersection, $\mathrm{Nrv}F = \{X \subset F : \bigcap_{A\in X} A \neq \emptyset\}$.

Obviously, the nerve $\mathrm{Nrv}F$ is an abstract simplicial complex. Indeed, if $\bigcap_{A\in X} A \neq \emptyset$ and $Y \subset X$, then $\bigcap_{A\in Y} A \neq \emptyset$. The nerve can be geometrically realized in some Euclidean space of appropriate dimension, which allows us to meaningfully discuss its topological and homotopy properties. Due to the geometric realization theorem, we can talk about the topology and homotopy of $\mathrm{Nrv}F$. The nerve theorem, whose early versions are attributed to[73],[14], and[103], is a basic result in algebraic and combinatorial topology.

Theorem 2.1: (Nerve)[5] Let F be a finite collection of closed, convex sets in Euclidean space. Then, the nerve of F and the union of the sets in F have the same homotopy type. Specifically, if $\bigcup_{A\in F} A$ is triangulable, each set in F is closed, and every non-empty intersection of sets in F is contractible, then $\mathrm{Nrv}F \sim \bigcup_{A\in F} A$.

Now, we can define Čech complexes and state the rationality of the definition.

Definition 2.2: (Čech Complexes) Let S be a finite set of points in $\mathbb{R}^d$ and $B_x(r) = x + r\mathbb{B}^d$ for the closed ball whose center is x and radius is r. The Čech complex of points cloud S and radius r is the nerve of this collection of $B_x(r)$ for all $x \in S$, but each ball is substituted with its center point, that is,

$$\mathcal{C}(S)_r := \{\sigma \subset S \mid \bigcap_{x\in\sigma} B_x(r) \neq \emptyset\}.$$

Based on the definition, we can derive an equivalent description of Čech complexes: $\sigma \in \mathcal{C}(S)_r$ if and only if there exists a point $y \in \mathbb{R}^d$ such that for any $x \in \sigma$, the distance $d(x, y) \leq r$. However, neither of the definitions of Čech complexes is straightforward to implement computationally. Therefore, we need a construction method for simplicial complexes that is computationally feasible, namely the Vietoris-Rips complexes.

Definition 2.3: (Vietoris-Rips Complexes) Let S be a finite set of points in $\mathbb{R}^d$. The

Vietoris-Rips complex of the point cloud S with radius r is defined as the collection of all subsets of S whose diameter is at most $2r$:

$$\mathcal{R}(S)_r = \{\sigma \subset S \mid \text{diam } \sigma \leq 2r\}.$$

From both definitions, we can deduce that if $r \leq s$, then $\mathcal{C}(S)_r$ and $\mathcal{R}(S)_r$ are simplicial subcomplexes of $\mathcal{C}(S)_s$ and $\mathcal{R}(S)_s$, respectively. It is evident that a Čech complex is the subcomplex of a Vietoris-Rips complex, i.e., $\mathcal{C}(S)_r \subset \mathcal{R}(S)_r$, because the latter includes every simplex warranted by the given simplices. Meanwhile, we have $\mathcal{R}(S)_r \subset \mathcal{C}(S)_{\sqrt{2}r}$, which can be proven with minimal effort.

The Nerve theorem guarantees that the Čech complex can describe the underlying space of the point cloud. If we assume that the finite point cloud S is randomly or uniformly sampled from a topological space X, then the topological space, $\bigcup_{x \in S} B_x(r)$ for some $r \geq 0$, can accurately reconstruct X. By the Nerve theorem, we know that $\mathcal{C}(S)_r \sim \bigcup_{x \in S} B_x(r)$, implies that the Čech complex is a reasonable combinatorial model for the underlying space X.

In addition to these two complexes, there are numerous methods for constructing simplicial complexes, including Alpha complexes, Witness complexes, and Neighbourhood complexes[65].

Persistent Homology and Persistence Diagrams

In the above statement, we can see that for any $r \geq 0$ and finite point cloud, a simplicial complex can be constructed, such as the Čech complex. A natural question arises: What happens to the corresponding simplicial complex as r increases? How can we describe this change? For a simplicial complex, we can use homology groups as an algebraic invariant to characterize its structure[82][55]. Thus, as r increases, we can describe the change of simplicial complexes by analyzing the change of homology groups of these simplicial complexes.

Let K be a filtration of simplicial complexes, that is, an increasing sequence of simplicial complexes:

$$\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n.$$

The inclusions between simplicial complexes are simplicial maps denoted $f_{i,i+1} : K_i \to K_{i+1}$. Additionally, we can assume that K_{i+1} has exactly one more simplex than K_i for all i. Then, the filtration induces a sequence of homology groups connected by homomorphisms,

$$0 = H_p(K_0) \overset{H_p(f_{0,1})}{\longrightarrow} H_p(K_1) \overset{H_p(f_{1,2})}{\longrightarrow} H_p(K_2) \overset{H_p(f_{2,3})}{\longrightarrow} \cdots \overset{H_p(f_{n-1,n})}{\longrightarrow} H_p(K_n)$$

again one for each dimension p. Note that the meaning of filtration varies slightly in different scenarios, but the idea is similar.

To describe the change of homology classes, we need to define certain algebraic invariants.

Definition 2.4: Let the p-th persistent homology groups $H_p(i,j) := \text{Im } H_p(f_{i,j})$, for $0 \leq i \leq j \leq n$, and the p-th persistent Betti numbers $\beta_p(i,j) = \text{rank } H_p(i,j)$.

In an analogous manner, we may extend the definition of persistent homology to the reduced setting by utilizing reduced homology groups, thereby obtaining reduced persistent homology groups and their associated reduced persistent Betti numbers. Here, we adopt the convention that $H_p(i,i)$ is isomorphic to the p-th homology group of the corresponding simplicial complex, that is, $H_p(i,i) := H_p(K_i)$. We know that the homology group of K_i is the quotient $H_p(K_i) = Z_p(K_i)/B_p(K_i)$, in which $Z_p(K_i)$ and $B_p(K_i)$ are the cycle and boundary of chain complexes $C_p(K_i)$, respectively. The persistent homology groups comprise those homology classes of the complex K_i that persist through the inclusion map to K_j, which can be formally expressed as the quotient group $H_p(i,j) = Z_p(K_i)/(B_p(K_j) \cap Z_p(K_i))$.

Let $\alpha \in H_p(K_i)$ be a homology class. we say it is born at K_i, if $\alpha \notin \text{Im } H_p(f_{i-1,i})$. Furthermore, we say that it dies K_j if it merges with another class as it goes from K_{j-1} to K_j. The birth and death of homology classes correspond to the addition and merging of simplices in the simplicial complex, respectively. If a homology class α is born at K_i and dies when it arrives exactly at K_j, then we call the difference $j - i$ the persistence of α. And if a homology class never dies, then we call that it persists to infinity.

Once we have defined persistent homology groups and persistent Betti numbers, we can characterize the structure of filtration of homology groups. We will represent the filtration of homology groups of simplicial complexes by a multiset, which is called **persistence diagram** or **barcode**. The elements of the multiset lie in the extended real plane $\bar{\mathbb{R}}^2 := (\mathbb{R} \cup \{\pm\infty\})^2$, since some homology classes never die and persist to infinity.

Let

$$\mu_p(i,j) := (\beta_p(i,j-1) - \beta_p(i,j)) - (\beta_p(i-1,j-1) - \beta_p(i-1,j))$$

for all $i < j$ and all p. From the above discussion, it can be seen that $\beta_p(i, j-1) - \beta_p(i, j)$ is the dimension of the linear space consisting of all homology classes that are born at or before K_i and die at K_j. Similarly, $(\beta_p(i-1, j-1) - \beta_p(i-1, j)$ is the dimension of the linear space consisting of all homology classes that are born at or before K_{i-1} and die at K_j. Thus, $\mu_p(i, j)$ is the dimension of the linear space consisting of all homology classes that are born at K_i and die at K_j.

Definition 2.5: For a filtration of simplicial complexes K, we can get a multiset called p-dimensional persistence diagram (p-dimensional persistence barcode)

$$dgm(H_p(K)) = \{((i, j), k) \mid k = \mu_p(i, j) \text{ and } (i, j) \in \bar{\mathbb{R}}^2\}$$

in which (i, j) is a element of the multiset and (i, j) appears $\mu_p(i, j)$ times.

If we interpret (i, j) as a point in $\bar{\mathbb{R}}^2$, the multiset is called the persistence diagram, dgm. If we interpret (i, j) as an interval, the multiset is called the persistence barcode or barcode $\mathcal{B}$. Due to some technical reasons that will be discussed in the next chapter, we will include the points on the diagonal in dgm, assigning them infinite multiplicity.

Lemma 2.1: (Fundamental Lemma of Persistent Homology)[40] For every pair of indices $0 \leq k \leq l \leq n$ and every dimension p, the p-th persistent Betti number is $\beta_p(k, l) = \sum_{i \leq k} \sum_{j > l} \mu_p(i, j)$.

This is an important result: the lemma states that the persistence diagram encodes all information about persistent homology groups. In the next section, we will restate persistence diagrams that are complete invariants from an algebraic perspective and provide an algebraic explanation of persistence diagrams. This will lead to a more intuitive understanding of the persistence diagram. Meanwhile, from the perspective of persistence modules, we can more clearly understand the core concept of persistence and the difficulties encountered in generalizing 1-parameter persistent homology to multi-parameter persistent homology.

2.2 1-Parameter Persistence Modules

Persistence modules are the categorization[17] of persistent homology. The purpose of studying persistence modules is to identify the discrete invariants of persistent homology, enabling its more effective application in topological data analysis.

In general, a persistence module can be defined as a functor $F : C \to D$, in which C

is a thin category and D is any category. A category C is said to be thin if, for every pair of objects $a, b \in \text{ob } C$, there exists at most one morphism from a to b. Sometimes, the category C is defined as a poset $\mathcal{P}$, where the ob $\mathcal{P}$ is the set $\mathcal{P}$, and morphisms are the partial order of $\mathcal{P}$. In specific studies, the thin category that we usually consider is $(\mathbb{R}, \leq)$ or $(\mathbb{Z}, \leq)$. In category $\mathbb{R}$, the object is a real number $r \in \mathbb{R}$, and the morphism $r \to s$ exists if and only if $r \leq s$. Similarly, we can define the category $(\mathbb{Z}, \leq)$.

Example 2.1: Examples of the persistence modules.

- $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}$
- $\mathbb{V} : (\mathbb{Z}, \leq) \to \mathbf{Vec}_{\Bbbk}$
- $\mathbb{W} : (\mathbb{R}^m, \leq) \to \mathbf{Vec}_{\Bbbk}$ in which $(a_i) \leq (b_i)$ if $a_i \leq b_i$ for $i = 1, 2, \cdots, m$.

In the above section, we know that if we have a filtration of topological spaces, then we can get a family of homology groups. The data of a family of homology groups contains homology groups and homomorphisms between homology groups. Thus, in this section and the next section, we will focus only on the family of homology groups over some field $\Bbbk$, which are vector spaces. Note that if the ground ring of homology groups is R, then the homology groups are also modules over R. Some knowledge of commutative algebra will be used without proof. The details can be found in[28,42,96].

This section mainly introduces the correspondence, classification, and parameterization of 1-parameter persistence modules[106]. For the convenience of discussion, in this section, we will mainly consider the definition of 1-parameter persistence modules as follows,

Definition 2.6: A 1-parameter persistence module is a functor $\mathbb{M} : (\mathbb{N}, \leq) \to \mathbf{Mod}_R$ from the category of natural numbers to the category of modules over R, for some ring R. Equivalently, a 1-parameter persistence module $\mathbb{M} = \{M_i, m_{i,j}\}_{0 \leq i \leq j}$ is also a collection of R-modules M_i equipped with homomorphisms $m_{i,j} : M_i \to M_j$.

We define $\mathbb{M}_i := \mathbb{M}(i)$ and $\mathbb{M}_{i,j} := \mathbb{M}(i \leq j)$. In discussion of interleaving, we will use $\mathbb{M}(k)$ to denote a new functor such that $\mathbb{M}(k)_i := \mathbb{M}_{k+i}$ and $\mathbb{M}(k)_{i,j} := \mathbb{M}_{i+k,j+k}$, rather than an object in a target category. Sometimes, we use also $\Sigma^k \mathbb{M}$ to denote the k-shift of $\mathbb{M}$. According to our definition, it is evident that $\mathbb{M}_i = M_i$ and $\mathbb{M}_{i,j} = m_{i,j}$.

Reviewing the discussion of persistent homology, this definition of 1-parameter persistence modules is reasonable. Meanwhile, we will suppose the 1-parameter persistence modules that we will discuss are of finite type, i.e.

Definition 2.7: A 1-parameter persistence module $\mathbb{M} = \{M_i, m_{i,j}\}_{0 \leq i \leq j}$ is of finite type if each component R-module M_i is a finite generated and if $m_{i,j}$ are isomorphic for $k \leq i \leq j$ for some k.

Consider a 1-parameter persistence module $\mathbb{M} = \{\mathbb{M}_i, m_{i,j}\}_{0 \leq i \leq j}$ over R. We endow $R[x]$ with the standard grading and construct a graded module over the polynomial ring $R[x]$ as follows:

$$\alpha(M) = \bigoplus_{i=0}^{\infty} M_i,$$

where the R-module structure is defined as the direct sum of the structures in each component, and the action of x is specified by

$$x \cdot (a_0, a_1, a_2, \cdots,) = (0, m_{0,1}(a_0), m_{1,2}(a_1), m_{2,3}(a_2), \cdots)$$

in which $a_i \in M_i$.

Theorem 2.2: (Correspondence)[23] The correspondence α defines an equivalence of categories between the category of 1-parameter persistence modules of finite type over some ring R and the category of finitely generated non-negatively graded modules over $R[x]$.

The correspondence theorem implies that there exists a simple classification of 1-parameter persistence modules if R is not a field, as in the case of $\mathbb{Z}$. This aligns with fundamental results in commutative algebra, which demonstrate that the classification of modules over $\mathbb{Z}[x]$ is inherently complex. Although meaningful invariants can be assigned to $\mathbb{Z}[x]$-modules, a straightforward classification remains unattainable and is unlikely to ever be achieved. In contrast, when the ground ring is a field $\Bbbk$, the correspondence theorem provides a simple and elegant decomposition. The graded ring $\Bbbk[x]$ is a principal ideal domain (PID), and its only graded ideals, (x^n), are homogeneous. Consequently, we have the classification theorem of $\Bbbk[x]$-modules.

Theorem 2.3: (Classification)[23] Any finitely generated non-negatively graded module over $\Bbbk[x]$ is isomorphic to

$$\left(\bigoplus_{i=1}^{n} \Sigma^{\beta_i} \Bbbk[x]\right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} \Bbbk[x]/(x^{n_j})\right)$$

where Σ^{α} denotes an α-shift upward in grading.

Thus, the classification theorem of graded modules over $\Bbbk[x]$ implies the complete classification of 1-parameter persistence modules of finite type.

In the above section, the feature of the point cloud we ultimately obtain is a persistence diagram dgm or barcode $\mathcal{B}$. Then, we want to obtain the representation of 1-parameter persistence modules, which is similar to the persistence diagram. This process is the parameterization of 1-parameter persistence modules.

We call ordered pairs (i, j) intervals with $0 \leq i < j \in \mathbb{Z} \cup \{\infty\}$, and define

$$Q(i,j) = \begin{cases} \Sigma^i \Bbbk[x]/(x^{j-i}), & \text{if } j \neq \infty, \\ \Sigma^i \Bbbk[x], & \text{otherwise} \end{cases} \tag{2-1}$$

If $S = \{(i_1, j_1), (i_2, j_2), \cdots, (i_m, j_m)\}$ is a multi-set of intervals, then we define that

$$Q(S) = \bigoplus_{k=1}^{m} Q(i_k, j_k).$$

Meanwhile, for any 1-parameter persistence module of finite type $\mathbb{V} : \mathbb{N} \to \mathbf{Vec}_{\Bbbk}$, we have that $\alpha(\mathbb{V}) \cong \left(\oplus_{i=1}^{n} \Sigma^{\beta_i} \Bbbk[x]\right) \oplus \left(\oplus_{j=1}^{m} \Sigma^{\gamma_j} \Bbbk[x]/(x^{n_j})\right)$ for some β_i, γ_j and n_j by classification theorem. Therefore we can parameterize the 1-parameter persistence modules $\mathbb{V} : \mathbb{N} \to \mathbf{Vec}_{\Bbbk}$.

Theorem 2.4: (Parameterization)[23] The correspondence $S \mapsto Q(S)$ establishes a bijection between the finite multisets of intervals and the finitely generated graded modules over $\Bbbk[x]$. Thus, the isomorphism classes of persistence modules of finite type $\mathbb{V} : \mathbb{N} \to \mathbf{Vec}_{\Bbbk}$ are bijective to the finite multisets of intervals.

We call the multi-set S the persistence diagram of 1-parameter persistence module $\mathbb{M}$ corresponding to $Q(S)$, denoted as $dgm(\mathbb{M})$ or $\mathcal{B}_{\mathbb{M}}$.

Because in specific studies, the point clouds are finite, the family of homology groups over the field $\Bbbk$ in persistent homology always satisfies the condition of finite type. The parameterization theorem states the fact that persistence diagrams are an almost perfect representation of persistent homology without considering the differences in the ways in which point clouds construct simplicial complexes and the difficulty of vectorization of persistence diagrams.

We can generalize the discussion on 1-parameter persistence modules $\mathbb{N} \to \mathbf{Vec}_{\Bbbk}$ to persistence modules $\mathbb{R} \to \mathbf{Vec}_{\Bbbk}$. We also refer to the latter as the 1-parameter persistent modules. Indeed, we may define that a 1-parameter persistence module is a functor $\mathbb{M} : (T, \leq) \to \mathbf{Vec}_{\Bbbk}$, $T \subset \mathbb{R}$. An interval in T is a subset $J \subset T$ such that if $r, t \in J$, $s \in T$ and

$r < s < t$ then $s \in J$. In the concept of 1-parameter persistent modules $(T, \leq) \to \mathbf{Vec}_{\Bbbk}$, we no longer discuss the correspondence theorem in detail, but we can still discuss the classification and the parameterization. The idea of classification is to decompose a 1-parameter persistence module into the direct sum of indecomposable components, where each indecomposable component is simple enough. For any nonempty subset $J \subset T$, the interval module $\Bbbk_J$ is defined to be the 1-parameter persistence module $\Bbbk_J : T \to \mathbf{Vec}_{\Bbbk}$

$$(\Bbbk_J)_t = \begin{cases} \Bbbk & \text{if } t \in J, \\ 0 & \text{otherwise,} \end{cases} \tag{2-2}$$

and linear maps

$$i_{s,t} = \begin{cases} id_{\Bbbk} & \text{if } t, s \in J \\ 0 & \text{otherwise} \end{cases}$$

The essence of the classification theorem is to decompose 1-parameter persistence modules into interval modules.

Theorem 2.5: (Interval Decomposition)[28] Suppose that $\mathbb{V}$ is a 1-parameter persistence module $T \to \mathbf{Vec}_{\Bbbk}$ with $T \subset \mathbb{R}$. In either of the following cases, $\mathbb{V}$ can be decomposed into a direct sum of interval modules:

- T is finite;
- all $\dim V_t$ is finite.

Conversely, there is a persistence module $\mathbb{Z} \to \mathbf{Vec}_{\Bbbk}$ which does not allow an interval decomposition.

Example 2.2: [28] Webb provides this example, which is indexed over the nonpositive integers $-\mathbb{N}$:

$$W_0 = \{\text{sequences } (x_1, x_2, x_3, \cdots) \text{ of scalars}\}$$
$$W_{-n} = \{\text{such sequences with } x_1 = x_2 = \cdots = x_n = 0\} \ (n \geq 0)$$

The w_{-m}^{-n} are the canonical inclusion maps for any $n \leq m \leq 0$. This module can be concisely denoted as an infinite product $\mathbb{W} = \prod_{n \geq 0} \Bbbk_{[-n,0]}$.

Suppose that $\mathbb{W}$ has an interval decomposition. Because every map w_{-n}^{-n-1} is an inclusion, all of the intervals of the interval decomposition must be of the form $[-n, 0]$ or $[-\infty, 0]$. Then the multiplicity of $[-n, 0]$ may be calculated by $\dim(W_{-n}/W_{-n-1}) = 1$. The multiplicity of $(-\infty, 0]$ is zero because any summand of that type requires a nonzero element of W_0 that is in the image of w_0^{-n} for all $n \geq 0$. However, there is no such element since $\bigcap_{n\geq 0} W_{-n} = 0$. All of this seems to indicate that $\mathbb{W} \cong \bigoplus_{n\geq 0} \mathbb{k}[-n, 0]$. But $\dim(W_0)$ is uncountable, which contradicts the results stated above. So $\mathbb{W}$ doesn't allow an interval decomposition.

After the interval decomposition theorem is established, parameterization can be given similar to the previous discussion about persistence modules $\mathbb{N} \to \mathbf{Vec}_{\mathbb{k}}$.

Remark 2.1: The parameterization argues that 1-parameter persistence modules can be represented by a complete discrete invariant.

In the study of persistence modules, we can roughly divide invariants into discrete and continuous ones. Discrete invariants refer to invariants such as the Betti number, which are always integers and come from a set that is independent of the coefficient field $\mathbb{k}$, giving them a finite parameterization. However, continuous invariants may exhibit uncountable cardinality or depend fundamentally on the choice of the coefficient field $\mathbb{k}$. Consequently, such invariants are generally unsuitable for computational purposes due to their inherent complexity and field dependence. It is crucial to emphasize that the classification of invariants as discrete or continuous is independent of the coefficient field $\mathbb{k}$ - that is, this distinction remains valid regardless of whether $\mathbb{k}$ is a continuous field (such as $\mathbb{R}$) or a discrete field (such as $\mathbb{Z}/p$ for a prime p).

2.3 Multi-Parameter Persistence Modules

In this section, we will discuss the correspondence, classification, and parameterization of multi-parameter persistence modules. Unfortunately, no satisfactory results have been found regarding the parameterization of multi-parameter persistence modules. What's more, it can be proved that the discrete complete invariant of multi-parameter persistence modules does not exist, even the 2-parameter persistence modules of finite type. Finally, we will display an enlightening example of a 2-parameter persistence module. However, we cannot find a discrete complete invariant of the 2-parameter persistence module. For more details, please refer to the paper[23].

We may regard $\mathbb{N}^n$ as a partially ordered set $(\mathbb{N}^n, \leq)$, with the partial order relation defined as follows:

$$\mathbf{v} = (v_i) \leq \mathbf{w} = (w_i), \text{ if } v_i \leq w_i \text{ for all } i.$$

Definition 2.8: A n-parameter persistence module is a functor $\mathbb{M} : (\mathbb{N}^n, \leq) \to \mathbf{Mod}_R$, for some ring R. Equivalently, a m-parameter persistence module $\mathbb{M} = \{M_{\mathbf{v}}, m_{\mathbf{v},\mathbf{w}}\}_{\mathbf{0}\leq\mathbf{v}\leq\mathbf{w}}$ is also a family of R-modules $M_{\mathbf{v}}$, together with homomorphisms $m_{\mathbf{v},\mathbf{w}} : M_{\mathbf{v}} \to M_{\mathbf{w}}$.

Since the $(\mathbb{N}^n, \leq)$ is a thin category, $m_{\mathbf{u},\mathbf{v}} \circ m_{\mathbf{v},\mathbf{w}} = m_{\mathbf{u},\mathbf{w}}$ whenever $\mathbf{u} \leq \mathbf{v} \leq \mathbf{w}$. Similar to 1-parameter persistence modules, we can also define multi-parameter persistence modules of finite type,

Definition 2.9: A n-parameter persistence module $\mathbb{M} = \{M_{\mathbf{v}}, m_{\mathbf{v},\mathbf{w}}\}_{\mathbf{0}\leq\mathbf{v}\leq\mathbf{w}}$ is of finite type if each component R-module $M_{\mathbf{v}}$ is a finite generated and if $m_{\mathbf{v},\mathbf{w}}$ are isomorphic whenever $\mathbf{v} = (v_i)$ and $\mathbf{w} = (w_i)$ satisfy $v_i = w_i$ for $i \neq i_0$ and $c \leq v_{i_0} \leq w_{i_0}$ for some $c \in \mathbb{N}$.

A monomial in $x_1, x_2, \cdots, x_n$ is a product of the form $x_1{}^{v_1} x_2{}^{v_2} \cdots x_n{}^{v_n}$ with $v_i \in \mathbb{N}$. We denote it $x^{\mathbf{v}}$, where $\mathbf{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{N}^n$. Let $A_n = \mathbb{k}[x_1, x_2, \cdots, x_n]$ be a n-graded ring, and $\mathbb{M}$ be any n-graded A_n-module. We may define a n-graded A_n-module $\Sigma^{\mathbf{v}}\mathbb{M}$ for any $\mathbf{v} \in \mathbb{Z}^n$ by defining its graded components as $(\Sigma^{\mathbf{v}}\mathbb{M})_{\mathbf{w}} := M_{\mathbf{w}-\mathbf{v}}$.

Let $\mathbb{M}$ be a persistence module, we may construct an n-graded module over A_n by defining

$$\alpha(\mathbb{M}) = \bigoplus_{\mathbf{v}} M_{\mathbf{v}}$$

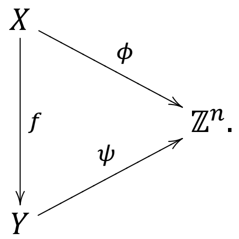
where the $\mathbb{k}$-module structure is given by the direct sum. Additionally, we require that the multiplication map $x^{\mathbf{v}-\mathbf{u}} : M_{\mathbf{u}} \to M_{\mathbf{v}}$ coincides with the morphism $m_{\mathbf{u},\mathbf{v}}$ in the persistence module $\mathbb{M} = \{M_{\mathbf{u}}, m_{\mathbf{u},\mathbf{v}}\}_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{v}}$ whenever any $\mathbf{u} \leq \mathbf{v} \in \mathbb{N}^n$.

The correspondence theorem is similar to the result of 1-parameter persistence modules.

Theorem 2.6: (Correspondence)[23] The correspondence α defines an equivalence of categories between the category of finite persistence modules over $\mathbb{k}$ and the category of finitely generated n-graded modules over $A_n = \mathbb{k}[x_1, \cdots, x_n]$.

Classification

Definition 2.10: A n-graded set (X, ϕ) is a set X with the grade $\phi : X \to \mathbb{Z}^n$. And the map f of n-graded sets satisfies the commutative diagram

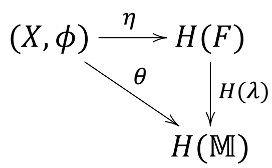


For any n-graded module $\mathbb{M} = \bigoplus_{\mathbf{v}} M_{\mathbf{v}}$ over A_n, $H(\mathbb{M}) := \bigcup_{\mathbf{v}\in\mathbb{Z}^n} M_{\mathbf{v}}$, then $H(\mathbb{M})$ is a n-graded set.

We define the free n-graded A_n-module on the graded set (X, ϕ) as an n-graded A_n-module F equipped with an inclusion map of n-graded sets

$$\eta : (X, \phi) \hookrightarrow H(F) \subset F,$$

satisfying the following universal property: for any n-graded A_n-module $\mathbb{M}$ and any map of n-graded sets $\theta : (X, \phi) \to H(\mathbb{M})$, there exists a unique homomorphism $\lambda : F \to \mathbb{M}$ of n-graded A_n-modules that makes the diagram



commutes.

Definition 2.11: The type of an n-graded vector space $\mathbb{V}$ is defined as the unique multiset isomorphic to a graded basis for $\mathbb{V}$, denoted by $\xi(\mathbb{V})$. Analogously, for any free n-graded module F, we define $\xi(F) := \xi(\mathbb{k} \otimes_{A_n} F)$.

Indeed, the type, $\xi(-)$, denotes the location and the number of generators of the object.

Example 2.3: If the type $\xi(\mathbb{V}) = \{((0,1),2), ((1,0),1), ((2,1),1)\}$ for some 2-graded vector space $\mathbb{V}$, the $\mathbb{V} \cong span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}, \mathbf{w}\}$ in which deg $\mathbf{u}_1 =$ deg $\mathbf{u}_2 = (0,1)$, deg $\mathbf{v} = (1,0)$ and deg $\mathbf{w} = (2,1)$.

If the type $\xi(F) = \{((0,1),2), ((1,0),1), ((2,1),1)\}$ for some 2-graded free module F over A_2, the F is isomorphic to the free A_2-module on the graded set $\{a_1, a_2, b, c\}$ in which $\deg a_1 = \deg a_2 = (0,1)$, $\deg b = (1,0)$ and $\deg c = (2,1)$.

Suppose that $\mathbb{M}$ is a finitely generated n-graded A_n-module. We define the finite-dimensional $\Bbbk$-vector space $\rho(\mathbb{M}) = \Bbbk \otimes_{A_n} \mathbb{M}$, where $\Bbbk$ is given the module structure where all the variables x_i act trivially, i.e., by zero. Let $\mathbb{V}$ be a n-graded vector space. Then there exists a free n-graded module $F(\mathbb{V})$ satisfying the isomorphism $\mathbb{V} \cong \rho(F(\mathbb{V})) = \Bbbk \otimes_{A_n} F(\mathbb{V})$. Meanwhile, we can also consider $\mathbb{V}(\xi)$ and $F(\xi)$ that are defined $\xi(\mathbb{V}(\xi)) = \xi$ and $\xi(F(\xi)) = \xi$, for any multi-set ξ.

For any n-graded module $\mathbb{M}$, we consider the minimal free resolution of $\mathbb{M}$

$$\cdots \to F_1 \to F_0 \to M \to 0$$

$\xi_0(\mathbb{M}) := \xi(F_0)$, $\xi_1(\mathbb{M}) := \xi(F_1)$. In fact, the free module F_0 is called the free hull of $\mathbb{M}$. The ξ_0 and ξ_1, which are multisets, are invariants of the isomorphism class of $\mathbb{M}$. If $\mathbb{M}$ is a 1-graded module over $A_1 = \Bbbk[x]$, that is the algebraic model of some 1-parameter persistence module $\mathbb{M}$, then $F_2 = F_3 = \cdots = 0$. Thus, 1-parameter persistence modules can be decided completely by persistence diagrams.

Suppose two multi-sets ξ_0 and ξ_1 satisfying $\xi_0 \cap \xi_1 = \emptyset$. We begin by constructing a free n-graded A_n-module F such that $\xi(\rho(F)) = \xi_0$. Then, we define that $\mathcal{S}(F,\xi) := \{L \mid L \text{ is a } A_n\text{-submodule of } F \text{ and } \xi(L) = \xi_1\}$ and $\mathcal{I}(\xi_0,\xi_1) := \{[\mathbb{M}] \text{ (isomorphic class)} \mid \xi_0(\mathbb{M}) = \xi_0,\ \xi_1(\mathbb{M}) = \xi_1\}$. Subsequently, we have the map

$$q : \mathcal{S}(F,\xi_1) \to \mathcal{I}(\xi_0,\xi_1)$$
$$L \mapsto [F/L]$$

The automorphism group of F acts on $\mathcal{S}(F,\xi_1)$ through the action defined by $g \cdot L = g(L)$ for any $g \in \mathrm{Aut}(F)$. Thus, it is obvious that $F/L \cong F/g(L)$. We represent the set of the orbits of action $Aut(F) \curvearrowright \mathcal{S}(F,\xi_1)$ as $G_F \setminus \mathcal{S}(F,\xi_1)$. The theorem is figured out easily,

Theorem 2.7: (Classification)[23] Let F be described as above, $\xi_0 \cap \xi_1 = \emptyset$, and let $G_F := Aut(F)$. The map q satisfies the formula $q(g \cdot L) = q(L)$ and thus induces a map $\bar{q} : G_F \setminus \mathcal{S}(F,\xi_1) \to I(\xi_0,\xi_1)$. Furthermore, $\bar{q}$ is bijective.

Remark 2.2: In the original version of this theorem (Theorem 9 in[23]), there was no condition $\xi_0 \cap \xi_1 = \emptyset$. But I made some minor modifications when stating it here. Because if we remove the condition $\xi_0 \cap \xi_1 = \emptyset$, we can easily provide a counterexample to state that the map q is not well defined.

Example 2.4: Given $F := A_1 \cdot a \oplus A_1 \cdot b \cong \Bbbk[x] \oplus \Sigma^1 \cdot \Bbbk[x]$ is a free 1-graded A_1-module, a and b are generators with deg $a = 0$, deg $b = 1$. Let $L_1 = A_1 \cdot b$, $M_1 = F/L_1 \cong A_1 \cong \Bbbk[x]$ and $L_2 = A_1 \cdot (xa)$, $M_2 = F/L_2 \cong A_1/(x) \oplus A_1 \cdot b \cong \Bbbk[x]/(x) \oplus \Sigma^1 \cdot \Bbbk[x]$.

Obviously, $\xi(L_1) = \xi(L_2) = \{((1),1)\}$, then $L_1, L_2 \in \mathcal{S}(F, \xi_1)$ with $\xi_1 = \{((1),1)\}$. However $\xi_0(M_1) = \{((0),1)\} \neq \xi_0(M_2) = \{((0),1),((1),1)\}$ and $\xi_1(M_1) = \emptyset \neq \xi_1(M_2) = \{((1),1)\}$.

In summary, for finitely generated n-graded A_n-modules, we can preliminarily classify them through ξ_0 and ξ_1. Meanwhile, if ξ_0 and ξ_1 satisfy the condition, $\xi_0 \cap \xi_1 = \emptyset$, then we may completely classify them by using the above theorem.

Parameterization

Our objective is to demonstrate that, in contrast to its 1-parameter counterpart, multi-parameter persistence modules do not have complete discrete invariants. Therefore, we need only to show that there is no complete discrete invariant for a subset of $\mathcal{I}(\xi_0, \xi_1)$.

Remark 2.3: The derivation of parameterization needs to use the theorem of classification, so we will suppose $\xi_0 \cap \xi_1 = \emptyset$. However, the original result[23] of parameterization has no condition $\xi_0 \cap \xi_1 = \emptyset$.

We begin by considering any n-graded A_n-module $\mathbb{M}$. For every $\mathbf{v} \in \mathbb{Z}^n$, we consider the $\Bbbk$-vector subspace

$$(I\mathbb{M})_{\mathbf{v}} = \sum_{\mathbf{v} > \mathbf{e}_i} x_i M_{\mathbf{v}-\mathbf{e}_i} \subset M_{\mathbf{v}}$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{Z}^n$. $\mathbf{v}$ is called a gap of $\mathbb{M}$ if $(I\mathbb{M})_{\mathbf{v}} \neq M_{\mathbf{v}}$. We define that $\Gamma(\mathbb{M})$ is the set of gaps of $\mathbb{M}$.

Remark 2.4: The gap $\mathbf{v}$ denotes that $M_{\mathbf{v}}$ contains the generators of $\mathbb{M}$. The module $I\mathbb{M}$ can be considered as leave, and the generators of $\mathbb{M}$ can be considered as roots.

Theorem 2.8: [23] If $\mathbb{M}$ is finitely generated, then $\Gamma(\mathbb{M})$ is finite. Additionally, the type of $\Bbbk \otimes_{A_n} \mathbb{M}$, denoted by $\xi(\mathbb{M})$, corresponds to the multi-set $(\Gamma(\mathbb{M}), \alpha_{\mathbb{M}})$, where

$$\alpha_{\mathbb{M}}(\mathbf{v}) = \dim\,(M_{\mathbf{v}}/(I\mathbb{M})_{\mathbf{v}}) = \dim\,(M_{\mathbf{v}}) - \dim\,((I\mathbb{M})_{\mathbf{v}}).$$

Theorem 2.9: [23] Suppose that F is a free n-graded A_n-module, and suppose L and L' are any n-graded submodules (note that L is not necessarily free). Then $L = L'$ if and only if $\Gamma(L) = \Gamma(L')$ and $L_{\mathbf{v}} = L'_{\mathbf{v}}$ for any $\mathbf{v}$ which are gaps of either.

The theorem shows the fact that if one wants to decide a submodule L of $\mathbb{M}$, then one only needs to decide the gaps $\mathbf{v}$ and $L_{\mathbf{v}}$.

Let $\xi = (V, \alpha)$ be an arbitrary multiset, and let $\delta : V \to \mathbb{Z}$ be any map. Let F be any finitely generated free n-graded module over A_n. We define that $ARR_{\xi,\delta}(F)$ denote the collection of all assignments $\mathbf{v} \mapsto L_{\mathbf{v}}$, where $\mathbf{v} \in V$ and $L_{\mathbf{v}}$ is a $\Bbbk$-linear subspace of $F_{\mathbf{v}}$, subject to the following three conditions:

- $\mathbf{v}' \leq \mathbf{v} \Rightarrow x^{\mathbf{v}-\mathbf{v}'} L_{\mathbf{v}'} \subset L_{\mathbf{v}}$,
- $\dim_{\Bbbk}(L_{\mathbf{v}}) = \delta(\mathbf{v})$,
- $\dim_{\Bbbk}(L_{\mathbf{v}} / \sum_{\mathbf{v}' \leq \mathbf{v}} x^{\mathbf{v}-\mathbf{v}'} L_{\mathbf{v}'}) = \alpha(\mathbf{v})$ for all $\mathbf{v} \in V$.

Remark 2.5: $\xi = (V, \alpha)$ denote the multi-set of gaps. Every designment L corresponds a submodule L satifying $\xi(L) = \xi$. The condition, $\mathbf{v}' \leq \mathbf{v} \Rightarrow x^{\mathbf{v}-\mathbf{v}'} L_{\mathbf{v}'} \subset L_{\mathbf{v}}$, corresponds to the condition that L is a submodule of F. The condition, $\dim_{\Bbbk}(L_{\mathbf{v}} / \sum_{\mathbf{v}' \leq \mathbf{v}} x^{\mathbf{v}-\mathbf{v}'} L_{\mathbf{v}'}) = \alpha(\mathbf{v})$ for all $\mathbf{v} \in V$, corresponds to the condition that $\xi(L) = \xi$. The condition, $\dim_{\Bbbk}(L_{\mathbf{v}}) = \delta(\mathbf{v})$, state that $ARR_{\xi,\delta}(F)$ is only a subset of $\mathcal{S}(F, \xi_1)$ generally.

Obviously, we have $ARR_{\xi,\delta}(F) \subseteq \mathcal{E} = \prod_{\mathbf{v} \in V} Gr_{\delta(\mathbf{v})}(F_{\mathbf{v}})$ from the condition $\dim_{\Bbbk}(L_{\mathbf{v}}) = \delta(\mathbf{v})$.

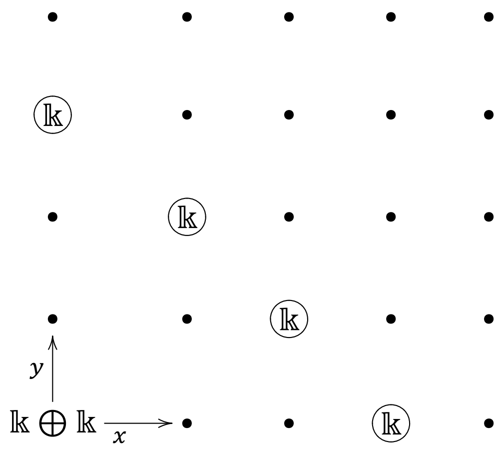
Theorem 2.10: [23] The set $ARR_{\xi,\delta}(F)$ is in bijective correspondence with a quasiprojective variety.

The theorem asserts that the set $\mathcal{S}(\xi_0, \xi_1)$ can be viewed as a subset of an algebraic variety. Moreover, when $\xi_0 \cap \xi_1 = \emptyset$, the action of $\mathrm{Aut}(F)$ on elements of $\mathcal{S}(\xi_0, \xi_1)$ constitutes an algebraic group action. Consequently, this portion of the classification emerges as a continuous invariant.

An Important Example

In this subsection, we will introduce an example that vividly demonstrates that even a portion of 2-parameter persistence modules may not possess discrete complete invariants.

This example was provided by Carlsson and Zomorodian in their paper[23].



Consider a set of 2-parameter persistence modules for which

$$\xi_0 = \{((0,0),2)\},$$
$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\},$$

as visualized on $\mathbb{N}^2$ in the above figure.

It is easy to build a bifiltered simplicial complex whose 1-dimensional homology groups correspond to this picture, that are persistence modules M generated by this 2-filtration of simplicial complexes satisfies $\xi_0(\mathbb{M}) = \xi_0$ and $\xi_1(\mathbb{M}) = \xi_1$.

At the point $(0,0)$, the complex consists of two loops, yielding $\mathbb{k} \oplus \mathbb{k}$. At each of the marked coordinates, we attach a distinct surface between the two loops, ensuring that no two complexes are identical. For instance, a cylinder can be attached at $(3,0)$, a punctured crosscap at $(2,1)$, and so forth. Notably, the discrete invariants ξ_0, ξ_1 fail to distinguish the differences between these aforementioned complexes.

To achieve the classification, we utilize the classification theorem. The generators of $F(\xi_0)$ are positioned together, enabling the complete group of automorphisms

$$GL(F(\xi_0)) = GL(\mathbb{k}^2) = GL_2(\mathbb{k}).$$

Classification:

$F(\xi_0) = A_2 \oplus A_2,\ GL(F(\xi_0))) = GL_2(\mathbb{k})$

For $\forall(\mathbf{v}, i) \in \xi_1$, dim $F(\xi_0)_\mathbf{v} = 2$ and dim $F(\xi_1)_\mathbf{v} = 1$, then $Gr_{\dim F(\xi_1)}(F(\xi_0))_\mathbf{v} = Gr_1(\mathbb{k}^2) = \mathbb{P}^1(\mathbb{k})$.

⇒ Classification: the orbit space of $GL_2(\mathbb{k}) \curvearrowright \mathbb{P}^1(\mathbb{k})^4$. (The action is evident.)

Parameterization:

Let Ω be the subspace of the orbits space $GL_2(\mathbb{k}) \curvearrowright \mathbb{P}^1(\mathbb{k})^4$ containing pairwise-distinct lines $GL_2(\mathbb{k}) \curvearrowright \{(l_1, l_2, l_3, l_4) \in \mathbb{P}^1(\mathbb{k})^4 \mid l_i \neq l_j \text{ for } i \neq j\}$.

We can transform the lines using matrices from $GL_2(\mathbb{k})$:

(1) l_1 transform into the x-axis,

(2) l_2 transform into the y-axis,

(3) l_3 transform into the line $\{x = y\}$,

Then $(l_1, l_2, l_3, l_4) \xrightarrow{GL_2(\mathbb{k})} (x\text{-axis}, y\text{-axis}, \text{diagonal}, \lambda_4)$, in which λ_4 is l_4 after the transformations.

$\Rightarrow \Omega \overset{1-1}{\longleftrightarrow} \mathbb{P}^1(\mathbb{k}) - \{0, 1, \infty\} = \mathbb{k} - \{0, 1\}$.

Therefore, it is impossible to obtain a complete discrete invariant in this example.

2.4 Harder-Narasimhan Filtrations of Persistence Modules

As discussed in the prior description, the difficulty in identifying discrete invariants for multi-parameter persistence modules has led some researchers to seek new approaches for discovering such invariants. A new method for finding discrete invariants of multi-parameter persistence modules will be introduced in this section, which involves computing invariants of the Harder-Narasimhan filtration[47-49] of the persistence modules. This method differs from the previously adopted approaches. Earlier, when discussing the decomposition of persistence modules, the primary focus was on the direct sum decomposition of persistence modules. However, this new perspective views the direct sum decomposition as merely a special case of decomposition. By drawing an analogy to the decomposition of topological spaces, specifically filtrations of topological spaces, we can interpret the decomposition of persistence modules as a filtration. Subsequently, by constructing filtrations for the persistence modules and then computing the invariants associated with the filtrations, we can get new discrete invariants of persistence modules $\mathbb{N}^n \to \mathbf{Vec}_{\mathbb{k}}$.

Before discussing filtrations of persistence modules $\mathbb{N}^n \to \mathbf{Vec}_{\mathbb{k}}$, we introduce a incomplete discrete invariant of persistence modules $\mathbb{M} : \mathbb{N}^n \to \mathbf{Vec}_{\mathbb{k}}$, the rank invariant $\rho_{\mathbb{M}}$. Define that $\dot{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\mathbb{D}^n := \{(\mathbf{u}, \mathbf{v}) | \mathbf{u} \in \mathbb{N}^n,\ \mathbf{v} \in \dot{\mathbb{N}}^n, \text{ and } \mathbf{u} \leq \mathbf{v}\}$.

Definition 2.12: Let $\mathbb{M} : \mathbb{N}^n \to \mathbf{Vec}_{\mathbb{k}}$ be any n-parameter persistence module, and be finitely generated if we regard $\mathbb{M}$ as an n-graded A_n module. The rank invariant $\rho_{\mathbb{M}}$ is a map from $\mathbb{D}$ to $\mathbb{N}$, defined as

$$\rho_{\mathbb{M}}(\mathbf{u}, \mathbf{v}) = \text{rank } (x^{\mathbf{v}-\mathbf{u}} : \mathbb{M}_{\mathbf{u}} \to \mathbb{M}_{\mathbf{v}}).$$

It is obvious that the map $\rho_{\mathbb{M}}$ is an invariant of persistence modules. And we can easily prove the following results.

Lemma 2.2: [48] If $\mathbf{u} \leq \mathbf{u}' \leq \mathbf{v}' \leq \mathbf{v}$, Then $\rho_{\mathbb{M}}(\mathbf{u}, \mathbf{v}) \leq \rho_{\mathbb{M}}(\mathbf{u}', \mathbf{v}')$.

Proof: Because $\mathbf{u} \leq \mathbf{u}' \leq \mathbf{v}' \leq \mathbf{v}$, we have $\mathbb{M}_{\mathbf{u}} \to \mathbb{M}_{\mathbf{u}'} \to \mathbb{M}_{\mathbf{v}'} \to \mathbb{M}_{\mathbf{u}}$. ■

When $\mathbb{M}$ a multi-parameter persistence modules $\mathbb{N}^n \to \mathbf{Vec}_{\mathbb{k}}$ with $n \geq 2$, the rank invariant $\rho_{\mathbb{M}}$ is not complete. However, when $n = 1$, the rank invariant $\rho_{\mathbb{M}}$ is complete.

Theorem 2.11: [48] The rank invariant $\rho_{\mathbb{M}}$ is complete for 1-parameter persistence modules.

Proof: In the first section of this chapter, we discussed persistent homology and persistence diagrams, and introduced the calculation formula of persistence diagrams

$$\mu_p(i,j) = (\beta_p(i,j-1) - \beta_p(i,j)) - (\beta_p(i-1,j-1) - \beta_p(i-1,j)),$$

where $\beta_p(i,j) = \rho_{\mathbb{M}}(i,j)$ if $\mathbb{M}$ is the persistence module as follows,

$$0 = H_p(K_0) \xrightarrow{H_p(f_{0,1})} H_p(K_1) \xrightarrow{H_p(f_{1,2})} H_p(K_2) \xrightarrow{H_p(f_{2,3})} \cdots \xrightarrow{H_p(f_{n-1,n})} H_p(K_n).$$

Meanwhile, we know that the persistence diagrams are complete invariants of persistent homology. Then the rank invariant $\rho_{\mathbb{M}}$ is complete for 1-parameter persistence modules. ■

We recall that a quiver Q is a multi-digraph, that is, a directed graph where loops and multiple arrows are allowed. In other words, a quiver Q consists of two sets, Q_0 and Q_1, where the elements of set Q_0 are called the vertices of the quiver Q, and the elements of Q_1 are called the edges of Q. Additionally, it is equipped with two maps called the source map and the target map, denoted as $s, t : Q_0 \to Q_1$, respectively. Each edge e may be denoted by an arrow $s(e) \to t(e)$. And we define a path in the quiver Q is a finite sequence of edges $p = (e_1, \cdots, e_n)$ satisfying $t(e_i) = s(e_{i+1})$ for any i. We call a path $p = (e_1, \cdots, e_n)$ is a loop if $s(e_1) = t(e_n)$. If a quiver $Q = (Q_0, Q_1)$ admits no loops, we call Q is acyclic. In this subsection, unless otherwise specified, we always assume that all quivers are acyclic and have only finitely many vertices and edges.

We recall a finite-dimensional representation $\mathbb{V}$ of a quiver $Q = (Q_0, Q_1)$ is a functor

$Q \to \mathbf{Vec}_{\Bbbk}$, where $\mathbb{V}_x$ is a vector space for any vertex $x \in Q_0$ and $\mathbb{V}_e$ is a linear map from $\mathbb{V}_{s(e)}$ to $\mathbb{V}_{t(e)}$. Suppose that $\mathbb{W}$ is another representation of Q. We say that $\mathbb{W}$ is a subrepresentation of $\mathbb{V}$ if there is a monomorphism $\phi : \mathbb{W} \hookrightarrow \mathbb{V}$. Fix a quiver $Q = (Q_0, Q_1)$, a representation of Q is called indecomposable if it does not admit any nontrivial direct sum decomposition in $\mathbf{Rep}(Q)$, where $\mathbf{Rep}(Q)$ is the category of representations of $Q = (Q_0, Q_1)$.

It is well-known[94] that for any finite-dimensional representation $\mathbb{V}$ of the quiver Q, there exists a unique multiset $(\mathrm{Ind}_Q(\mathbb{V}), d_{\mathbb{V}})$, where $d_{\mathbb{V}} : \mathrm{Ind}_Q(\mathbb{V}) \to \mathbb{N}$ is the multiplicity function, such that the representation $\mathbb{V}$ can be decomposed as follows:

$$\mathbb{V} \cong \bigoplus_I I^{d_{\mathbb{V}}(I)}$$

with I ranging over $\mathrm{Ind}_Q(\mathbb{V})$.

For any representation $\mathbb{V} : Q \to \mathbf{Vec}_{\Bbbk}$, we can define the dimension vector $\underline{\dim}_{\mathbb{V}} : Q_0 \to \mathbb{N}$ given by $x \mapsto \dim \mathbb{V}_x$. Gabriel[50] asserted that the collection of indecomposable objects within $\mathbf{Rep}(Q)$, corresponding to a specified dimension vector, is finite precisely when the undirected graph associated to Q is a finite union of simply laced Dynkin diagrams.

Next, let's formally introduce the Harder-Narasimhan filtration. Firstly, we present the concept of the Grothendieck group of an abelian category C. For any abelian category C, its Grothendieck group is an abelian group $K(C)$ freely generated by the isomorphism classes $[V]$ in C modulo a relation of the form $[V] = [U] + [W]$ if

$$0 \to U \to V \to W \to 0$$

in C.

A stability condition on C is a group homomorphism

$$Z : K(C) \to (\mathbb{C}, +)$$

and $Z(K(C) \setminus \{0\}) \subseteq \{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$. For a stability condition Z, the Z-slope of an object $V \neq 0$ is the real number

$$\mu_Z(V) := \frac{\mathrm{Im}\, Z(V)}{\mathrm{Re}\, Z(V)}.$$

We call V Z-semistable if $\mu_Z(U) \leq \mu_Z(V)$ for any subobject $U \subset V$ and $U \neq 0$, and V Z-stable if $\mu_Z(U) < \mu_Z(V)$ for any subobject $U \subset V$, and $U \notin \{0, V\}$. With these concepts in place, we can now proceed to elaborate on the Harder-Narasimhan filtration.

Theorem 2.12: [48] Suppose C is any abelian category satisfying the Noetherian and Artinian hypotheses. Fix Z a stability condition on C. If $V \neq 0$, there is a unique filtration $V^{\bullet}$ of finite length $n \geq 1$

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V$$

whose successive quotients $S^i := V^i/V^{i-1}$ are Z-semistable and strictly decreaing slopes:

$$\mu_Z(S^1) > \mu_Z(S^2) > \cdots > \mu_Z(S^n).$$

The filtration that appears in this theorem is precisely the Harder-Narasimhan filtration of V. And for any representation $\mathbb{V} \neq 0$ in **Rep**(Q), the Harder-Narasimhan filtration of $\mathbb{V}$ along a stability condition Z is denoted by $\mathbf{HN}_Z^{\bullet}(\mathbb{V})$. If the stability condition Z is a standard stability condition, $\mathbf{HN}_Z^{\bullet}(\mathbb{V})$ denoted by $\mathbf{HN}_{\alpha}^{\bullet}(\mathbb{V})$.

For any fixed quiver $Q = (Q_0, Q_1)$, we consider the Harder-Narasimhan filtration of the object of category **Rep**(Q), since the category **Rep**(Q) is abelian. For any stability condition on **Rep**(Q), $Z : K(\mathbf{Rep}(Q)) \to (\mathbb{C}, +)$, we may decide it by two functions $\alpha, \beta : Q_0 \to \mathbb{R}$, that is $Z(\mathbb{V}) = \sum_{x \in Q_0} (\beta(x) + \sqrt{-1}\alpha(x)) \cdot \dim \mathbb{V}_x$. Thus the Z-slope of $\mathbb{V}$, $\mu_Z(\mathbb{V}) = \frac{\operatorname{Im} Z(\mathbb{V})}{\operatorname{Re} Z(\mathbb{V})} = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim \mathbb{V}_x}{\sum_{x \in Q_0} \beta(x) \cdot \dim \mathbb{V}_x}$. If $\beta = 1$, then the Z-slope of $\mathbb{V}$, $\mu_Z(\mathbb{V}) = \frac{\sum_{x \in Q_0} \alpha(x) \cdot \dim \mathbb{V}_x}{\sum_{x \in Q_0} \dim \mathbb{V}_x}$, is determined by α and is denoted as $\mu_{\alpha}(\mathbb{V})$. Therefore, we call that the stability condition Z is a standard stability condition and call α the central charge of Z.

Lemma 2.3: [48] Let $\alpha : Q_0 \to \mathbb{R}$ be a function, and three objects $\mathbb{U}, \mathbb{V}, \mathbb{W} \in \operatorname{ob} \mathbf{Rep}(Q)$ satisfying the following short exact sequence

$$0 \to \mathbb{U} \to \mathbb{V} \to \mathbb{W} \to 0.$$

Then, one of the following inequalities must hold. Either

- $\mu_{\alpha}(\mathbb{U}) > \mu_{\alpha}(\mathbb{V}) > \mu_{\alpha}(\mathbb{W})$, or
- $\mu_{\alpha}(\mathbb{U}) = \mu_{\alpha}(\mathbb{V}) = \mu_{\alpha}(\mathbb{W})$, or
- $\mu_{\alpha}(\mathbb{U}) < \mu_{\alpha}(\mathbb{V}) < \mu_{\alpha}(\mathbb{W})$.

And when $\mu_{\alpha}(\mathbb{U}) = \mu_{\alpha}(\mathbb{V}) = \mu_{\alpha}(\mathbb{W})$, $\mathbb{V}$ is α-semistable if and only if $\mathbb{U}, \mathbb{W}$ are α-semistable.

Corollary 2.1: [48] If $\mathbb{U}, \mathbb{W}$ are α-semistable with the same α-slope μ, then $\mathbb{U} \oplus \mathbb{W}$ is also.

For any representation $\mathbb{V}$ of Q, we have defined the dimension vector $\underline{\dim}_{\mathbb{V}}$. Indeed, we may regard $\underline{\dim}$ as a group homomorphism $K(\mathbf{Rep}(Q)) \to \mathbb{Z}^{Q_0}$, that assigns a representation $\mathbb{V}$ to its dimension vector $\underline{\dim}_{\mathbb{V}}$.

Definition 2.13: The Harder-Narasimhan type of $\mathbb{V} \neq 0$ in $\mathbf{Rep}(\mathbb{V})$ along $\alpha : Q_0 \to \mathbb{R}$ is $T[\mathbb{V};\alpha]$, defined as

$$T[\mathbb{V};\alpha] := (\underline{\dim}_{S^1}, \underline{\dim}_{S^2}, \cdots, \underline{\dim}_{S^n})$$

where n is the length of the Harder-Narasimhan filtration $\mathbf{HN}^{\bullet}_{\alpha}(\mathbb{V})$ of $\mathbb{V}$, and $S^i = \mathbf{HN}^{i}_{\alpha}(\mathbb{V})/\mathbf{HN}^{i+1}_{\alpha}(\mathbb{V})$.

On the other hand, the Harder-Narasimhan type $T[\mathbb{V};\alpha]$ may be regarded as a map $T[\mathbb{V};\alpha] : \mathbb{R} \to \mathbb{Z}^{Q_0}$, defined as follows

$$T[\mathbb{V};\alpha](\lambda) = \begin{cases} \underline{\dim}_{S^i}, & \lambda = \mu_\alpha(S^i) \\ (0,0,\cdots,0), & \text{otherwise.} \end{cases} \tag{2-3}$$

Proposition 2.1: [48] If we regard $T[\mathbb{V};\alpha]$ as a map from $\mathbb{R}$ to $\mathbb{Z}^{Q_0}$, then for any representations $\mathbb{V}, \mathbb{W}$ in $\mathbf{Rep}(Q)$, $T[\mathbb{V} \oplus \mathbb{W};\alpha] = T[\mathbb{V};\alpha] + T[\mathbb{W};\alpha]$.

We have previously mentioned that the standard stability conditions are solely determined by α, and while there are various choices for α, we can prioritize the most distinctive ones, which are the delta functions.

Definition 2.14: we call the delta functions δ_x for any $x \in Q_0$ the skyscraper central charge at $x \in Q_0$, as follows

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases} \tag{2-4}$$

Meanwhile, we can define the skyscraper invariant $\delta_{\bullet}$ on $\mathbf{Rep}_Q$ that assigns each representation $\mathbb{V}$ in $\mathbf{Rep}(Q)$ to the collection of HN types $\delta_{\mathbb{V}} := \{T[\mathbb{V};\delta_x] | x \in Q_0\}$ along all skyscraper central charges δ_α for all $x \in Q_0$.

Theorem 2.13: [48] The skyscraper invariant $\delta_{\bullet}$ is strictly more discriminative than the rank invariant $\rho_{\bullet}$ in $\mathbf{Rep}(Q)$. Details are as follows

• Let $\mathbb{V}$ be any representation in $\mathbf{Rep}(Q)$ and $x \in Q_0$ be any a vertex. Suppose $0 = \mathbf{HN}^{0}_{\alpha}(\mathbb{V}) \subsetneq \mathbf{HN}^{1}_{\alpha}(\mathbb{V}) \subsetneq \cdots \subsetneq \mathbf{HN}^{n}_{\alpha}(\mathbb{V}) = \mathbb{V}$ is the HN filtration of $\mathbb{V}$ along δ_x. Then for any vertex $y \geq x$ in Q, we have

$$\rho_{\mathbb{V}}(x, y) = \sum_{k=1}^{j} \dim S_y^k$$

where $S^k = \mathbb{V}^k / \mathbb{V}^{k-1}$ and j is the smallest index satisfying $\mathbf{HN}_\alpha^j(\mathbb{V})_x$ equals $\mathbb{V}_x$.

- There are two representations $\mathbb{W}, \mathbb{W}'$ in $\mathbf{Rep}(Q)$ of the quiver Q

$$\begin{array}{ccc} c & \longrightarrow & d \\ \uparrow & & \uparrow \\ a & \longrightarrow & b \end{array}$$

such that $\rho_{\mathbb{W}} = \rho_{\mathbb{W}'}$ but $\delta_{\mathbb{W}} \neq \delta_{\mathbb{W}'}$.

Example 2.5: Let a quiver Q be as follows,

$$\begin{array}{ccc} c & \longrightarrow & d \\ \uparrow & & \uparrow \\ a & \longrightarrow & b \end{array}$$

We define the representations $\mathbb{W}$(left) and $\mathbb{W}'$(right) as follows

$$\begin{array}{ccc} \mathbb{k} & \longrightarrow & 0 \\ {\scriptstyle [0,1]}\uparrow & & \uparrow \\ \mathbb{k}^2 & \xrightarrow[{[1,0]}]{} & \mathbb{k} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{k} & \longrightarrow & 0 \\ {\scriptstyle [1,0]}\uparrow & & \uparrow \\ \mathbb{k}^2 & \xrightarrow[{[1,0]}]{} & \mathbb{k} \end{array}$$

By computing the rank invariants of $\mathbb{W}, \mathbb{W}'$, we know that $\rho_{\mathbb{W}} = \rho_{\mathbb{W}'}$.

Subsequently, we compute the skyscraper invariants $\delta_{\mathbb{W}}$ and $\delta_{\mathbb{W}'}$.

Given

$$\delta_{\mathbb{W}} = \{T[\mathbb{W}; \delta_a], T[\mathbb{W}; \delta_b], T[\mathbb{W}; \delta_c], T[\mathbb{W}; \delta_d]\},$$

$$\delta_{\mathbb{W}'} = \{T[\mathbb{W}'; \delta_a], T[\mathbb{W}'; \delta_b], T[\mathbb{W}'; \delta_c], T[\mathbb{W}'; \delta_d]\}.$$

We only check $T[\mathbb{W}; \delta_a]$ and $T[\mathbb{W}'; \delta_a]$. Firstly, we know that $\mu_{\delta_a}(\mathbb{W}) = \frac{1}{2}$ and $\mu_{\delta_a}(\mathbb{U}) \leq \frac{1}{2}$ for all subrepresentations $\mathbb{U}$ of $\mathbb{W}$. Then $\mathbb{W}$ is δ_a-semistable, we have $\mathbf{HN}_{\delta_a}^{\bullet}(\mathbb{W}) : 0 \subsetneq \mathbb{W}$. So,

$$T[\mathbb{W}; \delta_a] = (\underline{\dim}_{\mathbb{W}}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Similarly, we can compute $T[\mathbb{W}'; \delta_a]$. But the Harder-Narasimhan filtration of $\mathbb{W}'$

is not trivial, as follows,

$$0 \subsetneq \mathbb{V} \subsetneq \mathbb{W}'$$

where $\mathbb{V}$ is the subrepresentation of $\mathbb{W}'$,

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \Bbbk & \longrightarrow & 0 \end{array}$$

Therefore, through calculation, we have

$$T[\mathbb{W}'; \delta_a] = (\underline{\dim}_{\mathbb{V}}\,, \underline{\dim}_{\mathbb{W}'/\mathbb{V}}) = (\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}).$$

Clearly, $T[\mathbb{W}; \delta_a] \neq T[\mathbb{W}'; \delta_a]$, then $\delta_{\mathbb{W}} \neq \delta_{\mathbb{W}'}$.

CHAPTER 3 THE STABILITY OF PERSISTENCE MODULES

In this chapter, we introduce the most fundamental results in persistent homology theory: stability theorems[27]. Stability theorems assert that under small perturbations of the original dataset, the topological descriptors, which are persistence diagrams, do not undergo significant changes. Here, we employ the bottleneck distance d_B and Gromov-Hausdorff d_{GH} distance to quantify the differences between persistence diagrams and datasets, respectively. Subsequently, we discuss the generalization of the bottleneck distance d_B in the context of persistence modules, known as the interleaving distance d_I[44]. We demonstrate that for interval-decomposable 1-parameter persistence modules, the interleaving distance coincides entirely with the bottleneck distance. Finally, we will introduce a significant result by Blemberg and Lesnick[12] in the study of persistence modules: the homotopy interleaving distance d_{HI}. The homotopy interleaving distance serves as a homotopy-theoretic refinement of the interleaving distance.

The content of this chapter lays the groundwork for Chapter 4, where we will define and investigate persistence minimal free Lie models and discuss their stability results under d_I and d_{HI}.

3.1 Stability of Persistent Homology

Mathematicians developed persistent homology to identify the topological space from which a point cloud is sampled. When two different point clouds are sampled from distinct underlying spaces, we aim to distinguish these spaces by defining a metric or distance that quantifies the difference between the filtrations of simplicial complexes through the computation of homology groups. Furthermore, once a metric between two families of homology groups is defined, how can we justify that the definition of this metric is reasonable? The stability theorem[27] in persistent homology establishes criteria to evaluate whether the chosen metric is valid.

In this section, we will elaborate on the stability theorem of persistent homology, that is, the robustness of persistent homology[32][26][95].

Let $\mathcal{R}(S) = \{\mathcal{R}(S)_r\}_{r\geq 0}$ be a collection of Vietoris-Rips complexes constructed by a finite set S of points in $\mathbb{R}^d$. Due to the finiteness of point cloud S, as r increases, $\mathcal{R}(S)_r$ only changes at a finite number of values r. We call these values critical values or critical points. Thus, the filtration $\mathcal{R}(S)$ can be reduced to

$$S = \mathcal{R}(S)_{r_0} \subset \mathcal{R}(S)_{r_1} \subset \mathcal{R}(S)_{r_2} \subset \cdots \subset \mathcal{R}(S)_{r_n}$$

where $f_{r_i,r_j} : \mathcal{R}(S)_{r_i} \to \mathcal{R}(S)_{r_j}$ is a simplicial inclusion.

By computing the homology group of $\mathcal{R}(S)_r$ over some field $\mathbb{k}$, we obtain a sequence of finite-dimensional vector spaces $H_p(\mathcal{R}(S))$,

$$H_p(\mathcal{R}(S)_{r_0}) \xrightarrow{H_p(f_{r_0,r_1})} H_p(\mathcal{R}(S)_{r_1}) \xrightarrow{H_p(f_{r_1,r_2})} H_p(\mathcal{R}(S)_{r_2}) \xrightarrow{H_p(f_{r_2,r_3})} \cdots \xrightarrow{H_p(f_{r_{n-1},r_n})} H_p(\mathcal{R}(S)_{r_n}).$$

From this, we can construct the persistence diagram $dgm(H_p(\mathcal{R}(S))) = \{(r_i, r_j) : i < j\}$. For two different persistence diagrams, we use the bottleneck distance to quantify the extent of their difference.

Definition 3.1: Let X and Y be two persistence diagrams, and the bottleneck distance

$$d_B(X, Y) := \inf_{\eta:X\to Y} \sup_{x\in X} ||x - \eta(x)||_\infty$$

in which η is any bijection.

If X does not include the points on the diagonal with infinite multiplicity, then the bijection $\eta : X \to Y$ may not exist. Obviously, the bottleneck distance is only an extended pseudometric, as d_B does not satisfy the positivity condition.

Theorem 3.1: (Stability)[27] Let S and T be two finite set of points in $\mathbb{R}^d$, then we have

$$d_B(dgm(H_p(\mathcal{R}(S))), dgm(H_p(\mathcal{R}(T)))) \leq d_{GH}(S, T)$$

in which d_{GH} is the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance measures how far two compact metric spaces are from being isometric. Since S and T are finite sets of points in $\mathbb{R}^d$, they are compact metric spaces.

The same argument also holds for Čech complexes. That is, if we construct simplicial complexes through Čech complexes, the stability theorem still applies. In fact, there is a general statement about the stability theorem of persistent homology. Let K be a simplicial complex and $f : K \to \mathbb{R}$ be a function. We call f monotonic if $f(\tau) \leq f(\sigma)$ whenever τ is a face of σ for any simplex σ and τ. The monotonicity f ensures that for every real number $a \in \mathbb{R}$, the sublevel set $f^{-1}(-\infty, a]$ forms a subcomplex of the simplicial complex K.

Definition 3.2: Given a simplicial complex K and a monotonic function $f : K \to \mathbb{R}$, then we define the sublevel set filtration $\mathcal{S}(f) = \{\mathcal{S}(f)_r\}_{r\in\mathbb{R}}$, where $\mathcal{S}(f)_r = f^{-1}(-\infty, a]$.

For simplicity of description, we define $dgm_p(f) := dgm(H_p(\mathcal{S}(f)))$ the p-dimensional persistence diagram of $\mathcal{S}(f)$ for monotonic function $f : K \to \mathbb{R}$.

Theorem 3.2: (Stability)[40] Suppose that K is a simplicial complex and $f, g : K \to \mathbb{R}$ are two monotonic functions. For each dimension p, we have inequality

$$d_B(dgm_p(f), dgm_p(g)) \leq ||f - g||_\infty.$$

The stability reflects the resistance of persistent homology to noise. When noise is present in the original point cloud, the stability ensures that the difference between the persistence diagrams of the point clouds with and without noise remains small.

3.2 Interleaving Distance

In this section, we will consider morphisms and 'distance' between persistence modules, that is, the interleaving distance. The interleaving distance[44] between persistence modules can be seen as a generalization of the bottleneck distance between persistence diagrams. Note that unless otherwise specified, the persistence modules considered are always functors $\mathbb{R} \to C$, in which C is any category.

Recall that a persistence module $\mathbb{X}$ is a functor from a thin category C to a category D, $\mathbb{X} : C \to D$. For a morphism $a \to b$ in C, we denote $\mathbb{X}(a \to b)$ as $\mathbb{X}_{a,b}$ and denote $\mathbb{X}(a)$ as $\mathbb{X}_a$.

Definition 3.3: For persistence modules $\mathbb{X}$ and $\mathbb{Y}$, a morphism between $\mathbb{X}$ and $\mathbb{Y}$ is a natural transformation between $\mathbb{X}$ and $\mathbb{Y}$, $f : \mathbb{X} \Rightarrow \mathbb{Y}$.

The collection of all functors from C to D and all natural transformations between the functors is the category D^C.

If $C = \mathbb{R}$, we may think that persistence modules depict the evolution of objects in D over time. For instance, if persistence modules $\mathbb{X}, \mathbb{Y}$ satisfying $\mathbb{X}(t) = \mathbb{Y}(t + \delta)$ for some constant δ, then $\mathbb{X}$ and $\mathbb{Y}$ are same by shifting time δ. However, there is no isomorphism between persistence modules $\mathbb{X}$ and $\mathbb{Y}$, even any nontrivial morphism. Then, we need to expand the notations of morphisms and isomorphisms between persistence modules to the new version that contains the information of ϵ-shifting.

For $\delta \geq 0$, we define that the δ-interleaving category I^δ is the thin category such that ob $I^\delta := \mathbb{R} \times \{0, 1\}$ and there is the morphism $(r, i) \to (s, j)$ if and only if either

(1) $r + \delta \leq s$, or

(2) $i = j$ and $r \leq s$.

There exist two functors

$$E^0, E^1 : \mathbb{R} \to I^\delta$$

mapping $r \in \mathbb{R}$ to $(r, 0)$ and $(r, 1)$, respectively.

Definition 3.4: Let C be any category and $\mathbb{X}, \mathbb{Y} : \mathbb{R} \to C$ be any two functors. A δ-interleaving between $\mathbb{X}$ and $\mathbb{Y}$ is a functor

$$Z : I^\delta \to C$$

satisfying $Z \circ E^0 = \mathbb{X}$ and $Z \circ E^1 = \mathbb{Y}$.

We call persistence modules $\mathbb{X}, \mathbb{Y} : \mathbb{R} \to C$ are δ-interleaved, if there exists a functor $Z : I^\delta \to C$ that is a δ-interleaving between $\mathbb{X}$ and $\mathbb{Y}$.

Let $\mathbb{X}(\delta) : \mathbb{R} \to C$ be the functor by shifting $\mathbb{X}$ downward by δ, i.e., $\mathbb{X}(\delta)_r := \mathbb{X}_{r+\delta}$ and $\mathbb{X}(\delta)_{r,s} := \mathbb{X}_{r+\delta,s+\delta}$ for all $r \leq s \in \mathbb{R}$. Similarly, $f(\delta) : \mathbb{X}(\delta) \to \mathbb{Y}(\delta)$ is defined by $f(\delta)_{t,s} := f_{t+\delta,s+\delta}$, where $f : \mathbb{X} \to \mathbb{Y}$ is a morphism between persistence modules. Specially, we define the morphism $\phi^{\mathbb{X},\delta} : \mathbb{X} \to \mathbb{X}(\delta)$ for any $\mathbb{X} : \mathbb{R} \to C$, in which $\phi_t^{\mathbb{X},\delta} = \mathbb{X}_{t,t+\delta}$. A δ-interleaving Z between $\mathbb{X}$ and $\mathbb{Y}$ is characterized by a pair of natural transformations $f : \mathbb{X} \to \mathbb{Y}(\delta)$ and $g : \mathbb{Y} \to \mathbb{X}(\delta)$, satisfying the compatibility conditions $g(\delta)f = \phi^{\mathbb{X},2\delta}$ and $f(\delta)g = \phi^{\mathbb{Y},2\delta}$. On the other hand, $Z : I^\delta \to C$ is entirely determined by these natural transformations, which are referred to as δ-interleaving morphisms. When $\delta = 0$, these morphisms reduce to a pair of mutually inverse natural isomorphisms.

Definition 3.5: We define the interleaving distance d_I as a binary function

$$d_I : \text{ob } C^{\mathbb{R}} \times \text{ob } C^{\mathbb{R}} \to [0, \infty],$$

by taking

$$d_I(\mathbb{X}, \mathbb{Y}) := \inf \{\delta \mid \mathbb{X} \text{ and } \mathbb{Y} \text{ are } \delta\text{-interleaved}\}.$$

It is straightforward to verify that if $\mathbb{X}$ and $\mathbb{W}$ are δ-interleaved, and $\mathbb{W}$ and $\mathbb{Y}$ are

ϵ-interleaved, then $\mathbb{X}$ and $\mathbb{Y}$ are $\delta+\epsilon$-interleaved. Thus, we know that d_I satisfies the triangle inequality. Therefore, the d_I is obviously a pseudo-distance. What's more, if $\mathbb{X}, \mathbb{X}', \mathbb{Y} \in \text{ob}\, C^{\mathbb{R}}$ with $\mathbb{X} \cong \mathbb{X}'$, then $d_I(\mathbb{X}, \mathbb{Y}) = d_I(\mathbb{X}', \mathbb{Y})$, so function d_I defines a pseudo-distance on the isomorphism classes of objects in the category $C^{\mathbb{R}}$.

The interleaving distance d_I is a generalization of the bottleneck distance d_B.

Theorem 3.3: (Algebraic Stability[6]) Given a pair of persistence modules $\mathbb{M}, \mathbb{N} : \mathbb{R} \to \mathbf{Vec}_{\mathbb{k}}$ satisfying the condition each $\mathbb{M}_t, \mathbb{N}_t$ are finite-dimensional for all $t \in \mathbb{R}$, then

$$d_B(\mathcal{B}_{\mathbb{M}}, \mathcal{B}_{\mathbb{N}}) = d_I(\mathbb{M}, \mathbb{N}).$$

One of the most useful aspects of the categorical view of interleavings is that if we apply a functor to δ-interleaving, then the resulting diagrams are also δ-interleaving. That is,

Proposition 3.1: [18] Let $\mathbb{X}, \mathbb{Y} : \mathbb{R} \to C$ and $H : C \to D$. If $\mathbb{X}$ and $\mathbb{Y}$ are δ-interleaved, then so are $H\mathbb{X}$ and $H\mathbb{Y}$. Thus,

$$d_I(H\mathbb{X}, H\mathbb{Y}) \leq d_I(\mathbb{X}, \mathbb{Y}).$$

The process of composition of functors can be seen as the process of processing information, and information may be lost after processing, so the difference between the two persistence modules may be reduced. From this perspective, it is also easy to understand the actual meaning of the previous proposition. Meanwhile, there are scholars studying similar topics in this discussion, which is the change of interleaving distance when persistence modules compose some functors[7-9,52,77].

3.3 Homotopy Interleaving Distance

In this section, we will focus on the persistence modules $\mathbb{R} \to \mathbf{Top}_{\text{CGWH}}$, in which the category $\mathbf{Top}_{\text{CGWH}}$ refers to the category of compactly-generated weakly Hausdorff (CGWH) topological spaces. These persistence modules are called $\mathbb{R}$-spaces. Note that there is a model structure on $\mathbf{Top}_{\text{CGWH}}$, namely the Quillen-Serre model structure[62], and also a model structure on $\mathbf{Top}_{\text{CGWH}}^{\mathbb{R}}$, which is the projective model structure[61]. The two model structures are the primary ones discussed in this section.

The main results of this section come from the work of Blumberg and Lesnick[12]. For more details of model categories, please refer to references[30,59,79].

Review Theorem3.1, let $S, T \in \mathbb{R}^n$ be two finite sets of points. Then we have the inequality

$$d_B(dgm(H_p(\mathcal{R}(S))), dgm(H_p(\mathcal{R}(S)))) \leq d_{GH}(S,T)$$

in which d_{GH} is the Gromov-Hausdorff distance. A natural question arises as to whether the results for point clouds can derive a consequence of a topological result regarding the filtrations of simplicial complexes $\mathcal{R}(S)$ and $\mathcal{R}(T)$.

We hope to find out the pseudo-distance d defined on the $\mathbb{R}$-spaces that satisfies these conditions:

(1) For any metric spaces S and T,

$$d(\mathcal{R}(S), \mathcal{R}(T)) \leq d_{GH}(S,T)$$

(2) [homology bounding] For nay $\mathbb{R}$-spaces $\mathbb{X}$, $\mathbb{Y}$ and integer $i \geq 0$ satisfying $H_i(\mathbb{X})$ and $H_i(\mathbb{Y})$ are functors $(\mathbb{R}, \leq) \to \mathbf{Vec}_{\Bbbk}$,

$$d_B(dgm(H_i(\mathbb{X})), dgm(H_i(\mathbb{Y}))) \leq d(\mathbb{X}, \mathbb{Y}).$$

Definition 3.6: Let T be a topological space and $\gamma : T \to \mathbb{R}$ be a (not necessarily continuous) function. The sublevel set filtration $\mathcal{S}(\gamma) : \mathbb{R} \to \mathbf{Top}_{\mathrm{CGWH}}$ is constructed by defining

$$\mathcal{S}(\gamma)_t := \gamma^{-1}(-\infty, t]$$

for each $t \in \mathbb{R}$, where $\mathcal{S}(\gamma)_t$ is endowed with the subspace topology induced by its ambient space.

Definition 3.7: For any small category C and functors $\mathbb{X}, \mathbb{Y} : C \to \mathbf{Top}_{\mathrm{CGWH}}$, a natural transformation $f : \mathbb{X} \to \mathbb{Y}$ is called an (objectwise) weak equivalence if, for any $a \in \mathrm{ob}\, C$, the morphism $f_a : \mathbb{X}_a \to \mathbb{Y}_a$ is a weak homotopy equivalence.

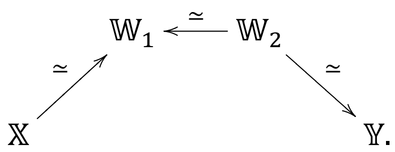
A weak equivalence from $\mathbb{X}$ to $\mathbb{Y}$ is denoted by $\mathbb{X} \xrightarrow{\simeq} \mathbb{Y}$.

If there exists a zigzag of weak equivalences

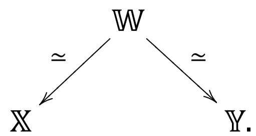
$$\mathbb{X} \xleftarrow{\simeq} \mathbb{W}_1 \xrightarrow{\simeq} \mathbb{W}_2 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \mathbb{W}_{n-1} \xleftarrow{\simeq} \mathbb{W}_n \xrightarrow{\simeq} \mathbb{Y}$$

connecting $\mathbb{X}$ and $\mathbb{Y}$ for some n, we call that $\mathbb{X}$ and $\mathbb{Y}$ are weakly equivalent, written as $\mathbb{X} \simeq \mathbb{Y}$.

This defines an equivalence relation on objects, though it is often cumbersome to work with. Indeed, in any model category D, $\mathbb{X} \simeq \mathbb{Y}$ holds precisely when there is the following diagram,



Furthermore, it is straightforward to verify that if every object in D is fibrant or every object is cofibrant, then $\mathbb{X} \simeq \mathbb{Y}$ holds exactly when there exists the following diagram of weak equivalences connecting $\mathbb{X}$ and $\mathbb{Y}$



In $\mathbf{Top}_{\mathrm{CGWH}}^{\mathbb{R}}$ with the projective model structure, all objects are cofibrant, and this property holds.

Proposition 3.2: [12] For any $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ which are δ-interleaved, there is a topological space T and two functions $\gamma^{\mathbb{X}}, \gamma^{\mathbb{X}} : T \to \mathbb{R}$ such that $\mathcal{S}(\gamma^{\mathbb{X}}) \simeq \mathbb{X}$, $\mathcal{S}(\gamma^{\mathbb{Y}}) \simeq \mathbb{Y}$, and $d_\infty(\gamma^X, \gamma^Y) \leq \delta$.

Indeed, the topological space $T \cong \varprojlim \mathbb{X} \cong \varprojlim \mathbb{Y}$. Thus the fact states that for δ-interleaved $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$, topological spaces $\mathbb{X}_t$ and $\mathbb{Y}_t$ are homoemorphic, when $t = \infty$.

In the above section, we know that d_B satisfies the general stability result about the filtration of sublevel simplicial complexes, Theorem3.2, and we believe that this property is worth preserving. Therefore, we hope that the pseudo-distance d we are looking for also satisfies this property. Meanwhile, d should be invariant under some continuous deformations.

Definition 3.8: We say a pseudo-distance d on $\mathbb{R}$-spaces is

(1) stable: if for any $T \in \mathrm{ob}\ \mathbf{Top}_{\mathrm{CGWH}}$ and functions $\gamma, \kappa : T \to \mathbb{R}$,

$$d(\mathcal{S}(\gamma), \mathcal{S}(\kappa)) \leq d_\infty(\gamma, \kappa),$$

(2) homotopy invariant: if $d(\mathbb{X}, \mathbb{Y}) = 0$ whenever $\mathbb{X} \simeq \mathbb{Y}$.

Based on previous discussions and results, we can infer that if d satisfies the stability, then it satisfies the inequality $d(\mathcal{R}(S), \mathcal{R}(T)) \leq d_{GH}(S, T)$.

Proposition 3.3: [12] For any stable and homotopy invariant pseudo-distance d on $\mathbb{R}$-spaces and for any metric spaces S and T, we have $d(\mathcal{R}(S), \mathcal{R}(T)) \leq d_{GH}(S, T)$, generalizing the Rips stability theorem (Theorem3.1) to a result at the filtration-level.

We will introduce the pseudo-distance that satisfies the stability, homotopy invariance, and homology bounding axiom, which is called the homotopy interleaving distance d_{HI}.

Definition 3.9: For any $\delta \geq 0$, $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ are called δ-homotopy-interleaved if there are $\mathbb{R}$-spaces $\mathbb{X}'$ and $\mathbb{Y}'$, so that $\mathbb{X}' \simeq \mathbb{X}$, $\mathbb{Y}' \simeq \mathbb{Y}$, and $\mathbb{X}'$ and $\mathbb{Y}'$ are δ-interleaved.

Definition 3.10: The homotopy interleaving distance between $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ is defined as

$$d_{HI}(\mathbb{X}, \mathbb{Y}) := \inf\{\delta \mid \mathbb{X}, \mathbb{Y} \text{ are } \delta\text{-homotopy-interleaved}\}$$

Theorem 3.4: [12] d_{HI} defines a pseudodistance on $\mathbb{R}$-spaces and satisfies the homotopy invariance, stability, and homology bounding.

There are several pseudo-distances on $\mathbb{R}$-spaces, besides d_{HI}. However, the interleaving distance d_{HI} is a canonical choice of such a pseudo-distance.

Theorem 3.5: (Universality)[12] If d is any stable and homotopy invariant distance on $\mathbb{R}$-spaces, then $d \leq d_{HI}$.

The homotopy interleaving distance and the concept of homotopy coherent diagrams are deeply interconnected. Homotopy coherent diagrams, intuitively, extend the notion of homotopy commutative diagrams by including specific choices of homotopies, higher-order homotopies between these homotopies, and so forth. Formally, homotopy coherent diagrams may be defined within the framework of simplicially enriched functors. For a small category $\mathbf{I}$, the category $\mathrm{Cho}(\mathbf{I})$ consists of homotopy coherent diagrams indexed by $\mathbf{I}$, with morphisms being homotopy classes of homotopy coherent natural transformations. Homotopy coherent diagrams address the critical question of what additional structure is needed to rectify a homotopy commutative diagram into a strictly commutative diagram[33][102].

Let $\widetilde{\mathrm{Ho}}(\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}})$ denote the localization of $\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}}$ with respect to objectwise homotopy equivalences, and recall that $\mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}})$ denotes the localization of $\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}}$ with respect to objectwise weak homotopy equivalences. Using Whitehead's theorem, it can be checked that two diagrams in $\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}}$ taking values in cofibrant spaces (e.g., CW complexes) are isomorphic in $\widetilde{\mathrm{Ho}}(\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}})$ if and only if they are isomorphic in $\mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}})$.

Vogt's theorem[102] gives an equivalence of categories

$$\mathrm{Coh}(\mathbf{I}) \to \widetilde{\mathrm{Ho}}(\mathbf{Top}^{\mathbf{I}}_{\mathrm{CGWH}}).$$

The theorem implies that homotopy coherent diagrams can be analyzed through strict commutative diagrams combined with zigzags of objectwise homotopy equivalences. Motivated by these insights, the homotopy-coherent definition of interleavings is proposed.

Definition 3.11: [12] For two $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$, we define a homotopy coherent δ-interleaving between $\mathbb{X}$ and $\mathbb{Y}$ as a homotopy coherent diagram $Z \in \mathrm{Coh}(\mathbf{I}^{\delta})$ satisfying $Z \circ E^0 \cong \mathbb{X}$ and $Z \circ E^1 \cong \mathbb{Y}$ in $\mathrm{Coh}(\mathbb{R})$.

By leveraging fundamental properties of the equivalence $\mathrm{Coh}(\mathbf{I}^{\delta}) \to \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}^{\delta}}_{\mathrm{CGWH}})$ established by Vogt's theorem, the following comparison can be readily verified.

Proposition 3.4: [12] The existence of homotopy coherent δ-interleaving between $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ implies the existence of δ-homotopy-interleaving between $\mathbb{X}$ and $\mathbb{Y}$. The converse holds as well if $\mathbb{X}$ and $\mathbb{Y}$ are objectwise cofibrant.

Building on Andrew's foundational work, several researchers have continued to investigate homotopy interleaving distances[13,70].

CHAPTER 4 PERSISTENCE RATIONAL HOMOTOPY

We know that persistence modules are functors $(\mathcal{P}, \leq) \to C$, in which $\mathcal{P}$ is a poset and C is a category, where $\mathcal{P}$ denotes a more general filtering method and C denotes a set of more general filtering objects. Thus, persistence modules can be seen as a generalization of the filtration of topological spaces.

In persistent homology, if we have already determined the method for constructing simplicial complexes from discrete point clouds, then the remaining issue is to establish algebraic models for these simplicial complexes. Currently, the most frequently used algebraic model is the homology groups over $\Bbbk$ for simplicial complexes. When we specify the coefficient field to be a field of characteristic 0, we can employ rational homotopy theory to establish a more refined algebraic model for simplicial complexes.

In rational homotopy theory, there are two significant algebraic models: the minimal Sullivan model and the minimal free Lie model. These serve as the associative algebra model and the Lie algebra model for simply connected rational spaces with homology groups of finite type, respectively. Next, we will introduce the essential knowledge of rational homotopy theory, as well as persistence rational homotopy theory.

A simply connected space X is called a rational space if X satisfies one of following equivalent conditions(Theorem 9.3 of the reference[45]):

- $\pi_*(X) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$
- $H_*(X, pt; \mathbb{Z}) \cong H_*(X, pt; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$
- $H_*(\Omega X, pt; \mathbb{Z}) \cong H_*(\Omega X, pt; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$

If $H_i(X, pt; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finitely dimensional vector space for all $i \in \mathbb{N}$, we call X is of finite type.

Definition 4.1: For a simply connected space X, a rationalization of X is a continuous map $\varphi : X \to X_{\mathbb{Q}}$ satisfying that φ induces an isomorphism

$$\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(X_{\mathbb{Q}}),$$

where $X_{\mathbb{Q}}$ is a simply connected rational space.

For any simply connected topological space X, we can always find a rational space $X_{\mathbb{Q}}$ such that $X_{\mathbb{Q}}$ is the rationalization of X.

Theorem 4.1: [45] (i) Let X be a simply connected space. Then there exists a relative CW complex $(X_{\mathbb{Q}}, X)$ that lacks 0-dimensional and 1-dimensional cells so that the inclusion $\varphi : X \to X_{\mathbb{Q}}$ is a rationalization.

(ii) Given $(X_{\mathbb{Q}}, X)$ as described in (i) and Y as any simply connected rational space. For any continuous map $f : X \to Y$, we may extend f to a continuous map $g : X_{\mathbb{Q}} \to Y$. Furthermore, if $g' : X_{\mathbb{Q}} \to Y$ extends $f' : X \to Y$ then any homotopy between f and f' can be extended to a homotopy between g and g'.

(iii) The rationalization specified in (i) is unique up to homotopy equivalence relative to X.

The theorem told us that every simply connected space can be rationalized and every continuous map $\varphi : X \to Y$ between simply connected spaces can induce the continuous map $\tilde{\varphi} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$.

In this chapter, we will focus on the category $\mathbf{Top}_{\mathbb{Q}}$ of simply connected rational spaces of finite type, and objects in $\mathbf{Top}_{\mathbb{Q}}$, that is simply connected rational spaces of finite type. Therefore, unless otherwise stated, all topological spaces encountered in this chapter are assumed to be simply connected rational spaces of finite type, and all numerical fields involved are assumed to be the field of rational numbers, $\mathbb{Q}$.

Specifically, we may notice that for any $X \in \text{ob}\, \mathbf{Top}_{\mathbb{Q}}$, $\pi_*(X)$ is a vector space over $\mathbb{Q}$. Then for a functor $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$, $\pi_*(\mathbb{X}), H_*(\mathbb{X}), H^*(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{grVec}_{\mathbb{Q}}$ are persistence modules, which is the most commonly encountered persistence module. In rational homotopy theory, we have more refined algebraic models than homotopy groups and homology groups, minimal Sullivan models, and minimal free Lie models.

4.1 Persistence Minimal Sullivan Models

We will introduce the basic definition and results of minimal Sullivan models[45] and persistence minimal Sullivan models[105].

Firstly, we will recall some definitions and results of commutative differential graded algebras ($cdga$) and Sullivan algebras.

Definition 4.2:

- A graded ring $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a ring satisfying $R_i \cdot R_j \subseteq R_{i+j}$.
- A graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ over a graded ring R is a module satisfying $R_i \cdot M_j \subseteq M_{i+j}$.
- A graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is both a graded module and a graded ring with

$1 \in A_0$.

• A differential graded algebra (dga) A is a graded algebra, equipped with a derivation $d : A \to A$, that is of degree $+1$, satisfying $d^2 = 0$ and $d(ab) = (da)b + (-1)^{\deg(a)} a(db)$ for any $a, b \in A$.

• A commutative differential graded algebra ($cdga$) is a dga A that is graded commutative: $ab = (-1)^{\deg a \deg b} ba$ for any homogeneous elements $a, b \in A$.

• A $cdga$ (A, d) is path-connected if $H^0(A, d) \cong \mathbb{Q}$, and is simply connected if $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$.

• A morphism $\varphi : A \to B$ of graded algebras is a degree-preserving homomorphism and satisfies $\varphi(1) = 1$.

The $cdga$ we focus on and deal with is mostly path-connected, so we assume **CDGA** is a category of path-connected $cdga$.

Example 4.1: Let V be a free graded module. Then, we define the tensor algebra TV as follows:

$$TV = \sum_{q=0}^{\infty} T^q V, \quad T^q V = \otimes^q V$$

Multiplication is given by $a \cdot b = a \otimes b$. Note that q is not the degree: elements $v_1 \otimes \cdots \otimes v_q \in T^q V$ have degree $= \Sigma \deg v_i$ and word length q.

The elements $v \otimes w - (-1)^{\deg v \deg w} w \otimes v (v, w \in V)$ generate an ideal $I \in TV$. The quotient

$$\Lambda V = TV/I$$

is called the exterior algebra (also the free commutative graded algebra) on V.

In homotopy theory, we focus more on homotopy relationships rather than simple equality or isomorphism, so we need the definition of quasi-isomorphism. A morphism $\varphi : (A, d) \to (A', d)$ of dga is called a quasi-isomorphism, denoted by $\simeq$, if $H(f) : H^*(A, d) \to H^*(A', d)$ is an isomorphism.

4.1.1 Minimal Sullivan Models

A Sullivan algebra is an external algebra that satisfies the certain nilpotence condition, which ensures that we can construct a Sullivan model for any $cdga$ (A, d) with $H^0(A, d) = \mathbb{Q}$.

Definition 4.3: A Sullivan algebra is a $cdga$ of the form $(\Lambda V, d)$ with $V^0 = 0$, where $V = \bigcup_{i=0} V(i)$, and $V(0) \subseteq V(1) \subseteq \cdots$ forms an ascending sequence of graded subspaces satisfying $d = 0$ in $V(0)$ and $dV(i) \subseteq \Lambda V(i-1)$ with $i \geq 1$.

A Sullivan algebra is called minimal if Im $d \subseteq \Lambda^{\geq 2} V$.

Definition 4.4: A Sullivan model for a $cdga$ (A, d) is a homomorphism of commutative differential graded algebras

$$m : (\Lambda V, d) \to (A, d)$$

satisfying m is a quasi-isomorphism.

Let X be a space in ob $\mathbf{Top}_{\mathbb{Q}}$. Then we define that a Sullivan model for X is a Sullivan model for $A_{PL}(X)$

$$m : (\Lambda V, d) \to A_{PL}(X).$$

If $(\Lambda V, d)$ is minimal, we call that m is a minimal Sullivan model, which we denote $m_X : M_{Su}(X) \to A_{PL}(X)$.

This definition utilizes the A_{PL}, which is a contravariant functor from the spaces to commutative differential graded algebras. Its specific structure will be introduced later.

Example 4.2: [46]

$M_{Su}(\mathbb{S}^{2n+1}) = (\Lambda u, 0), \quad \deg u = 2n+1,$

$M_{Su}(\mathbb{S}_{2n}) = (\Lambda(a, b), d), \quad da = 0, db = a^2, \quad \deg a = 2n,$

$M_{Su}(X \times Y) \cong M_{Su}(X) \otimes M_{Su}(Y)$, if one of $H(X)$ or $H(Y)$ is of finite type,

$M_{Su}(X \vee Y) \simeq M_{Su}(X) \oplus M_{Su}(Y),$

$M_{Su}(K(\mathbb{Z}, n)) = (\Lambda a, 0), \quad \deg a = n.$

Define augmentations $\epsilon_0, \epsilon_1 : \Lambda(t, dt) \to \mathbb{Q}$ by $\epsilon_0(t) = 0$, $\epsilon_1(t) = 1$. Then, we may define the homotopy in commutative differential graded algebras.

Definition 4.5: A homotopy between two morphisms $\varphi_0, \varphi_1 : (A, d) \to (A', d)$ of commutative differential graded algebras is a morphism

$$\Phi : (A, d) \to (A', d) \otimes (\Lambda(t, dt), d)$$

such that $(id \cdot \epsilon_i) \circ \Phi = \varphi_i$, $i = 0, 1$. We call that φ_0 and φ_1 are homotopic and denote this by $\varphi_0 \sim \varphi_1$.

In order to construct a Sullivan model of $X \in \text{ob}\,\mathbf{Top}_{\mathbb{Q}}$, we first need to establish an algebraic model for X. Therefore, we will briefly introduce the definition and properties of the A_{PL} functor in the following.

Definition 4.6: A simplicial differential algebra A is defined as a simplicial object within the category of differential algebras. More precisely, A is composed of a family of differential algebras $\{A_n\}_{n\geq 0}$ equipped with face and degeneracy morphisms that satisfy the necessary compatibility conditions.

The simplicial commutative cochain algebra, denoted by A_{PL}, is defined as following:

• differential graded algebra $(A_{PL})_n$ is given by

$$(A_{PL})_n = \frac{\Lambda(t_0, \cdots, t_n, y_0, \cdots, y_n)}{(\sum t_i - 1, \sum y_j)}$$

in which $\deg t_i = 0,\ \deg y_j = 1$ and $dt_i = y_i,\ dy_j = 0$.

• The face and degeneracy morphisms are the morphisms of differential graded algebras

$$\partial_i : (A_{PL})_{n+1} \to (A_{PL})_n \text{ and } s_j : (A_{PL})_n \to (A_{PL})_{n+1}$$

satisfying

$$\partial(t_k) = \begin{cases} t_k & ,k < i \\ 0 & ,k = i \\ t_{k-1} & ,k > i \end{cases} \text{ and } s_j(t_k) = \begin{cases} t_k & ,k < j \\ t_k + t_{k+1} & ,k = j \\ t_{k+1} & ,k > j \end{cases}$$

The definition of A_{PL} is actually a simulation of polynomial differential forms on the Euclidean simplex $\Delta_n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1 \text{ and } t_i \geq 0\}$.

Note that if we fix the degree $p \in \mathbb{N}$, the $(A_{PL})^p := \{(A_{PL})^p_n\}_{n\geq 0} \in \mathbf{SSet}$ is a simplicial set. in which **SSet** is the category of simplicial sets.

Let K be a simplicial set. Then $A_{PL}(K) = \{(A_{PL})^p(K)\}_{p\geq 0}$ is defined as the collection of simplicial set morphisms from K to $(A_{PL})^p$. Define

$$A_{PL}(K) := \bigoplus_{p\geq 0} \text{Hom}_{\mathbf{SSet}}(K, (A_{PL})^p),$$

which admits the structure of commutative differential graded algebra. In fact, we get a contravariant functor A_{PL} from simplicial sets to commutative differential algebras. In particular, for any topological space X and any continuous map f we apply this construction to the simplicial set $S_*(X)$ and $S_*(f)$, that is $A_{PL}(X) := A_{PL}(S_*(X))$.

Proposition 4.1: [45] Let (A, d) be a commutative differential graded algebra.

- If (A, d) is path-connected, that is $H^0(A, d) = \mathbb{Q}$, then there exists a Sullivan model

$$m : (\Lambda V, d) \xrightarrow{\simeq} (A, d).$$

- If (A, d) is a simply connected, that is $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$, then there is a minimal Sullivan model

$$m : (\Lambda W, d) \xrightarrow{\simeq} (A, d)$$

and minimal Sullivan models of (A, d) are all isomorphic.

Corollary 4.1: [45] For any simply connected rational space X, there exists a minimal Sullivan model of X

$$m : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X).$$

Proposition 4.2: [45] Let $\varphi : (A, d) \to (B, d)$ be a morphism between two simply connected commutative differential graded algebras, and let $m_A : (\Lambda V, d) \to (A, d)$ and $m_B : (\Lambda W, d) \to (B, d)$ be their respective minimal Sullivan models. Then there is a morphism $m_\varphi : (\Lambda V, d) \to (\Lambda W, d)$ such that

$$\begin{array}{ccc} (\Lambda V, d) & \xrightarrow{m_\varphi} & (\Lambda W, d) \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_B} \\ (A, d) & \xrightarrow[\varphi]{} & (B, d) \end{array}$$

commutes up to homotopy. The morphism m_φ is referred to as the Sullivan representative of φ.

Corollary 4.2: [45] Suppose that $f : X \to Y$ is a continuous map of rational spaces, and $m_X : (\Lambda V, d) \to A_{PL}(X)$ and $m_Y : (\Lambda W, d) \to A_{PL}(Y)$ are minimal Sullivan models. Then there is a morphism $m_f : (\Lambda W, d) \to (\Lambda V, d)$ such that

$$\begin{array}{ccc} (\Lambda W, d) & \xrightarrow{m_f} & (\Lambda V, d) \\ {\scriptstyle m_Y}\downarrow & & \downarrow{\scriptstyle m_X} \\ A_{PL}(Y) & \xrightarrow[A_{PL}(f)]{} & A_{PL}(X) \end{array}$$

commutes up to homotopy. We call the morphism m_f the Sullivan representative of f, and the homotopy class f uniquely determines the homotopy class of m_f.

Remark 4.1: [45] The above results actually tell us these maps

$$\begin{Bmatrix} \text{rational homotopy} \\ \text{type} \end{Bmatrix} \to \begin{Bmatrix} \text{isomorphism classes of} \\ \text{minimal Sullivan algebras over } \mathbb{Q} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \text{homotopy classes of} \\ \text{map } X \to Y \end{Bmatrix} \to \begin{Bmatrix} \text{homotopy classes of} \\ \text{morphisms } (\Lambda W, d) \to (\Lambda V, d) \end{Bmatrix}$$

Indeed, these maps are bijective, and we get a contravariant functor $M_{Su} : \mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}}) \to \mathbf{Ho(CDGA)}$. Therefore, in the framework of rational homotopy theory, the study of homotopy classes of spaces can be reduced to the study of minimal Sullivan models.

4.1.2 Persistence Minimal Sullivan Models

In this subsection, we will introduce these results[105] of Ling Zhou, persistence minimal Sullivan models, and their interleaving distance.

In the past, when scholars researched the persistence modules, the persistence modules mostly considered the functor $T \to \mathbf{Vec}_{\mathbb{Q}}$ from $\mathbb{R}$ or $\mathbb{N}$ to the category of finite-dimensional vector spaces over $\mathbb{k}$. However, Zhou Ling combined persistence with minimal Sullivan models to obtain the persistence minimal Sullivan models and discussed the stability of the persistence minimal Sullivan models, which brought new directions for the development of persistence homology.

Definition 4.7: Let $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$ be a persistence module, which is called rational $\mathbb{R}$-space, in which we denote $\mathbb{X}(r)$ as $\mathbb{X}_r$ and $\mathbb{X}(s \leq t)$ as $\mathbb{X}_{s\leq t}$. We define the persistence minimal Sullivan model of $\mathbb{X}$ to be a persistence minimal Sullivan algebra $M_{Su}(\mathbb{X})$ together with $cdga$ quasi-isomorphisms $m_{\mathbb{X}} := \{m_{\mathbb{X}_t} : M_{Su}(\mathbb{X}_t) \to A_{PL}(\mathbb{X}_t)\}$ such that

- for each t, $m_{\mathbb{X}_t} : M_{Su}(\mathbb{X}_t) \to A_{PL}(\mathbb{X}_t)$ is a minimal Sullivan model for $\mathbb{X}_t$
- for any $s \leq t \in \mathbb{R}$, the following diagram commutes up to homotopy

$$\begin{array}{ccc} M_{Su}(\mathbb{X}_t) & \xrightarrow{M_{Su}(\mathbb{X}_{s\leq t})} & M_{Su}(\mathbb{X}_s) \\ m_{\mathbb{X}_t} \downarrow \simeq & & \simeq \downarrow m_{\mathbb{X}_s} \\ A_{PL}(\mathbb{X}_t) & \xrightarrow[A_{PL}(\mathbb{X}_{s\leq t})]{} & A_{PL}(\mathbb{X}_s) \end{array}$$

Indeed, $m_{\mathbb{X}}$ induces a natural isomorphism between the functors $\mathbf{Ho} \circ M_{Su} \circ A_{PL} \circ \mathbb{X}$ and $\mathbf{Ho} \circ A_{PL} \circ \mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Ho(CDGA)}^{op}$.

Next, Ling Zhou's results describe the stability of $d_I^{\mathbf{Ho(CDGA)}}$ and provide upper and lower bounds for $d_I^{\mathbf{Ho(CDGA)}}$.

Theorem 4.2: [105] Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are two persistence spaces, and $M_{Su}(\mathbb{X})$ and $M_{Su}(\mathbb{Y})$ are persistence minimal Sullivan models of $\mathbb{X}$ and $\mathbb{Y}$, respectively. Then

$$d_I^{\mathbf{Ho(CDGA)}}(M_{Su}(\mathbb{X}), M_{Su}(\mathbb{Y})) \leq d_{HI}(\mathbb{X}, \mathbb{Y}).$$

If $\mathbb{X}$ and $\mathbb{Y}$ are Vietoris-Rips filtrations of points clouds X and Y, respectively. Then

$$d_I^{\mathbf{Ho(CDGA)}}(M_{Su}(VR_\bullet(X), M_{Su}(VR_\bullet(Y))) \leq 2 \cdot d_{GH}(X, Y).$$

Theorem 4.3: [105] Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two persistence $cdga$, then

$$d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H(\mathbb{A}), H(\mathbb{B})) \leq d_I^{\mathbf{Ho(CDGA)}}(\mathbb{A}, \mathbb{B}).$$

Suppose that $\Lambda\mathbb{V}$ and ΛW are two simply connected persistence minimal Sullivan algebras. Then

$$d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\mathbb{V}, \mathbb{W}) \leq d_I^{\mathbf{Ho(CDGA)}}(\Lambda\mathbb{V}, \Lambda\mathbb{W})(\leq d_I^{\mathbf{CDGA}}(\Lambda\mathbb{V}, \Lambda\mathbb{W})).$$

These results show that persistence minimal Sullivan models are an effective tool that promotes topological data analysis, although many challenges need to be overcome in practical applications.

4.2 Persistence Minimal Free Lie Models

In this section, we will first recall the properties and the definition of minimal free Lie models, which is another important algebraic model in rational homotopy theory. Then, we will generalize the persistence modules to persistence minimal free Lie models and discuss their properties.

In Quillen's paper[89], Quillen defined and used a sequence of functors that are Quillen equivalent, respectively, to assign to a simply connected rational space of finite type a differential graded Lie algebra (dgl),

$$X \mapsto \lambda X.$$

We call the functor

$$\lambda : \mathbf{Top}_{\mathbb{Q}} \to \mathbf{DGL}$$

Quillen functor where **DGL** is the category of connected dgl, that is $L = \{L_i\}_{i>0}$.

Before starting a detailed introduction to the free Lie model of rational spaces, we will first introduce the functors defined by Quillen and their main properties in homotopy theory.

We need to recall some notions of coalgebras and Lie algebras.

Definition 4.8: A graded coalgebra C consists of a graded module C equipped with two degree-preserving linear maps, one of which is called the comultiplication $\Delta : C \to C \otimes C$, and the other is referred to as the augmentation $\epsilon : C \to \mathbb{Q}$. These maps satisfy the coassociativity condition $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and the counit condition$(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = id_C$.

A graded coalgebra is called cocommutative if

$$\tau\Delta = \Delta$$

where $\tau : C \otimes C \to C \otimes C$ is the involution $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a$. We call a graded coalgebra co-augmented by the choice of an element $1 \in C_0$ so that $\epsilon(1) = 1$ and $\Delta(1) = 1 \otimes 1$. We can also say that co-augmentation is an embedding $\mathbb{Q} \hookrightarrow C$. For such coalgebra C, we write $\bar{C} = \text{Ker}\, \epsilon$, so that $C = \mathbb{Q} \oplus \bar{C}$ and define $\bar{\Delta} : \bar{C} \to \bar{C} \otimes \bar{C}$ with $\bar{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c$.

Example 4.3: The coalgebra ΛV is an instructive example, where comultiplication Δ is explicitly defined by the formula $\Delta v = v \otimes 1 + 1 \otimes v,\ v \in V$. And the augmented by $\epsilon : \Lambda^+ V \to 0,\ 1 \mapsto 1$ and co-augmented by $\mathbb{Q} = \Lambda^0 V$.

Definition 4.9: A graded Lie algebra L consists of a graded vector space $L = \{L_i\}_{i \in \mathbb{Z}}$ and a linear map of degree zero, $L \otimes L \to L$, denoted by $x \otimes y \mapsto [x, y]$ which satisfies the following conditions:

- $[x, y] = -(-1)^{\deg x \deg y}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \deg y}[y, [x, z]]$

The product [,] is called the Lie bracket.

We say a linear map of degree k, $\theta : L \to L$, is a A derivation of L of degree k if $\theta[x, y] = [\theta x, y] + (-1)^{k \deg x}[x, \theta(y)]$.

Example 4.4: Let V be a graded vector space. The tensor algebra TV on V carries a natural graded Lie algebra structure via the bracket operation $[x, y] := xy - (-1)^{\deg x \deg y} yx$. Then, the free graded Lie algebra $\mathbb{L}_V$ is defined as the smallest graded Lie subalgebra of

TV containing V. This object satisfies a universal property: any degree-preserving linear map $f : V \to L$ into another graded Lie algebra L may extend uniquely to a graded Lie algebra homomorphism $\mathbb{L}_V \to L$.

The free graded Lie algebra $\mathbb{L}_V$ naturally inherits a grading structure from the tensor algebra TV, which decomposes as the direct sum $\oplus_{k=0}^{\infty} T^k V$. Here, each homogeneous component $T^k V$ consists of tensors of degree k. Since $\mathbb{L}_V$ is generated by iterated Lie brackets of elements in V, its elements can be stratified by bracket length, defined as the number of generators (from V) involved in their construction.

- $\mathbb{L}_V = \oplus_{k\geq 1}(\mathbb{L}_V \cap T^k V)$;
- $x \in \mathbb{L}_V$ has bracket length k if and only if $x \in \mathbb{L}_V^k := \mathbb{L}_V \cap T^k V$.

Then we may decompose $\mathbb{L}_V = \oplus_{i\geq 1} \mathbb{L}_V^i$the differential $d = d_0 + d_1 + \cdots$, in which $d_k : V \to \mathbb{L}_V \cap T^{k+1} V$.

For any free Lie algebra $(\mathbb{L}_V, d = d_0 + \cdots)$, if $d_0 = 0$, then we call it **minimal**.

Next, we will review the two functors $C_* : \mathbf{DGL} \to \mathbf{CDGC}$ and $\mathcal{L} : \mathbf{CDGC} \to \mathbf{DGL}$ where **CDGA** is the category of 1-connected cocommutative differential graded coalgebras ($cdgc$), which played important roles in Quillen's work[89].

Suppose that (L, d_L) is a differential graded Lie algebra. The coderivations in ΛsL, where sL denotes the shift of degrees that is $(sL)_i = L_{i-1}$ for all i, are determined by the differential d_L and the Lie bracket $[\ ,\ ] : L \otimes L \to L$

$$d_0(sx_1 \wedge \cdots \wedge sx_k) = -\sum_{i=1}^{k} (-1)^{n_i} sx_1 \wedge \cdots \wedge sd_L x_i \wedge \cdots \wedge sx_k,$$

and

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1\leq i<j\leq k} (-1)^{\deg x_i + 1} (-1)^{n_{ij}} s[x_i, x_j] \wedge sx_1 \cdots s\hat{x}_i \cdots s\hat{x}_j \cdots sx_k$$

where $n_i = \sum_{j<i} \deg sx_j$, and $sx_1 \wedge \cdots \wedge sx_k = (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \cdots s\hat{x}_i \cdots s\hat{x}_j \cdots \wedge sx_k$. (Here, symbol ˆ means 'deleted'.)

By simple computation, we can know that $d = d_0 + d_1$ is a coderivation. In other words, $(\Lambda sL, d = d_0 + d_1)$ is a differential graded coalgebra.

Definition 4.10: The Cartan-Eilenberg-Chevalley construction on a dgl (L, d_L) is the $cdgc$ $C_*(L, d_L) = (\Lambda sL, d = d_0 + d_1)$.

The functor C_* assigns a dgl (L, d_L) a $cdgc$ $(\Lambda sL, d)$, and if $E = \{E_i\}_{i>0}$ and $L = \{L_i\}_{i>0}$, then $\varphi : E \to L$ is a quasi-isomorphism if and only if $C_*(\varphi)$ is a quasi-isomorphism[45].

There are some methods for constructing free Lie algebras, but we will introduce one that is closely related to C_*, Quillen's functor $\mathcal{L}$, which is the analog of the cobar construction.

Let $(C,d) = (\bar{C},d) \oplus \mathbb{Q}$ be any co-augmented $cdgc$. By the cobar consturction, $\Omega C = Ts^{-1}\bar{C}$. The differential has the form $d = d_0 + d_1$ with $d_0 : s^{-1}\bar{C} \to s^{-1}\bar{C}$ and $d_1 : s^{-1}\bar{C} \to s^{-1}\bar{C} \otimes s^{-1}\bar{C}$ that derives from the comultipliaction Δ of C. Since C is cocommutative, then we always express the $d_1(s^{-1}c)$ as the sum of commutators in $Ts^{-1}\bar{C}$. Let $\bar{\Delta}c = \sum a_i \otimes b_i$, then $\bar{\Delta}c = \sum (-1)^{\deg a_i \deg b_i} b_i \otimes a_i$. So

$$d_1(s^{-1}c) = \frac{1}{2}\sum_i (-1)^{\deg a_i}[s^{-1}a_i, s^{-1}b_i]$$

through simple calculations, then we can know that $d_1 : s^{-1}\bar{C} \to \mathbb{L}_{s^{-1}\bar{C}} \subseteq Ts^{-1}\bar{C}$. Hence, we have proven that $d = d_0 + d_1$ is the Lie derivation of the free Lie algebra $\mathbb{L}_{s^{-1}C}$.

Definition 4.11: The dgl $(\mathbb{L}_{s^{-1}\bar{C}}, d)$ is referred to as the Quillen construction on the co-augmented $cdgc$ (C,d) and it is denoted by $\mathcal{L}(C,d)$.

Theorem 4.4: [45] Let $(L = \{L_i\}_{i\geq 1}, d)$ be a connected dgl and $(C = \mathbb{Q} \oplus C_{\geq 2}, d)$ is a $cdgc$. Then, there exist natural quasi-isomorphisms

$$\varphi : (C,d) \to C_*\mathcal{L}(C,d) \text{ and } \psi : \mathcal{L}C_*(L,d) \to (L,d)$$

of $cdgc$'s (respectively, of dgl's).

The two functors, C_* and $\mathcal{L}$, we introduced above are adjoint to each other:

$$\mathcal{L} \dashv C_*.$$

What's more, the adjunction $(\mathcal{L} \dashv C_*)$ is a Quillen adjunction between the projective model structure on **DGL** and the model structure on **CDGC**.

For the category **DGL**, there is a model category structure $(\mathbf{DGL})_{proj}$ on the category **DGL** over $\mathbb{Q}$ so that

- the fibrations: surjective maps
- weak equivalences: the quasi-isomorphisms on the underlying chain complexes.

Meanwhile, for the category **CDGC**, there is a model category structure $(\mathbf{CDGC})_{Quillen}$ on the category **CDGC** over $\mathbb{Q}$ so that

- the cofibrations are the (degreewise) injections;
- the weak equivalences are those morphisms that become quasi-isomorphisms un-

der the functor $\mathcal{L}$, that is, quasi-isomorphisms if dgc is 1-connected.
Furthermore, Vladimir Hinich proved that the Quillen adjunction $(\mathcal{L} \dashv C_*)$ is a Quillen equivalence[60]. More generally, Quillen proved the following theorem

Theorem 4.5: [89] There exist equivalences of categories

$$\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}}) \xrightarrow{\lambda} \mathbf{Ho}(\mathbf{DGL}) \xrightarrow{C_*} \mathbf{Ho}(\mathbf{CDGC}).$$

4.2.1 Free Lie Models

In the previous section, we introduced functor $C_* : \mathbf{DGL} \to \mathbf{CDGC}$, which assigns a dgl to $cdgc$, and we know that $\mathbf{Hom}(C_*(L), \mathbb{Q})$ naturally becomes a $cdga$. Therefore, we define the functor $C^*(-) = \mathbf{Hom}(C_*(-), \mathbb{Q})$. Moreover, we have an important fact that $C^*(L)$ is a commutative $cdga$ because $C_*(L)$ is cocommutative. Moreover, if (L, d_L) is connected, then $C_*(L, d_L) = \Lambda sL = \mathbb{Q} \oplus \{C_i\}_{i\geq 2}$. The assertion that $C^*(L, d_L)$ is a Sullivan algebra follows from dualizing the Cartan-Eilenberg-Chevalley construction and leveraging properties of differential graded Lie algebras and Sullivan models.

Next, we will introduce the definition of the free Lie model for rational spaces, which is actually a Lie algebra model (L, d_L) for rational spaces X with the property $H_*(L, d_L) \cong (\pi_*(\Omega X), [\ ,\ ])$ where $[\ ,\ ]$ is determined by the Whitehead product $[\ ,\ ]_W$.

Definition 4.12: A free model of $(L, d) \in$ ob $\mathbf{DGL}$ is a quasi-isomorphism of differential graded Lie algebras

$$n : (\mathbb{L}_V, d) \xrightarrow{\simeq} (L, d)$$

with $V = \{V_i\}_{i\geq 1}$.

If $(\mathbb{L}_V, d)$ is minimal, we call $m : (\mathbb{L}_V, d) \xrightarrow{\simeq} (L, d)$ a minimal free Lie model of (L, d).

Definition 4.13: Let $X \in$ ob $\mathbf{Top}_{\mathbb{Q}}$. A Lie model for X is a quasi-isomorphism of differential graded algebras

$$n_X : C^*(L, d_L) \xrightarrow{\simeq} A_{PL}(X).$$

where (L, d_L) is a connected dgl of finite type. Sometimes, we also say that L is the Lie model of X. If $L = \mathbb{L}_V$, a free graded Lie algebra, we say (L, d_L) is a free Lie model for X.

Let $n_Y : C^*(E, d_E) \xrightarrow{\simeq} A_{PL}(Y)$ be a Lie model for the space Y, and $f : X \to Y$ be a continuous map. Then, a Lie representative for f is a morphism φ of differential graded Lie algebras such that $n_X C^*(\varphi) \sim A_{PL}(f) n_Y$.

In fact, the functor $\lambda : \mathbf{Top}_{\mathbb{Q}} \to \mathbf{DGL}$, which is constructed by Quillen, assigns a space X a Lie algebra λX which is a free Lie algebra. Thus, we call λX the free Lie model of X, and if a free Lie model $(\mathbb{L}_V, d)$ of X is minimal, then we call $(\mathbb{L}_V, d)$ a minimal free Lie model of X.

Example 4.5: The free Lie model of a sphere $\mathbb{S}^{n+1}$ with $n = 2k$ or $2k + 1$

$$\mathbb{L}(v) = \begin{cases} \mathbb{Q}v, & \deg v = 2k \\ \mathbb{Q}v \oplus \mathbb{Q}[v, v], & \deg v = 2k + 1. \end{cases}$$

and $d_L = 0$.

Proposition 4.3: [45] Any space $X \in$ ob $\mathbf{Top}_{\mathbb{Q}}$ has a minimal free Lie model $(\mathbb{L}_V, d)$, unique up to isomorphism. Suppose that $m_X : C^*(\mathbb{L}_V) \to A_{PL}(X)$ is the minimal free Lie model of X and $m_Y : C^*(\mathbb{L}_W) \to A_{PL}(Y)$ is the minimal free Lie model of Y. For any continuous map $f : X \to Y$, there is a Lie representative $n_f : (\mathbb{L}_V, d) \to (\mathbb{L}_W, d)$.

In rational homotopy theory, the following theorem establishes a correspondence between differential graded Lie algebras and the rational homotopy types of simply connected spaces:

Theorem 4.6: (Quillen's equivalence)[89] Every connected differential graded Lie algebra (L, d_L) of finite type serves as a Lie model for a simply connected CW complex X of finite rational type. Furthermore, this association is unique: two such CW complexes are rationally homotopy equivalent if and only if their corresponding differential graded Lie algebras are quasi-isomorphic.

4.2.2 Persistence Free Lie Models

Definition 4.14: Let $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$ be a rational $\mathbb{R}$-space. The persistence free Lie model of $\mathbb{X}$ is the functor $\lambda\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{DGL}$ with $(\lambda\mathbb{X})_t := \lambda\mathbb{X}_t$.

Indeed, through Theorem4.5, we can know that λ induces a functor $\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}})^{\mathbb{R}} \to \mathbf{Ho}(\mathbf{DGL})^{\mathbb{R}}$, since the morphism φ in $\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}})^{\mathbb{R}}$ is a set of $\{\varphi_a\}_{a\in\mathbb{R}}$, in which all φ_a are morphisms in $\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}})$ and λ induces the functor from $\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q}})$ to $\mathbf{Ho}(\mathbf{DGL})$[89].

Definition 4.15: Let $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$ be a rational $\mathbb{R}$-space. The persistence minimal free Lie model of $\mathbb{X}$ is the functor $M_{Qui}(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{DGL}$ with $M_{Qui}(\mathbb{X})_t$ is the minimal free Lie model of $\mathbb{X}_t$, and for any $s \leq t$, $M_{Qui}(X)_{s\leq t}$ is a Lie representative of $\mathbb{X}_{s\leq t}$.

Note that the definition of persistence minimal free Lie model is not well defined because we cannot promise the equation $M_{Qui}(X)_{r\leq t} = M_{Qui}(X)_{s\leq t} \circ M_{Qui}(X)_{r\leq s}$. However, if we focus on the homotopy category of **DGL**, **Ho**(**DGL**), then the definition of the persistence minimal free Lie model is meaningful.

Lemma 4.1: Let $n_X : C^*(\mathbb{L}_V) \to A_{PL}(X)$ and $n_Y : C^*(\mathbb{L}_W) \to A_{PL}(Y)$ be free Lie models of X and Y respectively. For any continuous map $f : X \to Y$, the Lie representative $n_f : (\mathbb{L}_V, d) \to (\mathbb{L}_W, d)$ is unique up to weak equivalence.

Proof: Given the following diagram

$$\begin{array}{ccc} A_{PL}(Y) & \xrightarrow{A_{PL}(f)} & A_{PL}(X) \\ {\scriptstyle n_Y}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle n_X} \\ C^*(\mathbb{L}_W) & \xrightarrow[C^*(n_f)]{} & C^*(\mathbb{L}_V) \end{array}$$

is commutative up to homotopy. If there is another Lie representative of f, m_f, then $C^*(n_f) \sim C^*(m_f)$. Because $C^*(\mathbb{L}_V)$ and $C^*(\mathbb{L}_W)$ are Sullivan models, $C^*(n_f)$ and $C^*(m_f)$ are two Sullivan representatives of f, $C^*(n_f) \sim C^*(m_f)$.

Note that $C_* : \mathbf{Ho}(\mathbf{DGL}) \to \mathbf{Ho}(\mathbf{CDGC})$ is a equivalence of categories, C^* induces a equivalence of categories $\mathbf{Ho}(\mathbf{DGL}) \to \mathbf{Ho}(\mathbf{CDGA})$ and we still use C^* to represent it. What's more, we know that if two morphisms in **CDGA** are homotopic, then these two morphisms are equivalent in **Ho**(**CDGA**), which is the homotopy category of **CDGA**, where weak equivalences are quasi-isomorphisms.

Therefore $n_f = m_f$ in **Ho**(**DGL**). ■

So, for any morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{Top}_{\mathbb{Q}}$, we have proven that $n_g \circ n_f = n_{gf}$ in **Ho**(**DGL**), where $n_f : \mathbb{L}_U \to \mathbb{L}_V$, $n_g : \mathbb{L}_V \to \mathbb{L}_W$, $n_{gf} : \mathbb{L}_U \to \mathbb{L}_W$ are Lie representatives of f, g, gf respectively, and $\mathbb{L}_U, \mathbb{L}_V, \mathbb{L}_W$ are minimal free Lie models of X, Y, Z respectively.

Theorem 4.7: For any rational $\mathbb{R}$-space $\mathbb{X} : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$, there exists a persistence minimal free Lie model $M_{Qui}(\mathbb{X}) : (\mathbb{R}, \leq) \to \mathbf{Ho}(\mathbf{DGL})$ such that $M_{Qui}(\mathbb{X})_t$ is a minimal free Lie model of $\mathbb{X}_r$ and $M_{Qui}(\mathbb{X})(s \leq t)$ is a Lie representative of $\mathbb{X}(s \leq t)$ up to weak equivalences.

For the persistence minimal free Lie model we construct, the post-composition of H_* and π_* computing the lower bound of the persistence minimal free Lie model $M_{Qui}(\mathbb{X})$ respectively, that is $H_*(M_{Qui}(\mathbb{X}))$ and $\pi_*(M_{Qui}(\mathbb{X}))$ are persistence modules that are functors from $(\mathbb{R}, \leq) \to \mathbf{grVec}_{\mathbb{Q}}$. Therefore, we can get the bounds of persistence minimal free Lie models. For any rational $\mathbb{R}$-space $\mathbb{X}$, we have the persistence minimal free Lie model $M_{Qui}(\mathbb{X})$. Here, we assume that Q is a map from free Lie algebras to vector spaces, $Q(\mathbb{L}_V) = V$. Obviously, any morphism of free Lie algebras $\varphi : \mathbb{L}_V \to \mathbb{L}_W$ can induces a morphism of vector spaces $Q(\varphi) : V \to W$ such that the diagram

$$\begin{array}{ccc} \mathbb{L}_V & \xrightarrow{\varphi} & \mathbb{L}_W \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ V & \xrightarrow[Q(\varphi)]{} & W \end{array}$$

is commutative.

Given $f : \mathbb{X} \to \mathbb{Y}$, then we have commutative diagram

$$\begin{array}{ccc} M_{Qui}(\mathbb{X}) & \xrightarrow{n_f} & M_{Qui}(\mathbb{Y}) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{V} & \xrightarrow[Q(n_f)]{} & \mathbb{W} \end{array}$$

where $\mathbb{V}_r := Q(M_{Qui}(\mathbb{X}_r))$ and $\mathbb{V}_{s \leq t} := Q(M_{Qui}(\mathbb{X})_{s \leq t})$.

Theorem 4.8: For any rational $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$, we have

- $d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y})) \leq d_{HI}(\mathbb{X}, \mathbb{Y}) \leq d_I(\mathbb{X}, \mathbb{Y})$
- $$\begin{aligned} d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) &= d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_* \circ M_{Qui}(\mathbb{X}), H_* \circ M_{Qui}(\mathbb{Y})) \\ &\leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y})) \end{aligned}$$
- $d_I^{\mathbf{grVec}_{\mathbb{Q}}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) = d_I^{\mathbf{grVec}_{\mathbb{Q}}}(\mathbb{V}, \mathbb{W}) \leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y}))$

To prove the theorem, we need some extra results.

Lemma 4.2: [45] Let (L, d) be a Lie model for $X \in \text{ob}\, \mathbf{Top}_{\mathbb{Q}}$. There exists a natural isomorphism $H_*(L) \xrightarrow{\cong} \pi_*(\Omega X)$ of graded Lie algebras, which converts the Lie bracket in $H_*(L)$ to the Whitehead product in $\pi_*(X)$ up to sign.

For any free Lie algebra $(\mathbb{L}_V, d)$, let $d_V : V \to V$ be the linear part of the differential

d, and $\bar{d} : sV \to sV$ be the suspension of d_V. And for any continuous map $f : X \to Y$, respective free Lie models $(\mathbb{L}_V, d)$ and $(\mathbb{L}_W, d)$ of X and Y, and a Lie representative n_f of f, we have know that $sH(V, d_V) \oplus \mathbb{Q} \cong H_*(X)$[45] and consider the linear part of the Lie representative n_f, $Q(n_f) : (sV \oplus \mathbb{Q}, d_V) \to (sW \oplus \mathbb{Q}, d_W)$.

We naturally pose the question: Is the morphism $H(Q(n_f))$ induced by $Q(n_f)$ 'equal' to the morphism $H_*(f)$? The following lemma provides an answer to our question.

Proposition 4.4: Suppose $(\mathbb{L}_V, d)$ is a free Lie model for X, then $sH(V, d_V) \oplus \mathbb{Q} \cong H_*(X)$ is a natural isomorphism of graded vector spaces.

To be more detailed, we have the following commutative diagram.

$$\begin{CD}
H_*(X) @>{H_*(f)}>> H_*(Y) \\
@V{\cong}VV @VV{\cong}V \\
sH(V, d_V) \oplus \mathbb{Q} @>>{H(Q(n_f))}> sH(W, d_W) \oplus \mathbb{Q}
\end{CD}$$

Specially, if $(\mathbb{L}_V, d)$ is minimal, then $H_*(X) \cong sV \oplus \mathbb{Q}$.

Proof: First, the morphism $C_*(\mathbb{L}_V, d) \xrightarrow{\simeq} A_{PL}(X)$ induces a cohomology isomorphism, that dualizes to an isomorphism $H_*(X) \xrightarrow{\cong} H_*(C_*(\mathbb{L}_V), d)$. Given that n_f is a Lie representative of $f : X \to Y$, then we have the following commutative diagram up to homotopy.

$$\begin{CD}
C^*(\mathbb{L}_V) @<{C^*(n_f)}<< C^*(\mathbb{L}_W, d) \\
@V{n_X}VV @VV{n_Y}V \\
A_{PL}(X) @<<{A_{PL}(f)}< A_{PL}(Y)
\end{CD}$$

Thus the diagram

$$\begin{CD}
H(C^*(\mathbb{L}_V, d)) @<{H\circ C^*(f)}<< H(C^*(\mathbb{L}_W, d)) \\
@V{\cong}VV @VV{\cong}V \\
H(A_{PL}(X)) @<<{H\circ A_{PL}(f)}< H(A_{PL}(Y)) \\
@V{\cong}VV @VV{\cong}V \\
H^*(X) @<<{H^*(f)}< H^*(Y)
\end{CD}$$

is commutative. Then, we get the following commutative diagram.

$$\begin{CD}
H(C^*(\mathbb{L}_V, d)) @>{H\circ C_*(n_f)}>> H(C^*(\mathbb{L}_W, d)) \\
@AAA @AAA \\
H_*(X) @<<{H_*(f)}< H_*(Y)
\end{CD}$$

Note that in[45], one provides a quasi-isomorphism $C_*(\mathbb{L}_V, d) \to (sV \oplus \mathbb{Q}, \bar{d})$ for any free Lie algebra $(\mathbb{L}_V, d)$. The quasi-isomorphism $C_*(\mathbb{L}_V, d) \to (sV \oplus \mathbb{Q}, \bar{d})$ is

$$C_*(\mathbb{L}_V, d) = \Lambda s\mathbb{L}_V \to s\mathbb{L}_V \oplus \mathbb{Q} \to sV \oplus \mathbb{Q},$$

where the first morphism annihilates $\Lambda^{\geq 2} s\mathbb{L}_V$ and the second morphism annihilates $s\mathbb{L}_v^{(\geq 2)}$. We obviously have the following commutative diagram

$$\begin{array}{ccc} C_*(\mathbb{L}_V) & \xrightarrow{C_*(n_f)} & C_*(\mathbb{L}_W) \\ \downarrow = & & \downarrow = \\ \Lambda s\mathbb{L}_V & & \Lambda s\mathbb{L}_W \\ \downarrow & & \downarrow \\ s\mathbb{L}_V \oplus \mathbb{Q} & \xrightarrow{n_f} & s\mathbb{L}_W \oplus \mathbb{Q} \\ \downarrow & & \downarrow \\ sV \oplus \mathbb{Q} & \xrightarrow{Q(n_f)} & sW \oplus \mathbb{Q} \end{array}$$

So we eventually get the following commutative diagram, which shows that $H_*(X) \xrightarrow{\cong} sH(V, d_V)$ is natural.

It is also easy to prove that $sH(V, d_V) \oplus \mathbb{Q} \xrightarrow{\cong} H_*(X)$ is natural. ■

With the two lemmas established above, we can now readily proceed to prove my theorem.

Proof: of Theorem4.8. This inequality $d_{HI}(\mathbb{X}, \mathbb{Y}) \leq d_I(\mathbb{X}, \mathbb{Y})$ is obvious and also an existing result. Suppose $d_{HI}(X, Y) = \delta$, then there is persistence spaces $\mathbb{X}'$ and $\mathbb{Y}' : (\mathbb{R}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$ such that $\mathbb{X} \simeq \mathbb{X}'$, $\mathbb{Y} \simeq \mathbb{Y}'$, and $d_I(\mathbb{X}', \mathbb{Y}') = \delta$.

$$\mathbb{X} \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \mathbb{X}' \qquad \mathbb{Y}' \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \mathbb{Y}$$

Consider their persistence minimal free Lie models in $\mathbf{Ho}(DGL)$,

$$M_{Qui}(\mathbb{X}) \xleftarrow{\cong} \bullet \xrightarrow{\cong} M_{Qui}(\mathbb{X}') \qquad M_{Qui}(\mathbb{Y}') \xleftarrow{\cong} \bullet \xrightarrow{\cong} M_{Qui}(\mathbb{Y})$$

where $M_{Qui}(\mathbb{X})$ is a object in category $\mathbf{Ho}(DGL)^{\mathbb{R}}$, $M_{Qui}(\mathbb{X}')$, so are $M_{Qui}(\mathbb{Y})$, and $M_{Qui}(\mathbb{Y}')$.

Suppose that $\mathbb{X}'$ and $\mathbb{Y}'$ are $(\delta + \epsilon)$-interleaved for any $\epsilon > 0$, a $(\delta + \epsilon)$-interleaving between $\mathbb{X}'$ and $\mathbb{Y}'$ induces a $(\delta + \epsilon)$-interleaving between $M_{Qui}(\mathbb{X}')$ and $M_{Qui}(\mathbb{Y}')$. Then $M_{Qui}(\mathbb{X})$ and $M_{Qui}(\mathbb{Y})$ are $(\delta + \epsilon)$-interleaved. Thus we have proven that $d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y})) \leq d_{HI}(\mathbb{X}, \mathbb{Y})$.

For the other two inequalities, $d_I^{\mathbf{Vec}}(H_* \circ M_{Qui}(\mathbb{X}), H_* \circ M_{Qui}(\mathbb{Y})) \leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y}))$ and $d_I^{\mathbf{Vec}}(\mathbb{V}, \mathbb{W}) \leq d_I^{\mathbf{Ho(DGL)}}(M_{Qui}(\mathbb{X}), M_{Qui}(\mathbb{Y}))$ are obvious. Lemma4.2 show that $d_I^{\mathbf{Vec}}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) = d_I^{\mathbf{Vec}}(H_* \circ M_{Qui}(\mathbb{X}))$ and Proposition4.4 show that $d_I^{\mathbf{Vec}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) = d_I^{\mathbf{Vec}}(\mathbb{V}, \mathbb{W})$. ■

From the proof process, we can see that apart from proving $d_I^{\mathbf{Vec}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) = d_I^{\mathbf{Vec}}(\mathbb{V}, \mathbb{W})$, we did not use the properties of the minimal free Lie model. Therefore, for any persistence free Lie model $\mathbb{L}_{\mathbb{V}}$ and $\mathbb{L}_{\mathbb{W}}$ of rational $\mathbb{R}$-spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively, we have the following results:

- $d_I^{\mathbf{Ho(DGL)}}(\mathbb{L}_{\mathbb{V}}, \mathbb{L}_{\mathbb{W}}) \leq d_{HI}(\mathbb{X}, \mathbb{Y})$,
- $d_I^{\mathbf{Vec}}(\pi_*(\mathbb{X}), \pi_*(\mathbb{Y})) = d_I^{\mathbf{Vec}}(H_* \circ \mathbb{L}_{\mathbb{V}}, H_* \circ \mathbb{L}_{\mathbb{W}}) \leq d_I^{\mathbf{Ho(DGL)}}(\mathbb{L}_{\mathbb{V}}, \mathbb{L}_{\mathbb{W}})$,
- $d_I^{\mathbf{Vec}}(H_*(\mathbb{X}), H_*(\mathbb{Y})) \leq d_I^{\mathbf{Vec}}(\mathbb{V}, \mathbb{W}) \leq d_I^{\mathbf{Ho(DGL)}}(\mathbb{L}_{\mathbb{V}}, \mathbb{L}_{\mathbb{W}})$.

What's more, we can prove easily that $d_I^{\mathbf{Ho(Top_{\mathbb{Q}})}}(\mathbb{X}, \mathbb{Y}) = d_I^{\mathbf{Ho(DGL)}}(\mathbb{L}_{\mathbb{V}}, \mathbb{L}_{\mathbb{W}})$.

In persistent homology, the persistence free Lie models have some special advantages.

Example 4.6: Let $\mathbb{X} : (\mathbb{N}, \leq) \to \mathbf{Top}_{\mathbb{Q}}$ be the filtration of skeletons of CW complex X satisfying $\mathbb{X}_r = X^r$ for $r \geq 2$ and $\mathbb{X}_0 = \mathbb{X}_1 = \emptyset$, where X is a simply connected CW complex so that $H_*(X; \mathbb{Q})$ is of finite type, and X^r is the r-dim skeleton of X. We know that $X^{r+1} = X^r \cup_{f_r} (\coprod_\alpha D_\alpha^{r+1})$, in which $f_r := \coprod_\alpha f_{r,\alpha} : \coprod_\alpha \mathbb{S}_\alpha^r \to X^r$. Next, we will construct a persistence free Lie model $Lie(\mathbb{X})$ for $\mathbb{X}$.

First, define $Lie(\mathbb{X})_0 = Lie(\mathbb{X})_1 = 0$ and $Lie(\mathbb{X})_2 = \lambda X^2$. Suppose that we have got $Lie(\mathbb{X})_r$ which is a free Lie model of X^r, that is $n_r : C^*(Lie(\mathbb{X})_r) \xrightarrow{\simeq} A_{PL}(X^r)$ is a quasi-isomorphism.

Without loss of generality, we assume that $Lie(\mathbb{X})_r = \mathbb{L}_V$. Because we have the isomorphism

$$\tau : sH(\mathbb{L}_V) \xrightarrow{\cong} \pi_*(X^r),$$

then the classes $[f_{r,\alpha}] \in \pi_*(X^r)$ determine the classes $s[z_\alpha] = \tau^{-1}[f_{r,\alpha}] \in sH(\mathbb{L}_V)$, where $z_\alpha \in \mathbb{L}_V$ are cycles.

We define that $Lie(\mathbb{X})_{r+1} := \mathbb{L}_{V \oplus W}$ and $dw_\alpha = z_\alpha$, in which W is a graded vector space with basis $\{w_\alpha\}$ with deg $w_\alpha = r$. We assert that $\mathbb{L}_{V \oplus W}$ is a free Lie model for X^{r+1}[45]. Therefore, we define a free Lie model $Lie(\mathbb{X})$ for $\mathbb{X}$, denoted as $\mathbb{L}_{\mathbb{V}}$ with $(\mathbb{L}_{\mathbb{V}})_r = \mathbb{L}_{\mathbb{V}_r} = Lie(\mathbb{X})_r$, where $\mathbb{V} : (\mathbb{N}, \leq) \to \mathbf{Vec}_{\mathbb{Q}}$ is a persistence module and any morphism

$\mathbb{V}_{s \leq t}$ is an embedding.

In addition to constructing persistence Lie models, we can also consider the persistence versions of Lie-infinity models[20,60] for rational spaces. Lie-infinity algebras inherently align more closely with the homotopy theory of topological spaces than classical Lie algebras. Indeed, while Quillen's construction provides a Lie-infinity model for a rational space X, bridging the gap to establish persistence Lie-infinity models and discuss their stability properties remains an open challenge. In fact, although Quillen's construction provides a Lie-infinity model for a rational space X, we still need a little work to overcome the difficulties if we consider persistence Lie-infinity models and the stability of persistence Lie-infinity models. And if we can construct minimal Lie-infinity models[69] for rational $\mathbb{R}$-spaces and prove that this construction satisfies functoriality, then I believe this model will have a unique advantage in theory and application of persistence modules.

CHAPTER 5 BLOCK-DECOMPOSABLE PERSISTENCE MODULES

Given $\mathbb{R}^3$ as a poset with the product order:

$$(x_1, x_2, x_3) \leq (y_1, y_2, y_3) \in \mathbb{R}^3 \Leftrightarrow x_i \leq y_i \text{ for all } i.$$

In the chapter, we consider 3-dimensional persistence modules are functors $\mathbb{R}^3 \to \mathbf{Vec}_{\Bbbk}$, where $\mathbf{Vec}_{\Bbbk}$ is the category of finitely dimensional vector spaces over $\Bbbk$.

To state our results, we need to define some notations. A cut on the real numbers $\mathbb{R}$ is a partition of $\mathbb{R}$ into two disjoint subsets c^+ and c^- such that for every $x \in c^-$ and $y \in c^+$, the inequality $x < y$ holds. This definition formalizes the idea of "splitting" $\mathbb{R}$ into a lower set c^- and an upper set c^+, where every element of c^- lies strictly below every element of c^+.

Example 5.1: Showing two different cuts:

- $c = (c^-, c^+)$ with $c^- = (-\infty, 1]$ and $c^+ = (1, +\infty)$;
- $c = (c^-, c^+)$ with $c^- = (-\infty, 1)$ and $c^+ = [1, +\infty)$

If $c^- = \emptyset$ or $c^+ = \emptyset$, we call the cut c trivial.

In $\mathbb{R}^3$, we can determine a cuboid C by 3 pairing cuts $(c_1, c^1), (c_2, c^2), (c_3, c^3)$ in which $c_1, c^1, c_2, c^2, c_3, c^3$ are cuts, so $C = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$. Every cuboid does not necessarily have to be open or closed. Some special cuboids, called blocks, will be detailed in the following.

These blocks can be divided into three major classes: layer block, birth block, and death block. The first major class is further divided into 3 sub-classes, each shown below. Let $C = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$,

- If all cuts except c_i, c^i are trivial, we call C a i-layer block;
- If c^1, c^2, c^3 are trivial, we call C a birth block;
- If c_1, c_2, c_3 are trivial, we call C a death block.

In this chater, we define $\mathbb{M}_{\mathbf{t}} := \mathbb{M}(\mathbf{t})$ and $\rho_{\mathbf{s}}^{\mathbf{t}} := \mathbb{M}(\mathbf{s} \leq \mathbf{t})$ for any persistence module $\mathbb{R}^n \to \mathbf{Vec}_{\Bbbk}$ and any $\mathbf{s} \leq \mathbf{t} \in \mathbb{R}^n$.

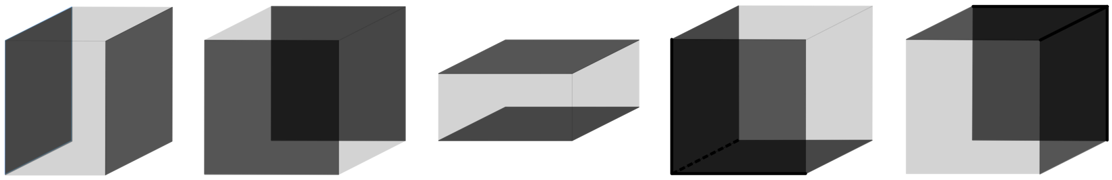


Figure 5-1 From left to right: three classes of layer blocks, birth blocks, death blocks

5.1 The Block-Decomposition of 2-Parameter Persistence Modules

Before continuing the discussion, we need to review some of Cochoy and Oudot's definitions and results[31]. Cochoy and Oudot considered the block-decomposition of 2-parameter persistence modules and proved the theorem of decomposition of **pfd** and strongly exact 2-parameter persistence modules.

In $\mathbb{R}^2$, we may also define 2-dimensional cuboids, rectangles R, by two pairing cuts, $R = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-)$. What's more, the special rectangles, which are blocks, are as follows:

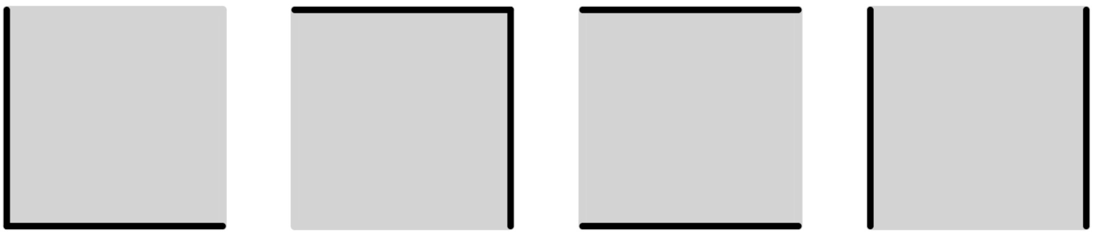


Figure 5-2 From left to right: birth blocks, death blocks, horizontal blocks, vertical blocks

In 2-parameter persistence modules $\mathbb{M} : (\mathbb{R}^2, \leq) \to \mathbf{Vec}_{\mathbb{k}}$, for any $(x_1, x_2) \leq (y_1, y_2) \in \mathbb{R}^2$, we have following commutative diagram:

$$\begin{array}{ccc} \mathbb{M}_{(x_1,y_2)} & \xrightarrow{\rho_{(x_1,y_2)}^{(y_1,y_2)}} & \mathbb{M}_{(y_1,y_2)} \\ \rho_{(x_1,x_2)}^{(x_1,y_2)} \uparrow & & \uparrow \rho_{(y_1,x_2)}^{(y_1,y_2)} \\ \mathbb{M}_{(x_1,x_2)} & \xrightarrow{\rho_{(x_1,x_2)}^{(y_1,x_2)}} & \mathbb{M}_{(y_1,x_2)} \end{array}$$

If for all $(x_1, x_2) \leq (y_1, y_2) \in \mathbb{R}^2$, the following sequence is exact, we call the 2-parameter persistence module $\mathbb{M}$ **2-parameter strongly exact**.

$$\mathbb{M}_{(x_1,x_2)} \xrightarrow{(\rho_{(x_1,x_2)}^{(x_1,y_2)}, \rho_{(x_1,x_2)}^{(y_1,x_2)})} \mathbb{M}_{(x_1,y_2)} \oplus \mathbb{M}_{(y_1,x_2)} \xrightarrow{\rho_{(x_1,y_2)}^{(y_1,y_2)} - \rho_{(y_1,x_2)}^{(y_1,y_2)}} \mathbb{M}_{(y_1,y_2)}$$

Theorem 5.1: [31] Let $\mathbb{M}$ be a pointwise finite-dimensional and strongly exact 2-parameter persistence module. Then, $\mathbb{M}$ decomposes uniquely (up to isomorphism and reordering of the terms) as a direct sum of block modules:

$$\mathbb{M} \cong \bigoplus_{B \in \mathcal{B}(\mathbb{M})} \mathbb{k}_B$$

where $\mathbb{k}_B$ is the block module associated with a block B, and $\mathcal{B}(\mathbb{M})$ is a multiset of blocks determined by $\mathbb{M}$.

For a block B, a block module $\mathbb{k}_B$ is defined as follows

$$(\mathbb{k}_B)_{\mathbf{t}} = \begin{cases} \mathbb{k}, & \mathbf{t} \in B \\ 0, & \mathbf{t} \notin B \end{cases} \tag{5-1}$$

and for any $\mathbf{s} \leq \mathbf{t}$, the morphisms $\rho_{\mathbf{s}}^{\mathbf{t}}$ in $\mathbb{k}_B$ are

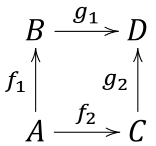
$$\rho_{\mathbf{s}}^{\mathbf{t}} = \begin{cases} id, & \text{if } \mathbf{s}, \mathbf{t} \in B \\ 0, & \text{otherwise.} \end{cases} \tag{5-2}$$

5.2 The Block-Decomposition of 3-Parameter Persistence Modules

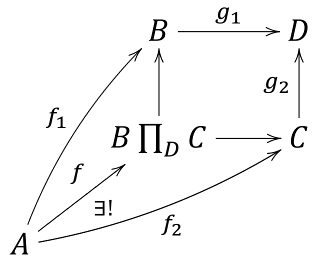
Before we begin this section, it is necessary to explain that the results presented in this section were obtained by us at the end of 2023, and at that time, we chose not to make them public. However, in 2024, Lerch et al.[71] published a more general solution to block-decomposability for multi-parameter persistence modules on arXiv. Despite this, I have decided to retain this content in my thesis because my proof method follows the approach[31] used by Oudot in solving the block-decomposability for 2-parameter persistence modules, and I believe this approach can be applied to the proof of the block-decomposition theorem for multi-parameter persistence modules. Additionally, our perspective on the generalization of the high-dimensional exactness condition differs from Lerch's, which is why I believe it is meaningful to include this part.

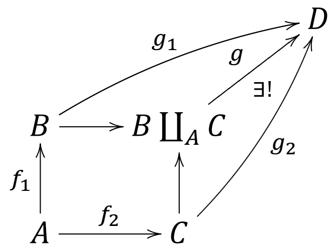
To solve the block-decomposition of 3-parameter persistence modules, we first need to generalize the strong exactness to 3-parameter cases.

Let the following diagram be a commutative diagram in **Vec**$_{\Bbbk}$



and deduce two commutative diagrams





in which $B \prod_D C = \{b + c \in B \oplus C : g_1(b) = g_2(c)\}$ and $B \coprod_A C = B \oplus C/\sim$, which $\sim$ is a deduced equivalent relation by $f_1(a) \sim f_2(a)$ for any $a \in A$.

Lemma 5.1: The following conditions are equivalence

- The sequence $A \xrightarrow{(f_1, f_2)} B \oplus C \xrightarrow{g_1 - g_2} D$ is exact;
- f is surjective;
- g is injective.

Proof: (1)⇒(2): For any $(b, c) \in B \prod_D C$, we can find a vector $a \in A$ such that $f_1(a) = b$ and $f_2(a) = c$ due to the strong exactness. So $f(a) = (b, c)$, f is surjective.

(2)⇒(3): For any $b \in B$ and $c \in C$ such that $[b + c] \in B \coprod_A C$, if $g([b + c]) = 0$, then we have $g([b]) = g([-c])$, that is $g_1(b) = g_2(-c)$. So $(b, -c) \in B \prod_D C$, and we can find out $a \in A$ such that $f_1(a) = b$ and $f_2(a) = -c$. So $[b + c] = 0 \in B \coprod_A C$. g is injective.

(3)⇒(1): For any $b \in B$ and $c \in C$ with $g_1(b) = g_2(c)$, $g([b - c]) = g_1(b) - g_2(c) = 0$. Since g is injective, $[b] = [c]$. Thus there is a vector $a \in A$ such that $f_1(a) = b$ and $f_2(a) = c$. ■

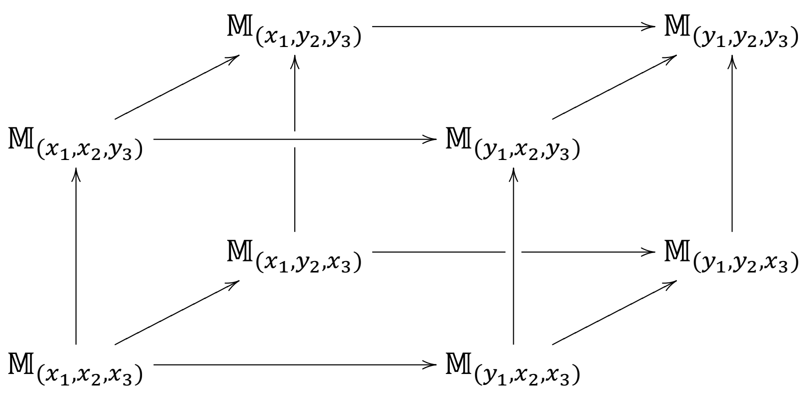
In the general case, we may also consider computing f and g similarly to the two-dimensional case. Let S be a finite set with $|S| = n$. The power set of S, $\mathcal{P}(S) = \{T : T \subseteq S\}$, is partially ordered set via inclusion. Let $\mathcal{P}_0(S) = \mathcal{P} \setminus \{\emptyset\}$ and $\mathcal{P}_1(S) = \mathcal{P}(S) \setminus \{S\}$. A functor $\mathcal{X} : \mathcal{P}(S) \to \mathbf{Vec}_{\mathbb{k}}$ is a commutative diagram shaped like a n-dim cube. What's more, we can get two morphisms $\psi : \mathcal{X}(\emptyset) \to \lim\limits_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ and $\varphi : \operatorname*{colim}\limits_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$ naturally.

Consider 2-parameter persistence modules M, then any $(x_1, x_2) \leq (y_1, y_2) \in \mathbb{R}^2$, we can get a commutative diagram

$$\begin{array}{ccc} \mathbb{M}_{(x_1,y_2)} & \xrightarrow{\rho_{(x_1,y_2)}^{(y_1,y_2)}} & \mathbb{M}_{(y_1,y_2)} \\ {\scriptstyle \rho_{(x_1,x_2)}^{(x_1,y_2)}} \uparrow & & \uparrow {\scriptstyle \rho_{(y_1,x_2)}^{(y_1,y_2)}} \\ \mathbb{M}_{(x_1,x_2)} & \xrightarrow{\rho_{(x_1,x_2)}^{(y_1,x_2)}} & \mathbb{M}_{(y_1,x_2)} \end{array}$$

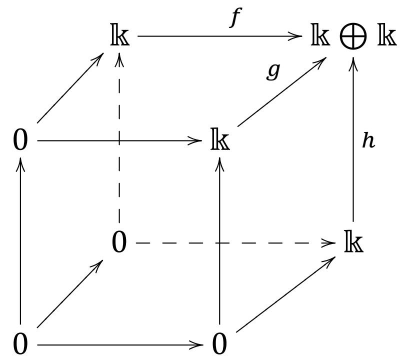
and the diagram deduces to a functor $\mathcal{X} : \mathcal{P}(S) \to \mathbf{Vec}_{\mathbb{k}}$ with $|S| = 2$. Thus for any functor $\mathcal{X} : \mathcal{P}(S) \to \mathbf{Vec}_{\mathbb{k}}$ obtained by the above method, $\psi : \mathcal{X}(\emptyset) \to \lim\limits_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ and $\varphi : \operatorname*{colim}\limits_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$ generated by the functor $\mathcal{X}$ are surjective and injective respectively if and only if M is strongly exact.

Now, we are considering block-decomposition of 3-dimensional persistence modules $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$, so we need to extend the strong exactness about 2-parameter persistence modules to the conditions about 3-dimensional persistence modules. Similar to the case of 2-parameter persistence modules, when considering the 3-dimensional persistence modules, for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3) \in \mathbb{R}^3$, there is a commutative diagram like 3-dim cube and the diagram induces the functor $\mathcal{X}(S) : \mathcal{P}(S) \to \mathbf{Vec}_{\mathbb{k}}$ with $|S| = 3$, resulting in two morphisms $\psi : \mathcal{X}(\emptyset) \to \lim\limits_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ and $\varphi : \operatorname*{colim}\limits_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$.



Thus, when we consider the block-decomposition of 3-parameter persistence modules, the strong exactness of 2-parameter block-decomposable persistence modules can be generalized to the following condition: the 3-parameter strong exactness.

Example 5.2: Consider the following 3-parameter persistence module $\mathbb{M} : \{0,1\}^3 \to$ $\mathbf{Vec}_{\mathbb{k}}$



where $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It is obvious that the 3-parameter persistence module $\mathbb{M}$ is not block-decomposable.

In the previous example, we know that only requiring φ to be injective does not guarantee that 3-parameter persistence modules are block-decomposable. What's more, in 3-parameter cases, the two conditions that φ is injective and ψ is subjective are not equivalent. Therefore, it is reasonable to assume that φ and ψ are injective and subjective, respectively.

Definition 5.1: We say that a 3-parameter persistence module $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$ is 3-parameter strongly exact if the following conditions are satisfied

- for any $r \in \mathbb{R}$, $\mathbb{M}|_{\{r\}\times\mathbb{R}\times\mathbb{R}}$, $\mathbb{M}|_{\mathbb{R}\times\{r\}\times\mathbb{R}}$, $\mathbb{M}|_{\mathbb{R}\times\mathbb{R}\times\{r\}}$ are among 2-parameter strongly exact.
- for any $(x_1, x_2, x_3) \leq (y_1, y_2, y_3) \in \mathbb{R}^3$, the associated morphisms ψ and φ is surjective and injective respectively.

We want to prove that if a 3-parameter persistence module $\mathbb{M}$ is strongly exact, then $\mathbb{M}$ can be decomposed as a direct sum of block modules. So we need to define the block modules and find all submodules of $\mathbb{M}$, which live exactly in blocks.

Definition 5.2: A persistence module $\mathbb{M}$ is called a block module if there is a block $\mathbb{M}$ such that $\mathbb{M} \cong \mathbb{k}_B$.

5.2.1 Some Basic Definitions and Results

In the 1-dimensional case, the interval modules of $\mathbb{M} : \mathbb{R} \to \mathbf{Vec}_{\mathbb{k}}$ can be easily found since $\mathbb{R}$ is a totally ordered set. By computing $V_{I,t}^{+} := \mathrm{Im}_{I,t}^{+} \cap \mathrm{Ker}_{I,t}^{+}$ and $V_{I,t}^{-} := \mathrm{Im}_{I,t}^{+} \cap \mathrm{Ker}_{I,t}^{-} + \mathrm{Im}_{I,t}^{-} \cap \mathrm{Ker}_{I,t}^{+}$ in which $I \ni t$ is a interval of $\mathbb{R}$ and

$$
\begin{aligned}
\mathrm{Im}_{I,t}^{+} &= \bigcap_{\substack{s\in I\\ s\leq t}} \mathrm{Im}\rho_s^t, & \mathrm{Im}_{I,t}^{-} &= \sum_{\substack{s\notin I\\ s\leq t}} \mathrm{Im}\rho_s^t \\
\mathrm{Ker}_{I,t}^{+} &= \bigcap_{\substack{u\notin I\\ u\geq t}} \mathrm{Ker}\rho_t^u, & \mathrm{Ker}_{I,t}^{-} &= \sum_{\substack{u\in I\\ u\geq t}} \mathrm{Ker}\rho_t^u
\end{aligned}
\tag{5-3}
$$

we can get $V_{I,t}^{+}/V_{I,t}^{-} \cong (\mathbb{k}_I)_t$. For $I = [a, b]$, $V_{I,t}^{+}/V_{I,t}^{-}$ denotes the vector space whose dimension equals the number of generators, which were born at a and died at b.

However $\mathbb{R}^n$, for $n \geq 2$, is not the totally ordered set, which results in $\mathrm{Im}_{B,t}^{-} \subseteq \mathrm{Im}_{B,t}^{+}$ and $\mathrm{Ker}_{B,t}^{-} \subseteq \mathrm{Ker}_{B,t}^{+}$, which hold in 1-dimensional persistence modules, not holding in high-dimensional persistence modules, in which $B \ni t$ is any block. Thus we need to redefine $\mathrm{Im}_{B,t}^{\pm}$ and $\mathrm{Ker}_{B,t}^{\pm}$ in which $B \subseteq \mathbb{R}^3$ is any block and $t \in B$.

Firstly, we can establish the following notation in any persistence modules $\mathbb{M} : P \to \mathbf{Vec}_{\mathbb{k}}$ in which P is a poset:

$$
\begin{aligned}
I_{P,t}^{+} &:= \bigcap_{\substack{s\in P\\ s\leq t}} \mathrm{Im}\ \rho_s^t, & I_{P,t}^{-} &:= \sum_{\substack{s\notin P\\ s\leq t}} \mathrm{Im}\ \rho_s^t \\
K_{P,t}^{+} &:= \bigcap_{\substack{u\notin P\\ u\geq t}} \mathrm{Ker}\ \rho_t^u, & K_{P,t}^{-} &:= \sum_{\substack{u\in P\\ u\geq t}} \mathrm{Ker}\ \rho_t^u
\end{aligned}
\tag{5-4}
$$

But we know that $I_{P,t}^{+} \nsubseteq I_{P,t}^{+}$ and $K_{P,t}^{-} \nsubseteq K_{P,t}^{+}$ from the above discussion. Thus, we define that

$$
\begin{aligned}
\mathrm{Im}_{P,t}^{+} &:= I_{P,t}^{+}, & \mathrm{Im}_{P,t}^{-} &:= I_{P,t}^{-} \cap I_{P,t}^{+}, \\
\mathrm{Ker}_{P,t}^{+} &:= K_{P,t}^{+} + K_{P,t}^{-}, & \mathrm{Ker}_{P,t}^{-} &:= K_{P,t}^{-}.
\end{aligned}
\tag{5-5}
$$

Obviously, $\mathrm{Im}_{C,\mathbf{t}}^{-} \subset \mathrm{Im}_{C,\mathbf{t}}^{+}$ and $\mathrm{Ker}_{C,\mathbf{t}}^{-} \subset \mathrm{Ker}_{C,\mathbf{t}}^{+}$.

When we consider the **pfd** 3-parameter persistence module $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$, the poset P is a cuboid in $\mathbb{R}^3$, which is determined by three pairing cuts $\{c_1, c^1, c_2, c^2, c_3, c^3\}$ that is $C = ({c_1}^{+} \cap {c^1}^{-}) \times ({c_2}^{+} \cap {c^2}^{-}) \times ({c_3}^{+} \cap {c^3}^{-})$.

For any $\mathbf{t} = (t_1, t_2, t_3) \in C$, We construct these limits by considering the restrictions of the module $\mathbb{M}$ along x-axis, y-axis, and z-axis, respectively

$$
\begin{aligned}
&\mathrm{Im}^+_{c_1,\mathbf{t}} = \bigcap_{\substack{x\in c_1{}^+\\ x\le t_1}} \mathrm{Im}\,\rho^{\mathbf{t}}_{(x,t_2,t_3)} && \mathrm{Im}^-_{c_1,\mathbf{t}} = \sum_{x\in c_1{}^-} \mathrm{Im}\,\rho^{\mathbf{t}}_{(x,t_2,t_3)}\\
&\mathrm{Im}^+_{c_2,\mathbf{t}} = \bigcap_{\substack{x\in c_2{}^+\\ x\le t_2}} \mathrm{Im}\,\rho^{\mathbf{t}}_{(t_1,x,t_3)} && \mathrm{Im}^-_{c_2,\mathbf{t}} = \sum_{x\in c_2{}^-} \mathrm{Im}\,\rho^{\mathbf{t}}_{(t_1,x,t_3)}\\
&\mathrm{Im}^+_{c_3,\mathbf{t}} = \bigcap_{\substack{x\in c_3{}^+\\ x\le t_3}} \mathrm{Im}\,\rho^{\mathbf{t}}_{(t_1,t_2,x)} && \mathrm{Im}^-_{c_3,\mathbf{t}} = \sum_{x\in c_3{}^-} \mathrm{Im}\,\rho^{\mathbf{t}}_{(t_1,t_2,x)}\\
&\mathrm{Ker}^+_{c^1,\mathbf{t}} = \bigcap_{x\in c^{1+}} \mathrm{Ker}\,\rho^{(x,t_2,t_3)}_{\mathbf{t}} && \mathrm{Ker}^-_{c^1,\mathbf{t}} = \sum_{\substack{x\in c^{1-}\\ x\ge t_1}} \mathrm{Ker}\,\rho^{(x,t_2,t_3)}_{\mathbf{t}}\\
&\mathrm{Ker}^+_{c^2,\mathbf{t}} = \bigcap_{x\in c^{2+}} \mathrm{Ker}\,\rho^{(t_1,x,t_3)}_{\mathbf{t}} && \mathrm{Ker}^-_{c^2,\mathbf{t}} = \sum_{\substack{x\in c^{2-}\\ x\ge t_2}} \mathrm{Ker}\,\rho^{(t_1,x,t_3)}_{\mathbf{t}}\\
&\mathrm{Ker}^+_{c^3,\mathbf{t}} = \bigcap_{x\in c^{3+}} \mathrm{Ker}\,\rho^{(t_1,t_2,x)}_{\mathbf{t}} && \mathrm{Ker}^-_{c^3,\mathbf{t}} = \sum_{\substack{x\in c^{3-}\\ x\ge t_3}} \mathrm{Ker}\,\rho^{(t_1,t_2,x)}_{\mathbf{t}}
\end{aligned}
\tag{5-6}
$$

Through simple computation, we can get

$$
\begin{aligned}
\mathrm{Im}^+_{C,\mathbf{t}} &= \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}\\
\mathrm{Im}^-_{C,\mathbf{t}} &= \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} + \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} + \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^-_{c_3,\mathbf{t}}\\
\mathrm{Ker}^+_{C,\mathbf{t}} &= \mathrm{Ker}^-_{c^1,\mathbf{t}} + \mathrm{Ker}^-_{c^2,\mathbf{t}} + \mathrm{Ker}^-_{c^3,\mathbf{t}} + \mathrm{Ker}^+_{c^1,\mathbf{t}} \cap \mathrm{Ker}^+_{c^2,\mathbf{t}} \cap \mathrm{Ker}^+_{c^3,\mathbf{t}}\\
\mathrm{Ker}^-_{C,\mathbf{t}} &= \mathrm{Ker}^-_{c^1,\mathbf{t}} + \mathrm{Ker}^-_{c^2,\mathbf{t}} + \mathrm{Ker}^-_{c^3,\mathbf{t}}
\end{aligned}
\tag{5-7}
$$

Note: If we do not make any special explanation, all the persistence modules we will discuss later are **pfd** 3-parameter persistence modules $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$ satisfying the 3-parameter strong exactness.

The following lemma allows these concepts, such as $\mathrm{Im}^{\pm}_{c_1,\mathbf{t}}$, $\mathrm{Ker}^{\pm}_{c^1,\mathbf{t}}$, involving infinity to be discussed concretely

Lemma 5.2: $\mathbb{M}$ can be extended to the persistence module over $[-\infty,+\infty]^3$ by defining $\mathbb{M}_{(\infty,\cdot,\cdot)} = \mathbb{M}_{(\cdot,\infty,\cdot)} = \mathbb{M}_{(\cdot,\cdot,\infty)} = 0$. Then

$$\mathrm{Im}^+_{c_1,\mathbf{t}} = \mathrm{Im}\,\rho^{\mathbf{t}}_{(x,t_2,t_3)} \text{ for some } x \in c_1{}^+ \cap (-\infty, t_1] \text{ and any lower } x \in c_1{}^+,$$

$$\mathrm{Im}^-_{c_1,\mathbf{t}} = \mathrm{Im}\,\rho^{\mathbf{t}}_{(x,t_2,t_3)} \text{ for some } x \in c_1{}^- \cup \{\infty\} \text{ and any greater } x \in c_1{}^-,$$

$$\mathrm{Ker}^{+}_{c^1,\mathbf{t}} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)} \text{ for some } x \in {c^1}^{+} \cup \{+\infty\} \text{ and any lower } x \in {c^1}^{+},$$

$$\mathrm{Ker}^{-}_{c^1,\mathbf{t}} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)} \text{ for some } x \in {c^1}^{-} \cap [t_1, +\infty) \text{ and any greater } x \in {c^x}^{-}.$$

The results for the cuts c_2, c^2, c_3, c^3 are similar to those for c_1, c^1, so we will not elaborate on them further.

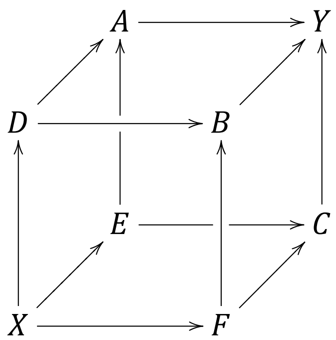
Due to the 3-parameter strong exactness, we can decompose the image and kernel in the 3-parameter persistence module $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$ into a simpler form.

Lemma 5.3: For any $\mathbf{s} \leq \mathbf{t} \in \mathbb{R}^3$, we have

$\mathrm{Im}\ \rho_{\mathbf{s}}^{\mathbf{t}} = \mathrm{Im}\ \rho_{(s_1,t_2,t_3)}^{\mathbf{t}} \cap \mathrm{Im}\ \rho_{(t_1,s_2,t_3)}^{\mathbf{t}} \cap \mathrm{Im}\ \rho_{(t_1,t_2,s_3)}^{\mathbf{t}}$,

$\mathrm{Ker}\ \rho_{\mathbf{s}}^{\mathbf{t}} = \mathrm{Ker}\ \rho_{\mathbf{s}}^{(t_1,s_2,s_3)} + \mathrm{Ker}\ \rho_{\mathbf{s}}^{(s_1,t_2,s_3)} + \mathrm{Ker}\ \rho_{\mathbf{s}}^{(s_1,s_2,t_3)}$.

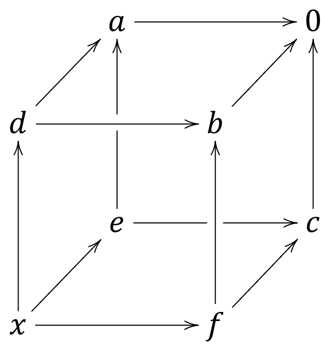
Proof: Let the following commutative diagram satisfy the 3-parameter strong exactness. We only need to prove $\mathrm{Im}\ \rho_X^Y = \mathrm{Im}\ \rho_A^Y \cap \mathrm{Im}\ \rho_B^Y \cap \mathrm{Im}\ \rho_C^Y$ and $\mathrm{Ker}\ \rho_X^Y = \mathrm{Ker}\ \rho_X^D + \mathrm{Ker}\ \rho_X^E + \mathrm{Ker}\ \rho_X^F$.



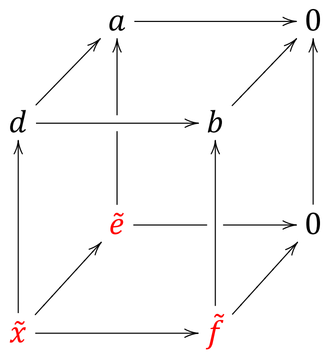
(1) Obviously, $\mathrm{Im}\ \rho_X^Y \subseteq \mathrm{Im}\ \rho_A^Y \cap \mathrm{Im}\ \rho_B^Y \cap \mathrm{Im}\ \rho_C^Y$. If $a \in A$, $b \in B$ and $c \in C$ such that $\rho_A^Y(a) = \rho_B^Y(b) = \rho_C^Y(c) = y$, we may find out $d \in D$, $e \in E$ and $f \in F$ by the 2-parameter strong exactness. Note that the construction of $\lim_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ and $\psi : \mathcal{X}(\emptyset) \to \lim_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$ is surjective, we can find out $x \in X$ so that $\rho_X^A(x) = a$, $\rho_X^B(x) = b$ and $\rho_X^C(x) = c$. Thus $\mathrm{Im}\ \rho_A^Y \cap \mathrm{Im}\ \rho_B^Y \cap \mathrm{Im}\ \rho_C^Y \subseteq \mathrm{Im}\ \rho_X^Y$. $\mathrm{Im}\ \rho_X^Y = \mathrm{Im}\ \rho_A^Y \cap \mathrm{Im}\ \rho_B^Y \cap \mathrm{Im}\ \rho_C^Y$.

(2) We can directly obtain that $\mathrm{Ker}\ \rho_X^D + \mathrm{Ker}\ \rho_X^E + \mathrm{Ker}\ \rho_X^F \subseteq \mathrm{Ker}\ \rho_X^Y$. Let $x \in \mathrm{Ker}\ \rho_X^Y$,

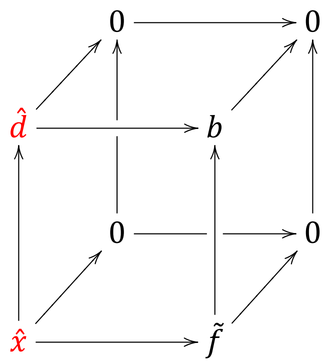
and the image of x at A, B, C, D, E, F be a, b, c, d, e, f respectively.



By the 2-parameter strong exactness, we can find out $\tilde{f} \in F$ and $\tilde{e} \in E$ such that $\rho_E^A(\tilde{e}) = a, \rho_E^C(\tilde{e}) = 0$ and $\rho_F^B(\tilde{f}) = b, \rho_F^C(\tilde{f}) = 0$. Then $\exists \tilde{x} \in X$ satisfies $\rho_X^D(\tilde{x}) = d, \rho_X^E(\tilde{x}) = \tilde{e}$ and $\rho_X^F(\tilde{x}) = \tilde{f}$. So $x - \tilde{x} \in Ker\rho_X^D$.



Similarly, we can find out $\hat{d} \in D$ and $\hat{x} \in X$ by $0 \in A$, $b \in B$, $0 \in E$, $0 \in C$ and $\tilde{f} \in F$. So $\hat{x} \in \text{Ker } \rho_X^E$.



Finally, we can easily prove that $\tilde{x} - \hat{x} \in \text{Ker } \rho_X^F$.

Thus $x = (x - \tilde{x}) + \hat{x} + (\tilde{x} - \hat{x})$. So $\text{Ker } \rho_X^Y \subseteq \text{Ker } \rho_X^D + \text{Ker } \rho_X^E + \text{Ker } \rho_X^F$, then $\text{Ker } \rho_X^Y = \text{Ker } \rho_X^D + \text{Ker } \rho_X^E + \text{Ker } \rho_X^F$. ■

With the decomposition of $\text{Im } \rho_{\mathbf{s}}^{\mathbf{t}}$ and $\text{Ker } \rho_{\mathbf{s}}^{\mathbf{t}}$ in the Lemma5.3, we can obtain the following crucial properties, which play an important role in finding submodules of M, which are block modules.

Lemma 5.4: Let $\mathbf{s} \leq \mathbf{t} \in \mathbb{R}^3$ and $\clubsuit, \spadesuit, \square \in \{+, -\}$. Then

$$\rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{c_1,\mathbf{s}}^{\clubsuit} \cap \mathrm{Im}_{c_2,\mathbf{s}}^{\spadesuit} \cap \mathrm{Im}_{c_3,\mathbf{s}}^{\square}) = \mathrm{Im}_{c_1,\mathbf{t}}^{\clubsuit} \cap \mathrm{Im}_{c_2,\mathbf{t}}^{\spadesuit} \cap \mathrm{Im}_{c_3,\mathbf{t}}^{\square},$$
$$(\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\mathrm{Ker}_{c^1,\mathbf{t}}^{\clubsuit} + \mathrm{Ker}_{c^2,\mathbf{t}}^{\spadesuit} + \mathrm{Ker}_{c^3,\mathbf{t}}^{\square}) = \mathrm{Ker}_{c^1,\mathbf{s}}^{\clubsuit} + \mathrm{Ker}_{c^2,\mathbf{s}}^{\spadesuit} + \mathrm{Ker}_{c^3,\mathbf{s}}^{\square}.$$

Proof: (1) The Lemme5.2 tells us that there exist $x \leq s_1 \leq t_1$, $y \leq s_2 \leq t_2$ and $z \leq s_3 \leq t_3$ (possibly equal to $-\infty$) such that

$$\mathrm{Im}_{c_1,\mathbf{s}}^{\clubsuit} = \mathrm{Im}\ \rho_{(x,s_2,s_3)}^{s} \text{ and } \mathrm{Im}_{c_1,\mathbf{t}}^{\clubsuit} = \mathrm{Im}\ \rho_{(x,t_2,t_3)}^{\mathbf{t}}$$
$$\mathrm{Im}_{c_2,\mathbf{s}}^{\spadesuit} = \mathrm{Im}\ \rho_{(s_1,y,s_3)}^{\mathbf{s}} \text{ and } \mathrm{Im}_{c_2,\mathbf{t}}^{\spadesuit} = \mathrm{Im}\ \rho_{(t_1,y,t_3)}^{\mathbf{t}}$$
$$\mathrm{Im}_{c_3,\mathbf{s}}^{\square} = \mathrm{Im}\ \rho_{(s_1,s_2,z)}^{\mathbf{s}} \text{ and } \mathrm{Im}_{c_3,\mathbf{t}}^{\square} = \mathrm{Im}\ \rho_{(t_1,t_2,z)}^{\mathbf{t}}$$

Then, we can directly compute

$$\begin{aligned}\mathrm{Im}_{c_1,\mathbf{t}}^{\clubsuit} \cap \mathrm{Im}\ {}_{c_2,\mathbf{t}}^{\beta} \cap \mathrm{Im}_{c_3,\mathbf{t}}^{\square} &= \mathrm{Im}\ \rho_{(x,t_2,t_3)}^{\mathbf{t}} \cap \mathrm{Im}\rho_{(t_1,y,t_3)}^{\mathbf{t}} \cap \mathrm{Im}\ \rho_{(t_1,t_2,z)}^{\mathbf{t}} \\ &= \mathrm{Im}\ \rho_{(x,y,z)}^{\mathbf{t}} = \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}\ \rho_{(x,y,z)}^{\mathbf{s}}) \\ &= \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}\ \rho_{(x,s_2,s_3)}^{s} \cap \mathrm{Im}\ \rho_{(s_1,y,s_3)}^{\mathbf{s}} \cap \mathrm{Im}\ \rho_{(s_1,s_2,z)}^{\mathbf{s}}) \\ &= \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{c_1,\mathbf{s}}^{\alpha} \cap \mathrm{Im}_{c_2,\mathbf{s}}^{\beta} \cap \mathrm{Im}_{c_3,\mathbf{s}}^{\square})\end{aligned}$$

(2) Similar to (1), we can find out $(x, y, z) \geq \mathbf{t} \geq \mathbf{s} \in \mathbb{R}^3$ (possibly x, y, z equal to $+\infty$) such that

$$\mathrm{Ker}_{c^1,\mathbf{s}}^{\clubsuit} = \mathrm{Ker}\ \rho_{\mathbf{s}}^{(x,s_2,s_3)} \text{ and } \mathrm{Ker}_{c^1,\mathbf{t}}^{\clubsuit} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)}$$
$$\mathrm{Ker}_{c^2,\mathbf{s}}^{\beta} = \mathrm{Ker}\rho_{\mathbf{s}}^{(s_1,y,s_3)} \text{ and } \mathrm{Ker}_{c^2,\mathbf{t}}^{\beta} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,y,t_3)}$$
$$\mathrm{Ker}_{c^3,\mathbf{s}}^{\square} = \mathrm{Ker}\ \rho_{\mathbf{s}}^{(s_1,s_2,z)} \text{ and } \mathrm{Ker}_{c^3,\mathbf{t}}^{\square} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,t_2,z)}$$

Then

$$\begin{aligned}&(\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)} + \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,y,t_3)} + \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,t_2,z)}) \\ =&(\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\rho_{\mathbf{t}}^{(x,y,z)}) = \mathrm{Ker}\ \rho_{\mathbf{s}}^{(x,y,z)} \\ =&\mathrm{Ker}\ \rho_{\mathbf{s}}^{(x,s_2,s_3)} + \mathrm{Ker}\ \rho_{s}^{(s_1,y,s_3)} + \mathrm{Ker}\ \rho_{\mathbf{s}}^{(s_1,s_2,z)}\end{aligned}$$

■

From the above Lemma, we may easily prove the following result.

Corollary 5.1:

$$\rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{\pm}) = \mathrm{Im}_{C,\mathbf{t}}^{\pm} \text{ and } (\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\mathrm{Ker}_{C,\mathbf{t}}^{\pm}) = \mathrm{Ker}_{C,\mathbf{s}}^{\pm}$$

Proof: We only need to pay attention to the facts that $f(U + V) = f(U) + f(V)$ and $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. ■

The following lemma states the relation between Ker and Im.

Lemma 5.5: If a 3-parameter persistence module $\mathbb{M}$ is **pfd** and satisfies the 3-parameter strong exactness, then

$$\mathrm{Ker}^{\clubsuit}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{\spadesuit}_{c_2,\mathbf{t}} \cap \mathrm{Im}^{\square}_{c_3,\mathbf{t}}$$

in which

- if ${c^1}^+ \neq \emptyset$, then $\clubsuit = +$, else $\clubsuit = -$;
- if ${c_2}^- \neq \emptyset$, then $\spadesuit = -$, else $\spadesuit = +$;
- if ${c_3}^- \neq \emptyset$, then $\square = -$, else $\square = +$.

Similarly, we have $\mathrm{Ker}^{\clubsuit}_{c^2,\mathbf{t}} \subseteq \mathrm{Im}^{\spadesuit}_{c_1,\mathbf{t}} \cap \mathrm{Im}^{\square}_{c_3,\mathbf{t}}$ and $\mathrm{Ker}^{\clubsuit}_{c^3,\mathbf{t}} \subseteq \mathrm{Im}^{\spadesuit}_{c_1,\mathbf{t}} \cap \mathrm{Im}^{\square}_{c_2,\mathbf{t}}$.

Proof: We only prove $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{+}_{c_2,\mathbf{t}} \cap \mathrm{Im}^{+}_{c_3,\mathbf{t}}$, others are similar to $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{+}_{c_2,\mathbf{t}} \cap \mathrm{Im}^{+}_{c_3,\mathbf{t}}$.

Let $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3$. From the Lemma5.2, we can find out $x \in {c^1}^-$ and $y \in {c_2}^+$ such that $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} = \mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)}$ and $\mathrm{Im}^{+}_{c_2,\mathbf{t}} = \mathrm{Im}\ \rho^{\mathbf{t}}_{(t_1,y,t_3)}$. Because $\mathbb{M}|_{\mathbb{R}\times\mathbb{R}\times\{t_3\}}$ satisfies 2-parameter strong exactness, we may consider the following commutative diagram

$$\begin{array}{ccc}
\mathbb{M}_{(t_1,t_2,t_3)} & \longrightarrow & \mathbb{M}_{(x,t_2,t_3)} \\
\uparrow & & \uparrow \\
\mathbb{M}_{(t_1,y,t_3)} & \longrightarrow & \mathbb{M}_{(x,y,t_3)}
\end{array}$$

For any $\alpha \in \mathrm{Ker}^{-}_{c^1,\mathbf{t}}$, we can find out a common antecedent $\beta \in \mathbb{M}_{(t_1,y,t_3)}$ with $0 \in \mathbb{M}_{(x,y,t_3)}$, then $\alpha \in \mathrm{Im}^{+}_{c_2,\mathbf{t}}$. So $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{+}_{c_2,\mathbf{t}}$.

Similarly, we easily prove $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{+}_{c_3,\mathbf{t}}$. Therefore, $\mathrm{Ker}^{-}_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^{+}_{c_2,\mathbf{t}} \cap \mathrm{Im}^{+}_{c_3,\mathbf{t}}$. ■

5.2.2 Find Block Submodules in $\mathbb{M}$

Next, we will try to find out all the block modules $\Bbbk_B$, which are submodules of $\mathbb{M}$. If $\mathbb{M}$ may be decomposed as a direct sum of block modules $\Bbbk_B$, the block modules $\Bbbk_B$ are exactly the submodules of $\mathbb{M}$, which the elements are born at the birth boundary of B and die at death boundary of B.

For any cuboid $C = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$, we define $V^{+}_{C,\mathbf{t}} = \mathrm{Im}^{+}_{C,\mathbf{t}} \cap \mathrm{Ker}^{+}_{C,\mathbf{t}}$ and $V^{-}_{C,\mathbf{t}} = \mathrm{Im}^{+}_{C,\mathbf{t}} \cap \mathrm{Ker}^{-}_{C,\mathbf{t}} + \mathrm{Im}^{-}_{C,\mathbf{t}} \cap \mathrm{Ker}^{+}_{C,\mathbf{t}}$. Obviously, $V^{-}_{C,\mathbf{t}} \subseteq V^{+}_{C,\mathbf{t}}$.

According to our supposition, $V^{+}_{C,\mathbf{t}}/V^{-}_{C,\mathbf{t}}$ is isomorphic to the vector space, whose elements are exactly survive in C, and combined with the previous results, we can obtain the following lemma.

Lemma 5.6: If a 3-parameter persistence module $\mathbb{M}$ is **pfd** and satisfies the 3-parameter strong exactness, then $\rho_{\mathbf{s}}^{\mathbf{t}}(V_{C,\mathbf{s}}^{\pm}) = V_{C,\mathbf{t}}^{\pm}$ and the induced morphism $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}} : V_{C,\mathbf{s}}^{+}/V_{C,\mathbf{s}}^{-} \to V_{C,\mathbf{t}}^{+}/V_{C,\mathbf{t}}^{+}$ is an isomorphism.

Proof: We can easily prove that

$$\begin{aligned}\rho_{\mathbf{s}}^{\mathbf{t}}(V_{C,\mathbf{s}}^{+}) &= \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{+} \cap \mathrm{Ker}_{C,\mathbf{s}}^{+}) \\ &\subseteq \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{+}) \cap \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Ker}_{C,\mathbf{s}}^{+}) \\ &= \mathrm{Im}_{C,\mathbf{t}}^{+} \cap \mathrm{Ker}_{C,\mathbf{t}}^{+} = V_{C,\mathbf{t}}^{+},\end{aligned}$$

$$\begin{aligned}\rho_{\mathbf{s}}^{\mathbf{t}}(V_{C,\mathbf{s}}^{-}) &= \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{+} \cap \mathrm{Ker}_{C,\mathbf{s}}^{-} + \mathrm{Im}_{C,\mathbf{s}}^{-} \cap \mathrm{Ker}_{C,\mathbf{s}}^{+}) \\ &\subseteq \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{+}) \cap \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Ker}_{C,\mathbf{s}}^{-}) + \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Im}_{C,\mathbf{s}}^{-}) \cap \rho_{\mathbf{s}}^{\mathbf{t}}(\mathrm{Ker}_{C,\mathbf{s}}^{+}) \\ &= \mathrm{Im}_{C,\mathbf{t}}^{+} \cap \mathrm{Ker}_{C,\mathbf{t}}^{-} + \mathrm{Im}_{C,\mathbf{t}}^{-} \cap \mathrm{Ker}_{C,\mathbf{s}}^{+} = V_{C,\mathbf{t}}^{-}.\end{aligned}$$

So we have $\rho_{\mathbf{s}}^{\mathbf{t}}(V_{C,\mathbf{s}}^{\pm}) \subseteq V_{C,\mathbf{t}}^{\pm}$ and $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}} : V_{R,\mathbf{s}}^{+}/V_{R,\mathbf{s}}^{-} \to V_{R,\mathbf{t}}^{+}/V_{R,\mathbf{t}}^{+}$. Sequently, we prove that $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}}$ is injective and surjective.

Surjectivity: For any $\beta \in V_{C,\mathbf{t}}^{+} = \mathrm{Im}_{C,\mathbf{t}}^{+} \cap \mathrm{Ker}_{C,\mathbf{t}}^{+}$, we can find out $\alpha \in \mathrm{Im}_{C,\mathbf{s}}^{+}$ such that $\beta = \rho_{\mathbf{s}}^{\mathbf{t}}(\alpha)$. Note that $\alpha \in (\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\beta) \subseteq (\rho_{\mathbf{s}}^{\mathbf{t}})^{-1}(\mathrm{Ker}_{C,\mathbf{t}}^{+}) = \mathrm{Ker}_{C,\mathbf{s}}^{+}$, then $\alpha \in V_{C,\mathbf{s}}^{+}$. Thus $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}}$ is surjective.

Injectivity: Let $\beta = \rho_{\mathbf{s}}^{\mathbf{t}}(\alpha) \in V_{R,\mathbf{t}}^{-}$ in which $\alpha \in V_{C,\mathbf{s}}^{+}$. We have $\beta = \beta_1 + \beta_2$ with $\beta_1 \in \mathrm{Im}_{C,\mathbf{t}}^{-} \cap \mathrm{Ker}_{C,\mathbf{t}}^{+}$ and $\beta_2 \in \mathrm{Im}_{C,\mathbf{t}}^{+} \cap \mathrm{Ker}_{C,\mathbf{t}}^{-}$. By the same argument as before, $\beta_1 = \rho_{\mathbf{s}}^{\mathbf{t}}(\alpha_1)$ for some $\alpha_1 \in \mathrm{Im}_{C,\mathbf{s}}^{-} \cap \mathrm{Ker}_{C,\mathbf{s}}^{+}$. Because $\rho_{\mathbf{s}}^{\mathbf{t}}(\alpha - \alpha_1) = \beta_2 \in \mathrm{Ker}_{C,\mathbf{t}}^{-}$, then $\alpha - \alpha_1 \in \mathrm{Ker}_{C,\mathbf{s}}^{-}$. Note that $\alpha, \alpha_1 \in \mathrm{Im}_{C,\mathbf{s}}^{+}$, then $\alpha - \alpha_1 \in \mathrm{Im}_{C,\mathbf{s}}^{+}$. So $\alpha \in V_{C,\mathbf{s}}^{-}$ and $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}}$ is injective. It implies that $\bar{\rho}_{\mathbf{s}}^{\mathbf{t}}(V_{C,\mathbf{s}}^{-}) = V_{C,\mathbf{t}}^{-}$. ■

The above lemma tells us that we can find out the vectors that exactly live in the cuboid C by using $V_{C,t}^{+}/V_{C,t}^{-}$. However, we do not want to the vector space $V_{C,\mathbf{t}}^{+}/V_{C,\mathbf{t}}^{-}$ to depend on the selection of position $\mathbf{t}$. So we need to define $\mathcal{CF}_C(\mathbb{M}) := \varprojlim_{\mathbf{t} \in C} V_{C,\mathbf{t}}^{+}/V_{C,\mathbf{t}}^{-}$. The counting functor $\mathcal{CF}$ plays a central role in the decomposition of persistence modules. Specifically, it is an additive functor[31] that determines the multiplicity of the summand $\mathbb{k}_C$ in the decomposition of the module $\mathbb{M}$ into a direct sum.

Lemma 5.7: Let $\mathbb{M}$ be **pfd** and decompose into a direct sum of cuboid modules. For any cuboid C, the dimension of the vector space $\mathcal{CF}_C(\mathbb{M})$ precisely equals the multiplicity of the summand $\mathbb{k}_C$ in the decomposition of $\mathbb{M}$ into a direct sum.

Proof: Because $\mathcal{CF}$ is an additive functor, the proof can be reduced to demonstrating the result for a single summand $\mathbb{k}_{C'}$. Suppose $C = ({c_1}^{+} \cap {c^1}^{-}) \times ({c_2}^{+} \cap {c^2}^{-}) \times ({c_3}^{+} \cap {c^3}^{-})$,

$C' = ({c_{1'}}^{+} \cap {c^{1'}}^{-}) \times ({c_{2'}}^{+} \cap {c^{2'}}^{-}) \times ({c_{3'}}^{+} \cap {c^{3'}}^{-})$, and $C \neq C'$. We can find a cut that is different between C and C'. Without loss of generality, let $c_1 \neq c_{1'}$. For any $\mathbf{t} \in C \cap C'$, $\mathrm{Im}^{+}_{c_1,t}(\mathbb{k}_{C'}) = \mathrm{Im}^{-}_{c_1,t}(\mathbb{k}_{C'})$, then $\mathrm{Im}^{-}_{C,\mathbf{t}}(\mathbb{k}_{C'}) = \mathrm{Im}^{+}_{C,\mathbf{t}}(\mathbb{k}_{C'})$. Thus $V^{-}_{C,\mathbf{t}}(\mathbb{k}_{C'}) = V^{+}_{C,\mathbf{t}}(\mathbb{k}_{C'})$, that is $V^{+}_{C,\mathbf{t}}(\mathbb{k}_{C'})/V^{-}_{C,\mathbf{t}}(\mathbb{k}_{C'}) = 0$. What's more, for any $\mathbf{t} \in C - C'$, we have $(\mathbb{k}_{C'})_{\mathbf{t}} = 0$, then $V^{+}_{C,\mathbf{t}}(\mathbb{k}_{C'})/V^{-}_{C,\mathbf{t}}(\mathbb{k}_{C'}) = 0$. So $\mathcal{CF}_C(\mathbb{k}_{C'}) = 0$. If we consider that $c^1 \neq c^{1'}$, then we may get the same result by computing $\mathrm{Ker}^{\pm}_{C,\mathbf{t}}(\mathbb{k}_{C'})$.

Secondly, we suppose that $C = C'$. For any $\mathbf{t} \in C$, we easily get $V^{+}_{C,\mathbf{t}} = \mathbb{k}$ and $V^{-}_{C,\mathbf{t}} = 0$, thus $\mathcal{CF}_C(\mathbb{k}_{C'}) = \mathbb{k}$. ■

In order to obtain the submodule $\mathbb{M}_B$, which is a submodule of $\mathbb{M}$ and exactly distribute in the block B, we need to define $V^{\pm}_B(\mathbb{M}) := \varprojlim_{\mathbf{t}\in B} V^{\pm}_{B,\mathbf{t}}$ that is independent of the selection of position $\mathbf{t}$.

Lemma 5.8: If the 3-parameter persistence module $\mathbb{M}$ is **pfd** and satisfies the 3-parameter strong exactness, then $\mathcal{CF}_B(\mathbb{M}) \cong V^{+}_B/V^{-}_B$.

Proof: Obviously, $V^{-}_{B,\mathbf{s}} \subseteq V^{-}_{B,\mathbf{t}}$ for all $\mathbf{s} \leq \mathbf{t} \in B$. We easily know that $\{V^{-}_{B,\mathbf{s}}, \rho^{\mathbf{t}}_{\mathbf{s}}\}_{\mathbf{s}\leq\mathbf{t}\in B}$ is an inverse system, and the Mittag-Leffler condition holds for the inverse system since every space $V^{-}_{B,\mathbf{s}}$ is finite-dimensional. Meanwhile, B contains a countable subset that is coinitial for the product order $\leq$, and the collection of sequences is exact

$$0 \to V^{-}_{B,\mathbf{t}} \to V^{+}_{B,\mathbf{t}} \to V^{+}_{B,\mathbf{t}}/V^{-}_{B,\mathbf{t}} \to 0.$$

Thus, the limit sequence

$$0 \to V^{-}_B(\mathbb{M}) \to V^{+}_B(\mathbb{M}) \to \mathcal{CF}_B(\mathbb{M}) \to 0$$

is exact by Proposition 13.2.2 of the reference[54]. ■

Let $\pi_{\mathbf{t}} : V^{+}_B(\mathbb{M}) \to V^{+}_{B,\mathbf{t}}$ denote the natural morphism induced by the universal property of $\varprojlim_{\mathbf{t}\in B} V^{+}_{B,\mathbf{t}}$. Then we may get $V^{-}_B(\mathbb{M}) = \bigcap_{\mathbf{t}\in B} \pi_{\mathbf{t}}^{-1}(V^{-}_{B,\mathbf{t}})$ and $V^{+}_B(\mathbb{M}) = \bigcap_{\mathbf{t}\in B} \pi_{\mathbf{t}}^{-1}(V^{+}_{B,\mathbf{t}})$. Thus we have $V^{-}_B(\mathbb{M}) \subset V^{+}_B(\mathbb{M})$.

Lemma 5.9: If a 3-parameter persistence module $\mathbb{M}$ is **pfd** and satisfies the 3-parameter strong exactness, then

$$\bar{\pi}_{\mathbf{t}} : V^{+}_B(\mathbb{M})/V^{-}_B(\mathbb{M}) \to V^{+}_{B,\mathbf{t}}/V^{-}_{B,\mathbf{t}}$$

is isomorphic.

Proof: Referring to the proof of Lemma 5.2 of[31]. ■

After obtaining the previous results, we can select the appropriate subspace M_B^0 from V_B^+, so that the submodules $\mathbb{M}_B$ are generated through $\pi_{\mathbf{t}}(M_B^0)$.

Proposition 5.1: If the 3-parameter persistence module $\mathbb{M}$ is **pfd** and satisfies the 3-parameter strong exactness, then the subspace V_B^- has a complementary space M_B^0 in V_B^+ such that the following persistence module

$$(\mathbb{M}_B)_{\mathbf{t}} = \begin{cases} \pi_{\mathbf{t}}(M_B^0), & \mathbf{t} \in B \\ 0, & \mathbf{t} \notin B \end{cases} \tag{5-8}$$

is a submodule $\mathbb{M}_B$ of $\mathbb{M}$.

Proof: We will discuss the proof of the results in three cases: birth blocks, death blocks, and layer blocks. For a fixed block B, regardless of the choice of subspace M_B^0 satisfying the decomposition $V_B^+(\mathbb{M}) = M_B^0 \oplus V_B^-(\mathbb{M})$, the following statements will hold:

- for any $\mathbf{s}, \mathbf{t} \in B$ satisfying $\mathbf{s} \le \mathbf{t}$, $\rho_{\mathbf{s}}^{\mathbf{t}}((\mathbb{M}_B)_{\mathbf{t}}) \subseteq (\mathbb{M}_B)_{\mathbf{t}}$, since $\rho_{\mathbf{s}}^{\mathbf{t}} \circ \pi_{\mathbf{s}} = \pi_{\mathbf{t}}$ by the definition of π.
- for any $\mathbf{s} \notin B, \mathbf{t} \in B$ satisfying $\mathbf{s} \le \mathbf{t}$, $\rho_{\mathbf{s}}^{\mathbf{t}}((\mathbb{M}_B)_{\mathbf{s}}) = \rho_{\mathbf{s}}^{\mathbf{t}}(0) = 0 \subseteq (\mathbb{M}_B)_{\mathbf{t}}$.

There only remains to show that, for any $\mathbf{s} \le \mathbf{t}$, $\mathbf{s} \in B$ and $\mathbf{t} \notin B$, $\rho_{\mathbf{s}}^{\mathbf{t}}((\mathbb{M}_B)_{\mathbf{s}}) = 0$. Therefore, we need to choose a suitable subspace M_B^0 that satisfies the condition.

Case B is birth block: $C = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$ in which ${c^1}^+ = {c^2}^+ = {c^3}^+ = \emptyset$.

For any choice of subspace M_B^0, the condition can be satisfied.

Case B is death block: $C = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$ in which ${c_1}^- = {c_2}^- = {c_3}^- = \emptyset$.

Let $K_{B,\mathbf{s}}^+ = \mathrm{Ker}_{c^1,\mathbf{s}}^+ \cap \mathrm{Ker}_{c^2,\mathbf{s}}^+ \cap \mathrm{Ker}_{c^3,\mathbf{s}}^+$ for all $\mathbf{s} \in B$. The collection of these vector spaces, combined with the transition maps $\rho_{\mathbf{s}}^{\mathbf{t}}$ for $\mathbf{s} \le \mathbf{t} \in B$ forms an inverse system. Because $K_{B,\mathbf{s}}^+ \subseteq \mathrm{Im}_{B,\mathbf{s}}^+$ by Lemma5.4 and $K_{B,\mathbf{s}}^+ \subseteq \mathrm{Ker}_{B,\mathbf{s}}^+$, then $K_{B,\mathbf{s}}^+ \subseteq V_{B,\mathbf{s}}^+$. Thus

$$K_B^+(\mathbb{M}) = \varprojlim_{\mathbf{s} \in B} K_{B,\mathbf{t}}^+ = \bigcap_{\mathbf{s} \in B} \pi_{\mathbf{s}}^{-1}(K_{B,\mathbf{s}}^+) \subseteq V_B^+(\mathbb{M}).$$

And for any $\mathbf{s} \in B$, following equation holds:

$$\begin{aligned} V_{B,\mathbf{s}}^+ &= \mathrm{Im}_{B,\mathbf{s}}^+ \cap \mathrm{Ker}_{B,\mathbf{s}}^+ = \mathrm{Im}_{B,\mathbf{s}}^+ \cap (\mathrm{Ker}_{B,\mathbf{s}}^- + K_{B,\mathbf{s}}^+) \\ &= \mathrm{Im}_{B,\mathbf{s}}^+ \cap \mathrm{Ker}_{B,\mathbf{s}}^- + \mathrm{Im}_{B,\mathbf{s}}^+ \cap K_{B,\mathbf{s}}^+ = V_{B,\mathbf{s}}^- + K_{B,\mathbf{s}}^+. \end{aligned}$$

In other words, for any $\mathbf{s} \in B$, the following sequence is exact:

$$0 \to V_{B,\mathbf{s}}^- \cap K_{B,\mathbf{s}}^+ \xrightarrow{\alpha \mapsto (\alpha, -\alpha)} V_{B,\mathbf{s}}^- \oplus K_{B,\mathbf{s}}^+ \xrightarrow{(\alpha,\beta) \mapsto \alpha+\beta} V_{B,\mathbf{s}}^+ \to 0.$$

This system of exact sequences satisfies the Mittag-Leffler condition, since every space $V^-_{B,\mathbf{s}} \cap K^+_{B,\mathbf{s}}$ is finite-dimensional, and so, by Proposition13.2.2 of[54], the limit sequence is exact. Note that $\varprojlim_{\mathbf{s}\in B} V^-_{B,\mathbf{s}} \cap K^+_{B,\mathbf{s}} = V^-_B(\mathbb{M}) \cap K^+_B(\mathbb{M})$ in $V^+_B(\mathbb{M})$, and the canonical morphism $V^-_B(\mathbb{M}) \oplus K^+_B(\mathbb{M}) \to \varprojlim_{\mathbf{s}\in B} V^-_{B,\mathbf{s}} \oplus K^+_{B,\mathbf{s}}$ is an isomorphism, then the following sequence is exact:

$$0 \to V^-_B(\mathbb{M}) \cap K^+_B(\mathbb{M}) \xrightarrow{\alpha\mapsto(\alpha,-\alpha)} V^-_B(\mathbb{M}) \oplus K^+_B(\mathbb{M}) \xrightarrow{(\alpha,\beta)\mapsto\alpha+\beta} V^+_B(\mathbb{M}) \to 0$$

which implies that $V^-_B(\mathbb{M}) + K^+_B(\mathbb{M}) = V^+_B(\mathbb{M})$. Thus we only need to choose a complement subspace M^0_B of $V^-_B(\mathbb{M})$ inside $K^+_B(\mathbb{M})$.

Case B is strict layer block: We only need to consider one case that is ${c_2}^- = {c^2}^+ = {c_3}^- = {c^3}^+ = \emptyset$ and ${c_1}^- \neq \emptyset \neq {c^1}^+$, the rest are similar. Let $K^+_{B,\mathbf{s}} = \mathrm{Im}^+_{c_1,\mathbf{s}} \cap Ker^+_{c^1,\mathbf{s}}$ for any $\mathbf{s} \in B$.

We have $V^+_{B,\mathbf{s}} = \mathrm{Im}^+_{B,\mathbf{s}} \cap \mathrm{Ker}^+_{B,\mathbf{s}} = \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Im}^+_{c_3,\mathbf{s}} \cap (\mathrm{Ker}^+_{c^1,\mathbf{s}} + \mathrm{Ker}^-_{c^2,\mathbf{s}} + \mathrm{Ker}^-_{c^3,\mathbf{s}})$. Because $\mathrm{Ker}^-_{c^2,\mathbf{s}} \subseteq \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Im}^+_{c_3,\mathbf{s}}$, $\mathrm{Ker}^-_{c^3,\mathbf{s}} \subseteq \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Im}^+_{c_2,\mathbf{s}}$ and $\mathrm{Ker}^-_{c^1,\mathbf{s}} \subseteq \mathrm{Ker}^+_{c^1,\mathbf{s}} \subseteq \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Im}^+_{c_3,\mathbf{s}}$ by Lemma5.4, then we get

$$\begin{aligned} V^+_{B,\mathbf{s}} &= \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Im}^+_{c_3,\mathbf{s}} \cap (\mathrm{Ker}^+_{c^1,\mathbf{s}} + \mathrm{Ker}^-_{c^2,\mathbf{s}} + \mathrm{Ker}^-_{c^3,\mathbf{s}}) \\ &= \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Ker}^+_{c^1,\mathbf{s}} + \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Ker}^-_{c^2,\mathbf{s}} + \mathrm{Im}^+_{c_3,\mathbf{s}} \cap \mathrm{Ker}^-_{c^3,\mathbf{s}} \\ &= K^+_{B,\mathbf{s}} + \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Ker}^-_{c^2,\mathbf{s}} + \mathrm{Im}^+_{c_3,\mathbf{s}} \cap \mathrm{Ker}^-_{c^3,\mathbf{s}} \end{aligned}$$

And we have

$$\begin{aligned} V^-_{B,\mathbf{s}} &= \mathrm{Im}^+_{B,\mathbf{s}} \cap \mathrm{Ker}^-_{B,\mathbf{s}} + \mathrm{Im}^-_{B,\mathbf{s}} \cap \mathrm{Ker}^+_{B,\mathbf{s}} \\ &= \mathrm{Im}^+_{c_1,\mathbf{s}} \cap \mathrm{Ker}^-_{c^1,\mathbf{s}} + \mathrm{Im}^+_{c_2,\mathbf{s}} \cap \mathrm{Ker}^-_{c^2,\mathbf{s}} + \mathrm{Im}^+_{c_3,\mathbf{s}} \cap \mathrm{Ker}^-_{c^3,\mathbf{s}} + \mathrm{Im}^-_{B,\mathbf{s}} \cap \mathrm{Ker}^+_{B,\mathbf{s}} \end{aligned}$$

Thus $V^+_{B,\mathbf{s}} = V^-_{B,\mathbf{s}} + K^+_{B,\mathbf{s}}$. Following a similar argument to the preceding case, we conclude that the limits satisfy $V^+_B(\mathbb{M}) = V^-_B(\mathbb{M}) + K^+_B(\mathbb{M})$. Therefore, we may select the vector space complement M^0_B inside $K^+_B(\mathbb{M})$, guaranteeing that $\pi_{\mathbf{s}}(M^0_B) \subseteq K^+_{B,\mathbf{s}}$ for any $\mathbf{s} \in B$. ■

Because $V^+_B(\mathbb{M}) = V^-_B \oplus M^0_B$, and $\pi_{\mathbf{t}}$ and $\pi_{\mathbf{t}}|_{V^-_B}$ are isomophisms, $M^0_B \cong (\mathbb{M}_B)_{\mathbf{t}}$ and $V^+_{B,\mathbf{t}} \cong V^-_{B,\mathbf{t}} \oplus (\mathbb{M}_B)_{\mathbf{t}}$.

Corollary 5.2: For every block B, $\mathbb{M}_B \cong \bigoplus_{\dim \mathcal{CF}_B(\mathbb{M})} \mathbb{k}_B$.

5.2.3 The Direct Sum Decomposition

Before we prove the direct sum decomposition of $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$, which satisfies the 3-parameter strong exactness, we need to introduce some definitions and results of disjointness and covering of sections[34].

In a vector space U, a section consists of two subspaces (F^-, F^+) such that $F^- \subset F^+ \subset U$. First, we introduce the notations of the disjointness of sections. We call that a collection of sections $\{(F_\lambda^-, F_\lambda^+)\}_{\lambda\in\Lambda}$ in U is said to be disjoint, if whenever $\lambda \neq \mu$, one of the inclusions $F_\lambda^+ \subseteq F_\mu^-$ or $F_\mu^+ \subseteq F_\lambda^-$ is satisfied.

Lemma 5.10: [34] Let $\{(F_\lambda^-, F_\lambda^+)\}_{\lambda\in\Lambda}$ be a collection of sections in U, that is disjoint. For any $\lambda \in \Lambda$, suppose that V_λ is a subspace satisfying $F_\lambda^+ = M_\lambda \oplus F_\lambda^-$, then $\sum V_\lambda \cong \oplus V_\lambda$.

Lemma 5.11: [34] Given a fixed $\mathbf{t} \in \mathbb{R}^3$, each of the collections $\{(\mathrm{Im}_{c_1,\mathbf{t}}^-, \mathrm{Im}_{c_1,\mathbf{t}}^+)\}_{c_1:t_1\in c_1{}^+}$, $\{(\mathrm{Ker}_{c^1,\mathbf{t}}^-, \mathrm{Ker}_{c^1,\mathbf{t}}^+)\}_{c^1:t_1\in c^1{}^-}$, $\{(\mathrm{Im}_{c_2,\mathbf{t}}^-, \mathrm{Im}_{c_2,\mathbf{t}}^+)\}_{c_2:t_2\in c_2{}^+}$, $\{(\mathrm{Ker}_{c^2,\mathbf{t}}^-, \mathrm{Ker}_{c^2,\mathbf{t}}^+)\}_{c^2:t_2\in c^2{}^-}$, $\{(\mathrm{Im}_{c_3,\mathbf{t}}^-, \mathrm{Im}_{c_3,\mathbf{t}}^+)\}_{c_3:t_3\in c_3{}^+}$ and $\{(\mathrm{Ker}_{c^3,\mathbf{t}}^-, \mathrm{Ker}_{c^3,\mathbf{t}}^+)\}_{c^3:t_3\in c^3{}^-}$ is disjoint in $M_\mathbf{t}$.

Lemma 5.12: [34] Let the collection of sections in U, $\mathcal{F} = \{(F_\lambda^-, F_\lambda^+)\}_{\lambda\in\Lambda}$, be disjoint, and $\mathcal{G} = \{(G_\sigma^-, G_\sigma^+)\}_{\sigma\in\Sigma}$ be any collection of sections in U. Then the collection of sections in U

$$\{(F_\lambda^- + G_\sigma^- \cup F_\lambda^+, F_\lambda^- + G_\sigma^+ \cup F_\lambda^+)\}_{(\lambda,\sigma)\in\Lambda\times\Sigma}$$

is disjoint.

Because $\mathcal{V}_\mathbf{t} := \{(V_{B,\mathbf{t}}^-, V_{B,\mathbf{t}}^+)\}_{B:block\ni\mathbf{t}}$ is not disjoint, we cannot directly study the direct sum decomposition of $\mathbb{M}$ by considering $\mathcal{V}_\mathbf{t}$. Thus, we define the disjoint section $\mathcal{F}_\mathbf{t} := \{F_{B,\mathbf{t}}^-, F_{B,\mathbf{t}}^+\}_{B:block\ni\mathbf{t}}$

$$F_{B,\mathbf{t}}^+ = \mathrm{Im}_{B,\mathbf{t}}^- + V_{B,\mathbf{t}}^+ = \mathrm{Im}_{B,\mathbf{t}}^- + \mathrm{Ker}_{B,\mathbf{t}}^+ \cap \mathrm{Im}_{B,\mathbf{t}}^+,$$
$$F_{B,\mathbf{t}}^- = \mathrm{Im}_{B,\mathbf{t}}^- + V_{B,\mathbf{t}}^- = \mathrm{Im}_{B,\mathbf{t}}^- + \mathrm{Ker}_{B,\mathbf{t}}^- \cap \mathrm{Im}_{B,\mathbf{t}}^+.$$

Lemma 5.13: $F_{B,\mathbf{t}}^+ = F_{B,\mathbf{t}}^- \oplus (\mathbb{M}_B)_\mathbf{t}$

Proof: From the definition of $F_{B,\mathbf{t}}^\pm$, we can easily know that $F_{B,\mathbf{t}}^+ = F_{B,\mathbf{t}}^- + (\mathbb{M}_B)_\mathbf{t}$. And, we have $(\mathbb{M}_B)_\mathbf{t} \subseteq V_{B,\mathbf{t}}^+$, so

$$F_{B,\mathbf{t}}^- \cap (\mathbb{M}_B)_\mathbf{t} = F_{B,\mathbf{t}}^- \cap V_{B,\mathbf{t}}^+ \cap (\mathbb{M}_B)_\mathbf{t} = V_{B,\mathbf{t}}^- \cap (\mathbb{M}_B)_\mathbf{t} = 0$$

■

From this lemma, we see that we can study the direct sum of $\mathbb{M}$ by considering $\mathcal{F}_{\mathbf{t}}$.

We divide these type of blocks $B = ({c_1}^+ \cap {c^1}^-) \times ({c_2}^+ \cap {c^2}^-) \times ({c_3}^+ \cap {c^3}^-)$ into following 5 types:

- $\mathcal{B}_1 = \{B | {c_2}^- = {c_2}^+ = {c_3}^- = {c^2}^+ = \emptyset$ and ${c^1}^+ \neq \emptyset\}$;
- $\mathcal{B}_2 = \{B | {c_1}^- = {c_1}^+ = {c_3}^- = {c^2}^+ = \emptyset$ and ${c^2}^+ \neq \emptyset\}$;
- $\mathcal{B}_3 = \{B | {c_1}^- = {c_1}^+ = {c_2}^- = {c^2}^+ = \emptyset$ and ${c^3}^+ \neq \emptyset\}$;
- $\mathcal{B}_4 =$ the set of all death blocks $\setminus \bigcup_{i=1}^{3} \mathcal{B}_i$;
- $\mathcal{B}_5 =$ the set of all birth blocks.

We first prove that in each individual type, the decomposition is a direct sum decomposition. The proof process for the first four types is easy, but we need to make some small efforts to prove the fifth type.

Proposition 5.2: Let $\mathcal{B}_i$ be a fixed block type. The submodules $\mathbb{M}_B$, where B ranges over all blocks of the block type $\mathcal{B}_i$, are in direct sum, that is $\sum\limits_{B \in \mathcal{B}_i} \mathbb{M}_B \cong \bigoplus\limits_{B \in \mathcal{B}_i} \mathbb{M}_B$.

Proof: Let $\mathbf{t} \in \mathbb{R}^3$. We only need to prove the equation, $\sum\limits_{B \in \mathcal{B}_i} (\mathbb{M}_B)_{\mathbf{t}} \cong \bigoplus\limits_{B \in \mathcal{B}_i} (\mathbb{M}_B)_{\mathbf{t}}$

Case $\mathcal{B}_i$ with $i = 1, 2, 3$: We only need to prove the case of $i = 1$, and the proof for $i = 2, 3$ is similar. From Lemma5.11, we can know that $\{(\mathrm{Im}^-_{c_1,\mathbf{t}}, \mathrm{Im}^+_{c_1,\mathbf{t}})\}_{{c_1}^+ \ni t_1}$ is disjoint. Taking the intersection of all the spaces in this collection with $\mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}$, we deduce that

$$\{(\mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}, \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}})\}_{{c_1}^+ \ni t_1} = \{(\mathrm{Im}^-_{B,\mathbf{t}}, \mathrm{Im}^+_{B,\mathbf{t}})\}_{B:\mathcal{B}_1 \ni \mathbf{t}}$$

is also disjoint. Hence, by Lemma5.12, the collection of subspaces $\{(\mathbb{M}_B)_{\mathbf{t}}\}_{B:\mathcal{B}_1 \ni \mathbf{t}}$ is in direct sum.

Case $\mathcal{B}_4$: Consider any finite collection of distinct death quadrants $B_1, B_2, \cdots, B_m$ that contain $\mathbf{t}$. Since all of them are distinct, there must exist one (denoted as B_1) that is not contained within the union of the others. Therefore, there exists some $\mathbf{u} \geq \mathbf{t}$ such that $\mathbf{u} \in B_1 - \bigcup_{i \geq 2} B_i$. Suppose there is some relation $\sum_{i=1}^{m} \alpha_i = 0$ with $\alpha_i \in (\mathbb{M}_{B_i})_{\mathbf{t}}$ non-zero for all i. Due to the linearity of $\rho_{\mathbf{t}}^{\mathbf{u}}$, it follows that $\sum_{i=1}^{m} \rho_{\mathbf{t}}^{\mathbf{u}}(\alpha_i) = 0$. However, $\rho_{\mathbf{t}}^{\mathbf{u}}(\alpha_i) = 0$ for any $i \geq 2$ and $\rho_{\mathbf{t}}^{\mathbf{u}}(\alpha_1) \neq 0$ due to $\mathbf{u} \in B_1 - \bigcup_{i \geq 2} B_i$. This raises a contradiction.

Case $\mathcal{B}_5$: It suffices to show that, for any finite collection of different birth quadrants $B_1, B_2, \cdots, B_m$, there exists at least one of them (e.g., $\mathbb{B}_1$) whose corresponding subspace $(\mathbb{M}_{B_1})_{\mathbf{t}} \subseteq M_{\mathbf{t}}$ is in direct sum with the subspaces corresponding to the other blocks in the collection. Therefore, the result is obtained by a straightforward induction on the size m of the collection.

Let $B_1, B_2, \cdots, B_m$ be such a collection and each block $B_i = {c_{1,i}}^+ \times {c_{2,i}}^+ \times {c_{3,i}}^+$. By reordering if necessary, we can suppose that B_1 satisfies

$$ {c_{1,1}}^+ \subseteq \bigcap_{i>1} {c_{1,i}}^+, {c_{2,1}}^+ \subseteq \bigcap_{\substack{i>1 \\ c_{1,i}=c_{1,1}}} {c_{2,i}}^+, {c_{3,1}}^+ \subseteq \bigcap_{\substack{i>1 \\ c_{1,i}=c_{1,1} \\ c_{2,i}=c_{2,1}}} {c_{3,i}}^+. $$

From the assumption of B_1, we can get that B_1 does not contain any other blocks. Therefore, by reordering, we can divide these blocks into two subcollections: the ones (denoted as $B_2, \cdots, B_k$) contain B_1, while the others (denoted as $B_{k+1}, \cdots, B_m$) neither contain B_1 nor are not contained by B_1.

In a manner analogous to the proof of Proposition 6.6 in Cochoy and Oudot's work[31], we deduce that $(\mathbb{M}_{B_1})_{\mathbf{t}} \cap (\sum_{i=2}^{m} (\mathbb{M}_{B_i})_{\mathbf{t}}) \subseteq F^-_{B_1,\mathbf{t}}$. Note $(\mathbb{M}_{B_1})_{\mathbf{t}} \cap F^-_{B_1,\mathbf{t}} = 0$, then the result follows. ■

Proposition 5.3: The submodules $\bigoplus_{B:\mathcal{B}_1} \mathbb{M}_B$, $\bigoplus_{B:\mathcal{B}_2} \mathbb{M}_B$, $\bigoplus_{B:\mathcal{B}_3} \mathbb{M}_B$, $\bigoplus_{B:\mathcal{B}_4} \mathbb{M}_B$ and $\bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B$ are in direct sum, that is

$$ \bigoplus_{B:\mathcal{B}_1} \mathbb{M}_B + \bigoplus_{B:\mathcal{B}_2} \mathbb{M}_B + \bigoplus_{B:\mathcal{B}_3} \mathbb{M}_B + \bigoplus_{B:\mathcal{B}_4} \mathbb{M}_B + \bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B \cong \bigoplus_{B:\text{blocks}} \mathbb{M}_B $$

Proof: We will divide the proof into four parts,

- $(\bigoplus_{B:\mathcal{B}_5} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_1} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{t}) = 0$
- $(\bigoplus_{B:\mathcal{B}_1} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$
- $(\bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$
- $(\bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$

(1) prove that $(\bigoplus_{B:\mathcal{B}_5} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_1} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$

Note that if $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ is large enough, then we can know that $\mathbf{u} \in B$ for any block $B \in \mathcal{B}_5$ but $\mathbf{u}$ is not in any other blocks. we only need demand u_1, u_2, u_3 are large enough.

Let α be a non-zero vector and be in the intersection. It can be decomposed as a linear combination of non-zero vectors $\alpha_1, \cdots, \alpha_n$ from the summands of a finite number of blocks $B_1, B_2, \cdots, B_n$ in $\mathcal{B}_5$. Simultaneously, α can be decomposed as a linear combination of non-zero vectors $\beta_1, \cdots, \beta_m$ from the summands of a finite number of blocks $B'_1, \cdots, B'_m$ of other types: $\sum_{i=1}^{n} \alpha_i = \alpha = \sum_{j=1}^{m} \beta_j$.

Select a point $\mathbf{u} \in \mathbb{R}^3$ so that $\mathbf{u}$ is sufficiently large to lie outside the blocks $B_1', \cdots, B_n'$. What's more, $\mathbf{u}$ still lies in the birth quadrants $B_1, \cdots, B_n$. Thus $\rho_{\mathbf{t}}^{\mathbf{u}}(\sum_{i=1}^{n} \alpha_i) \neq 0$ but $\rho_{\mathbf{t}}^{\mathbf{u}}(\sum_{j=1}^{m} \beta_i) = 0$. This is a contradiction.

(2) prove that $(\bigoplus_{B:\mathcal{B}_1} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$, $(\bigoplus_{B:\mathcal{B}_2} (M_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_3} (M_B)_{\mathbf{t}} \oplus \bigoplus_{B:\mathcal{B}_4} (M_B)_{\mathbf{t}}) = 0$ and $(\bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$.

Similar to (1), we can also choose a point $\mathbf{u} \in \mathbb{R}^3$ so that it lies outside the blocks in $\mathcal{B}_1$ but is not in the blocks in $\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$. we need only to demand u_1, u_2 are large enough. The remaining processes are almost identical to (1).

(3) prove that $(\bigoplus_{B:\mathcal{B}_2} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$ and $(\bigoplus_{B:\mathcal{B}_3} (\mathbb{M}_B)_{\mathbf{t}}) \cap (\bigoplus_{B:\mathcal{B}_4} (\mathbb{M}_B)_{\mathbf{t}}) = 0$.

They are treated similarly to (2). ∎

Subsequently, we will prove that $\mathbb{M} = \sum_{B:block} \mathbb{M}_B$. Then we need the notation of covering of sections[34]. For any collection of sections $\{(F_\lambda^-, F_\lambda^-)\}_{\lambda \in \Lambda}$, we say that $\{(F_\lambda^-, F_\lambda^-)\}_{\lambda \in \Lambda}$ covers a vector space U if for every proper subspace $X \subsetneq U$ there exists a $\lambda \in \Lambda$ satisfying

$$X + F_\lambda^- \neq X + F_\lambda^+.$$

This collection is said to strongly cover U, if for all subspaces $X \subsetneq U$ and $Z \nsubseteq X$ there exists a $\lambda \in \Lambda$ so that

$$X + (F_\lambda^- \cap Z) \neq X + (F_\lambda^+ \cap Z).$$

The validity of employing covering sections is substantiated by the subsequent lemma from the reference[34].

Lemma 5.14: [34] Let $\{(F_\lambda^-, F_\lambda^+)\}_{\lambda \in \Lambda}$ be a collection of sections that covers U. For every $\lambda \in \Lambda$, suppose V_λ is a subspace of U satisfying $F_\lambda^+ = V_\lambda \oplus F_\lambda^-$. It follows that, $U = \sum_{\lambda \in \Lambda} V_\lambda$.

Lemma 5.15: [34] Let $\{(F_\lambda^-, F_\lambda^+)\}_{\lambda \in \Lambda}$ and $\{G_\sigma^-, G_\sigma^+\}_{\sigma \in \Sigma}$ be two collections of sections, where the former covers U and the latter strongly covers U. Then the following collection covers U:

$$\{(F_\lambda^- + G_\sigma^- \cap F_\lambda^+, F_\lambda^- + G_\sigma^+ \cap F_\lambda^+)\}_{(\lambda \times \sigma) \in \Lambda \times \Sigma}.$$

Lemma 5.16: [34] Given a fixed $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3$, for any subsets $X \subsetneq \mathbb{M}_{\mathbf{t}}$ and

$Z \nsubseteq X$, there is a cut c_1 with $t_1 \in {c_1}^+$ such that $\mathrm{Im}^-_{c_1,\mathbf{t}} \cap Z \subseteq X \nsupseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap Z$. Similarly, there are cuts c_2 with $t_2 \in {c_2}^+$ and c_3 with $t_3 \in {c_3}^+$ such that $\mathrm{Im}^-_{c_2,\mathbf{t}} \cap Z \subseteq X \nsupseteq \mathrm{Im}^+_{c_2,\mathbf{t}} \cap Z$ and $\mathrm{Im}^-_{c_3,\mathbf{t}} \cap Z \subseteq X \nsupseteq \mathrm{Im}^+_{c_3,\mathbf{t}} \cap Z$. Same for kernels.

Next, we will prove that the 3-parameter persistence module $\mathbb{M}$, which is **pfd** and satisfies the 3-parameter strong exactness, is the direct sum of block modules.

Before proving the main theorem, we need to redivide blocks.

- $\mathcal{B}_1 = \{B | {c_2}^- = {c^2}^+ = {c_3}^- = {c^3}^+ = \emptyset$ and ${c_1}^- \neq \emptyset \neq {c^1}^+\}$;
- $\mathcal{B}_2 = \{B | {c_1}^- = {c^1}^+ = {c_3}^- = {c^3}^+ = \emptyset$ and ${c_2}^- \neq \emptyset \neq {c^2}^+\}$;
- $\mathcal{B}_3 = \{B | {c_1}^- = {c^1}^+ = {c_2}^- = {c^2}^+ = \emptyset$ and ${c_3}^- \neq \emptyset \neq {c^3}^+\}$;
- $\mathcal{B}_4 =$ the set of all death blocks $\setminus \{\mathbb{R}^n\}$
- $\mathcal{B}_5 =$ the set of all birth blocks

To prove that the direct sum decomposition of $\mathbb{M}$, we need to define a new 3-parameter persistence module $\tilde{\mathbb{M}} : (\mathbb{R}^3, \leq) \to \mathbf{Vec}_{\mathbb{k}}$, which is a submodule of $\mathbb{M}$, defined as $\tilde{\mathbb{M}}_{\mathbf{t}} := F^-_{\mathbb{R}^3,\mathbf{t}} = V^-_{\mathbb{R}^3,\mathbf{t}} = \mathrm{Im}^+_{\mathbb{R}^3,\mathbf{t}} \cap \mathrm{Ker}^-_{\mathbb{R}^3,\mathbf{t}}$. Let $X = \tilde{\mathbb{M}}_{\mathbf{t}} + \sum_{B:\text{birth and layer}} (\mathbb{M}_B)_{\mathbf{t}}$. Based on the definition of $\tilde{\mathbb{M}}$, it is natural to conjecture that the submodule $\tilde{\mathbb{M}}$ is spanned by the block modules corresponding to with death blocks that are proper subsets of $\mathbb{R}^3$.

Proposition 5.4: $\mathbb{M} = \tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_5} \mathbb{M}_B$.

Proof: Given a fixed $\mathbf{t} \in \mathbb{R}^3$, let $X = \tilde{\mathbb{M}}_{\mathbf{t}} + \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_5} (\mathbb{M}_B)_{\mathbf{t}}$. Suppose for a contraction that $X \subsetneq \mathbb{M}_{\mathbf{t}}$. Then apply Lemma5.16 with $Z = \mathbb{M}_{\mathbf{t}}$ to get a cut c_1 such that $t_1 \in {c_1}^+$ and $\mathrm{Im}^-_{c_1,\mathbf{t}} \subseteq X \nsupseteq \mathrm{Im}^+_{c_1,\mathbf{t}}$. Again, use Lemma5.16 with $Z = \mathrm{Im}^+_{c_1,\mathbf{t}}$ to get a cut c_2 such that $t_2 \in {c_2}^+$ and $\mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_2,\mathbf{t}} \subseteq X \nsupseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}}$. Again, use Lemma5.16 with $Z = \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}}$ to find a cut c_3 so that $t_3 \in {c_3}^+$ and $\mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^-_{c_3,\mathbf{t}} \subseteq X \nsupseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}$.

If ${c_1}^- = {c_2}^- = {c_3}^- = \emptyset$, then

$$\mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} = \mathrm{Im}^+_{\mathbb{R}^3,\mathbf{t}} = F^+_{\mathbb{R}^3,\mathbf{t}} = F^-_{\mathbb{R}^3,\mathbf{t}} + (\mathbb{M}_{\mathbb{R}^3})_{\mathbf{t}} = \tilde{\mathbb{M}}_{\mathbf{t}} + (\mathbb{M}_{\mathbb{R}^3})_{\mathbf{t}} \subseteq X.$$

However, our selection of c_1, c_2, c_3 ensures that $\mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \nsubseteq X$. This is a contradiction. Thus ${c_1}^- \neq \emptyset$ or ${c_2}^- \neq \emptyset$ or ${c_3}^- \neq \emptyset$.

We distinguish these cases below: These cases are divided as follows:

Case ${c_1}^- \neq \emptyset, {c_2}^- \neq \emptyset, {c_3}^- \neq \emptyset$. Let the block $B = {c_1}^+ \times {c_2}^+ \times {c_3}^+$. We have

$\mathrm{Im}^+_{B,\mathbf{t}} = \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} = F^+_{B,\mathbf{t}} \nsubseteq X$. But

$$\begin{aligned} F^-_{B,\mathbf{t}} &= \mathrm{Im}^-_{B,\mathbf{t}} + \mathrm{Ker}^-_{B,\mathbf{t}} \cap \mathrm{Im}^+_{B,\mathbf{t}} \\ &\subseteq \mathrm{Im}^-_{B,\mathbf{t}} + (\mathrm{Im}^-_{c_2,\mathbf{t}} \cap \mathrm{Im}^-_{c_3,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_3,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_2,\mathbf{t}}) \cap \mathrm{Im}^+_{B,\mathbf{t}} \\ &\subseteq \mathrm{Im}^-_{B,\mathbf{t}} \subseteq \mathrm{Im}^-_{c_1,\mathbf{t}} + \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_2,\mathbf{t}} + \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^-_{c_3,\mathbf{t}} \subseteq X \end{aligned}$$

by Lemma5.4. Note that $F^+_{B,\mathbf{t}} = F^-_{B,\mathbf{t}} \oplus (\mathbb{M}_B)_\mathbf{t}$. Then we get a contradiction, $(\mathbb{M}_B)_\mathbf{t} \nsubseteq X$.

Case ${c_1}^- \neq \emptyset, {c_2}^- \neq \emptyset, {c_3}^- = \emptyset$. Let the block $B = {c_1}^+ \times {c_2}^+ \times {c_3}^+$. We have $\mathrm{Im}^+_{B,\mathbf{t}} = \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} = F^+_{B,\mathbf{t}} \nsubseteq X$, but $F^-_{B,\mathbf{t}} = \mathrm{Im}^-_{B,\mathbf{t}} + \mathrm{Ker}^-_{B,\mathbf{t}} \cap \mathrm{Im}^+_{B,\mathbf{t}} \subseteq \mathrm{Im}^-_{B,\mathbf{t}} + (\mathrm{Im}^-_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^-_{c_2,\mathbf{t}}) \cap \mathrm{Im}^+_{B,\mathbf{t}} \subseteq \mathrm{Im}^-_{B,\mathbf{t}} \subseteq X$ by Lemma5.4. Note that $F^+_{B,\mathbf{t}} = F^-_{B,\mathbf{t}} \oplus (\mathbb{M}_B)_\mathbf{t}$. Thus, $(\mathbb{M}_B)_\mathbf{t} \nsubseteq X$. This is a contradiction.

Similarly, we can prove these cases that ${c_1}^- \neq \emptyset, {c_2}^- = \emptyset, {c_3}^- \neq \emptyset$ and ${c_1}^- = \emptyset, {c_2}^- \neq \emptyset, {c_3}^- \neq \emptyset$.

Case ${c_1}^- \neq \emptyset, {c_2}^- = \emptyset, {c_3}^- = \emptyset$. By Lemma5.16, applied with $Z = \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}$, there is a cut c^1 such that $\mathbf{t} \in {c^1}^-$ and

$$\mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap \mathrm{Ker}^-_{c^1,\mathbf{t}} \subseteq X \nsupseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap \mathrm{Ker}^+_{c^1,\mathbf{t}}.$$

Let the block $B = ({c_1}^+ \cap {c^1}^-) \times c_2 \times c_3$. Using Lemma5.4, we have $\mathrm{Ker}^+_{c^1,\mathbf{t}} \subseteq \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}$, $\mathrm{Ker}^-_{c^2,\mathbf{t}} \subseteq \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}}$ and $\mathrm{Ker}^-_{c^3,\mathbf{t}} \subseteq \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}}$. Then

$$\mathrm{Im}^-_{B,\mathbf{t}} = \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \subseteq X$$

$$\mathrm{Ker}^+_{B,\mathbf{t}} \cap \mathrm{Im}^+_{B,\mathbf{t}} \supseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap \mathrm{Ker}^+_{c^1,\mathbf{t}} \nsubseteq X$$

$$\begin{aligned} \mathrm{Im}^+_{B,\mathbf{t}} \cap \mathrm{Ker}^-_{B,\mathbf{t}} &= \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap (\mathrm{Ker}^-_{c^1,\mathbf{t}} + \mathrm{Ker}^-_{c^2,\mathbf{t}} + \mathrm{Ker}^-_{c^3,\mathbf{t}}) \\ &\subseteq \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap (\mathrm{Ker}^-_{c^1,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}}) \\ &= \mathrm{Im}^+_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \cap \mathrm{Ker}^-_{c^1,\mathbf{t}} + \mathrm{Im}^-_{c_1,\mathbf{t}} \cap \mathrm{Im}^+_{c_2,\mathbf{t}} \cap \mathrm{Im}^+_{c_3,\mathbf{t}} \subseteq X \end{aligned}$$

Thus, $F^-_{B,\mathbf{t}} \subseteq X \nsupseteq F^+_{B,\mathbf{t}}$. Hence, $(\mathbb{M}_B)_\mathbf{t} \nsubseteq X$. This is a contradiction.

Similarly, we can prove these cases that ${c_1}^- = \emptyset, {c_2}^- \neq \emptyset, {c_3}^- = \emptyset$ and ${c_1}^- = \emptyset, {c_2}^- = \emptyset, {c_3}^- \neq \emptyset$. ■

Lemma 5.17: $(\tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3} \mathbb{M}_B) + \bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B = (\tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3} \mathbb{M}_B) \oplus \bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B$

Proof: Assume the opposite, and let $\mathbf{t} \in \mathbb{R}^3$ so that $(\tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3} \mathbb{M}_B)_\mathbf{t} \cap (\bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B)_\mathbf{t} \neq \emptyset$. Then there exist $\alpha \in \tilde{\mathbb{M}}_\mathbf{t}$, $\alpha_1 \in (\mathbb{M}_{B_1})_\mathbf{t}, \cdots, \alpha_r \in (\mathbb{M}_{B_r})_\mathbf{t}$ and $\alpha_{r+1} \in (\mathbb{M}_{B_{r+1}})_\mathbf{t}, \cdots, \alpha_n \in (\mathbb{M}_{B_n})_\mathbf{t}$, such that $B_1, \cdots, B_r$ are in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, $B_{r+1}, \cdots, B_n$ are in $\mathcal{B}_5$, and we have

$$\alpha + \sum_{i=1}^{r} \alpha_i = \sum_{j=r+1}^{n} \alpha_j \neq 0.$$

Because of the shape of these blocks in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, we may find out some $\mathbf{u} \geq \mathbf{t}$ such that $\mathbf{u} \notin \bigcup_{i=1}^{r} B_i$. What's more, since $\alpha \in \tilde{\mathbb{M}}_{\mathbf{t}} = \mathrm{Im}^{+}_{\mathbb{R}^3,\mathbf{t}} \cap \mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{t}} \subseteq \mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{t}} = \mathrm{Ker}^{-}_{c^1,\mathbf{t}} + \mathrm{Ker}^{-}_{c^2,\mathbf{t}} + \mathrm{Ker}^{-}_{c^3,\mathbf{t}}$, we have $\alpha = \alpha_1' + \alpha_2' + \alpha_3'$ for some $\alpha_1' \in \mathrm{Ker}^{-}_{c_1,\mathbf{t}}$, $\alpha_2' \in \mathrm{Ker}^{-}_{c_2,\mathbf{t}}$ and $\alpha_3' \in \mathrm{Ker}^{-}_{c_3,\mathbf{t}}$. By Lemma5.2, there are finite coordinates $x \geq t_1$, $y \geq t_2$ and $z \geq t_3$ such that $\alpha_1' \in \mathrm{Ker}\ \rho_{\mathbf{t}}^{(x,t_2,t_3)}$, $\alpha_2' \in \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,y,t_3)}$ and $\alpha_3' \in \mathrm{Ker}\ \rho_{\mathbf{t}}^{(t_1,t_2,z)}$. Let $\mathbf{v}$ be a point with coordinates $(\max\{u_1,x\}, \max\{u_2,y\}, \max\{u_3,z\})$. Then we obtain

$$\rho_{\mathbf{t}}^{\mathbf{v}}(\alpha + \sum_{i=1}^{r} \alpha_i) = 0.$$

However, because $\rho_{\mathbf{t}}^{\mathbf{v}}$ restricted to $\oplus_{r+1}^{n}(\mathbb{M}_{B_i})_r$ is injective, we have $\rho_{\mathbf{t}}^{\mathbf{v}}(\sum_{i=r+1}^{n})\alpha_i \neq 0$. This is a contradiction. Thus $(\tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3} \mathbb{M}_B) \cap \bigoplus_{B:\mathcal{B}_5} \mathbb{M}_B = 0$. ■

Lemma 5.18: $\tilde{\mathbb{M}} + \bigoplus_{B:\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3} \mathbb{M}_B = \tilde{\mathbb{M}} \oplus \bigoplus_{B:\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3} \mathbb{M}_B$

Proof: Assume the opposite, and let $\mathbf{t} \in \mathbb{R}^3$ so that $(\tilde{\mathbb{M}})_{\mathbf{t}} \cap (\bigoplus_{B:\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3} \mathbb{M}_B)_{\mathbf{t}} \neq \emptyset$. Then there exist $\alpha \in \tilde{\mathbb{M}}_{\mathbf{t}}$, $\alpha_i \in (\mathbb{M}_{B_i})_{\mathbf{t}}$ with $i = 1,2,\cdots,n$ such that $B_1, B_2, \cdots, B_n$ are in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and

$$\alpha = \sum_{i}^{n} \alpha_i \neq 0.$$

Assume that $B_1,\cdots,B_k$ are in $\mathcal{B}_1$, $B_{k+1},\cdots,B_r$ are in $\mathcal{B}_2$ and $B_{r+1},\cdots,B_n$ are in $\mathcal{B}_3$. And assume that none of the α_i's are zero. Because of the shape of these blocks, we may find a point $\mathbf{u} = (u_1,u_2,u_3) = (x,y,t_3) \in \mathbb{R}^3$ such that $\mathbf{u} \notin \bigcup_{i=k+1}^{n} B_i$, then $\rho_{\mathbf{t}}^{\mathbf{u}}(\sum_{i=k+1}^{n} \alpha_i) = 0$. Since the restriction of $\rho_{\mathbf{t}}^{\mathbf{u}}$ to $\oplus_{i=1}^{k}(\mathbb{M}_{B_i})_{\mathbf{t}}$ is injective, $\rho_t^u(\sum_{i=1}^{k} \alpha_i) \neq 0$.

Let $\beta = \rho_{\mathbf{t}}^{\mathbf{u}}(\alpha) \in \tilde{\mathbb{M}}_{\mathbf{u}}$ and $\beta_i = \rho_{\mathbf{t}}^{\mathbf{u}}(\alpha_i) \in (\mathbb{M}_{B_i})_{\mathbf{u}}$ for $i = 1,\cdots,k$

$$\beta = \sum_{i=1}^{k} \beta_i \neq 0.$$

Now, we have $\tilde{\mathbb{M}}_{\mathbf{u}} \subseteq \mathrm{Im}^{+}_{\mathbb{R}^3,\mathbf{u}} = F^{+}_{\mathbb{R}^3,\mathbf{u}}$. From the proof of Lemma5.2, the collection of sections $\{(F^{-}_{B_1,\mathbf{u}}, F^{+}_{B_1,\mathbf{u}}), \cdots, (F^{-}_{B_k,\mathbf{u}}, F^{+}_{B_k,\mathbf{u}})\}$ is disjoint. Note that $F^{+}_{\mathbb{R}^3,\mathbf{u}} \subset F^{-}_{B_i,\mathbf{u}}$ for every i, then the collection of sections $\{(0, F^{+}_{\mathbb{R}^3,\mathbf{u}}), (F^{-}_{B_1,\mathbf{u}}, F^{+}_{B_1,\mathbf{u}}), \cdots, (F^{-}_{B_k,\mathbf{u}}, F^{+}_{B_k,\mathbf{u}})\}$ is disjoint. Then according to Lemma5.10, $F^{+}_{\mathbb{R}^3,\mathbf{u}}$ is in direct sum with $\oplus_{i=1}^{k}(\mathbb{M}_{B_i})_{\mathbf{u}}$. Because $\tilde{\mathbb{M}}_u$ is a subspace of $F^{+}_{\mathbb{R}^3,\mathbf{u}}$ and $F^{+}_{\mathbb{R}^3,\mathbf{u}}$ is in direct sum with $\oplus_{i=1}^{k}(\mathbb{M}_{B_i})_{\mathbf{u}}$, the result contradicts

$\beta = \sum_{i=1}^{r} \beta_i \neq 0$. ■

Corollary 5.3: $\mathbb{M} = \tilde{\mathbb{M}} \oplus \bigoplus_{B:\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_5} \mathbb{M}_B$

Through the above discussion, we have extracted all the block submodules of $\mathbb{M}$ except for dead blocks and proved that they are in direct sum. To prove the main theorem, we only need to prove that the submodule $\tilde{\mathbb{M}}$ can also be decomposed as the direct sum of block modules.

However, we do not directly decompose $\tilde{\mathbb{M}}$ but rather decompose the duality $\tilde{\mathbb{M}}^*$. Let $\tilde{\mathbb{M}}^*$ be the pointwise dual of $\tilde{\mathbb{M}}$, that is $(\tilde{\mathbb{M}}^*)_{\mathbf{t}} = Hom(\tilde{\mathbb{M}}_{\mathbf{t}}, \mathbb{k})$. Since the duality is a contravariant functor, $\tilde{\mathbb{M}}^* : ((\mathbb{R}^{op})^3, \geq) \to \mathbf{Vec}_{\mathbb{k}}$ is a persistence module, where $\mathbb{R}^{op}$ denoted the poset $\mathbb{R}$ with the opposite order $\geq$.

So we need the following result.

Lemma 5.19: $\tilde{\mathbb{M}}^*$ is **pfd** and satisfies the 3-parameter strong exactness.

Proof: Our proof is mainly divided into two parts. The first part is to prove that for any $r \in \mathbb{R}$, $\tilde{\mathbb{M}}_{\{r\}\times\mathbb{R}\times\mathbb{R}}$ satisfies 2-parameter strong exactness, and the proof method for $\tilde{\mathbb{M}}|_{\mathbb{R}\times\{r\}\times\mathbb{R}}$ and $\tilde{\mathbb{M}}|_{\mathbb{R}\times\mathbb{R}\times\{r\}}$ are similar. The second part is to prove that for any $(s_1, s_2, s_3) \leq (t_1, t_2, t_3) \in \mathbb{R}^3$, the morphism φ and ψ associated with the persistence module $\tilde{\mathbb{M}}$ are injective and surjective respectively.

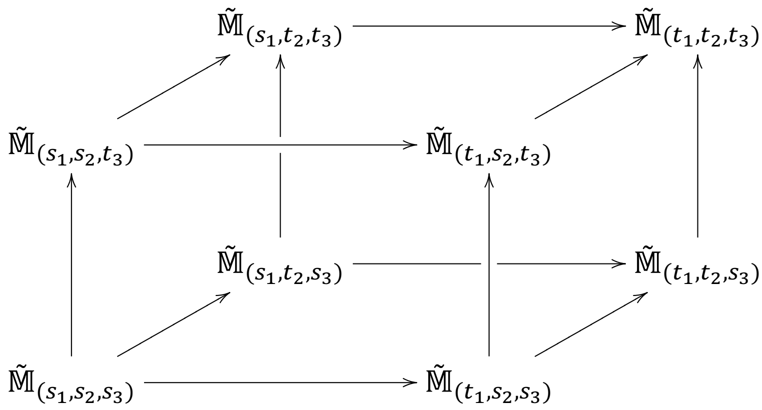
Obviously, $\tilde{\mathbb{M}}$ is **pfd**, then $\tilde{\mathbb{M}}^*$ is **pfd**.

$$\begin{array}{ccc} N_{(r,s_2,t_3)} & \longrightarrow & N_{(r,t_2,t_3)} \\ \uparrow & & \uparrow \\ N_{(r,s_2,s_3)} & \longrightarrow & N_{(r,t_2,s_3)} \end{array}$$

Firstly, let $(r, s_2, s_3) \leq (r, t_2, t_3) \in \mathbb{R}^3$ and take an element $\delta \in \tilde{\mathbb{M}}_{(r,t_2,t_3)}$ that has preimages $\beta \in \tilde{\mathbb{M}}_{(r,t_2,s_3)}$ and $\gamma \in \tilde{\mathbb{M}}_{(r,s_2,t_3)}$. Then, by the 3-parameter strong exactness of $\mathbb{M}$, β and γ have a shared preimage $\alpha \in \mathbb{M}_{(r,s_2,s_3)}$. Indeed, we can prove that $\alpha \in \tilde{\mathbb{M}}_{(r,s_2,s_3)}$. Obviously, we know that $\alpha \in (\rho_{(r,s_2,s_3)}^{(r,t_2,t_3)})^{-1}\tilde{\mathbb{M}}_{(r,t_2,t_3)} \subseteq (\rho_{(r,s_2,s_3)}^{(r,t_2,t_3)})^{-1}(\mathrm{Ker}^{-}_{\mathbb{R}^3,(r,t_2,t_3)}) = \mathrm{Ker}^{-}_{\mathbb{R}^3,(r,s_2,s_3)}$. What's more, because of $\beta \in \tilde{\mathbb{M}}_{(r,t_2,s_3)} \subseteq \mathrm{Im}^{+}_{\mathbb{R}^3,(r,t_2,s_3)}$, for any $\mathbf{u} < (r, t_2, s_3) \in \mathbb{R}^3$ with $u_1 = r$ and $u_2 = t_2$ there is some preimage $\beta_{\mathbf{u}}$ of β in $\mathbb{M}_{(r,t_2,u_3)}$ by the 3-parameter strong exactness, implies that there exists a shared preimage $\alpha_{\mathbf{u}}$ of α and $\beta_{\mathbf{u}}$ in $\mathbb{M}_{\mathbf{u}}$. Thus $\alpha \in \mathrm{Im}^{+}_{c_3,(r,s_2,s_3)}$, where c_3 is the trivial cut that is $c_3{}^{-} = \emptyset$. Similarly, we can know that $\alpha \in \mathrm{Im}^{+}_{c_1,(r,s_2,s_3)}$ and $\alpha \in \mathrm{Im}^{+}_{c_2,(r,s_2,s_3)}$, in which $c_1{}^{-} = c_2{}^{-} = \emptyset$. So $\alpha \in \mathrm{Im}^{+}_{\mathbb{R}^3,(r,s_2,s_3)}$, and therefore $\alpha \in \tilde{\mathbb{M}}_{(r,s_2,s_3)}$.

In other words, $\tilde{\mathbb{M}}|_{\{r\}\times\mathbb{R}\times\mathbb{R}}$ satisfies the 2-parameter strong exactness. Thus, $\tilde{\mathbb{M}}^*|_{\{r\}\times\mathbb{R}^{op}\times\mathbb{R}^{op}}$ satisfies the 2-parameter strong exactness[31].

Secondly, for any $(s_1, s_2, s_3) \leq (t_1, t_2, t_3) \in \mathbb{R}^3$, we get a commutative diagram



We will denote it as $\mathcal{X} : \mathcal{P}(S) \to \mathbf{Vec}_{\mathbb{k}}$ that S is a set with $|S| = 3$, and get the morphism $\varphi : \operatorname{colim}_{T\in\mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$ and morphism $\psi : \mathcal{X}(\emptyset) \to \lim_{T\in\mathcal{P}_0(S)} \mathcal{X}(T)$.

Note that for any $\mathbf{s} \leq \mathbf{t} \in \mathbb{R}^3$, we have $\tilde{\mathbb{M}}_{\mathbf{t}} = \mathrm{Im}^+_{\mathbb{R}^3,\mathbf{t}} \cap \mathrm{Ker}^-_{\mathbb{R}^3,\mathbf{t}}$, $\rho^{\mathbf{t}}_{\mathbf{s}}(\mathrm{Im}^+_{\mathbb{R},\mathbf{s}}) = \mathrm{Im}^+_{\mathbb{R},\mathbf{t}}$ and $(\rho^{\mathbf{t}}_{\mathbf{s}})^{-1}(\mathrm{Ker}^-_{\mathbb{R}^3,\mathbf{t}}) = \mathrm{Ker}^-_{\mathbb{R}^3,\mathbf{s}}$. Thus for any $\alpha \in \tilde{\mathbb{M}}_{\mathbf{t}}$, we always can find out some $\beta \in \tilde{\mathbb{M}}_{\mathbf{s}}$ such that $\rho^{\mathbf{t}}_{\mathbf{s}}(\beta) = \alpha$. Given $\operatorname{colim}_{T\in\mathcal{P}_1(S)} \mathcal{X}(T) = \tilde{\mathbb{M}}_{(s_1,t_2,t_3)} \oplus \tilde{\mathbb{M}}_{(t_1,s_2,t_3)} \oplus \tilde{\mathbb{M}}_{(t_1,t_2,s_3)}/\sim$. Then for any $[\alpha + \beta + \gamma] \in \operatorname{colim}_{T\in\mathcal{P}_1(S)} \mathcal{X}(T)$ satisfying $\varphi([\alpha + \beta + \gamma]) = 0$ in which $\alpha \in \tilde{\mathbb{M}}_{(s_1,t_2,t_3)}$, $\beta \in \tilde{\mathbb{M}}_{(t_1,s_2,t_3)}$ and $\gamma \in \tilde{\mathbb{M}}_{(t_1,t_2,s_3)}$, we may find out some $\tilde{\gamma} \in \tilde{\mathbb{M}}_{(t_1,t_2,s_3)}$ such that $[\alpha + \beta + \gamma] = [\tilde{\gamma}]$. Since $\varphi([\tilde{\gamma}]) = 0$, then $\rho^{(t_1,t_2,t_3)}_{(t_1,t_2,s_3)}(\tilde{\gamma}) = 0$. Therefore, we can find out some common preimage of $\tilde{\gamma}$ and $0 \in \tilde{\mathbb{M}}_{(t_1,s_2,t_3)}$, then $[\alpha + \beta + \gamma] = [\tilde{\gamma}] = 0 \in \operatorname{colim}_{T\in\mathcal{P}_1(S)} \mathcal{X}(T)$. Thus, φ is injective. Obviously, φ^* is surjective.

To proving that ψ is surjective, suppose $\alpha_1 \in \tilde{\mathbb{M}}_{(t_1,s_2,s_3)}, \alpha_2 \in \tilde{\mathbb{M}}_{(s_1,t_2,s_3)}, \alpha_3 \in \tilde{\mathbb{M}}_{(s_1,s_2,t_3)}$. Because of the 3-parameter strong exactness of $\mathbb{M}$, we may find out $\alpha \in \mathbb{M}_{(s_1,s_2,s_3)}$ such that α is the common preimage of $\alpha_1, \alpha_2, \alpha_3$. Given that

$$\alpha_1 \in \tilde{\mathbb{M}}_{(t_1,s_2,s_3)} = \mathrm{Im}^+_{\mathbb{R}^3,(t_1,s_2,s_3)} \cap \mathrm{Ker}^-_{\mathbb{R}^3,(t_1,s_2,s_3)},$$
$${\rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}}^{-1} \mathrm{Ker}^-_{\mathbb{R}^3,(t_1,s_2,s_3)} = \mathrm{Ker}^-_{\mathbb{R}^3,(s_1,s_2,s_3)},$$
$$\mathrm{Im}^+_{\mathbb{R}^3,(t_1,s_2,s_3)} = \mathrm{Im}^+_{c_1,(t_1,s_2,s_3)} \cap \mathrm{Im}^+_{c_2,(t_1,s_2,s_3)} \cap \mathrm{Im}^+_{c_3,(t_1,s_2,s_3)}.$$

Because of Lemma5.2, we may prove that ${\rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}}^{-1} \mathrm{Im}^+_{c_2,(t_1,s_2,s_3)} = \mathrm{Im}^+_{c_2,(s_1,s_2,s_3)}$. Lemma5.2 told us that there exists some $y \leq s_2$ such that

$$\mathrm{Im}^+_{c_2,(t_1,s_2,s_3)} = \rho^{(t_1,s_2,s_3)}_{(t_1,y,s_3)} \mathbb{M}_{(t_1,y,s_3)},$$
$$\mathrm{Im}^+_{c_2,(s_1,s_2,s_3)} = \rho^{(s_1,s_2,s_3)}_{(s_1,y,s_3)} \mathbb{M}_{(s_1,y,s_3)}.$$

For any $\beta \in {\rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}}^{-1} \mathrm{Im}^{+}_{c_2,(t_1,s_2,s_3)}$, we may find out $\gamma \in \mathbb{M}_{(t_1,y,s_3)}$ such that $\rho^{(s_1,s_2,s_3)}_{(t_1,y,s_3)}(\gamma) = \rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}(\beta)$. By the 2-parameter strong exactness, there is a preimage $\delta \in \mathbb{M}_{(s_1,y,s_3)}$ of β and γ, then $\beta \in \rho^{(s_1,s_2,s_3)}_{(s_1,y,s_3)} M_{(s_1,y,s_3)} = \mathrm{Im}^{+}_{c_2,(s_1,s_2,s_3)}$. So we have proven ${\rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}}^{-1} \mathrm{Im}^{+}_{c_2,(t_1,s_2,s_3)} = \mathrm{Im}^{+}_{c_2,(s_1,s_2,s_3)}$. Similarly, we can prove that ${\rho^{(t_1,s_2,s_3)}_{(s_1,s_2,s_3)}}^{-1} \mathrm{Im}^{+}_{c_3,(t_1,s_2,s_3)} = \mathrm{Im}^{+}_{c_3,(s_1,s_2,s_3)}$, then $\alpha \in \mathrm{Im}^{+}_{c_2,(s_1,s_2,s_3)} \cap \mathrm{Im}^{+}_{c_3,(s_1,s_2,s_3)}$. In the same way, by considering α as a preimage of α_2 and α_3 respectively, we can prove that $\alpha \in \mathrm{Im}^{+}_{c_1,(s_1,s_2,s_3)} \cap \mathrm{Im}^{+}_{c_3,(s_1,s_2,s_3)}$ and $\alpha \in \mathrm{Im}^{+}_{c_1,(s_1,s_2,s_3)} \cap \mathrm{Im}^{+}_{c_2,(s_1,s_2,s_3)}$. We have proven that $\alpha \in \mathrm{Im}^{+}_{c_1,(s_1,s_2,s_3)} \cap \mathrm{Im}^{+}_{c_2,(s_1,s_2,s_3)} \cap \mathrm{Im}^{+}_{c_3,(s_1,s_2,s_3)} = \mathrm{Im}^{+}_{\mathbb{R}^3,(s_1,s_2,s_3)}$. Thus $\alpha \in \mathrm{Im}^{+}_{\mathbb{R}^3,\mathbf{s}} \cap \mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{s}} = \tilde{\mathbb{M}}_{\mathbf{s}}$, and the morphism ψ is surjective. Obviously, the duality of ψ, ψ^*, is injective.

So $\tilde{\mathbb{M}}^*$ satisfies the 3-parameter strong exactness.

■

By the above lemma, we know that $\tilde{\mathbb{M}}^*$ can also be decomposed like the above decomposition of $\mathbb{M}$.

Lemma 5.20: For any $\mathbf{t} \in (\mathbb{R}^{op})^3$, $\mathrm{Im}^{+}_{(\mathbb{R}^{op})^3,\mathbf{t}}(\tilde{\mathbb{M}}^*) = 0$.

Proof: Let $X^{\perp}$ denote the annihilator of any subspace $X \subseteq \tilde{\mathbb{M}}_{\mathbf{t}}$:

$$X^{\perp} = \{\phi(\alpha) = 0 \text{ for all } \alpha \in X\}.$$

Because the annihilator operation transforms sums into intersections and kernels into images, then

$$\begin{aligned}(\mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{t}}(\tilde{\mathbb{M}}))^{\perp} &= (\mathrm{Ker}^{-}_{c^1,\mathbf{t}}(\tilde{\mathbb{M}}) + \mathrm{Ker}^{-}_{c^2,\mathbf{t}}(\tilde{\mathbb{M}}) + \mathrm{Ker}^{-}_{c^3,\mathbf{t}}(\tilde{\mathbb{M}}))^{\perp} \\ &= \mathrm{Im}^{+}_{c^1,\mathbf{t}}(\tilde{\mathbb{M}}^*) \cap \mathrm{Im}^{+}_{c^2,\mathbf{t}}(\tilde{\mathbb{M}}^*) \cap \mathrm{Im}^{+}_{c^3,\mathbf{t}}(\tilde{\mathbb{M}}^*) = \mathrm{Im}^{+}_{(\mathbb{R}^{op})^3,\mathbf{t}}(\tilde{\mathbb{M}}^*)\end{aligned}$$

Note that $\tilde{\mathbb{M}}_{\mathbf{t}} = \mathrm{Im}^{+}_{\mathbb{R}^3,\mathbf{t}}(\mathbb{M}) \cap \mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{t}}(\mathbb{M})$, so $\mathrm{Ker}^{-}_{\mathbb{R}^3,\mathbf{t}}(\tilde{\mathbb{M}}) = \tilde{\mathbb{M}}_{\mathbf{t}}$. Thus $\mathrm{Im}^{+}_{(\mathbb{R}^{op})^3,\mathbf{t}}(\tilde{\mathbb{M}}^*) = (\tilde{\mathbb{M}}_{\mathbf{t}})^{\perp} = 0$. ■

Based on the previous results, we know that the module $\tilde{\mathbb{M}}^*$ can be decomposed into the direct sum of block modules, which are of the type $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_5$. Then the submodule $\tilde{\mathbb{M}}$ can be decomposed into the direct sum of block modules, which are the type of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.

Thus, we have proved our main theorem.

Theorem 5.2: $\mathbb{M} \cong \bigoplus_{B:\text{block}} \mathbb{M}_B$ in which $M_B \cong \oplus_{i=1}^{n_B} \mathbb{k}_B$ in which n_B are determined by the counting functor $\mathcal{CF}$, i.e. Corollary5.2.

CONCLUSION

In this chapter, we will review the main contributions of our work on persistence modules. Our research addresses two central topics in persistence modules: decomposition and stability. These results can advance our understanding of persistent modules.

Our main contributions are summarized below:

(1) Using the language of category theory, we have reformulated the strong exactness condition for 2-parameter persistence modules, which allows us to effortlessly generalize the strong exactness condition for 2-parameter persistence modules to the 3-parameter case.

(2) Prove the block-decomposition theorem of 3-parameter persistence modules $\mathbb{M} : \mathbb{R}^3 \to \mathbf{Vec}_{\mathbb{k}}$. This laid the foundation for our future research on the rectangle decomposition of 3-parameter persistence modules.

(3) Define the persistence minimal free Lie model $M_{Qui}(\mathbb{X})$ for any rational $\mathbb{R}$-space $\mathbb{X} : \mathbb{R} \to \mathbf{Top}_{\mathbb{Q}}$ and prove the existence of persistence minimal free Lie models. This result indicates that we can consider more algebraic models for $\mathbb{R}$-spaces, and such algebraic models are also persistent modules, which are more refined than the algebraic models obtained by directly computing the homology or homotopy groups of these spaces. Moreover, as demonstrated by the examples we provided, we can concretely construct persistence free Lie models for some rational $\mathbb{R}$-spaces.

(4) Discuss and prove the stability of persistence free Lie models.

We still have some issues that we haven't discussed yet. Botnan et al.[15] have proposed and demonstrated the necessary and sufficient conditions for the rectangle decomposition of 2-parameter persistence modules. However, we still do not know the necessary and sufficient conditions for the rectangle decomposition of higher-dimensional persistence modules, and even the 3-parameter case. Therefore, our next step is to investigate the rectangle decomposition of 3-parameter persistence modules and attempt to extend this research to the case of n-parameter with any $n \geq 3$.

On the other hand, for rational $\mathbb{R}$-space $\mathbb{X} : \mathbb{R} \to \mathbf{Top}_{\mathbb{Q}}$, we have defined and proven the existence of the persistence minimal free Lie model $M_{Qui}(\mathbb{X}) : \mathbb{R} \to \mathbf{DGL}$. However, we are aware that L_∞-algebras are algebraic models that are closer to homotopy than Lie algebras. We aim to attempt the construction of a persistence L_∞ model for rational $\mathbb{R}$-

spaces and to discuss its stability.

Moreover, regarding the decomposition problem of multi-parameter persistence modules, since the direct sum decomposition is a special case of filtration, some scholars believe that we can consider the filtration of persistence modules like how we consider the filtration of topological spaces. By doing so, we hope to obtain decomposition theorems for persistence modules, thereby deriving discrete invariants of persistence modules.

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

Yi Siheng was born in August 1994, in Leshan, Sichuan, China.

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- Electric Energy Trading Center (under establishment) of China Three Gorges Corporation, Intern (June 2019 - July 2019).
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- Siheng Yi. Block-Decomposition for 3-Parameter Persistence Modules. arXiv:2505.08391 [math.AT], 2025.
- Siheng Yi. Persistence Minimal Free Lie Model. arXiv:2505.08373 [math.AT], 2025.

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