博士学位论文

$K(n-1)$ 局部化的 Morava E-理论上的幂运算

POWER OPERATIONS ON $K(n-1)$ LOCALIZED MORAVA E-THEORY

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摘 要

自从周期现象在某些类型空间或谱的同伦群计算中显现以来，色展同伦理论已得到长足发展。该理论在代数拓扑对象（复定向谱）与代数几何对象（形式群）之间架起了桥梁。色展同伦理论的核心研究对象是 Morava K-理论和 Morava E-理论，它们分别对应高度 n 的形式群及其形变。另一方面，上同调运算（如 Steenrod 运算、Adams 运算）是代数拓扑研究的重要工具，其基础而简洁的性质揭示了代数拓扑中许多高度非平凡的结果，这类运算都可看成幂运算的特殊形式。

Ando、Hopkins 和 Strickland 对 Morava E-理论的幂运算进行了系统的研究，他们的工作建立了 Morava E-理论全幂运算与带等级结构的形式群形变之间的深刻联系。特别地，Strickland 计算了对称群 $B\Sigma_k$ 的 Morava E-理论环 $E^*B\Sigma_k$，证明其对偶是对称群的完全 E-同调群，由此可计算 $K(n)$-局部 E-代数上的 Dyer-Lashof 代数。然而 E-代数不一定总是 $K(n)$-局部的。

基于此，我们研究了 $K(n-1)$-局部 E-代数（等价于 $K(n-1)$-局部 $L_{K(n-1)}E_n$ -代数）。本论文计算了对称群在 $L_{K(n-1)}E_n$ 系数下的上同调环，证明其在 $\pi_*L_{K(n-1)}E_n$ 上是自由的。进而将 $L_{K(n-1)}E_n$ 上的全幂运算与不完美域（$k((x))$）上形式群的增广形变与其上有限子群结构联系起来。通过 Tate 上同调理论，我们证明了对称群 $L_{K(n-1)}E_n$-上同调与相应完备同调间的对偶关系，从而将 $K(n-1)$-局部 E-代数上的 Dyer-Lashof 代数实现为 $L_{K(n-1)}E_n^*B\Sigma_k$ 的对偶。

在高度 n=2 情形下，我们给出了 $L_{K(1)}E_2$ 全幂运算的具体公式，并解释了该计算与椭圆曲线的联系。简言之，该全幂运算由万有 p 次同源目标曲线的 Hasse 不变量所表示；从模形式视角看，它是来源曲线 Hasse 不变量经 Atkin-Lehner 对合映射的像。

关键词：代数拓扑；形式群；增广形变；椭圆曲线；模形式

ABSTRACT

Chromatic homotopy theory has been well developed since the periodic phenomenon manifested in computations of homotopy groups for certain types of spaces or spectra. It builds a bridge between obejcts in algebraic topology–complex oriented spectra and objects in algebraic geometry–formal groups. The crucial objects in chromatic homotopy theory are Morava K-theories and Morava E-theories. They correspond to height n formal groups and their deformations. On the other hand, cohomology operations such as Steenrod operations, Adam operations are crucial tools in the study of algebraic topology. Their basic and simple properties have illustrated many highly nontrivial results in algebraic topology. These operations are special types of power operations.

The study of power operations on Morava E-theory has been well developed by Ando, Hopkins and Strickland. Their works established a deep connection between the total power operations on Morava E-theory and deformations of formal groups with level structures. In particular, Strickland calculated the Morava E-theory of symmetric groups $E^*B\Sigma_k$, and showed its dual is the completed E-homology of symmetric groups. Hence one can compute the Dyer-Lashof algebra of $K(n)$-local E-algebras. However, an E-algebra need not to be $K(n)$-local.

From this point of view, we study the $K(n-1)$-local E-algebras, which are equivalent to $K(n-1)$-local $L_{K(n-1)}E_n$-algebras. In this dissertation, we calculate the $L_{K(n-1)}E_n$ cohomology of symmetric groups. It is free over $\pi_*L_{K(n-1)}E_n$. Then we interpret the total power operation on $L_{K(n-1)}E_n$ in terms of augmented deformations of formal groups over imperfect fields, $k((x))$ to be explicit, and subgroups. We also using the theory of Tate cohomology deduce an analogue duality between $L_{K(n-1)}E_n$-cohomology of symmetric groups and the corresponding completed homology. This allow us to interpret the Dyer-Lashof algebra over $K(n-1)$-local E-algebras as the dual of $L_{K(n-1)}E_n^*B\Sigma_k$.

In the height $n = 2$ case, we calculate an explicit formula of the total power operation on $L_{K(1)}E_2$ and explained the relation between our computations and elliptic curves. Roughly speaking, this total power operation is encoded in the Hasse invariant of the target curve along a universal degree p isogeny. From the modular forms point of view, it is the image of the Atkin-Lehner involution of the Hasse invariant of the source curve.

Keywords: Algebraic topology; Formal groups; Augmented deformation; elliptic curves; modular forms

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LIST OF SYMBOLS AND ACRONYMS

E	a complex oriented cohomology theory
F	$K(n-1)$ localized E theory
Σ_k	permutation group on k letters
BG	classifying space of group G
E^*, E_*	coefficients ring for E
$\mathbb{G}$	a formal group
$\mathscr{M}_{FG}$	moduli stack of formal groups
$\mathscr{M}_{FG}^{\leq n}$	moduli stack of formal groups of height less than or equal to n
$\mathscr{M}_{FG}^{n}$	moduli stack of formal groups of height n
C	an elliptic curve
$\widehat{C}$	formal group associated to C
$\mathscr{M}_{ell}$	moduli stack of elliptic curves
$\mathscr{M}_{ell}^{ord}$	moduli stack of ordinary elliptic curves
$\mathscr{M}_{N}$	moduli stack (scheme) of elliptic curves with an N torsion point
$\mathscr{M}_{N,p}$	moduli stack (scheme) of elliptic curves with an N torsion point and a degree p subgroup
$\Gamma_0(p)$	level structure of degree p subgroups
$\Gamma_1(N)$	level structure of N torsion points
$\Gamma(N)$	level structure of Drinfeld basis

CHAPTER 1 INTRODUCTION

Cohomology operations are crucial in algebraic topology. It equips cohomology rings with more richer algebraic structures. Many non-trivial results have roots in these operations. For instance, the *Steenrod mod-p operations* in mod$-p$ cohomology[1,2], the *Adams operations*[3] in the topological $K-$theory[4] are related directly to the *Hopf invariant one* problem[5,6] and the image of the stable J-homomorphism[7]. These operations are examples of *power operations*.

Let E be a multiplicative cohomology theory, or equivalently, a homotopy commutative ring spectrum. Suppose E admits an E_∞ structure[8] (Chapter 1,2), i.e., its multiplication structure $E \wedge E \xrightarrow{\mu} E$ commutes not only up to homotopy, but up to higher homotopy coherence. In this case, one can define the *total power operation*

$$P_n : E^0 \longrightarrow E^0 B\Sigma_n,$$

where $B\Sigma_n$ is the classifying space of the $n-$letters permutation groups. See Section 3.2 for detailed constructions. Some of the many important applications of power operations can be found in[9–11] etc.

For a spectrum X, the chromatic fracture square and the chromatic convergence theorem suggest that one can break X into pieces, $L_{K(n)}X$ namely, lying in each chromatic layer and recover itself via patching all these pieces together, at least in good circumstance. *Transchromatic homotopy theory* studies such chromatic layers. A fundamental and vital object in transchromatic homotopy theory is the $K(t)-$localized Morava $E-$theory $L_{K(t)}E_n$ for $t \leq n$.

Various work has been devoted to the study of $L_{K(t)}E_n$. In[12], Stapleton constructed associated character theory over it using $p-$divisible groups. He and Schlank also gave a transchromatic proof of Strickland's theorem based on such characters and inertia groupoid functors[13]. The spectrum $L_{K(t)}E_n$ itself is quite complicated. The coefficient ring $\pi_0 L_{K(t)}E_n$ is obtained by first inverting u_t in

$$\pi_0 E_n = W(k)[[u_1, \dots, u_{n-1}]]$$

then partially completing with the ideal $(p, u_1, \dots, u_{t-1})$. This ring is known to be excellent[14]. Various different topology could be defined over it. From this point of view, Mazel-Gee, Peterson and Stapleton proposed a modular interpretation of $\pi_* L_{K(t)} E_n$ in terms of pipe rings and pipe formal groups[15]. While when $t = n - 1$, things are slightly easier, $\pi_0 L_{K(n-1)} E_n$ is still a complete local Noetherian ring with an imperfect residue field $k((u_{n-1}))$. Torii compared $L_{K(n-1)} E_n$ with E_{n-1} by studying the associated formal groups[16,17] and Vankoughnett gave a modular interpretation of $\pi_0 L_{K(n-1)} E_n$ using augmented deformations[18], which is basically deformations of formal groups together with a choice of the last Lubin-Tate coordinates. (See Section 2.3 for definitions.)

Motivated by Vankoughnett's result, in this paper, we investigate the modular interpretation of the total power operation on the $K(n-1)$ -localized Morava E-theory $L_{K(n-1)} E_n$ at height n.

Let $F = L_{K(n-1)} E_n$, and $\mathbb{G}_F$ be the associated formal group. Let $\mathbb{G}_F^0$ be the fiber of $\mathbb{G}_F$ over the residue field $k((u_{n-1}))$ of F^0. We showed that

Theorem A (Theorem 3.3): *The ring $R_m = F^0 B\Sigma_{p^m}/I$ classifies augmented deformations of $\mathbb{G}_F^0$ together with a degree p^m subgroup, which means for any complete local Noetherian ring R, we have a bijection*

$$\mathrm{Map}_{cts}(R_m, R) = \{(\mathbb{K}, H)\}$$

between the set of continuous maps from R_m to R and the set of pairs consisting of an augmented deformation $\mathbb{K}$ of $\mathbb{G}_F^0$ over R and a degree p^m subgroup H of $\mathbb{K}$.

Moreover, the additive total power operation

$$\psi_F^p : F^0 \longrightarrow F^0 B\Sigma_p/I$$

behaves like take the quotient, i.e.

$$\mathrm{Map}_{cts}(R_m, R) \xrightarrow{(\psi_F^p)^*} \mathrm{Map}_{cts}(F^0, R)$$
$$(\mathbb{K}, H) \longmapsto \mathbb{K}/H.$$

Our result is an attempt to understand the full picture of power operations in chromatic setting by studying power operations on each associated $K(n)$-local monochromatic layer. The author also guesses that the algebraic information about the choice of the last Lubin-Tate coordinates has its own modular interpretation in terms of the étale part of G_E, when considered as a p-divisible group over $\pi_0 L_{K(n-1)} E_n$.

Let R be an E_∞ ring. The homotopy group of an R-algebra A possesses a module structure over the *Dyer-Lashof* algebra DL_R. The Dyer-Lashof algebra is a generalization of Steenrod algebra in generalized cohomology setting, which governs all homotopy operations.

Our second result concerns Dyer-Lashof algebra over $K(n-1)$-local E_n-algebras. Using Tate spectrum, we show the $L_{K(n-1)}E_n$ theory for symmetric groups is self-dual which leads to the following.

Theorem B (Proposition 3.10): The Dyer-Lashof algebra over $K(n-1)$-local E_n-algebras is

$$\mathrm{DL}_{LE} = \bigoplus_{m \geq 1} \widehat{F}_0 B\Sigma_m$$

where $F = L_{K(n-1)}E_n$. Moreover we have

$$\widehat{F}_0 B\Sigma_m = F^0 B\Sigma_m$$

and thus we have

$$\widehat{F}_0 B\Sigma_m = \mathrm{Hom}_{F^0}(F^0 B\Sigma_m, F^0)$$

is the dual of $F^0 B\Sigma_m$. The *primitives* in the left hand side correspond to *indecomposables* in the right hand side.

This allows us to find a presentation of DL_{LE} in terms of coefficients of the total power operation ψ^p_F, as what have been done in the $K(n)$-local E_n-algebra setting.

When the height $n = 2$, there is a connection between the spectrum $L_{K(1)}E_2$ and elliptic curves. Recall that there is a sheaf of E_∞ rings $\mathcal{O}^{top}$ defined over the étale site $(\overline{\mathcal{M}}_{ell})_{\acute{e}t}$, which assigns each elliptic curve

$$C : \mathrm{Spec}(R) \longrightarrow \overline{\mathcal{M}}_{ell}$$

an E_∞ ring E^C_R [19]. The p-completed stack $(\overline{\mathcal{M}}_{ell})_p$ can be decomposed into supersingular part $\overline{\mathcal{M}}_{ell}^{ss}$ and ordinary part $\overline{\mathcal{M}}_{ell}^{ord}$. Hence we have a chromatic fracture square for p-completed Tmf [20]:

$$\begin{array}{ccc} Tmf_p & \longrightarrow & L_{K(2)}Tmf \\ \downarrow & & \downarrow \\ L_{K(1)}Tmf & \longrightarrow & L_{K(1)}L_{K(2)}Tmf. \end{array}$$

In the above diagram, the right up corner $L_{K(2)}Tmf$ is a variation of Morava E-theories of height 2 and the right lower corner is thus a kind of $K(1)$-localized E_2. The spectrum $L_{K(1)}E_2$ can be viewed as the intersection of $\mathcal{O}^{top}$ over $\overline{\mathcal{M}}_{ell}^{ss}$ and $\overline{\mathcal{M}}_{ell}^{ord}$, which corresponds to a punctured formal neighborhood of a supersingular point, as illustrated in the following picture.

Our third result concerns the total power operation on $L_{K(1)}E_2$. The Dyer-Lashof algebra structure over $L_{K(1)}E_2$ is clear. It is a free θ−algebra on one generator as stated

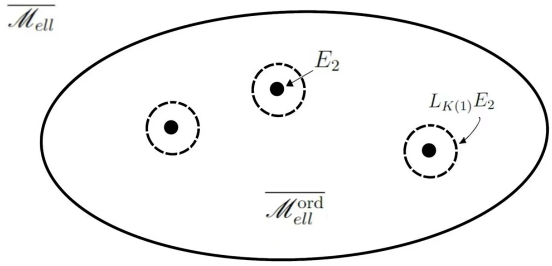


Figure 1-1 Spectra on p completed stack $\overline{\mathscr{M}_{ell}}$

in[21]. Via the naturality of total power operations, we obtained the explicit formula of the total power operation on $L_{K(1)}E_2$.

Theorem C (Theorem 4.6): *Let F be a $K(1)$-local Morava E-theory at height 2. The total power operation ψ_F^p on F^0 is determined by*

$$\psi_F^p(h) = \alpha^* + \sum_{i=0}^{p}(\alpha^*)^i \sum_{\tau=1}^{p} w_{\tau+1} d_{i,\tau}, \tag{1.0.1}$$

where

$$\alpha^* = (-1)^{p+1} p \cdot h^{-1} + \left(1 + (-1)^{p+1}\frac{p(p-1)}{2}\right) p^3 \cdot h^{-3} + \textit{lower terms} \tag{1.0.2}$$

is the unique solution of

$$w(h,\alpha) = (\alpha - p)(\alpha + (-1)^p)^p - (h - p^2 + (-1)^p)\alpha$$

in $W(\overline{\mathbb{F}}_p)((h))_p^{\wedge} \cong F^0$.

The other coefficients w_i and $d_{i,\tau}$ are defined as

$$w_i = (-1)^{p(p-i+1)} \left[\binom{p}{i-1} + (-1)^{p+1} p \binom{p}{i}\right]$$

and

$$d_{i,\tau} = \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\cdots m_{\tau-n}=\tau+i \\ 1\le m_s\le p+1 \\ m_{\tau-n}\ge i+1}} w_{m_1}\cdots w_{m_{\tau-n}}.$$

In particular, ψ_F^p satisfies the Frobenius congruence, i.e.

$$\psi_F^p(h) \equiv h^p \bmod p$$

.

Remark 1.1: The element α^* in 1.0.2 is the restriction of a modular form α of $\Gamma_0(p)$ over the punctured formal neighborhood of a supersingular point, which parametrizes subgroups of elliptic curves, and h is a lift of Hasse invariant.

The modular form α is sometimes called a *norm parameter* for $\Gamma_0(p)$ [22] (7.5.2), [23] (Section 4.3). Choosing a coordinate u on elliptic curves with a $\Gamma_0(p)$ structure, i.e. a degree p subgroup, one has

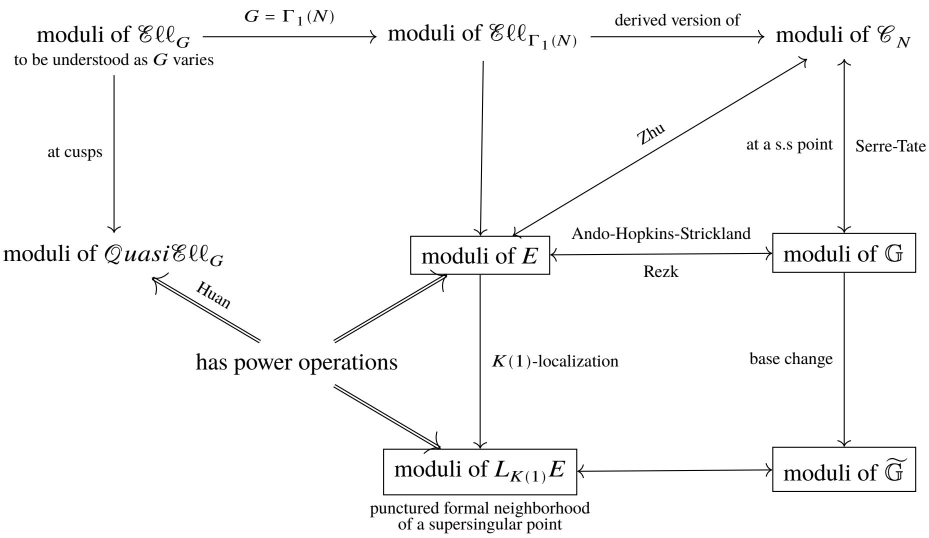
$$\alpha := \prod_{Q \in \mathscr{G}^{(p)} - O} u(Q)$$

where $\mathscr{G}^{(p)}$ is a degree p subgroup. Note that this implies α depends on the choice of u. (Remark 1.2)

The element $\psi_F^p(h)$ is actually the image of h under the *Atkin-Lehner involution*, see Section 3.2 and[22] (Chapter 11) for details.

Remark 1.2: Our computation depends on a specific model for Morava E-theories[24] (Definition 2.23). The extent of this dependence can be found in[24] (Remark 2.25).

This result can be viewed as a first step toward to the total power operation on the p-completed Tmf. Our analysis fits in the boxed regions in the diagram below[24] (Page 3).



Here E is a height 2 Morava E-theory, $\mathscr{C}_N$ is a universal elliptic curve with an N torsion point and $\mathbb{G}$ is the universal deformation of the associated height 2 formal group of $\mathscr{C}_N$ at a supersingular point.

1.1 Literature review

Calculating the stable homotopy group of spheres is always a central scene in algebraic topology. The original method was due to Serre[25,26] using the Hurewicz theorem[27] and Serre spectral sequence[28]. It was founded that there are some periodic families in the stable homotopy group of spheres by Toda[29], Smith and Toda[30,31], Miller, Ravenel and Wilson[32], etc. In the recent progress of computing stable stems[33–36] and in particular, the motivic homotopy theory involved[37,38], there are also periodic families which beyond the framework of classical settings[39–42].

Recently, with the development of technique of algebraic Novikov spectral sequence and the synthetic method, Lin-Wang and Xu had successfully computed the 126th stable homotopy group of sphere, at least its 2-component. This result directly suggests that there are some manifolds with Kervaire invariant 1 in dimension 126, which completes the final piece of the problem of Kervaire invariants, a 64 years lasted problem. However, using spectral sequence, we could compute homotopy groups prime by prime, while most consequences of stable homotopy groups are only stated in $p = 2$ case. For $p = 3$ or other cases, good references are[43] and[44].

To have a conceptual understanding of these periodic phenomenons, the theory of formal group laws and chromatic spectral sequences were introduced[45]. It is inspiring when the complex cobordism MU spectrum was introduced[46,47]. Despite of its geometric interpretation of cobordism classes of manifolds, Quillen's work showed the connection between MU_* and formal group laws[48–50]. This is the beginning of the *Chromatic homotopy theory*. On the other hand, algebraic topologists were interested in creating new cohomology theories from formal group laws. This was suggested by the work of Conner and Floyd[51,52] and finally landed in the Landweber exact functor theorem[53,54]. Lurie also has an refined version[55–57] using techniques of spectral algebraic geometry[58,59] such as spectral p-divisible groups and spectral elliptic curves.

The stack of formal groups has been well studied[60,61].Over an algebraically closed field, formal groups are classified by heights. For each p and height n, there is a spectrum $K(n)$, called Morava K-theory, which corresponds to the height n formal group[62]. There is also a Morava E-theory which parameterizes deformations of such a formal group. Bousfield introduced his localization technique in[63,64]. In this setting, localizing at E_n behaves like restricting to the open substack of $\mathscr{M}_{FG}^{\leq n}$ and localizing at $K(n)$ behaves like completing along the locally closed stack $\mathscr{M}_{FG}^{n}$. And also, using the homotopy fixed point spectral sequence[65,66], Devinatz and Hopkins showed the homotopy fiexed point spectral sequence of Morava stabilizer group converges to the $K(n)$-local sphere[67,68].

Morava E theories admit essentially unique E_∞ structure[69], hence admit power operations. Surprisingly, the total power operation over it has a strong connection with the moduli of subgroups of formal groups and subgroups of elliptic curves, and it has been well studied by Ando[70,71], Hopkins[72], Strickland[73] and Rezk[74–76], etc.

1.2 Outline of the paper

In section 3.1, we calculate the $K(n-1)$-localized Morava E-theory of symmetric groups using the generalized character map. Then we deduce the modular interpretation of the total power operation ψ_F^p in section 3.2.

In[72], the similar moduli interpretation of modified power operations in terms of level structures associated to *abelian* group is confirmed. Though it is claimed in[72] (Remark 3.12) that it is not necessary to use abelian group, there is not a direct proof in nonabelian cases. Section 3.2 is devoted to such a proof. Seasoned readers can take it for granted and skip it.

In section 3.3 we combine our analysis with augmented deformations and obtain the Theorem A.

In section 3.4, we show the self-dualness of $L_{K(n-1)}E_n$ and compute the Dyer-Lashof algebra structure on $K(n-1)$-local E_n-algebras, which is Theorem B.

The chapter 4 is devoted to the connection between our analysis and theory of elliptic curves when $n = 2$. In section 4.2, we review the method developed for computing total power operations on E-theory, and in 4.3 explain how to interpret these topological elements involved in terms of moduli schemes and modular curves. In section 4.4, we calculate an explicit formula for the total power operation using the naturality of power operations in section 4.2 and explain how these ideas are related to elliptic curve, modular forms and p-divisible groups in section 4.3.

The chapter 5 is somehow independent from the main line. We investigate a family of spectra, so called augmented deformation spectra and show that there underlying spectra are independent of the choices of formal groups.

CHAPTER 2 COMPLEX ORIENTED COHOMOLOGY THEORIES AND DEFORMATIONS OF FORMAL GROUPS

In this section, we review some basic ideas of complex oriented cohomology theory and the relation to formal groups. These objects are fundamental in *chromatic homotopy theory*.

2.1 Complex Oriented Cohomology Theory

Let E be a multiplicative cohomology theory, or equivalently, a homotopy commutative ring spectrum.

Definition 2.1: We say E is complex oriented if for any complex vector bundle $\xi \to X$ of dimension n, there is a class

$$U_\xi \in \widetilde{E}^{2n}(X^\xi)$$

such that

- For each $x \in X$, we have the image of U_ξ under the composition

$$\widetilde{E}^{2n}(X^\xi) \to \widetilde{E}^{2n}(*^\xi) \to \widetilde{E}^{2n}(S^{2n}) \to E^0(*)$$

is the canonical element $1 \in E^*$.

- Suppose $f : X \to Y$ is a map, then we have

$$f^* U_\xi = U_{f^*\xi}$$

If η is another bundle over X, we have

$$U_{\xi\oplus\eta} = U_\xi \cdot U_\eta$$

A useful criterion to determine whether a generalized cohomology theory is complex oriented is as follow.

Proposition 2.1: A generalized cohomology theory is complex oriented if and only if there is a class $x_E \in \widetilde{E}^2(\mathbb{C}P^\infty)$, called a *complex orientation*, which restricts to the identity 1 via the map

$$\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^1) = \widetilde{E}^2(S^2) \cong E^0(*)$$

Remark 2.1: Note that the above proposition gurantees that the Atiyah-Hirzebruch spectral sequence for $E^*(\mathbb{C}P^\infty)$

$$E_2^{p,q} = H^p\left(\mathbb{C}P^\infty; E^q(*)\right)$$

collapses, and hence we have an isomorphism

$$E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$$

Conversely, if the E_2 page of the spectral sequence collapses, one can easily deduce that E is complex oriented.

Example 2.1: By this criteria, some of our familiar cohomology theories are complex oriented.

- Ordinary cohomology theory with coefficients a commutative ring R, i.e. HR is complex oriented.
- The complex K-theory, KU is complex oriented.
- Any spectrum X with $\pi_{\text{odd}}X$ vanishing is complex oriented.
- The real K-theory KO is *not* complex oriented. For one can check that for the tautological line bundle ξ over $\mathbb{C}P^\infty$ the composition in Definition 2.1 is multiplication by 2.

Now suppose E is complex oriented. Consider the tensor product of line bundles, which is classified by the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

It induces a map

$$E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

in the E cohomology theory. Since E is complex oriented, the above map can be identified as

$$E^*[[t]] \longrightarrow E^*[[x, y]]$$

Let $F(x, y)$ denote the image of t in the right hand side. Since tensor products of line bundles satisfies associativity, commutativity etc., the power series F should also satisfy these relations. To be explicit, we have the following definition.

Definition 2.2: A *formal group law* over a ring R is a power series $F \in R[[x, y]]$ which satisfies:

- Identity: $F(x, 0) = F(0, x) = x$,
- Commutativity: $F(x, y) = F(y, x)$ and

- Associativity: $F(F(x,y),z) = F(x,F(y,z))$

From the above discussion, we know that any complex oriented cohomology theory has an associated formal group law.

Example 2.2: Ordinary cohomology theories have the associated formal group law $F(x,y) = x + y$, which is called the additive formal group law.

The complex K-theory has $F(x,y) = x + y + xy$, which is called the multiplicative formal group law.

Let $f : R \to S$ be a ring map and $F = \sum a_{ij}x^i y^j$ be a formal group law over R. It is clear that the power series

$$f^*F = \sum f(a_{ij})x^i y^j$$

is a new formal group law over the ring S. A natural question it that does there exist a ring L which carries a universal formal group law F^u in the sense that for any formal group law F over R, one has

$$F = f^*F^u$$

for some map $f : L \to R$.

The answer is positive and not too hard. One can just define L to be the ring

$$L = \mathbb{Z}[a_{ij}]/I$$

where I is the ideal generated by relations constrained by identity, associativity and commutativity. The universal formal group is just

$$F^u = \sum a_{ij}x^i y^j$$

The ring L is called the *Lazard ring*, which was first introduced by Lazard in[77]. He also showed that L has a simple structure despite of its horrible construction.

Theorem 2.1 (Lazard, 1955): The ring L is isomorphic to

$$\mathbb{Z}[x_1, x_2, \dots]$$

where each a_{ij} has degree $(i + j - 1)$ and x_i has degree i.

Moreover in[48], Quillen showed the following result.

Theorem 2.2 (Quillen, 1969): The complex cobordism spectrum MU is complex oriented, with coefficients

$$\pi_*MU = \mathbb{Z}[x_1, x_2, \dots]$$

where x_i has degree $2i$.

Moreover, the canonical map

$$L \longrightarrow MU_*$$

which classifies the formal group law over MU_* is an isomorphism.

Hence we may regard MU as the universal complex oriented cohomology theory and a ring map

$$x_E : MU \longrightarrow E$$

establishes a complex orientation of E.

Suppose x_E and x'_E are two complex orientations of E, i.e. elements in $E^*(\mathbb{C}P^\infty)$ and F_E, F'_E are corresponding formal groups laws respectively. There is a power series $f(t) \in E^*[[t]]$ such that

$$x'_E = f(x_E)$$

The following diagrams

$$\begin{tikzcd} E^*[[x_E]] \arrow[r] \arrow[d] & E^*[[x,y]] \arrow[d] \\ E^*[[x'_E]] \arrow[r] & E^*[[x',y']] \end{tikzcd} \qquad \begin{tikzcd} f(x_E) \arrow[r, mapsto] \arrow[d, mapsto] & f(F_E(x,y)) \arrow[d, mapsto] \\ x'_E \arrow[r, mapsto] & F'_E(x',y') \end{tikzcd}$$

imply that

$$f(F_E(x,y)) = F'_E(f(x),f(y))$$

Definition 2.3: Let F, G be two formal group laws over R. A homomorphism from F to G is a power series $f = \sum_{i=1}^{\infty} b_i t^i \in R[[t]]$, such that

$$f(F(x,y)) = G(f(x),f(y))$$

We say f is an isomorphism if f is invertible, i.e. b_1 is a unit. We say f is a *strict* isomorphism if $b_1 = 1$.

Thus different choices of complex orientations yield isomorphic formal group laws for a given complex oriented cohomology theory E. Hence we can consider formal group laws more intrinsically, which are *formal groups*.

In algebraic geometry, a *formal scheme* is used to detect the local behavior around a closed point. For example, Let $R = k[x]$ for some field k. The maximal ideal (x) corresponds to the closed point $[0]$. To study the local behavior around this closed point, one has a sequence

$$\operatorname{Spec} k \cong \operatorname{Spec} k[x]/x \to \operatorname{Spec} k[x]/x^2 \to \cdots \to \operatorname{Spec} k[x]/x^n \to \cdots .$$

In each stage, Spec $k[x]/x^n$ has the underlying space $[0]$, but the functions are more. This indicates us to take the colimit of this sequence. Unfortunately, the category **Sch** of schemes does not have all limits and colimits.

Remark 2.2: The category of locally ringed spaces has all limits and colimits. The category **Aff** of affine schemes is the opposite category of **Ring**. We have the adjunction

$$\Gamma : \mathbf{Sch} \leftrightarrows \mathbf{Ring}^{op} : \mathrm{Spec}$$

with Spec being a right adjoint. Hence it preserves the limits in $\mathbf{Ring}^{op}$, or equivalently, colimits in **Ring**.

No matter in what cases, there is no evidence that the colimit should exist. Hence we have the following definition. Now we say a scheme means an affine scheme in all of the notes, and denote the category of affine scheme by $\mathfrak{X}$, the full subcategory of $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$ consisting of representable functors.

Definition 2.4: A formal scheme X is a small filtered colimit of scheme X_i. As we already explained, this colimit may not exist in $\mathfrak{X}$. We can embed $\mathfrak{X}$ into $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$, where the later always has colimits, pointwisely.

To be more concrete, for each ring R, we have

$$X(R) = \mathrm{colim} X_i(R).$$

Definition 2.5: Let $X = \mathrm{colim} X_i$ and $Y = \mathrm{Y_j}$ be formal schemes. Define

$$\widehat{\mathfrak{X}}(X, Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i, Y_j).$$

We denote $\widehat{\mathfrak{X}}(X, \mathbb{A}^1)$ by $\mathcal{O}_X$, where $\mathbb{A}^1 = \mathbf{Ring}(\mathbb{Z}[t], -)$. To be precise,

$$\mathcal{O}_X = \lim \mathcal{O}_{X_i}.$$

Remark 2.3: From the definition, $\widehat{\mathfrak{X}}$ is actually the same as $\mathcal{I}nd(\mathfrak{X})$. Note that in general, one has

$$[\mathrm{colim}\, X_i, Y] = \lim [X_i, Y].$$

By the definition of colim Y_j, we have

$$\widehat{\mathfrak{X}}(X, Y) = \lim_i \widehat{\mathfrak{X}}(X_i, Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i, Y_j).$$

This is how we define morphisms in $\widehat{\mathfrak{X}}$.

Example 2.3: Let $N_i = \mathrm{Spec}\ \mathbb{Z}[x]/x^n$. The resulting formal scheme is denoted by $\widehat{\mathbb{A}}^1$. Note that $\widehat{\mathbb{A}}^1(R) = \mathrm{colim}\mathbf{Ring}(\mathbb{Z}[x]/x^n, R)) = \mathrm{Nil}(R)$. And $\mathcal{O}_{\widehat{\mathbb{A}}^1} = \mathbb{Z}[[x]]$.

The category $\widehat{\mathfrak{X}}$ has better categorical properties than $\mathfrak{X}$.

(1) $\widehat{\mathfrak{X}}$ has all small colimits and finite limits.

(2) finite limits commute with small colimits in $\widehat{\mathfrak{X}}$.

There are a special kind of formal schemes, called *solid* formal scheme, $\operatorname{Spf} R$, which we will define right now.

Definition 2.6: A linear topologized ring is a ring R equipped with a neighborhood system around 0 consisting of open ideals, which forms a topological basis under translation. The category of such rings and continuous maps is denoted by **LRing**.

We can equip any ring S with discrete topology, which yields a fully faithful embedding $\mathbf{Ring} \to \mathbf{LRing}$. Suppose $R \in \mathbf{LRing}$, $S \in \mathbf{Ring}$, f is a continuous map from R to S. We must have $f^{-1}(0) = J$ an open ideal in R. Hence f is equivalent to a map $R/J \to S$ between rings. All open ideals in R form a cofiltered system under inclusion maps. Hence we have

$$\mathbf{LRing}(R, S) = \operatorname*{colim}_J \mathbf{Ring}(R/J, S).$$

Therefore we define $\operatorname{Spf} R \in \mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$ by

$$\operatorname{Spf} R(S) = \operatorname*{colim}_J \mathbf{Ring}(R/J, S).$$

Definition 2.7: A solid formal scheme is a formal scheme which is isomorphic to $\operatorname{Spf} R$ for some linearly topologized ring R. The solid formal schemes form a full subcategory $\widehat{\mathfrak{X}}_{sol}$ of $\widehat{\mathfrak{X}}$.

Given a linearly topologized ring R, we have the related cofiltered system $\{R/J\}$, where J runs through all open ideals. The limit of this system is denoted by $\widehat{R}$, called the completion of R. The ring $\widehat{R}$ automatically inherits a topological structure from R. The preimage $\overline{J}$ of J under the natural map $\widehat{R} \to R/J$ forms a neighborhood system around 0 in $\widehat{R}$. It is easy to check $\operatorname{Spf} \widehat{R} = \operatorname{Spf} R$. A ring R is complete or a formal ring if $R = \widehat{R}$. The category of formal rings is denoted by **FRing**, which is a full subcategory of **LRing**.

Note that

$$\widehat{\mathfrak{X}}(X, \operatorname{Spf} R) = \lim_i \widehat{\mathfrak{X}}(X_i, \operatorname{Spf} R) = \lim_i \mathbf{LRing}(R, \mathcal{O}_{X_i}) = \mathbf{LRing}(R, \mathcal{O}_X).$$

Hence we have the adjoint pairs:

$$\mathcal{O} : \widehat{\mathfrak{X}} \leftrightarrows \mathbf{LRing}^{op} : \operatorname{Spf}.$$

We have the unit map $X \to \operatorname{Spf} \mathcal{O}_X$, and the counit $R \to \widehat{R}$, which is just the completion.

Proposition 2.2: We have the following propositions.

(1) X is a solid fomal scheme then $\mathcal{O}_X$ is a formal ring.

(2) X is solid iff $X \to \operatorname{Spf} \mathcal{O}_X$ is an isomorphism.

(3) The inclusion functor $\widehat{\mathfrak{X}}_{sol} \to \widehat{\mathfrak{X}}$ is right adjoint to $X \to \operatorname{Spf} \mathcal{O}_X$.

(4) The inclusion functor **FRing** $\to$ **LRing** is right adjoint to taking completion.

Proof: (1) Obvious.

(2) X is solid, then X is isomorphic to $\operatorname{Spf} R$ for some R. Therefore $\mathcal{O}_X$ is isomorphic to $\widehat{R}$, which yields the conclusion. The converse is obvious.

(3) The functor $\mathbf{FRing}^{op} \to \widehat{\mathfrak{X}}_{sol}$ sending R to $\operatorname{Spf} R$ is fully faithful. Suppose R, S are two formal rings, then

$$\widehat{\mathfrak{X}}_{sol}(\operatorname{Spf} R, \operatorname{Spf} S) = \lim_{J} \widehat{\mathfrak{X}}_{sol}(\operatorname{Spec} R/J, \operatorname{Spf} S) = \lim \mathbf{LRing}(S, R/J) = \mathbf{FRing}(S, R).$$

Therefore by (2), this functor defines an equivalence. The equation

$$\widehat{\mathfrak{X}}(X, \operatorname{Spf} R) = \mathbf{LRing}(R, \mathcal{O}_X) = \widehat{\mathfrak{X}}_{sol}(\operatorname{Spf} \mathcal{O}_X, \operatorname{Spf} R)$$

implies the inclusion functor being right adjoint to $X \to \operatorname{Spf} \mathcal{O}_X$.

(4) The same argument holds.

$$\mathbf{LRing}(R, \hat{S}) = \operatorname{Spf} R(\hat{S}) = \operatorname{Spf} \widehat{R}(\hat{S}) = \mathbf{FRing}(\widehat{R}, \hat{S}).$$

■

Now we are ready to define what so called formal groups.

Definition 2.8: A formal group G over a formal scheme X is a group object in $\widehat{\mathfrak{X}}_X$. We also require that G is isomorphic to $X \times \widehat{\mathbb{A}}^1$ in $\widehat{\mathfrak{X}}_X$. A map $u : G \to \widehat{\mathbb{A}}^1$ makes G isomorphic to $X \times \widehat{\mathbb{A}}^1$ is called a coordinate on G.

Suppose X is solid. Then $X \times \widehat{\mathbb{A}}^1$ is again solid. From the equivalence of categories, we have $X \times \widehat{\mathbb{A}}^1$ is isomorphic to the Spf of coproduct of $\mathcal{O}_X$ and $\mathbb{Z}[[t]]$ in **FRing**, which is the completion of $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[[t]] \cong \mathcal{O}_X[[t]]$. Therefore G is solid as well with $\mathcal{O}_G \cong \widehat{\mathcal{O}_X[[t]]}$.

Moreover, if we further assume X is just a scheme, then

$$\mathcal{O}_G \cong \mathcal{O}_X \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[t]] \cong \mathcal{O}_X[[t]]$$

for now $\mathcal{O}_X$ is equipped with the discrete topology. A coordinate on G is the same as an isomorphism from G to $X \times \widehat{\mathbb{A}}^1$, which corresponds to a continuous map

$$u : \mathbb{Z}[[t]] \to \mathcal{O}_G$$

which induces an isomorphism

$$\mathcal{O}_X[[t]] \to \mathcal{O}_G.$$

Now since G is a group object, we have a map $G \times_X G \xrightarrow{\mu} G$, which corresponds to

$$\mathcal{O}_G \to \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G$$

of $\mathcal{O}_X$ modules. We also call the latter map μ, and it satisfies following properties.

Identity There is a map $X \xrightarrow{e} G$ such that the composite

$$X \to G \to X$$

is identity. Moreover we require the composition

$$G \cong X \times_X G \xrightarrow{e \times id} G \times_X G \xrightarrow{\mu} G$$

equals the identity from G to itself.

Equivalently, there is a map $e : \mathcal{O}_G \to \mathcal{O}_X$, such that

$$\mathcal{O}_X \to \mathcal{O}_G \to \mathcal{O}_X$$

is identity and

$$\mathcal{O}_G \xrightarrow{\mu} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \xrightarrow{e \otimes id} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_G \cong \mathcal{O}_G$$

is identity.

Associativity

$$\begin{tikzcd} G \times_X G \times_X G \arrow[r, "\mu \times id"] \arrow[d, "id \times \mu"'] & G \times_X G \arrow[d, "\mu"] \\ G \times_X G \arrow[r, "\mu"] & G \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_G \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \arrow[d, "id \otimes \mu"] \\ \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \arrow[r, "\mu \otimes id"] & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G. \end{tikzcd}$$

Commutativity

$$\begin{tikzcd} G \times_X G \arrow[rr, "T"] \arrow[dr, "\mu"'] & & G \times_X G \arrow[dl, "\mu"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \arrow[rr, "T"] \arrow[dr, "\mu"'] & & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \arrow[dl, "\mu"] \\ & \mathcal{O}_G & \end{tikzcd}$$

where T is transposition.

If we choose a coordinate on G, then we have an isomorphism from $\mathcal{O}_G$ to $\mathcal{O}_X[[t]]$. The map μ now is

$$\mathcal{O}_X[[t]] \longrightarrow \mathcal{O}_X[[x, y]]$$

between $\mathcal{O}_X$ modules, which is determined by $f(x, y) = \mu(t) \in \mathcal{O}_X[[x, y]]$. Such power series $f(x, y)$ is just a formal group law over $\mathscr{O}_X$, which satisfies

- $f(0, y) = y$;
- $f(x, f(y, z)) = f(f(x, y), z)$;
- $f(x, y) = f(y, x)$.

Remark 2.4: The identity $\mathscr{O}_X[[t]] \to \mathscr{O}_X$ can only be $t \mapsto 0$. Since this map is continuous between $\mathscr{O}_X$ modules, hence is determined by the image of t which is a nilpotent element n in $\mathscr{O}_X$. Note that

$$f(x, y) = \sum f_i(x) y^i = \sum f_i(y) x^i$$

by commutativity. Hence we have

$$f(n, y) = \sum f_i(n) y^i = y$$

which implies that $f_0(n) = 0$. Hence $f_0(x)$ is divided by x^k for some k. So does $f_0(y)$, that is

$$f(x, y) = f_0(x) + f_0(y) + \cdots.$$

Hence k must be 1 and $n = 0$.

Example 2.4: The additive formal group $\mathbb{G}_a = \operatorname{Spf} \mathbb{Z}[[t]]$ is a formal group over $\mathbb{Z}$. For any ring R, $\mathbb{G}_a(R) \to \operatorname{Spec} \mathbb{Z}(R)$ is given by the inclusion of rings $\operatorname{Nil}(R) \to *$. The group structure on $\mathbb{G}_a$ is given by

$$\operatorname{Nil}(R) \times \operatorname{Nil}(R) \to \operatorname{Nil}(R), \ (a, b) \mapsto a + b.$$

If we choose a coordinate $id : \mathbb{Z}[[t]] \to \mathbb{Z}[[t]]$, the nwe get a formal group law $f(x, y) = x + y$.

The multiplicative formal group $\mathbb{G}_m$ over $\mathbb{Z}$ has the same underlying formal scheme. The group structure is given by

$$\operatorname{Nil}(R) \times \operatorname{Nil}(R) \to \operatorname{Nil}(R), \ (a, b) \mapsto a + b + ab.$$

Use the same coordinate, we have a formal group law $f(x, y) = (1 + x)(1 + y) - 1$.

A morphism between two formal groups $\mathbb{G}$ over X and $\mathbb{H}$ over Y is just a commutative diagram in $\widehat{\mathfrak{X}}$ which respects the group structures of $\mathbb{G}$ and $\mathbb{H}$.

$$\begin{tikzcd} \mathbb{G} \arrow[r, "q"] \arrow[d] & \mathbb{H} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[d] \\ \mathscr{O}_{\mathbb{H}} \arrow[r, "q^*"] & \mathscr{O}_{\mathbb{G}} \end{tikzcd}$$

Let x, y be coordinates on $\mathbb{G}$ and $\mathbb{H}$, then we have isomorphisms $\mathscr{O}_{\mathbb{G}} \cong \mathscr{O}_X[[x]]$ and

$\mathscr{O}_{\mathbb{H}} \cong \mathscr{O}_Y[[y]]$ respectively. The morphism q^* sending y to a series $f(x) \in \mathscr{O}_X[[x]]$, which satisfies

$$f(x +_{\mathbb{G}} x') = f(x) +_{\mathbb{H}} f(x').$$

Such series is called a homomorphism between formal group laws.

Example 2.5: A crucial endomorphism from $\mathbb{G}$ to itself is multiplication by p. It is induced by

$$[p] : \mathbb{G} \xrightarrow{\Delta} \underbrace{G \times_X \cdots \times_X G}_{p\ times} \xrightarrow{\mu} G.$$

where Δ is the diagonal. Choose a coordinate, we have $[p](x) = x +_{\mathbb{G}} \cdots +_{\mathbb{G}} x$,

Suppose now X is over Spec $\mathbb{F}_p$ and $q : \mathbb{G} \to \mathbb{H}$ over X with x, y are coordinates on them. Then there must be $a \neq 0 \in \mathscr{O}_X$ and r such that

$$q^*(y) = ax^r \bmod x^{r+1}.$$

Since q is a homomorphism, we have

$$a(x_0^r + x_1^r) = a(x_0 + x_1)^r \bmod (x_0, x_1)^{r+1}.$$

Let $r = p^n m$, we have

$$x_0^r + x_1^r = (x_0^{p^n} + x_1^{p^n})^m = x_0^r + m x_0^{r-p^n} x_1^{p^n} + \cdots \bmod (x_0, x_1)^{r+1}.$$

Hence m must be 1 and $r = p^n$ is a power of p.

Definition 2.9: We call such n the strict height of q. We also let $\mathrm{ht}(q)$ to be the strict height of $\tilde{q} : \mathbb{G}_0 \to \mathbb{H}_0$ over the special fiber. Finally, we define $\mathrm{ht}(\mathbb{G})$ to be $\mathrm{ht}([p] : \mathbb{G} \to \mathbb{G})$.

Remark 2.5: Strict height is always not greater then height obviously. Moreover, we have $q^*(y) = g(x^{p^n})$. This is because $q^*(y) = f(x)$ must have no constant term due to the continuity. If $f'(0) \neq 0$, which means $f(x) = x + \cdots$, already meets the requirement. If $f'(0) = 0$, then the group law will force $f'(x) = 0$, which implies $f(x) = g(x^p)$.

There is a geometric way to think of the strict height of a morphism $f : \mathbb{G} \to \mathbb{H}$ over X. Since X is over Spec $\mathbb{F}_p$, we have a Frobenius map $F_X : X \to X$. The pullback $F_X^* \mathbb{G}$ is also a formal group. If we choose a coordinate x on $\mathbb{G}$ and the induced coordinate y on $F_X^* \mathbb{G}$, then the formal group law on $F_X^* \mathbb{G}$ is given by $g^{(p)}(y, y')$, where g is the formal group law of $\mathbb{G}$ under the coordinate x and $g^{(p)}$ is the series obtained from replacing coefficients g_{ij} in g by g_{ij}^p.

$$\begin{tikzcd}
\mathbb{G} \arrow[dr, dashed, "F_{\mathbb{G}/X}"] \arrow[drr, bend left, "F_W"] \arrow[ddr, bend right, "q"'] & & \\
& F_X^*\mathbb{G} \arrow[r] \arrow[d, "q"] & \mathbb{G} \arrow[d, "q"] \\
& X \arrow[r, "F_X"] & X
\end{tikzcd}$$

The commutativity of Frobenius maps induces a map $F_{\mathbb{G}/X} : \mathbb{G} \to F_X^*\mathbb{G}$, which is also a group homomorphism. Using the coordinates above, we have $F_{\mathbb{G}/X}^*(y) = x^p$.

Now suppose $f : \mathbb{G} \to \mathbb{H}$ is a group homomorphism with $f^*(y) = g(x^p)$ where x, y are coordinates on $\mathbb{G}$ and $\mathbb{H}$ respectively. From the expression of $f^*(y)$, we know that f factors through

$$\mathbb{G} \xrightarrow{F_{\mathbb{G}/X}} F_X^*\mathbb{G} \to \mathbb{H}.$$

The strict height of f corresponds to the height of the tower:

$$\begin{tikzcd}
\mathbb{G} \arrow[d] \arrow[dddr, "f"] & \\
F_X^*\mathbb{G} \arrow[d] \arrow[ddr, "f_1"] & \\
\vdots \arrow[d] & \\
(F_X^n)^*\mathbb{G} \arrow[r, "f_n"] & \mathbb{H}
\end{tikzcd}$$

Proposition 2.3: Let $f : \mathbb{G} \to \mathbb{H}$ be a nonzero homomorphism over X with $\mathrm{ht}(\mathbb{G})$ finite. Then $\mathrm{ht}(\mathbb{G}) = \mathrm{ht}(\mathbb{H})$ and $\mathrm{ht}(f)$ is finite.

Proof: Just a direct computation. ■

Example 2.6: Using heights of formal groups, we can distinguish different cohomology theories. For example, $H\mathbb{F}_p$ and the mod p K-theory $K(-; \mathbb{F}_p)$ is not equivalent as ring spectrum. For $\mathbb{G}_a$ has height ∞ and $\mathbb{G}_m$ has height 1 over $\mathbb{F}_p$.

Formal groups over different characteristic behave quite different.

Proposition 2.4: Suppose F is a formal group over $\mathbb{Q}$. Then there is a unique *strict* isomorphism from F to the additive formal group $\mathbb{G}_a$.

Proof: Let f be a corresponding formal group law of F. Suppose l is such an isomorphism, i.e. $l(x) = x + \cdots$. We have

$$l(f(x, y)) = l(x) + l(y)$$

Taking derivative respect to y and set $y = 0$ implies that

$$l'(x) f_y(x, 0) = 1$$

Hence $l(x) = \int \frac{1}{f_y(x,0)}$, which is well-defined over $\mathbb{Q}$. ■

While over fields with $p = 0$, things are more interesting.

Theorem 2.3 (Lubin-Tate, Lazard): Let K be a field of characteristic p. For each n, there exists a formal group of height n over K.

If $\overline{K} = K$, then any two formal groups with the same height are isomorphic.

Example 2.7: For each n, there is a height n formal group H, called the *Honda formal group*, which is determined by its p-series

$$[p]_H(x) = x^{p^n}$$

2.2 The Moduli Stack of Formal Groups

In this section, we review the language of *Hopf algebroids* and *stacks*. Then we discuss the quasicoherent sheaves over the stack $\mathscr{M}_{FG}$.

Let E be a complex oriented spectrum, x_E and x'_E be two complex orientations. Then we have the following diagram

$$\begin{tikzcd}
MU \wedge S \arrow[dr, "Id\wedge 1"] \arrow[drrr, bend left, "x_E"] & & & \\
& MU \wedge MU \arrow[r, "x_E \wedge x'_E"] & E \wedge E \arrow[r, "\mu"] & E \\
S \wedge MU \arrow[ur, "1\wedge Id"'] \arrow[urrr, bend right, "x'_E"'] & & &
\end{tikzcd}$$

This diagram implies that the composite

$$MU \wedge MU \to E$$

classifies an isomorphism between the associated formal group laws F_E and F'_E, or equivalently, an isomorphism of the associated formal group $\mathbb{G}_E$. In fact, this classification is universal.

Proposition 2.5 (Proposition 6.5[78]): Let η_L and η_R be the map $Id \wedge 1$ and $1 \wedge Id$ from MU to $MU \wedge MU$ respectively. The ring $MU_*MU = \pi_*MU \wedge MU$, which is flat over MU_*, carries a universal isomorphism of formal groups. To be explicit, a morphism

$$f : MU_*MU \longrightarrow R$$

gives an isomorphism from formal group F to G over R, where F is $(f \circ \eta_L)^* F^u$ and G is $(f \circ \eta_R)^* F^u$.

The pair (MU_*, MU_*MU) is an example of *Hopf algebroid*, which corepresents a groupoid object in the category of rings. For each ring R, the set $\mathbf{Ring}(MU_*, R)$ is the set of objects and $\mathbf{Ring}(MU_*MU, R)$ is the set of isomorphisms.

The formal definition is as follow.

Definition 2.10: A *Hopf algebroid* is a pair of rings (A, Γ), together with the following data:

(1) Left unit/Source map: $\eta_L : A \longrightarrow \Gamma$;

(2) Right unit/Target map: $\eta_R : A \longrightarrow \Gamma$;

(3) Coproduct/Composition map: $\Delta : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$;

(4) Augmentation/Identity map: $\epsilon : \Gamma \longrightarrow A$;

(5) Conjugation/Inverse map: $c : \Gamma \longrightarrow \Gamma$,

such that the following conditions hold.

- The identity map has the same source and target.

$$\epsilon \circ \eta_L = \epsilon \circ \eta_R = Id_A$$

- Composing with the identity leaves morphisms unchanged.

$$(Id_\Gamma \otimes \epsilon) \circ \Delta = (\epsilon \otimes Id_\Gamma) \circ \Delta = Id_\Gamma$$

- Associativity of composition.

$$(Id_\Gamma \otimes \Delta) \circ \Delta = (\Delta \otimes Id_\Gamma) \circ \Delta$$

- Inverting a morphism interchanges the source and target.

$$c \circ \eta_L = \eta_R, \quad c \circ \eta_R = \eta_L$$

- Composing with the inverse yields identity maps.

$$\begin{tikzcd}
\Gamma & \Gamma \otimes \Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] \arrow[d] & \Gamma \\
& \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

The above relations guarantee that the functor pair $\mathbf{Ring}(A,-)$ and $\mathbf{Ring}(\Gamma,-)$ takes values in **Grpd**. The maps $s = \eta_L^*$ and $t = \eta_R^*$ stand for taking source and target of a morphism respectively.

$$\mathbf{Ring}(\Gamma,-) \underset{t}{\overset{s}{\rightrightarrows}} \mathbf{Ring}(A,-)$$

Remark 2.6: It is clear from the definition that the ring Γ has two A module structure, which we refer to ${}_L\Gamma_R$, via the map η_L and η_R. The morphism c in (5) turns the source ${}_L\Gamma_R$ into ${}_R\Gamma_L$.

A stack is something analogue to a Hopf algebroid but satisfies a more strong gluing property.

Recall the definition of *Grothendieck topology*.

Definition 2.11: A Grothendieck topology over a category $\mathscr{C}$ is a collection $\mathscr{J}$ consisting of a collection of sets of morphisms $\{V_i \to V\}$ called *coverings*, such that

- Isomorphisms are coverings.
- If $\{U_i \to U\}$ and $\{U_{ij} \to U_i\}$ are coverings, then the composite $\{U_{ij} \to U\}$ is a covering.
- If $\{U_i \to U\}$ is a covering and $V \to U$ is a map, then $\{V \times_U U_i \to V\}$ is a covering.

The category $\mathscr{C}$ equipped with a Grothendieck topology $\mathscr{J}$ is called a *site*.

Example 2.8: Here's some basic and useful examples.

(1) The usual coverings of topological spaces in **Top**.

(2) The flat topology on affine schemes **Aff**. Coverings are $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$ such that

•$R \to R_i$ is flat.

•If an R module M satisfies $M \otimes R_i = 0$ for all i, then $M = 0$.

Definition 2.12: A sheaf (of sets) $\mathscr{F}$ over a site $(\mathscr{C}, \mathscr{J})$ is a contravariant functor from $\mathscr{C}$ to **Sets**, such that

$$\mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \times_U U_j)$$

is an equilizer diagram.

Definition 2.13: A sheaf of groupoids is a pair of sheaves (X_0, X_1) over $\mathscr{C}$ together with two sheaf maps

$$s, t : X_1 \longrightarrow X_0$$

standing for taking source and target.

Clearly, a Hopf algebroid (A, Γ) is a sheaf of groupoids over the site **Aff**.

Suppose (X_0, X_1) is a sheaf of groupoids and $\{U_i \to U\}$ is a covering. Let $Desc_{\{U_i\}}$ be the cateogry of *descent datum*, which consisting of

- **Objects** : An object is a collection of pairs $\{(E_i, \alpha_{ij})\}_{i,j}$, where $E_i \in X_0(U_i)$ and $\alpha_{ij} : E_i|_{U_i \times_U U_j} \to E_j|_{U_i \times_U U_j} \in X_1(U_i \times_U U_j)$ which satisfies the cocycle condition:

$$\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$$

- **Morphisms** : A morphism between $\{(E_i, \alpha_{ij})\}_{i,j}$ and $\{(E_i', \alpha_{ij}')\}_{i,j}$ is a collection $\{f_i\}$, with each $f_i : E_i \to E_i' \in X_1(U_i)$, which makes the diagram commutes

$$\begin{tikzcd}
E_i|_{U_i \times_U U_j} \arrow[r, "f_i"] \arrow[d, "\alpha_{ij}"'] & E_i'|_{U_i \times_U U_j} \arrow[d, "\alpha_{ij}'"] \\
E_j|_{U_i \times_U U_j} \arrow[r, "f_j"] & E_j'|_{U_i \times_U U_j}
\end{tikzcd}$$

Here we are now ready to define what so called stacks.

Definition 2.14: A stack is a sheaf of groupoids (X_0, X_1), which satisfies the *descent condition*:

$$(X_0, X_1)(U) \longrightarrow Desc_{\{U_i\}}$$

is an equivalence of groupoids, for all coverings $\{U_i \to U\}$.

We can obtain a stack from a sheaf of groupoids. To be explicit, let (X_0, X_1) be a sheaf of groupoids, there is an associated stack $\mathscr{M}_{(\mathscr{X}_0, \mathscr{X}_1)}$, which is defined by the universal property that for each stack $\mathscr{N}$, we have

$$ShGroupoids((X_0, X_1), \mathscr{N}) = Stacks(\mathscr{M}_{(X_0, X_1)}, \mathscr{N})$$

Remark 2.7: Since stacks take values in 2-category, then above equality is actually an equivalence of categories.

The stack $\mathscr{M}_{(X_0, X_1)}$ is the stackification of (X_0, X_1). Over each object U, the corresponding groupoid is

$$\mathscr{M}_{(X_0, X_1)}(U) = \operatorname*{colim}_{\{U_i \to U\}} Desc_{\{U_i\}}$$

where the colimit is taken over all coverings of U.

Definition 2.15: The moduli stack of formal groups $\mathscr{M}_{FG}$ is the associated stack of the Hopf algebroid (MU_*, MU_*MU). For each ring R, $\mathscr{M}_{FG}(R)$ is the groupoid of formal groups over R with isomorphisms. We also let $\mathscr{M}^s_{FG}$ denote the stack of formal groups with *strict* isomorphisms.

There is another convenient description of $\mathscr{M}_{FG}$ and $\mathscr{M}^s_{FG}$.

Let $G = \operatorname{Spec}\mathbb{Z}[b_1, b_2, \dots]$. The R-point $G(R)$ is the set of power series $g(t) = t + b_1 t + \cdots \in R[[t]]$. The group structure is given by the composition of power series. There is a natural action of G on $\operatorname{Spec} L$

$$G \times \operatorname{Spec} L \longrightarrow \operatorname{Spec} L$$

which is given by

$$(g,f) \longmapsto gf(g^{-1}(x), g^{-1}(y))$$

for all $g \in R[[t]]$ and formal group law f over R.

Therefore $\mathcal{M}^s_{FG}$ can be identified with the quotient stack $\operatorname{Spec} L/G$.

Let $G^+ = \operatorname{Spec}\mathbb{Z}[b_0^{\pm}, b_1, \dots]$. The action of G on $\operatorname{Spec} L$ extends over G^+ with the same formula and $\mathcal{M}_{FG}$ can be identified with $\operatorname{Spec} L/G^+$. Also note that $G + /G = \operatorname{Spec}\mathbb{Z}[b_0^{\pm}] = \mathbb{G}_m$.

According to Proposition 2.4, we know that

$$\mathcal{M}^s_{FG} \times \operatorname{Spec}\mathbb{Q} = \operatorname{Spec}\mathbb{Q}$$

and

$$\mathcal{M}_{FG} \times \operatorname{Spec}\mathbb{Q} = B\mathbb{G}_m$$

where $\mathbb{G}_m$ is the multiplicative groups scheme.

While in the p-local case, the stack $\mathcal{M}_{FG} \times \operatorname{Spec}\mathbb{Z}_{(p)}$ has a straightification along the height. There is a closed stack

$$\mathcal{M}^{\geq n}_{FG} = \operatorname{Spec}\left(L_{(p)}/(v_0, v_1, \dots, v_{n-1})\right)/G^+$$

which classifies formal groups of height at least n. The elements v_i's here classifies the height of a formal group. They are the coefficients of the term x^{p^i} in the p-series and has degree $2(p^i - 1)$.

Between successive two such closed substack, there is a locally closed substack

$$\mathcal{M}^{n}_{FG} = \mathcal{M}^{\geq n}_{FG} - \mathcal{M}^{\geq n+1}_{FG} = \operatorname{Spec}\left(L_{(p)}[v_n^{-1}]/(v_0, v_1, \dots, v_{n-1})\right)/G^+$$

In fact, the p-local $\mathcal{M}_{FG}$ is the associated stack of (BP_*, BP_*BP), where BP is the *Brown Peterson* specctrum, which is a direct summand of $MU_{(p)}$ with

$$\pi_* BP = \mathbb{Z}_{(\mathbb{p})}[v_1, v_2, \cdots], \ \ |v_i| = 2(p^i - 1)$$

Over $\overline{\mathbb{F}}_p$, the locally closed stack $\mathcal{M}^n_{FG}$ has a simple description

$$\mathcal{M}^n_{FG} = \operatorname{Spec}\overline{\mathbb{F}}_p/\mathbb{G}_n$$

where $\mathbb{G}_n$ is a profinite group

$$\mathbb{G}_n = \mathrm{Aut}(\overline{\mathbb{F}}_p, f)$$

called the *nth extended Morava stabilizer group*, where f is any formal group law of height n over $\overline{\mathbb{F}}_p$ (for they are all isomorphic). The profinite group $\mathrm{Aut}(\overline{\mathbb{F}}_p, f)$ consists of pairs (η, α), where η is an automorphism of $\overline{\mathbb{F}}_p$ and $\alpha : f \to \eta^* f$ is an isomorphism. It is also easy to see that $\mathbb{G}_n$ fis into a short exact sequence:

$$0 \to S_n \to \mathbb{G}_n \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

with S_n being the subgroup of $\mathbb{G}_n$, called the *nth Morava stabilizer group*, which fixes $\overline{\mathbb{F}}_p$.

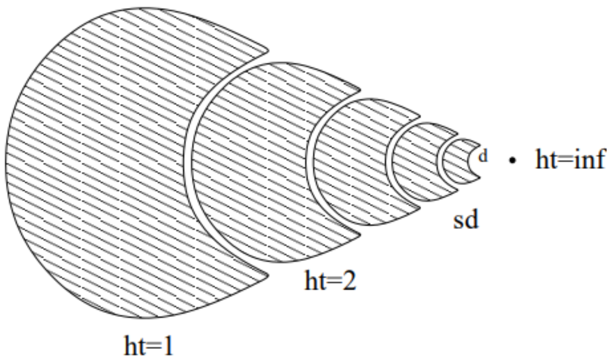


Figure 2-1 The p-local $\mathscr{M}_{FG}$,[79]

Now, back to topology, let X be any spectrum and E be a ring spectrum with E_*E is flat over E^*. The composite

$$E \wedge X \xrightarrow{\eta_L \wedge Id_X} E \wedge E \wedge X$$

induces

$$E_*X \to E_*(E \wedge X) \to E_*(E) \otimes_{E_*} E_*(X)$$

which exhibits a *comodule structure* over the Hopf algebroid (E_*, E_*E) on $E_*(X)$.

Definition 2.16: A comodule M over a Hopf algebroid (A, Γ) is an A module equipped with an A module map

$$\psi : M \longrightarrow \Gamma \otimes_A M$$

The map ψ is required to satisfies

(1) Counity: $M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\epsilon \otimes Id_M} M$ is identiy.

(2) Coassociativity:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A M \arrow[d, "Id_\Gamma \otimes \psi"] \\ \Gamma \otimes_A M \arrow[r, "\Delta \otimes Id_M"] & \Gamma \otimes_A \Gamma \otimes_A M \end{tikzcd}$$

Definition 2.17: a quasicohorent sheaf $\mathscr{F}$ over $\mathscr{M}_{fg}$ is a rule, which assigns each p : $\operatorname{Spec} R \to \mathscr{M}_{fg}$ an R-module $\mathscr{F}(p)$, such that for each commutative diagram

$$\begin{tikzcd} \operatorname{Spec} S \arrow[rr, "f"] \arrow[dr, "q"'] & & \operatorname{Spec} R \arrow[dl, "p"] \\ & \mathscr{M}_{fg} & \end{tikzcd}$$

in 2−category, there is an isomorphism

$$\alpha_{pq} : \mathscr{F}(q) \longrightarrow f^* \mathscr{F}(p) = S \otimes_R \mathscr{F}(p).$$

Example 2.9: A basic example is the structure sheaf $\mathscr{O}_{\mathscr{M}_{FG}}$, which assigns every q : $\operatorname{Spec} R \to \mathscr{M}_{FG}$ the R−module R itself and isomorphisms being the identity maps.

Proposition 2.6: The category $Qcoh(\mathscr{M}_{FG})$ consisting of all quasicoherent sheaves over $\mathscr{M}_{FG}$ is an abelian category.

Proposition 2.7: There is an equivalence

$$\varphi : \mathrm{Comod}_{(A,\Gamma)} \longrightarrow Qcoh(\mathscr{M}_{FG})$$

between the category of comodules over the Hopf algebroid $(A, \Gamma) = (MU_*, MU_*MU)$ and the category of quasicoherent sheaves over $\mathscr{M}_{FG}$.

Proof: We define the inverse functor ϕ to be

$$\phi : \mathscr{F} \mapsto \mathscr{F}(\operatorname{Spec} A \to \mathscr{M}_{FG}) := M,$$

where $\operatorname{Spec} A \to \mathscr{M}_{FG}$ is the canonical map which classifies the universal formal group law. The diagram

$$\begin{tikzcd} \operatorname{Spec} \Gamma \arrow[r, "\eta_L"] \arrow[d, "\eta_R"] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \mathscr{M}_{FG} \end{tikzcd}$$

implies there is a unique isomorphism as Γ modules

$$\alpha : \Gamma \underset{A,\eta_L}{\otimes} M = \eta_L^* M \to \eta_R^* M = \Gamma \underset{A,\eta_R}{\otimes} M.$$

Since Γ is an A−module via the map η_L, we can obtain an A−module map

$$\psi : M \to \Gamma \underset{A,\eta_L}{\otimes} M \to \Gamma \underset{A,\eta_R}{\otimes} M,$$
$$\psi : m \mapsto 1 \otimes m \mapsto \alpha(1 \otimes m).$$

Counitality: Consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[dr, "\epsilon"] \arrow[drr, dashed, bend left, "id"] \arrow[ddr, dashed, bend right, "id"'] & & \\
& \operatorname{Spec} \Gamma \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \operatorname{Spec} A \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \mathscr{M}_{FG}.
\end{tikzcd}$$

Solid arrows yield an isomorphism

$$id \otimes \alpha : A \underset{\Gamma,\epsilon}{\otimes} \Gamma \underset{A,\eta_L}{\otimes} M \to A \underset{\Gamma,\epsilon}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M,$$

which equals to the identity isomorphism $id : M \to M$ induced by dashed arrows, by the commutativity. The map

$$\epsilon \otimes id : \Gamma \underset{A,\eta_R}{\otimes} M \to M$$

is the same as the canonical map

$$\Gamma \underset{A,\eta_R}{\otimes} M \to A \underset{\Gamma,\epsilon}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M.$$

Therefore the composite

$$M \xrightarrow{\psi} \Gamma \underset{A,\eta_R}{\otimes} M \to A \underset{\Gamma,\epsilon}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M$$

will send m to

$$m \mapsto \psi(m) = \alpha(1 \otimes m) \mapsto 1 \otimes \alpha(1 \otimes m) = m,$$

which yields the counitality.

Coassociativity: Consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} \Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma \arrow[rr, dashed, "p_2"] \arrow[dr, "\Delta"] \arrow[dd, dashed, "p_1"'] & & \operatorname{Spec} \Gamma \arrow[d, dashed, "\eta_R"] \\
& \operatorname{Spec} \Gamma \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} \Gamma \arrow[r, dashed, "\eta_L"] & \operatorname{Spec} A \arrow[r] & \mathscr{M}_{FG}
\end{tikzcd}$$

Again, the solid arrows give an isomorphism of $\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma-$modules:

$$(\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,\Delta}{\otimes} \Gamma \underset{A,\eta_L}{\otimes} M \to (\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,\Delta}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M,$$

which sends m to $(\Delta \otimes id) \circ \psi(m)$. On the other hand, we have another isomorphism which equals to the previous one, induced by dashed arrows:

$$(\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_1}{\otimes} \Gamma \underset{A,\eta_L}{\otimes} M \to (\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_2}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M.$$

To calculate it, we use the following diagram

$$\begin{tikzcd}
\operatorname{Spec} \Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma \arrow[r, "p_2"] \arrow[d, "p_1"] & \operatorname{Spec} \Gamma \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \operatorname{Spec} A \arrow[dd] \\
\operatorname{Spec} \Gamma \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \operatorname{Spec} A \arrow[dr] & \\
\operatorname{Spec} A \arrow[rr] & & \mathscr{M}_{FG}
\end{tikzcd}$$

Now, the desired isomorphism is

$$\begin{tikzcd}
(\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_1}{\otimes} \Gamma \underset{A,\eta_L}{\otimes} M \arrow[r] & (\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_1}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M \arrow[d] & \\
& (\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_2}{\otimes} \Gamma \underset{A,\eta_L}{\otimes} M \arrow[r] & (\Gamma \underset{\eta_R,A,\eta_L}{\otimes} \Gamma) \underset{\Gamma,p_2}{\otimes} \Gamma \underset{A,\eta_R}{\otimes} M
\end{tikzcd}$$

which sends m to $(id \otimes \psi) \circ \psi(m)$, hence yields the coassociativity.

Now we define the functor φ. Suppose M is an (A, Γ) comodule and $p : \operatorname{Spec} A \to \mathscr{M}_{FG}$ is the canonical map classifying the universal formal group over A. We define a quasicoherent sheaf $p_* M$ as follow. If $\operatorname{Spec} R \xrightarrow{q} \mathscr{M}_{FG}$ factors through

$$\operatorname{Spec} R \xrightarrow{f} \operatorname{Spec} A \to \mathscr{M}_{fg}$$

then we define $p_* M(q) = f^* M = R \underset{A,f}{\otimes} M$.

In general case, we can take a cover $\{\operatorname{Spec} U_i \to \operatorname{Spec} R\}$ of $\operatorname{Spec} R$, such that each $\operatorname{Spec} U_i$ factors through $\operatorname{Spec} A$. we can define each $p_* M(U_i)$, then patch them together.

The isomorphisms can be checked on the universal case

$$\begin{tikzcd}
\operatorname{Spec} \Gamma \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \mathscr{M}_{FG}
\end{tikzcd}$$

Since M is an $(A, \Gamma)-$comodule, there is an $A-$module map $\psi : M \to \Gamma_R \otimes_A M$, which extends to a $\Gamma-$module map

$$\widetilde{\psi} : \Gamma_L \otimes_A M \longrightarrow \Gamma_R \otimes_A M$$
$$\widetilde{\psi} : \quad \gamma \otimes m \longmapsto \gamma \cdot \psi(m).$$

It is an isomorphism indeed, with inverse

$$\Gamma_R \otimes_A M = \Gamma \underset{\Gamma,c}{\otimes} \Gamma_L \otimes_A M \xrightarrow{id\otimes\widetilde{\psi}} \Gamma \underset{\Gamma,c}{\otimes} \Gamma_R \otimes_A M = \Gamma_L \otimes_A M$$

which sends

$$\gamma \cdot (1 \otimes m) = \gamma \otimes m = \gamma \otimes 1 \otimes m \mapsto \gamma \otimes \widetilde{\psi}(m) = \gamma \cdot (c \otimes id)(\widetilde{\psi}(m)),$$

where c is the conjugation $c : {}_{\eta_L}\Gamma_{\eta_R} \to {}_{\eta_R}\Gamma_{\eta_L}$ (standing for taking inverse of isomorphisms.)

Now the composite

$$\begin{aligned}(id \otimes c \otimes id) \circ (id \otimes \widetilde{\psi}) \circ \widetilde{\psi} &= (id \otimes c \otimes id) \circ (\Delta \otimes id) \circ \widetilde{\psi} \\ &= (((id \otimes c) \circ \Delta) \otimes id) \circ \widetilde{\psi} \\ &= ((\eta_L \circ \epsilon) \otimes id) \circ \widetilde{\psi}\end{aligned}$$

is a $\Gamma-$module map. Evaluating at $(1 \otimes m)$ yields

$$\begin{aligned}((\eta_L \circ \epsilon) \otimes id) \circ \widetilde{\psi}(1 \otimes m) &= ((\eta_L \circ \epsilon) \otimes id) \circ \psi(m) \\ &= (\eta_L \otimes id) \circ (\epsilon \otimes id) \circ \psi(m) \\ &= (\eta_L \otimes id)(1 \otimes m) \\ &= (1 \otimes m)\end{aligned}$$

which completes the proof. ■

So far, we start with a complex oriented cohomology theories from topology, and end up with formal groups in algebraic setting. Could we invert this progress, which means start with a formal group and result a spectrum?

To be explicit, let F_R be a formal group (law) over a graded ring R, is there a ring spectrum E_R such that,

- $\pi_* E_R = R$
- $\mathrm{Spf}(E_R^*(\mathbb{C}P^\infty)) = F_R$

Remark 2.8: If R is ungraded, we simply replace R with $R[\beta^\pm]$, where $|\beta| = 2$ and require that $\pi_* E_R = R[\beta^\pm]$, and $\mathrm{Spf}\, E_R^0(\mathbb{C}P^\infty) = F_R$.
Without many efforts, the problem can be reduced to when $MU_*(-) \otimes_{MU_*} R$ is a homology theory. Then applying Brown representation theorem to get the desired spectrum.

At first glance, the ring R is required to be flat over MU_* to preserve exact sequences in the axioms of homology theory. But this condition is too strong. On the other hand, since we could regard $MU_*(X)$ as a quasicoherent sheaf over the stack $\mathscr{M}_{FG}$, we could only ask the map

$$q : \operatorname{Spec} R \longrightarrow \mathscr{M}_{FG}$$

which classifies F_R, being flat. Then the exactness is guaranteed by the functor

$$q^* : Qcoh(\mathscr{M}_{FG}) \longrightarrow Qcoh(\operatorname{Spec} R) = \mathrm{Mod}_R$$

is exact. This is precisely the *Landweber exact functor theorem*.

Theorem 2.4 (Lecture 15,[80]): Let M be a graded module over MU_*. If M is flat over $\mathscr{M}_{FG}$, then the functor

$$X \longmapsto MU_*(X) \otimes_{MU_*} M$$

is a homology theory.

There is also a convenient criteria on the flatness of M over $\mathscr{M}_{FG}$.

Theorem 2.5 (Landweber, 76): The MU_* module M is flat over $\mathscr{M}_{FG}$ if and only if the sequence $(v_0 = p, v_1, \cdots)$ acts regularly on M for all p, which means that

$$v_i : M/(v_0, \dots, v_{i-1}) \xrightarrow{\cdot v_i} M/(v_0, \dots, v_{i-1})$$

is an injection.

Example 2.10: The complex K-theory is a first example of Landweber exact functor theorem, such that

$$MU_*(X) \otimes_{MU_*} K_* \to K_*(X)$$

is an isomorphism, called *Conner-Floyd isomorphism*[81].

2.3 Deformation of Formal Groups

In this section, we review the deformation theory of formal groups over perfect and imperfect fields.

The completion of a scheme X along a closed point captures the local behavior around that closed point. From this point of view, we study deformations of a height n formal group over field k of characteristic p.

Definition 2.18: Let k be a field with $p = 0$ and Γ be a formal group of height n over k. A deformation of Γ over R consists of a triple (R, G, α):

- R is a complete local Noetherian ring with residue field $R/\mathfrak{m}$ a k-algeba.
- G is a formal group over R.
- $\alpha : \Gamma \otimes_k R/\mathfrak{m} \to G \otimes_R R/\mathfrak{m}$ is an isomorphism over $R/\mathfrak{m}$.

We say two deformations G and G' over R are $\star$-isomorphic, if there is an isomorphism

$$f : G \to G'$$

over R, which makes the diagram commute

$$\begin{tikzcd} \Gamma \otimes_k R/\mathfrak{m} \arrow[r, "\alpha"] \arrow[d, "Id"] & G \otimes_R R/\mathfrak{m} \arrow[d, "f"] \\ \Gamma \otimes_k R/\mathfrak{m} \arrow[r, "\alpha'"] & G' \otimes_R R/\mathfrak{m} \end{tikzcd}$$

Suppose k is a perfect field and R is a complete local Noetherian ring with the residue field $R/\mathfrak{m}$ a k algebra. Then there is a ring $W(k)$, called the *ring of Witt vectors*, and a *unique* ring map from $W(k)$ to R, makes the diagram commute.

$$\begin{tikzcd} W(k) \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r] & R/\mathfrak{m} \end{tikzcd}$$

The moduli space $\mathrm{Def}_G(R)$ classifying $\star$-isomorphism classes of deformations of Γ over R is a groupoid for there could be nontrivial $\star$-automorphism. The following proposition rules out this probability, hence it is actually discrete, which means $\mathrm{Def}_G(R)$ is a set.

Proposition 2.8 (Lemma 1.1.2[82]): Let F_0, F_1 be lifts (deformations) of a formal group F over $A \in \mathbf{Art}_k$; then the homomorphism

$$\rho_A : \mathrm{Hom}_A(F_0, F_1) \longrightarrow \mathrm{End}_k(F)$$

defined by reduction modulo $\mathfrak{m}_A$ is *injective*.

Moreover, the following theorem says that this moduli probelem is representable.

Theorem 2.6 (Lubin-Tate, 66): Suppose k is a perfect field of characteristic p and Γ is a formal group of height n over k. Then there is an bijection between sets.

$$\mathrm{Hom}_{cts}(W(k)[[v_1, \dots, v_{n-1}]], R) = \mathrm{Def}_\Gamma$$

There is also a formal group Γ_u defined over $W(k)[[v_1, \dots, v_{n-1}]]$ and for any deformation F of Γ over R, there is a continuous map

$$f : A_0 = W(k)[[v_1, \dots, v_{n-1}]] \to R$$

such that F is $\star$-isomorphic to $f^*\Gamma_u$.

The map from MU_* or BP_* to $W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm}]$

$$v_i \mapsto v_i \beta^{1-p^i} \text{ for } i < n;$$
$$v_n \mapsto \beta^{1-p^n};$$
$$v_i \mapsto 0 \text{ for } i > n$$

satisfies the Landweber celebrated criteria and hence there is a corresponding spectrum, called the *Morava E*-theory of height n, denoted by E_n, with

$$\pi_* E_n = W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm}], \ |\beta| = 2$$

Remark 2.9: In[83], the ring spectrum E_n has been shown that it admits an *essential unique* E_∞ structure using the obstruction theory. After that Lurie gave a constructional proof of this in his elliptic cohomology Part II.

Example 2.11: When the height $n = 1$, the corresponding A_0 is just $W(k)$. Since any Artinian ring R with residue field a k algebra has a unique $W(k)$ algebra structure, any two deformations are $\star$−isomorphic.

Example 2.12: Let $k = \mathbb{F}_p$ and $\Gamma = x + y + xy$ be the multiplicative formal group law. The corresponding spectrum E_1 has

$$\pi_* E_1 = \mathbb{Z}_p[\beta^{\pm}]$$

In fact, the spectrum E_1 is homotopy equivalent to the p-completed K-theory, $\widehat{K_p}$.

Example 2.13: When the height $n = 2$, the corresponding E_2 with

$$\pi_* E_2 = W(k)[[h]][\beta^{\pm}]$$

is an example of *elliptic cohomology*. It corresponds to formal neighborhoods of supersingular elliptic curves on the stack $\mathscr{M}_{ell}$ of elliptic curves. The element h stands for the *Hasse invariant*, which parameterizes the supersingularity of elliptic curves.

As mentioned in the beginning of this section, the deformation of formal groups characterizes a small neighborhood of a closed point on p-local $\mathscr{M}_{FG}$. The data of deformations then tells how to patch these open strata $\mathscr{M}^n_{FG}$ together and there are spectra E_n associated to these small neighborhood.

While there is also a spectrum $K(n)$, called *Morava K-theory*, associated to a height n closed point, with

$$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm}], \ \ |v_n| = 2(p^i - 1)$$

Remark 2.10: The construction of $K(n)$ is a little subtle. It is not Landweber exact and it is not an E_∞ ring for all n.

Having discussed the deformation theory of formal groups over perfect fields, let us consider the case over imperfect fields, in particular over the field $k((x))$, where k is a perfect field. Such deformation theory is essential to the power operation on $L_{K(n-1)}E_n$.

Let us first recall the Lubin-Tate theory, which can be restated as follow.

Theorem 2.7 (Lubin-Tate): Let $A \in \mathbf{Art}_k$, and Γ be a height n formal group over k. Then the moduli problem of deformations of Γ over Artinian A algebra is represented by

$$\mathrm{Def}_\Gamma^A(R) = \mathrm{Hom}_{cts}^A(A[[v_1, \dots, v_{n-1}]], R)$$

where the right hand Hom is required to be continuous A algebra map.

The Theorem 2.6 is a direct conculsion of 2.7, for any Artinian local ring with residue field a k-algebra has a unique structure of $W(k)$ algebra, when k is perfect.

While the field k is not perfect, we have the following proposition.

Proposition 2.9: Suppose R is a complete local ring with maximal ideal $\mathfrak{m}$. There is a map from $C(k)$ to R filling the diagram

$$\begin{tikzcd} C(k) \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r] & R/\mathfrak{m} \end{tikzcd}$$

where $C(k)$ is a complete local ring with residue field k, called the *Cohen ring* of k.

Remark 2.11: In general, the map filling the diagram is *not* unique. While the field k is perfect, $C(k)$ is isomorphic to the $W(k)$ and the map is unique.

Example 2.14: The cohen ring for $k((x))$ is non-cononically isomorphic to $\Lambda = W(k)((x))_p^\wedge$, which is the p-adic completion of the ring of Laurent series in variable x over the coefficients $W(k)$.

From this point of view, Vankoughnett developed the theory of augmented deformations.

Definition 2.19 (Definition 4.9[18]): Let $\mathbb{H}$ be a height $n-1$ formal group over $k((x))$. An augmented deformation of $\mathbb{H}$ consists of a triple $(G/R, i, \alpha)$.

- G is a formal group over R and R is a complete Noetherian local ring.
- A continuous map $i : \Lambda = W(k)((x))_p^\wedge \to R$
- An isomorphism $\alpha : \mathbb{H} \otimes_{k((x))} R/\mathfrak{m} \to G \otimes_R R/\mathfrak{m}$.

We say two augmented deformations $(G/R, i, \alpha)$ and $(G'/R, i', \alpha')$ are $\star$-isomorphic if

- $i_1 = i_2$
- there is an isomorphism $f : G \to G'$ over R

- the following diagram commutes.

$$\begin{array}{ccc} \mathbb{H} \otimes_{k((x))} R/\mathfrak{m} & \xrightarrow{\alpha} & G \otimes_R R/\mathfrak{m} \\ \downarrow Id & & \downarrow f \\ \mathbb{H} \otimes_{k((x))} R/\mathfrak{m} & \xrightarrow{\alpha'} & G' \otimes_R R/\mathfrak{m} \end{array}$$

Remark 2.12: Note that in the second condition of the above definition, we are not only required a $k((x))$ algebra structure of $R/\mathfrak{m}$, as which in Definition 2.18, but also a continuous map $\Lambda \to R$, which induces a $k((x))$ algebra structure of $R/\mathfrak{m}$. This is because, as stated in Remark 2.11, the map from Λ to R inducing a given $k((x))$ algebra structure on $R/\mathfrak{m}$ is not unique. Hence we must specify one.

The moduli problem of augmented deformations is also representable.

Theorem 2.8 (Vankoughnett, 22): The moduli problem $\mathrm{Def}_{\mathbb{H}}^{\mathrm{aug}}$, which classifies augmented deformations of $\mathbb{H}$ upto $\star$-isomorphism is represented by

$$\Lambda[[v_1, \dots, v_{n-1}]] = W(k)((x))_p^{\wedge}[[v_1, \dots, v_{n-1}]]$$

To be explicit, let R be complete Noetherian local ring, we have

$$\mathrm{Def}_{\mathbb{H}}^{\mathrm{aug}}(R) = \mathrm{Hom}_{cts}(\Lambda[[v_1, \dots, v_{n-1}]], R)$$

2.4 Bousfield Localizations

In this section, we review the theory of Bousfield localization, and introduce the main object $L_{K(n-1)}E_n$ we concerned.

Let us begin with an example first. The rational homology can only recognize information up to $\mathbb{Q}$, but has no ideas about torsion parts. This can be viewed as a blindness of $H\mathbb{Q}$ in the sense of[84].

Let C be a full subcategory of **Sp**, which is stable under shift and homotopy colimits. The inclusion $C \to \mathbf{Sp}$ preserves homotopy colimits, and hence we have an right adjoint:

$$G : \mathbf{Sp} \to C$$

We let $L(X)$ denote the cofiber

$$G(X) \to X \to L(X)$$

It is easily verified that for any $Y \in C$, we have an equivalence

$$Map(Y, G(X)) \cong Map(Y, X)$$

Therefore $Map(Y, L(X)) \cong *$.

Let $C^{\perp}$ be the full subcategory of **Sp** consisting of all spectra L such that $Map(Y, L) \cong *$ for all $Y \in C$. Suppose $Z \in C\ \perp$, then we have

$$Map(L(X), Z) \cong Map(X, Z)$$

Hence we can regard L as a left adjoint of the inclusion $C^{\perp} \to \mathbf{Sp}$.

Definition 2.20: Let E be a spectrum. Let C be the category of E-acyclic spectra and spectra in $C^{\perp}$ are called E-local spectra. The functor $L = L_E$ can be chracterized as follow:

（1）$L_E(X)$ is an E-local spectrum.

（2）there is a natrual transformation $\eta : Id \to L$, which is an E-equivalence for all $X \to L_E(X)$.

A map f is an E-equivalence if $E \wedge f$ is an equivalence.

Example 2.15: 1. $E = M\mathbb{Z}/p$, then we have $L_E(X) = X_p^{\wedge}$, where

$$X_p^{\wedge} = \text{holim}\{\cdots \to X \wedge M\mathbb{Z}/p^2 \to X \wedge M\mathbb{Z}/p\}$$

2. $E = M\mathbb{Q} = H\mathbb{Q}$, then $L_E = L_{\mathbb{Q}}$ is the rationalization of X.

Example 2.16: The most important example is the Bousfield localization respect to $K(n)$[79,85]. Let E be a p-local complex oriented ring spectrum, then the $K(n)-$localization of E is

$$L_{K(n)}E = \underset{I \in \mathbb{N}^n}{\text{holim}}\ v_n^{-1}E/(p, v_1, \ldots, v_{n-1})^I$$

When E is torsion free and concentrated in even degrees, we have

$$\pi_* E = v_n^{-1}(\pi_* E)^{\wedge}_{(p, v_1, \ldots, v_{n-1})}$$

In particular, we have the $K(n-1)$-localized E_n has coefficients

$$\pi_* L_{K(n-1)}E_n = W(k)((u_{n-1}))_p^{\wedge}[[u_1, \ldots, u_{n-2}]][\beta^{\pm}]$$

and

$$\pi_* L_{K(1)}E_2 = W(k)((h))_p^{\wedge}[\beta^{\pm}]$$

It follows from Theorem 2.8 that $\pi_0 L_{K(n-1)}E_n$ classifies augmented deformations of a height $n-1$ formal group over the field $k((u_{n-1}))$.

CHAPTER 3 $K(n-1)$-LOCALIZED E-THEORY OF SYMMETRIC GROUPS

Let E be the Morava E-theory associated to a height n formal group over a field k, and F be the $K(n-1)$-localization of E. The coefficient ring

$$F^* = W(k)((u_{n-1}))^{\wedge}_{p}[[u_1, \dots, u_{n-2}]][u^{\pm}]$$

is a Noetherian complete local ring with the maximal ideal $(p, u_1, \dots, u_{n-2})$. It satisfies the conditions in[86] (Section 1.3), in particular, $p^{-1}F^* \neq 0$ by direct computation.

In this chapter, we calculate the ring $F^*B\Sigma_k$ and $F^*B\Sigma_k/I$, where I is the ideal generated by images of proper transfers, using the framework in[73] and give an interpretation of the additive total power operation ring $F^*B\Sigma_k/I$ in terms of subgroups of a certain formal group. The method for showing this is first show that $F^*B\Sigma_k$ is free over F^*, this relies on a version of the theory of good groups[86] (Section 7). Then we identify $F^*B\Sigma_k/I$ as indecomposables in $F^*B\Sigma_k$. The rank formula follows from Theorem C in[86]. After that, we identify the ring $F^*B\Sigma_{p^k}/I$ as the ring of functions of $\mathrm{Sub}_k(\mathbb{G}_F)$ and using Corollary 10.12 in[87] to deduce its rank.

3.1 Calculations of $F^*B\Sigma_k$ and $F^*B\Sigma_k/I$

Our conclusion is as follows:

Theorem 3.1: The ring $F^0B\Sigma_k$ is a Noetherian local ring which is free over F^0 of rank $d(n-1,k)$, which is defined to be the number of isomorphism classes of order k sets with an action of $\mathbb{Z}_p^{n-1}$.

To start, we shall prove two lemmas.

Definition 3.1: Let K be a spectrum, which is a graded field, i.e. every K-module splits as a direct sum of copies of K. We say an element x in $K^*(BG)$ is good, if x is a transferred Euler class of a subrepresentation of G, i.e. $x = \mathrm{Tr}_H^G(e(\rho))$. A group G is good if $K^*(BG)$ is generated by good elements over K^*. Of course, $K^*(BG)$ is concentrated in even degrees if G is good.

Example 3.1: The most important example of K needed in this paper is

$$K = K_{u_{n-1}} := u_{n-1}^{-1}E/(p, u_1, \dots, u_{n-2})$$

with

$$K_{u_{n-1}*} = k((u_{n-1}))[u^{\pm}]$$

Proposition 3.1: The following properties for being good hold.

(1) Every finite abelian group is good.

(2) G is good if its Sylow p-subgroup is good.

(3) If x_1 is a good element in $K^*(BG_1)$ and x_2 is good in $K^*(BG_2)$, then so is their product in $K^*(BG_1 \times BG_2)$.

(4) If $f : H \to G$ is any homomorphism and x is good in $K^*(BG)$, then $f^*(x)$ is a linear combination of good elements in $K^*(BH)$.

(5) If x and y are both good, then their cup product xy is a sum of good elements.

Proof:

(1) We only need to consider p-components of G, then reduce the case to $G = \mathbb{Z}/p$. While $K^*(B\mathbb{Z}/p) = K^*[x]/[p]_F(x)$ and x is the Euler class of any line bundle corresponding to a generator of the character group $\mathbb{Z}/p^*$.($\alpha : \mathbb{Z}/p \to S^1$ will induce a map $B\mathbb{Z}/p \to BS^1 = \mathbb{C}P^\infty$, and x is the Euler class of the corresponding line bundle.)

(2) The map $\mathrm{Tr}^* : K^*(BG_p) \to K^*(BG)$ is surjective.

(3) Suppose $x_1 = \mathrm{Tr}_{H_1}^{G_1}(e(\rho_1))$ and $x_2 = \mathrm{Tr}_{H_2}^{G_2}(e(\rho_2))$. We have

$$x_1 \times x_2 = \mathrm{Tr}_{H_1 \times H_2}^{G_1 \times G_2}(e(\rho_1 \oplus \rho_2))$$

.

(4) Suppose $x = \mathrm{Tr}_H^G(e(\rho))$. We have

$$\begin{array}{ccc} \prod BK_\alpha & \xrightarrow{\prod f_\alpha} & BH \\ \downarrow & & \downarrow \\ BK & \xrightarrow{Bf} & BG \end{array}$$

The naturality of transfer maps yield

$$f^*(x) = \sum \mathrm{Tr}(e(f_\alpha^*(\rho))).$$

(5) If x and y are good, then $x \times y$ is good in $K^*(B(G \times G))$. Composing with the diagonal $\Delta : G \to G \times G$ gives the cup product xy.

■

Remark 3.1: Not all groups are good. In fact, let $K = K(n)$ for each p, there are examples of groups which are not good. For $p > 2$, $n \geq 2$, the Sylow p-subgroup $P = (\mathbb{Z}/p)^4 \rtimes (\mathbb{Z}/p)^2$ of $GL_4(\mathbb{F}_p)$ works. See[88] for detail calculations.

To show a group G is good, we may consider its Sylow p-subgroups. In practice, a lot of such groups have wreath product expressions. For example, Sylow p-subgroups of $B\Sigma_k$ is a (product) of iterated wreath product of $\mathbb{Z}/p$ with itself. Thus it is wonderful if the following is true.

Lemma 3.1 (The Wreath Product Lemma): If G is good, then so does the wreath product $G \wr \mathbb{Z}/p$.

Proof: We shall first recall the proof for $K = K(n)$ in[86].

Let W denote $G \wr \mathbb{Z}/p$. Consider the sequence

$$1 \to G^p \to W \to \mathbb{Z}/p \to 1$$

which induces a fiber sequence

$$BG^p \to BW \to B\mathbb{Z}/p.$$

We have the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}(BW) = H^*(B\mathbb{Z}/p, K(n)^*(BG^p)) \Rightarrow K(n)^*(BW)$$

The action of $\mathbb{Z}/p$ over G^p is a cyclic permutation, hence $\mathbb{Z}/p$ acts on $K(n)^*(BG^p) = \otimes K(n)^*(BG)$ by permutation too. Since $K(n)^*(BG)$ is finitely generated, we can choose a basis $\{x_i\}$ of $K(n)^*(BG)$, then

$$K(n)^*(BG^p) = F \oplus T$$

The module F is a free $\mathbb{Z}/p$ module, with basis $\{x_{i_1} \otimes \cdots \otimes x_{i_p}\}$ such that not all i_j are same. The module T has trivial $\mathbb{Z}/p$ action. Therefore, the E_2 page can be identified with

$$E_2 = H^*(B\mathbb{Z}/p, F \oplus T) = H^*(\mathbb{Z}/p, F) \oplus H^*(\mathbb{Z}/p, T).$$

A simple calculation implies

$$H^*(\mathbb{Z}/p, F) = \begin{cases} F^{\mathbb{Z}/p}, & * = 0 \\ 0 \quad , & \text{else} \end{cases}$$

and

$$H^*(\mathbb{Z}/p, T) = H^*(B\mathbb{Z}/p) \otimes T$$

with

$$H^*(B\mathbb{Z}/p) = E(u) \otimes P(x)$$

where $|u| = 1$ and $|x| = 2$.

From above analysis, we find that the part $E_2^{\geq 1,*}$ of the E_2 page is $H^{\geq 1}(B\mathbb{Z}/p) \otimes T$.

Since we alredy know that the spectral sequence

$$E_2^{*,*}(B\mathbb{Z}/p) = H^*(B\mathbb{Z}/p, K(n)^*) \Rightarrow K(n)^*(B\mathbb{Z}/p) = K(n)^*[x]/v_n x^{p^n},$$

the only nonzero differential is

$$d_{2p^n-1}(u) = v_n x^{p^n}.$$

Hence we conclude that for $r \geq 2$, the $E_r^{\geq 1,*}(BW)$ page is isomorphic to $E_2^{\geq 1,*}(B\mathbb{Z}/p) \otimes T$. In particular, when $r \geq 2p^n$, there are no differentials in this area.

If the elements in $E_2^{0,*}(BW)$ are all permanent cycles, which means there are also no differentials starting from the 0th column, and then we have

$$E_r^{*,*}(BW) = H^0(\mathbb{Z}/p, F) \oplus (E_r^{*,*}(B\mathbb{Z}/p) \otimes T),$$

and

$$K(n)^*(BW) = F^{\mathbb{Z}/p} \oplus (K(n)^*(B\mathbb{Z}/p) \otimes T).$$

The last identity implies W is a good group directly. ■

Lemma 3.2: The elements in $E_2^{0,*}(BW)$ are all permanent cycles, which are linear combinations of good elements.

Proof: The proof falls into two parts.

An element in $F^{\mathbb{Z}/p} \subset K(n)^*(BG^p)$ is a permanent cycles if and only if it is an image of the restriction map $K(n)^*(BW) \to K(n)^*(BG^p)$. Note that $F^{\mathbb{Z}/p}$ is generated by $\sigma(x) = \sum_{\sigma_i \in \mathbb{Z}/p} \sigma_i(x)$, $x \in K(n)^*(BG^p)$, i.e. the sum of orbits of x. The composite

$$K(n)^*(BG^p) \xrightarrow{\mathrm{Tr}} K(n)^*(BW) \xrightarrow{\mathrm{Res}} K(n)^*(BG^p)$$

will send x to $\sigma(x)$.

An element in T is of the form $x \otimes \cdots \otimes x$ for $x \in K(n)^*(BG)$. We can assume $x = \mathrm{Tr}_H^G(e(\rho))$ is a transferred Euler class. The representation $\rho \oplus \cdots \oplus \rho$ is a representation of H^p, which extends to a representation $\widehat{\rho}$ of $H \wr \mathbb{Z}/p$. The result follows from the diagram.

$$\begin{tikzcd}
K(n)^*(BH^p) \arrow[d, "\mathrm{Tr}"] & K(n)^*(B(H \wr \mathbb{Z}/p)) \arrow[l, "\mathrm{Res}"'] \arrow[d, "\mathrm{Tr}"] \\
K(n)^*(BG^p) & K(n)^*(BW) \arrow[l, "\mathrm{Res}"']
\end{tikzcd}$$

■

Proof of Lemma 3.1: From the proof in $K = K(n)$, we find that the only properties used in the proof are K unneth formulas, $K(n)^* = \mathbb{F}_p[u^{\pm}]$ and the *additive* structure of the Atiyah Hirzebruch spectral sequence of $K(n)^*(B\mathbb{Z}/p)$. Hence we can replace $K(n)$ by any field spectrum K of chromatic height n, since different choices of height n formal groups only effect the multiplicative strucutre on the spectral sequence. The K uneth formula is proved below ■

Proposition 3.2: There is a linearly duality between F-homology and cohomology, i.e.

$$F^*(X) = \mathrm{Hom}_{F_*}(F_*(X), F^*).$$

For spectra X and Y, we have a Künneth homeomorphism:

$$F^*(X)\widehat{\otimes}F^*(Y) \xrightarrow{\sim} F^*(X \wedge Y)$$

Proof: There are two ways to see the first statement. One way is applying the universal coefficient spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{F_*}^{*,*}(F_*(X), F^*) \Rightarrow F^*(X).$$

Since all things are free F_* modules. The E_2 page collapses and we only has the 0th column, i.e. the Hom part.

The second way is to look at the Serre spectral sequences. The homological and cohomological spectral sequences are dual to each other (both terms and differentials), which yields the conclusion.

The second statement is[89] (Theorem 4.19). ■

There is also a finiteness property on even periodic field spectra.

Proposition 3.3: For each finite group G, $K^*(BG)$ is finite as K^* modules.

Proof: This is proved for $K = K(n)$ in [Rav82] of which I haven't found the citation link. We will recall his proof in our settings.

First, we may assume G is a p-group, for the surjectivity of transfer maps. We can find a normal subgroup H of G with index p, and a group $\widehat{G}$ with $\widehat{G}/H \cong \mathbb{Z}$.

Assume $F^*(BH)$ is finite. The fiber sequence

$$BH \to B\widehat{G} \to S^1$$

implies $F^*(B\widehat{G})$ is finite.

Consider the map between fiber sequences

$$\begin{tikzcd}
B\widehat{G} \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d, "\mathrm{Id}"] \\
S^1 \arrow[r] & B\mathbb{Z}/p \arrow[r] & \mathbb{C}P^\infty
\end{tikzcd}$$

The Atiyah Hirzebruch spectral sequence for the bottom row implies there is a differential killing x^{p^n} for $K^*(B\mathbb{Z}/p) = F^*[x]/g(x)$, where $g(x)$ is the degree p^n Weierstrass polynomial associated to $[p]_K(x)$.

Finally, we see that $E_r^{*,*}(BG)$ is a module over $E_r^{*,*}(B\mathbb{Z}/p)$. Hence x^{p^n} is killed in $E_r^{*,*}(BG)$. The finiteness of $K^*(BG)$ follows from it of $K^*(B\widehat{G})$. ■

Now we can start our calculations on $F^*B\Sigma_k$ and $F^*B\Sigma_k/I$.

Proposition 3.4: $F^*B\Sigma_k$ is finitely generated over F^*.

Proof: This is a consequence of[90] (Corollary 4.4). We need to verify F is admissible in the sense of[90] (Definition 2.1). E^0 is Noetherian and both localization and completion preserve Noetherianess. Hence F^0 is Noetherian and all other conditions are satisfied automatically. ■

Proposition 3.5: $F^*B\Sigma_k$ is free over F^*, concentrated in even degrees.

Proof: From[73] (Proposition 3.6), we know that E^*BG is concentrated in even degrees. Let $u_{n-1}^{-1}E$ be the homotopy colimit

$$u_{n-1}^{-1}E = \operatorname{hocolim} E \xrightarrow{u_{n-1}} E \xrightarrow{u_{n-1}} E \to \cdots$$

where u_{n-1} is the corresponding element in E^0 and let $u_{n-1}^{-1}E/(p, u_1, \dots, u_{n-2})$ be the successive cofiber, denoted by $K_{u_{n-1}}$, with

$$K_{u_{n-1}*} = k((u_{n-1}))[u^{\pm}]$$

We claim that $K_{u_{n-1}}^*B\Sigma_k$ is concentrated in even degrees and free. Since $\pi_*K_{u_{n-1}}$ is a graded field $k((u_{n-1}))[u^{\pm}]$, $K_{u_{n-1}}^*B\Sigma_k$ is automatically free. In[86] (Section 7), it has been shown that Σ_k is a good group respect to Morava K-theory. The argument is still valid if one replaces $K(n)$ with any even periodic field spectrum, which implies our claim.

Now let $F_i = F/(p, u_1, \dots, u_{i-1})$, and let $F_0 = F$. By construction, we have $F_{n-1} = K_{u_{n-1}}$. We will show that if $F_i^*B\Sigma_k$ is free and concentrated in even degrees, the same is true for $i-1$ as well. Consider the long exact sequence of cohomology groups

$$F_{i-1}^*B\Sigma_k \to F_{i-1}^*B\Sigma_k \to F_i^*B\Sigma_k$$

obtained from the cofibration

$$F_{i-1} \xrightarrow{u_i} F_{i-1} \to F_i.$$

Each $F_i^*B\Sigma_k$ is finitely generated by Proposition 3.4. Since $F_i^*B\Sigma_k$ is concentrated in even degrees, multiplying u_i on $F_{i-1}^{\text{odd}}B\Sigma_k$ is a surjective. Hence by Nakayama's lemma, $F_{i-1}^{\text{odd}}B\Sigma_k = 0$. From this, we know the action of u_i on $F_{i-1}^{\text{even}}B\Sigma_k$ is regular, and $F_{i-1}^*B\Sigma_k/u_i = F_i^*B\Sigma_k$ which implies that $F_{i-1}^*B\Sigma_k$ is a free F^* module. Note in particular, we have shown that

$$K^*_{u_{n-1}}B\Sigma_k = K^*_{u_{n-1}} \otimes_{F^*} F^*B\Sigma_k.$$

■

Proof of Theorem 3.1: Applying[86] (Theorem C), we have the rank of

$$p^{-1}F^*B\Sigma_k$$

over $p^{-1}F^*$ is just $d(n-1,k)$. By Proposition 3.5, this rank must equal to the rank of $F^*B\Sigma_k$ over F^*. ■

Proposition 3.6: The ring $F^0B\Sigma_k/I = 0$ for $k \neq p^m$ and $R_m := F^0B\Sigma_{p^m}/I$ is a free module over F^0 of rank $\overline{d}(n-1,m)$, where I is the transfer ideal and $\overline{d}(n-1,m)$ denotes the number of lattices of index p^m in $\mathbb{Z}_p^{n-1}$.

Proof: For the first sentence, there is a standard argument in[73] (Lemma 8.10). For the second, using the method in[91] we see that $L(DS^0) := \prod L \otimes_{F^0} F^0B\Sigma_k$ is a Hopf ring, which can be identified with the ring of functions $F(\mathbb{B}, L)$, where L is a ring extension of F^0 with p^{-1} and all roots of the p-series of the formal group law over F^0 added and $\mathbb{B}$ is the Burnside semiring.

The $\times$−indecomposables $\mathrm{Ind}L(DS^0) = \prod L \otimes_{F^0} F^0B\Sigma_k/I_k$ is identified with $F(\mathbb{L}, L)$, where $\mathbb{L}$ is the set if all lattices in $\mathbb{Z}_p^{n-1}$ and I_k is the transfer. Hence we have an isomorphism $L \otimes_{F^0} F^0B\Sigma_k/I_k \cong F(\mathbb{L}_k, L)$, with $\mathbb{L}_k$ being the set of such lattices of index k. This implies the rank of R_m over F^0 is $\overline{d}(n-1,m)$. ■

3.2 Modular interpretation of ψ_F^p

Let $f \in E^0(X)$, which is represented by a map

$$f : X \longrightarrow E.$$

Then we obtained the nth power f^n via the composition

$$X \xrightarrow{\Delta} \underbrace{X \wedge \cdots \wedge X}_{n} \xrightarrow{f^{\wedge n}} \underbrace{E \wedge \cdots \wedge E}_{n} \xrightarrow{\mu} E.$$

It is clear that the symmetric group Σ_n acts on this map and preserves it. Hence it factors as

$$f^n : X \to X \times B\Sigma_n \to E^{\wedge n}_{h\Sigma_n} \xrightarrow{\mu} E,$$

where the first map is including the base point. The latter composition is an element in $E^0(X \times B\Sigma_n)$, denoted by $\psi^n(f)$. Thus we have obtained a refined nth power operation

$$\Psi^n : E^0(X) \longrightarrow E^0(X \times B\Sigma_n)$$

called *the nth total power operation*. Composing with including the base point of $B\Sigma_n$ yields the ordinary nth power operation. Note that Ψ^n is only multiplicative. To obtain a ring map, we can further mod out non-additive terms:

$$\psi^n : E^0(X) \longrightarrow E^0(X \times B\Sigma_n) \longrightarrow E^0(X \times B\Sigma_n)/I$$

where I is the transfer ideal generated by the images of the transfer maps

$$\mathrm{Tr} : E^0\left(X \times (B\Sigma_i \times B\Sigma_j)\right) \to E^0(X \times B\Sigma_n)$$

for all $i + j = n$,[92] (Section 11.3).

Let $\mathbb{G}_E$ and $\mathbb{G}_F$ be the formal groups over $\mathrm{Spf}(E^0)$ and $\mathrm{Spf}(F^0)$ respectively. In[73] (Section 9), the scheme $\mathrm{Spf}(E^0 B\Sigma_{p^k}/I)$ is identified with the subgroup scheme $\mathrm{Sub}_m(\mathbb{G}_E)$[87] (Theorem 10.1) over $\mathrm{Spf}(E^0)$.

The same procedure can be carried through with E replaced by F without harm.

Proposition 3.7: There is a canonical isomorphism $\mathrm{Spf}(F^0 B\Sigma_{p^m}/I) \to \mathrm{Sub}_m(\mathbb{G}_F)$. That is, the ring $F^0 B\Sigma_{p^m}/I$ classifies degree p^m subgroups of $\mathbb{G}_F$.

Proof: There is a canonical map from $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G}_F)}$ to $F^0 B\Sigma_{p^m}/I$ as constructed in[73] (Proposition 9.1). Note that, these two rings has the same rank over F^0. So we proceed as[73] (Theorem 9.2), by showing

$$k((u_{n-1})) \otimes_{F^0} \mathcal{O}_{\mathrm{Sub}_m(\mathbb{G}_F)} \to k((u_{n-1})) \otimes_{F^0} F^0 B\Sigma_{p^m}/I$$

is injective. The key ingredient here is to show $b_m = c_{p^m}^{(p^{n-1}-1)/(p-1)} \neq 0$ in

$$k((u_{n-1})) \otimes_{F^0} F^0 B\Sigma_{p^m} = K^0_{u_{n-1}} B\Sigma_{p^m},$$

where $c_{p^m} = e(V_{p^m} - 1)$ is the Euler class of representation $V_{p^m} - 1$ in $F^0 B\Sigma_{p^m}$ and V_{p^m} is the standard complex representation of Σ_{p^m}. This follows from[73] (Theorem 3.2) with K replaced by $K_{u_{n-1}}$. The rest follows[73] (Theorem 9.2). ■

Remark 3.2: We can not obtain this result directly from[87] (Theorem 10.1) which asserts that

$$\mathrm{Spf}\, F^0 \times_{\mathrm{Spf}\, E^0} \mathrm{Sub}_m(\mathbb{G}_E) = \mathrm{Sub}_m(\mathrm{Spf}\, F^0 \times_{\mathrm{Spf}\, E^0} \mathbb{G}_E) = \mathrm{Sub}_m(\mathbb{G}_F).$$

The failure of this equation is because the map $E^0 \to F^0$ is not continuous.

In order to figure out how the total power operation

$$\psi_F^p : F^0 \longrightarrow F^0 B\Sigma_p / I$$

interacts with the modular interpretation of $F^0 B\Sigma_p/I$, we shall recall some constructions from[72] (Section 3).

Let Y denote the function spectrum $F(\mathbb{C}P^\infty, F)$, we have

$$\pi_0 Y = F^0 \mathbb{C}P^\infty = F^0[[x]]$$

which is a complete local Noetherian ring, with maximal ideal $(p, u_1, \dots, u_{n-2}, x)$ and the canonical map $\pi_0 F \to \pi_0 Y$ is continuous with respect to their maximal ideal topology.

Proposition 3.8: The ring $Y^0 B\Sigma_p/J$ is free over Y^0 and equal to $Y^0 \otimes_{F^0} F^0 B\Sigma_p/I$, where I and J are transfer ideals respectively.

Proof: For each k, we have

$$Y^* B\Sigma_k = [\Sigma_+^\infty B\Sigma_k, F(\mathbb{C}P^\infty, F)] = [\Sigma_+^\infty (B\Sigma_k \wedge \mathbb{C}P^\infty), F] = F^*(B\Sigma_k \wedge \mathbb{C}P^\infty).$$

By the Atiyah Hirzebruch spectral sequence, we have

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, F^q B\Sigma_k) \Rightarrow Y^{p+q} B\Sigma_k$$

Since $F^* B\Sigma_k$ is concentrated in even degrees, we conclude that

$$Y^* B\Sigma_k = Y^* \otimes_{F^*} F^* B\Sigma_k.$$

It follows that $Y^0 \otimes_{F^0} I = J$, and hence

$$Y^0 B\Sigma_p / J = Y^0 \otimes_{F^0} F^0 B\Sigma_p / I.$$

which completes the proof. ■

In the language of algebraic geometry, $\operatorname{Spf} Y^0 = \mathbb{G}_F$ and the above proposition can be summarized as the pullback diagram.

$$\begin{tikzcd} \operatorname{Spf}(Y^0 B\Sigma_p/J) = i^*\mathbb{G}_F \arrow[r] \arrow[d] & \mathbb{G}_F \arrow[d] \\ \operatorname{Spf}(F^0 B\Sigma_p/I) \arrow[r] & \operatorname{Spf} F^0 \end{tikzcd}$$

Together with the naturality of the total power operation:

$$\begin{tikzcd} i^*\mathbb{G}_F \arrow[r, "\psi_Y^*"] \arrow[d] & \mathbb{G}_F \arrow[d] \\ \operatorname{Spf}(F^0 B\Sigma_p/I) \arrow[r, "\psi_F^*"] & \operatorname{Spf} F^0 \end{tikzcd}$$

we obtain a map $\psi^*_{Y/F} : i^*\mathbb{G}_F \to (\psi^p_F)^*\mathbb{G}_F$ over the ring $F^0B\Sigma_p/I$, as indicated in the diagram.

$$\begin{tikzcd}
i^*\mathbb{G}_F \arrow[dr, dashed, "\psi^*_{Y/F}"] \arrow[drr, bend left, "\psi^*_Y"] \arrow[ddr, bend right] & & \\
& (\psi^p_F)^*\mathbb{G}_F \arrow[r] \arrow[d] & \mathbb{G}_F \arrow[d] \\
& \mathrm{Spf}(F^0B\Sigma_p/I) \arrow[r, "(\psi^p_F)^*"] & \mathrm{Spf}\, F^0
\end{tikzcd}$$

Proposition 3.9: The isogeny $\psi^*_{Y/F} : i^*\mathbb{G}_F \to (\psi^p_F)^*\mathbb{G}_F$ is of degree p over $F^0B\Sigma_p/I$, with kernel the universal degree p subgroup K of $\mathbb{G}_F$ over $F^0B\Sigma_p/I$.

Proof: Choosing a coordinate x on $\mathbb{G}_F$, ψ^*_Y sends x to x^p in $Y^0B\Sigma_p/J = \mathcal{O}_{i^*\mathbb{G}_F}$ modulo maximal ideal of Y^0. This follows from

$$\pi_0 Y \xrightarrow{D_p} \pi_0 Y^{B\Sigma_p^+} \xrightarrow{S^0 \to B\Sigma_p^+} \pi_0 Y$$

sending x to x^p. Since $(\psi^p_F)^*(x) = x$, we conclude that $\psi^*_{Y/F}$ is of degree p. Therefore the kernel of $\psi^*_{Y/F}$ is of rank p.

To show the kernel is precisely the universal degree p subgroup K of $\mathbb{G}_F$ over $F^0B\Sigma_p/I$, we need to recall the construction of K from[73] (Proposition 9.1) (in which K is denoted by H_k). Let V_p be the standard permutation representation of Σ_p. There is a divisor $\mathbb{D}(V_p)$ of degree p over $F^0B\Sigma_p$, whose base change to $F^0B\Sigma_p/I$ is K. Let A be a transitive abelian p subgroup of Σ_p, we have a composition of maps

$$\mathrm{Level}(A^*, \mathbb{G}_F) \to \mathrm{Hom}(A^*, \mathbb{G}_F) = \mathrm{Spf}\, F^0BA \to \mathrm{Spf}\, F^0B\Sigma_p.$$

The divisor $\mathbb{D}(V_p)$ becomes a subgroup divisor $\Sigma_{a\in A^*}[\ell(a)]$ with ℓ the universal level-A^* structure of $\mathbb{G}_F$ on $\mathrm{Level}(A^*, \mathbb{G}_F)$(See[72] (Section 3) for definition). It is claimed in[73] (Proposition 9.1) that the map

$$\mathrm{Level}(A^*, \mathbb{G}_F) \to \mathrm{Spf}\, F^0B\Sigma_p$$

factors through $\mathrm{Spf}\, F^0B\Sigma_p/I$ and the union of the images of these maps for all such A is actually $\mathrm{Spf}\, F^0B\Sigma_p/I$. Hence it is sufficient to show the base change of $\ker\psi^*_{Y/F}$ to $\mathrm{Level}(A^*, \mathbb{G}_F)$ is $\Sigma_{a\in A^*}[\ell(a)]$.

Now Let $D(A) = \mathcal{O}_{\mathrm{Level}(A^*,\mathbb{G}_F)}$, the following diagram

$$\begin{tikzcd} F^0 \arrow[r, "D_A"'] \arrow[rr, bend left, "\psi^\ell_F"] \arrow[d, equal] & F^0BA \arrow[r] & D(A) \\ F^0 \arrow[r, "D_p"] \arrow[rr, bend right, "\psi^p_F"'] & F^0B\Sigma_p \arrow[r] \arrow[u] & F^0B\Sigma_p/I \arrow[u, dashed] \end{tikzcd}$$

implies the composition of the total power operation ψ^p_F and the dashed arrow is ψ^ℓ_F (See[72] (Definition 3.9)). Hence after base change to $\mathrm{Level}(A^*, \mathbb{G}_F)$, the map $\psi^*_{Y/F}$ becomes $\psi^{Y/F}_\ell$[72] (diagram 3.14). According to[72] (Proposition 3.21), the kernel of $\psi^{Y/F}_\ell$ is precisely $\ell[A] = \Sigma_{a\in A^*}[\ell(a)]$. ■

3.3 Augmented deformations

In this section, we combine our analysis about $F^0B\Sigma_p/I$ and the modular interpretation of F^0 in terms of augmented deformations. Recall that there is a formal group $\mathbb{G}_F$ over F^0, which is the base change of the universal deformation $\mathbb{G}_E$. Let $\mathbb{G}^0_F$ be the special fiber of $\mathbb{G}_F$, which is the base change of $\mathbb{G}_F$ over the residue field $k((u_{n-1}))$ of F^0.

The formal group $\mathbb{G}^0_F$ has height $n-1$ over $k((u_{n-1}))$. At first glance, one would like to construct the deformation theory of $\mathbb{G}^0_F$ as[93] does. However, the problem arises immediately for the field $k((u_{n-1}))$ being imperfect. A way to avoid the imperfectness is the treatment stated in[18]. We shall recall these constructions.

Definition 3.2: An augmented deformation of a formal group $\mathbb{H}$ over $k((u_{n-1}))$ consists of a triple $(\mathbb{K}/R, i, \alpha)$ where

- R is a complete local ring and $\mathbb{K}$ is a formal group over R,
- A local homomorphism $i : \Lambda \to R$ fits into the commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r, "i"] \arrow[d] & R \arrow[d] \\ k((u_{n-1})) \arrow[r, "\bar{i}"] & R/\mathfrak{m} \end{tikzcd}$$

- and an isomorphism $\alpha : \mathbb{H} \otimes^{\bar{i}}_{k((u_{n-1}))} R/\mathfrak{m} \simeq \mathbb{K} \otimes_R R/\mathfrak{m}$,

where $\Lambda = W(k)((u_{n-1}))^\wedge_p$ is a Cohen ring with residue field $k((u_{n-1}))$.

Remark 3.3: There always exists such a local homomorphism $i : \Lambda \to R$ filling the diagram

$$\begin{tikzcd} \Lambda \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k((u_{n-1})) \arrow[r] & R/\mathfrak{m} \end{tikzcd}$$

due to the property of Cohen rings. Note that such morphisms may not be unique.

This is the main difference between deformation theories over perfect fields and imperfect fields. When the field on the left lower corner is perfect, there is a unique local homomorphism from the Witt ring $W(k)$ to R and consequently a unique $W(k)$-algebra structure over R. While this is not true for Cohen rings of imperfect fields. Hence one must specify a Λ-algebra structure when discussing deformations in the imperfect context.

Theorem 3.2 ([18], Theorem 1.1): Let $\mathbb{H}$ be any height $n-1$ formal groups over $k((u_{n-1}))$. The ring F^0 classifies augmented deformations of $\mathbb{H}$. To be precise, let $\mathrm{Def}^{\mathrm{aug}}_{\mathbb{H}}(R)$ denote the groupoid of augmented deformations of $\mathbb{H}$ together with isomorphisms. Then we have

$$\mathrm{Def}^{\mathrm{aug}}_{\mathbb{H}}(R) = \mathrm{Maps}_{cts}(F^0, R).$$

In particular, this implies the moduli problem of classifying augmented deformation is discrete.

Combining our previous analysis on $F^B\Sigma_k$ and the modular interpretation of F^0, we have the following theorem.

Theorem 3.3: The ring $F^0B\Sigma_{p^m}/I$ is free over F^0 of rank $\overline{d}(m, n-1)$. It classifies augmented deformations of $\mathbb{G}^0_F$ together with a subgroup of degree p^m.

$$\mathrm{Maps}_{cts}(F^0B\Sigma_{p^m}/I, R) = \{(\mathbb{K}/R, H)\}$$

To be precise, for any complete local ring R, there is a bijection between the set of continuous maps from $F^0B\Sigma_{p^m}/I$ to R and the set of all pairs $(\mathbb{K}/R, H)$, where $\mathbb{K}$ is an augmented deformations of $\mathbb{G}^0_F$ and H is a degree p^m subgroup of $\mathbb{K}$.

Equivalently, $F^0B\Sigma_{p^m}/I$ classifies augmented deformations of mth Frobenius[74] (Section 11.3), with the universal example

$$\psi^*_{Y/F} : i^*\mathbb{G}_F \to (\psi^{p^m}_F)^*\mathbb{G}_F$$

defined in the Proposition 3.9.

Proof: Combines Proposition 3.7, 3.9 and Theorem 3.2 ■

3.4 Dyer-Lashof algebra of $K(n-1)$-local E_n-algebras

We begin with the general theory of algebraic theories.

Recall that a group is a set G, equipped with maps

- $m : G \times G \to G$

- $i : G \to G$
- $e : * \to G$

making certain diagrams commute.

We can abstract this, which is called an algebraic theory.

Definition 3.3: An *algebraic theory* T is a small category with objects $\{T^0, T^1, \dots\}$. And there are maps $\pi_i : T^n \to T^1$ for all $n \geq 0, 1 \leq i \leq n$, such that $T(T^k, T^n) \xrightarrow{\pi_i} \prod_{i=1}^n T(T^k, T^1)$ is a bijection.

Remark 3.4: Note that these π_i make T^n isomorphic to n-fold product of T^1.

Example 3.2: The algebraic theory of groups. T is a subcategory of Grps^{op}. T^0 is the set of single element. $T^i =< x_1, \dots, x_i >$ is the free group of i generators. Structure maps are given by

$$\pi_i : T(T^n, T^1) = \mathrm{Grps}(< x_1 >, < x_1, \dots, x_n >)$$
$$x_1 \longmapsto x_i$$

and

- Multiplication: $x_1 \mapsto x_1 x_2 \in T(T^2, T^1)$.
- Inversion: $x_1 \mapsto x_1^{-1} \in T(T^1, T^1)$.
- Identity: $x_1 \mapsto * \in T(T^0, T^1)$.

Example 3.3: The algebraic theory C_R of commutative R algebras. Let F be the full subcategory of Alg_R, with obejects $\{F_0, F_1, \dots\}$, $F_i = R[x_1, \dots, x_i]$. Let $T = F^{op}$. Let π_i be the following:

$$T^n \xrightarrow{\pi_i} T^1$$
$$R[x_1, \dots, x_n] \leftarrow R[x_1]$$
$$x_i \leftarrow x_1.$$

The R algebra structure is characterized by

- Addition: $x_1 \mapsto x_1 + x_2 \in T(T^2, T^1)$
- Multiplication: $x_1 \mapsto x_1 x_2 \in T(T^2, T^1)$
- Inversion: $x_1 \mapsto -x_1 \in T(T^1, T^1)$
- Scalar: $x_1 \mapsto 1 \in T(T^0, T^1)$

Definition 3.4: A *model* for an algebraic theory T is a functor $F : T \to \mathrm{Sets}$, which preserves finite products. A morphism ϕ between two models is simply a natural transformation between them.

Example 3.4: By the definition above, a model F for the group theory can be identified with a group, represented by $F(T^1)$. The category of all models for group theory is the category Grps of groups.

Hence given an algebraic theory T and a model A, we usually abbreviate the notation $A(T^1)$ by A.

Example 3.5: The functor $F_T(n) = T(T^n, -)$ itself is a model for the algebraic theory T, which is called the *free model on n generators*.

For example, in the algebraic theory of commutative R-algebras:

$$F_T(n)(T^1) = T(T^n, T^1) = Alg_R(R[x], R[x_1, \dots, x_n]) \cong R[x_1, \dots, x_n].$$

Definition 3.5: A morphism of theories is a functor $\phi : T \to T'$, which preserve products and sends the projection maps to projection maps.

Definition 3.6: A *commutative operation theory* (COT), is a triple (T, R, ϕ), which consists of

- An algebraic theory T
- A ring R and the algebraic theory of R-algebras
- A map $\phi : C_R \to T$ between algebraic theories, such that

$$\phi^* : Model_T \to Model_{C_R}$$

commutes with finite coproducts.

Remark 3.5: This means a T model A has an underlying structure of R algebras, and the coproduct of T models can be computed as tensor product of their underlying R algebras.

Example 3.6: Let G be a monoid. Consider the category of R algebras with an action of G. Intuitively, this theory is a COT, and it does. The free model of n generators is represented by

$$R[x_i^g : 1 \leq i \leq n, g \in G]$$

Example 3.7: Let R be a ring. Consider the category of R algebras with an R-derivation, i.e. $\partial : A \to A$ with the Leibnitz rule

$$\partial(xy) = \partial(x)y + x\partial(y)$$

The corresponding algebraic theory is a COT, with free model on one generator given by

$$R[x, \partial x, \partial^2 x, \cdots]$$

To work more conveniently with topological objects, we need the graded algebraic theory.

Definition 3.7: Let C be a fixed set, and $\mathbb{N}[C]$ is the set generated over C. A *graded* theory T is a category with objects $\{T^d\}_{d\in\mathbb{N}[C]}$, together with, for each $d = \Sigma_{c\in C} d_c[c] \in \mathbb{N}[C]$, a specified identification of T^d with the product $\prod(T^{[c]})^{\times d_c}$.

Example 3.8: Let R_* be a graded ring. Let C_{R_*} be the $\mathbb{Z}$ graded theory with free model $C_{R_*}([c_1] + \cdots + [c_m])$ given by

$$R_*[x_1, \dots, x_m]$$

with each x_i has degree c_i.

Definition 3.8: A graded COT is a graded theory T with a map $\phi : C_{R_*} \to T$, such that

$$\phi^* : Model_T \to Model_{C_{R_*}}$$

preserves coproducts.

Example 3.9: Consider the Eilenberg-Maclane spectrum HR for some R. Consider the following graded theory T, with

$$T(T^{[c_1]+\cdots+[c_m]}, T^{[d_1]+\cdots+[d_n]}) = [K(R, c_1) \times \cdots \times K(R, c_m), K(R, d_1) \times \cdots \times K(R, d_n)]$$

where the right hand side is the homotopy classes of maps.

If R is a field, then by Künneth formula, this is a graded COT.

Instead of focusing on the whole algebraic theory, we now focus on individual operations carried by an algebraic theory.

Definition 3.9: Let $f \in F_T(n)(T^1)$, and $a_1, \dots, a_n \in A(T^1)$, where A is any model of T.

Let $f \propto (a_1, \dots, a_n)$ denote the image of f under the map $F_T(n) \to A$ sending x_i to a_i. We call the function:

$$f \propto : A^n \to A$$

the *operation* associated to f.

Let $T = C_R$, then we know that

$$F_T(n)(T^1) = Alg_R(R[x], R[x_1, \dots, x_n]) \cong R[x_1, \dots, x_n].$$

We abbreviate $F_T(n)(T^1)$ by $R\{x_1, \dots, x_n\}$.

Hence

$$f \propto (a_1, \dots, a_n)$$

is just $f(a_1, \dots, a_n)$.

If T is a COT, then we have

$$F_T(n) \cong F_T(1) \otimes_R \cdots \otimes_R F_T(1).$$

Hence we may focus on operations in $F_T(1)$.

These operation satisfies

$$\begin{aligned} x \propto a &= a \\ (f \propto g) \propto a &= f \propto (g \propto a) \\ (f+g) \propto a &= f \propto a + g \propto a \\ (fg) \propto a &= (f \propto a)(g \propto a) \\ r \propto a &= r \end{aligned}$$

Furthermore, if $f \in R\{x\}$ satisfies

$$f \propto (a_1 + a_2) = f \propto a_1 + f \propto a_2$$

We say it is an *additive operation*, denoted the set of all additive operations by $\mathscr{A}$.

Remark 3.6: $\mathscr{A}$ is an associative ring with product $\propto$, but not commutative in general.

Considering the following diagram.

$$\begin{tikzcd} F_T(2)(T^1) \arrow[r, "(a,b)"] & A(T^1) \\ F_T(1)(T^1) \arrow[u, "x+y"] \arrow[r] \arrow[dr, dashed, "x\cdot 1+1\cdot y"'] & F_T(2)(T^1) \arrow[u, "(a,b)"'] \arrow[d, equal] \\ & F_T(1)(T^1) \otimes F_T(1)(T^1) \end{tikzcd}$$

$R\{x\}$ has a additive coproduct $\Delta : R\{x\} \to R\{x_1, x_2\}$ given by $x \mapsto x_1 + x_2$, corresponds to the structure map under addition. Hence additive operations are those elements with *primitive* image.

Now we focus on the algebraic theory on E_∞ rings. Let R be a commutative S-algebra, i.e. an E_∞ ring. We want an algebraic theory describes the structure of $\pi_* A$ for an R algebra A.

There is a pair of adjoint functor

$$\mathbb{P} : \mathrm{Mod}_S \Longleftrightarrow \mathrm{Alg}_S : U$$

where $\mathbb{P}$ is the free algebra functor and U is the forgetful functor. We have

$$\mathbb{P}(X) = \bigvee \mathbb{P}^m(X) = \bigvee X^m/\Sigma_m$$

The functor $\mathbb{P}$ has the following properties

- $\mathbb{P}^m(\Sigma^\infty T) = \Sigma^\infty(T^m_{h\Sigma_m})$ for spaces T
- $\mathbb{P}^m(S^0) = \Sigma^\infty_+ B\Sigma_m$
- $\mathbb{P}^m(S^d) = \Sigma^\infty_+ B\Sigma_m^{dV_m}$

where $d \in \mathbb{Z}$, V_m is $\mathbb{R}^m$ equipped with the Σ_m action by permuting the coordinates and dV_m is a virtual bundle over $B\Sigma_m$.

Similarly, we have adjoint functors

$$\mathbb{P}_R : \mathrm{Mod}_R \Longleftrightarrow \mathrm{Alg}_R : U_R$$

With these notations, we have

$$\begin{aligned}\pi_q(A) &\cong h\mathrm{Mod}_S(S^q, A)\\ &\cong h\mathrm{Mod}_R(R \wedge S^q, A)\\ &\cong h\mathrm{Alg}_S(\mathbb{P}(S^q), A)\\ &\cong h\mathrm{Alg}_R(\mathbb{P}_R(R \wedge S^q), A)\end{aligned}$$

Thus we have the following definition

Definition 3.10: For any E_∞ ring or S-module R, we define the $\mathbb{Z}$ graded algebraic theory DL_R by

$$T(T^{[c_1]+\cdots[c_m]}, T^{[d_1]+\cdots[d_n]}) = h\mathrm{Alg}_R(\mathbb{P}_R(R\wedge(S^{d_1}\vee\cdots\vee S^{d_n})), \mathbb{P}_R(R\wedge(S^{c_1}\vee\cdots\vee S^{c_m})))$$

Note that the free model evaluated at $[d]$ is given by

$$\begin{aligned}T(T^{[c_1]+\cdots[c_m]}, T^{[d]}) &= h\mathrm{Alg}_R(\mathbb{P}_R(R \wedge S^d), \mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee S^{c_m})))\\ &= h\mathrm{Mod}_R(R \wedge S^d, \mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee S^{c_m})))\\ &= \pi_d(\mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee S^{c_m})))\\ &= \pi_d(R \wedge \mathbb{P}(S^{c_1} \vee \cdots \vee S^{c_m}))\end{aligned}$$

Also note that DL_R is a COT iff when $R_*P(S^c)$ is flat over R_*, in this case, we can obtain the coproduct preserving property via the Künneth formula.

Hence we can also view taking homotopy group as a functor

$$\pi_* : \mathrm{Alg}_R \to Model_{\mathrm{DL}_R}$$

For the simplicity, we now only focus on degree zero part of the whole Dyer-Lashof theory. Let E be an E_∞ ring and A be an E-algebra. In section 3.2, we have seen that there is a canonical map

$$\psi^n : \pi_0 A \to \pi_0 A^{B\Sigma_m}$$

which sends $f : E \to A$ to the composite

$$E \wedge B\Sigma_n \cong E^{\wedge n}_{h\Sigma_n} \xrightarrow{f^n} A^{\wedge n}_{h\Sigma_n} \xrightarrow{\mu} A$$

between E-modules.

Composing with any element $\eta : E \to E \wedge B\Sigma_n \in E_0 B\Sigma_n$ yields a map

$$Q_\eta : \pi_0 A \to \pi_0 A$$

which sends $f \in \pi_0 A$ to the composite

$$Q_\eta(f) : E \xrightarrow{\eta} E \wedge B\Sigma_n \xrightarrow{\psi^n(f)} A$$

Note that these Q_η need not to be additive or multiplicative.

The algebra generated by these Q_η over E_0 is called the *Dyer-Lashof* algebra of E, denoted by DL_E. It is clear that

$$\mathrm{DL}_E = \bigoplus_n E_0 B\Sigma_n$$

Remark 3.7: What we have just defined is actually only the degree 0 part of the full Dyer-Lashof algebra. In fact, one can start with $f : S^d \wedge E \to A \in \pi_d A$ and obtain an element

$$\psi^n(f) : E \wedge B\Sigma_n^{dV_n} \to A$$

where dV_m is d copies of V_m and V_m is the vector space $\mathbb{R}^m$ with permutation action of Σ_m. Thus the full Dyer-Lashof algebra for E is generated by elements in

$$\bigoplus_{n,d} E_* B\Sigma_n^{dV_n}$$

The Dyer-Lashof algebra DL_E governs homotopy operations on E-algebras. For example, let X be a spectrum, then $E \wedge X$ is an E-algebra. We thus recover homology operations from $E_0(X)$ to itself. If we take the function spectrum $F(X, E) = E^X$ which is an E-algebra, then we recover cohomology operations on $E^0 X$.

Now let E be a height n Morava E-theory. In[73], it is shown that

$$\widehat{E}_0 B\Sigma_n = E^0 B\Sigma_m$$

for all m, where

$$\widehat{E}_0(X) = \pi_0 L_{K(n)}(E \wedge X)$$

is called the completed E homology of X.

Thus for $K(n)$-local E-algebra, their Dyer-Lashof algebra is referred to the modified version

$$\mathrm{DL}_E = \bigoplus_n \widehat{E}_0 B\Sigma_n$$

Elements in $\oplus_n \widehat{E}_0 B\Sigma_n$ can be decomposed into primitives, which correspond to indecomposables $E^0 B\Sigma_n / I$ in $E^0 B\Sigma_n$.

So what is going on for $K(n-1)$-local E-algebras? First, one obseves that a $K(n-1)$-local E-algebra is the same as a $K(n-1)$-local F-algebra, where $F = L_{K(n-1)}E_n$ is the $K(n-1)$-localized E_n. This is because the structure map

$$E \to A$$

of E-algebra structure on A factors through

$$E \to L_{K(n-1)}E \to A$$

uniquely (up to homotopy) for $L_{K(n-1)}E$ being the closest $K(n-1)$-local object to E.

Thus the calculation of the Dyer-Lashof algebra on $K(n-1)$-local E-algebra demands the calculation of $F_0 B\Sigma_m$, where $F = L_{K(n-1)}E$, as usual. To obtain good values on Dyer-Lashof algebra, we modify our notation with the completed homology

$$\widehat{F}_0 B\Sigma_m = \pi_0 L_{K(n-1)}(F \wedge B\Sigma_m)$$

as what we done in the $K(n)$-local Morava E case.

Proposition 3.10: There is an isomorphism

$$\widehat{F}_0 B\Sigma_m \to F^0 B\Sigma_m$$

Hence the Dyer-Lashof algebra over $K(n-1)$-local E-algebra is generated by the coefficients of the total power operation ψ_F^p.

Proof: As explained above, the second statement is straightforward if the isomorphism holds. Consider the Greenlees-May's cofiber sequence[94]

$$k \wedge EG \to F(EG, k) \to t_G(k)$$

Taking homotopy pixed point yields a cofiber sequence

$$K \wedge BG \to F(BG, K) \to t_G(i_* K)^G$$

where K is any spectrum and $k = i_* K$ is the G-equivariant version of K[95] (Section 1.1).

We will show that the spectrum $t_G(i_* F)^G$ is $K(n-1)$ acyclic and hence we have an equivalence

$$L_{K(n-1)}(F \wedge BG) \to L_{K(n-1)} F(BG, F) = F(BG, F)$$

which induces the desired isomorphism.

Our strategy is as follow. Choose a generalized Moore spectrum M of type $n-1$, such that $M \wedge F$ is generated by $K_{u_{n-1}}$, which is defined in Proposition 3.5. Then we show

$$t_G(i_* K_{u_{n-1}})^G = 0 \tag{3.4.1}$$

Hence by property of Tate cohomology, we have

$$t_G(i_*(M \wedge F))^G = M \wedge t_G(i_* F)^G = 0$$

Applying $K(n-1)$ homology and Kunneth formula implies

$$K(n-1)_*(t_G(i_* F)^G) = 0$$

According to[95] (Proposition 3.1), the equation 3.4.1 holds if for all finite group G, $K_{u_{n-1}*} BG$ is finite generated as $K_{u_{n-1}*}$ module. This is true because $K_{u_{n-1}}$ is admissible and the finiteness follow from[90] (Corollary 4.4), or one can directly check that

$$K_{u_{n-1}*}(-) = K_*(-) \otimes_{K_*} K_{u_{n-1}*}$$

and the condition of being finite is valid for K, where K is the even periodic version of $K(n)$. ■

Example 3.10: As a simple application, we consider the $n = 2$ case. From the calculation in 3.6 we know that

$$F^0 B\Sigma_k / I = F^0$$

for $k = p^m$, and is zero for other k. Hence the corresponding Dyer-Lashof algebra is freely generated by Q_i for each $i = p^m$. This fits in the general framework over $K(1)$-local setting. There is a theorem of McClure which asserts that the Dyer-Lashof algebra of $K(1)$-local E_∞ rings is freely generated as a θ algebra over one generator[96].

CHAPTER 4 CONNECTIONS WITH ELLIPTIC CURVES

In this chapter we focus on the height 2 case and hence we let E be a Morava E-theory at height 2 and $F = L_{K(1)}E_2$ be the $K(1)$-localization of E.

4.1 Elliptic curves and p-divisible groups

We begin with elliptic curves.

Definition 4.1: Let S be a scheme. An elliptic curve C over S is a proper smooth curve with geometrically connected, genus one fibers and a given section 0.

For our purpose, only elliptic curves over affine schemes are involved. Hence for an elliptic C over a ring R, C is defined by a Weierstrass equation

$$C : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

and over the affine chart $z \neq 0$, we have

$$C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where all a_i's belong to R.

Not all curves defined by this equation are elliptic curves. Associated to this equation, there are some quantities:

- $b_2 = a_1^2 + 4a_2, b_4 = 2a_4 + a_1a_3, b_6 = a_3^2 + 4a_6$
- $b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$
- $c_4 = b_2^2 - 24b_4, c_6 = -b_2^3 + 36b_2b_4 - 216b_6$
- $\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$

where the last term Δ is called the discriminant of C. A cubic curve defined by the Weierstrass equation is elliptic if and only iff the descriminant $\Delta \neq 0$. Otherwise, the curve is singular, which means it has singular points.

Over fields K with characteristc not 2, by replacing y with $\frac{1}{2}(y - a_1x - a_3)$, we have

$$C : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

If we further assume $3 \neq 0 \in K$, via the transformation

$$(x, y) \longmapsto (\frac{x - 3b_2}{36}, \frac{y}{108})$$

the equation can be written as

$$C : y^2 = 4x^3 - 27c_4x - 54c_6$$

One can directly check that

$$1728\Delta = c_4^3 - c_6^2$$

Remark 4.1: When $\Delta = 0$, there are two types of these singular curves. The first type is nodal, which satisfies $\Delta = 0$ but $c_4 \neq 0$. In this case, there is a point on C, which has two distinct tangent directions. Another type is cuspidal. In this case, we have $\Delta = c_4 = 0$ and there is a point with two opposite tangent directions.

Both cases contain only one such singular point.

Example 4.1: Here are two examples of singualr curves. The curve $y^2 = x^3$ is cuspidal and $y^2 = x^3 + x^2$ is nodal. See the picture below[97] (Section 3, Figure 3.2).

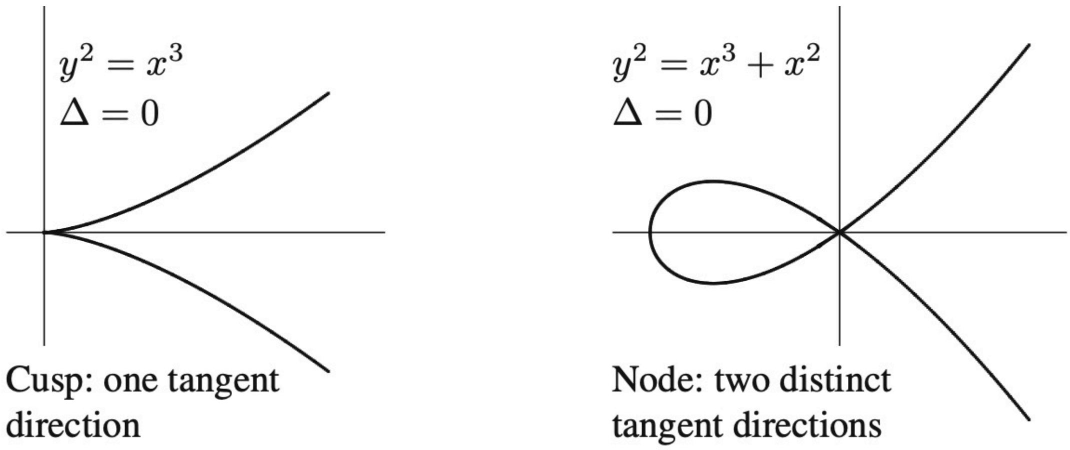


Figure 4-1 Two singular curves

Now let C be an elliptic curve over a field K, with $\mathrm{char}K = p$. It is known that C admits an abelian group structure. Moreover let C_{ns} be the nonsingular part of C, if C is nodal, then C_{ns} is the multiplicative group (under some field extension over K) and if C is cuspidal , we have C_{ns} is the additive group. In each case, we can define an isogeny

$$[m] : C \to C$$

which sends P to the m times P under the group operation.

Let $C[m]$ denote the kernel of $[m]$, which consists of m torsion points on C.

Proposition 4.1 (Corollary 6.4[97]): If $m \neq \mathrm{char}K = p$, then

$$C[m] = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$$

If $m = p^e$, only one following situation will happen

(1) $C[p^e] = \mathbb{Z}/p^e\mathbb{Z}$ or,

(2) $C[p^e] = 0$

Definition 4.2: With the above notation, if C/K satisfies the second condition, i.e. $C[p^e] = 0$, then we say C is supersingular.

Remark 4.2: There are also other characterization of supersingular elliptic curves. The first is the *Hasse invariant* of C equals 0. The another is the formal group assocaited to C has height 2.

The following theorem provides a useful criterion for being supersingular.

Theorem 4.1: Let $K = \mathbb{F}_q$ be a field of characteristic $p \geq 3$. Let C be an elliptic curve over K with the Weierstrass form

$$C : y^2 = f(x)$$

The coefficient of x^{p-1} in $f(x)^{\frac{p-1}{2}}$ is the Hasse invariant of C, which vanishes if and only if C is supersingular.

It is more convenient to consider things like subgroup from the scheme-theoretic point of view.

Definition 4.3: Let S be a scheme, and C/S is an elliptic curve. An effective Cartier divisor D on C is a closed subscheme of C, which is finite flat over S and the ideal sheaf $\mathcal{O}_D$ is an invertible $\mathcal{O}_X$-module. In this case $\mathcal{O}_D$ is a locally free sheaf over $\mathcal{O}_S$. We denote the rank of $\mathcal{O}_D$ by $\deg D$.

Definition 4.4: A subgroup of C is an effective Cartier divisor D, which is also a group scheme, and the closed embedding $D \hookrightarrow C$ is a group homomorphism.

Example 4.2: It is easy to see that $C[N]$ is a subgroup of C with degree N^2. In fact, for any abelian variety A with dimension d, the subgroup $A[N]$ has degree N^{2d}.

Definition 4.5: Let S be a scheme. A *p-divisible group* $G = (G_v, i_v)$ is an inductive limit

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} \cdots$$

such that

- G_v is a finite group scheme over S with degree p^{vh}
- G_v can be identified with the kernel

$$0 \to G_v \xrightarrow{i_v} G_{i_{v+1}} \xrightarrow{p^v} G_{i_{v+1}}$$

The natural number h is called the height of G.

A homomorphism between G and G' is a collection of $(f_v : G_v \to G'_V)$ compatible with the structure map.

Example 4.3: The trivial case is

$$(\mathbb{Q}_p/\mathbb{Z}_p)^h = \operatorname{colim} \mathbb{Z}/p^{\nu h}\mathbb{Z}$$

which is the constant p-divisible group over S.

Another example is

$$\mathbb{G}_m[p^\infty] = (\mu_{p^\nu}, i_\nu)$$

which is a p-divisble group of height 1.

Example 4.4: The essential case is that for any abelian variety A of dimension d, the associated p-divisble group $A[p^\infty]$ is of height $2d$. In particular, we have $C[p^\infty]$ is of height 2 for each elliptic curve C.

Proposition 4.2: Suppose R is a complete Noetherian local ring and G is a p-divisible group over R. Then, there is a natural short exact sequence between p-divisible groups

$$0 \to G^\circ \to G \to G_{\acute{e}t} \to 0$$

Proof: We can put it in every finite stage, while this is just the connected-étale short exact sequence of algebraic group over complete local ring. If each G_ν is affine, with $\mathcal{O}_G$ being a finite generated R-algebra over the base scheme $\operatorname{Spec} R$. Then $\mathcal{O}_{G_{\acute{e}t}}$ is the maximal étale subalgebra in $\mathcal{O}_G$, and $\mathcal{O}_{G^\circ}$ is the quotient of $\mathcal{O}_G$ by its maxiaml étale subalgebra. ■

Remark 4.3: If R is a perfet field, then the above sequence is split.

Remark 4.4: Let G be an affine algebraic group over a field K, then the number of connected components of G is the rank of its maximal étale subalgebra over K.

Proposition 4.3: We have the equality

$$\text{height} G = \text{height} G^\circ + \text{height} G_{\acute{e}t}$$

Proof: Obvious. ■

Example 4.5: The constant p-divisible group $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ is clearly étale.

Consider the multiplicative group $\mathbb{G}_m$ over R, which is represented by $\operatorname{Spec} R[x, x^{-1}]$ or equivalently $\operatorname{Spec} R[x, y]/(xy - 1)$. The p^nth kernel $\mathbb{G}_m[p^n]$ is represented by $\operatorname{Spec} R[x]/(x^{p^n} - 1)$. For simplicity, we assume R is a field. Since $\mathbb{G}_m[p^\infty]$ is of height 1, we know it is either connected or étale at all. If char $\neq p$, then $f = x^{p^n} - 1$ is separale over R. Hence let L be the splitting field over f over R, we have

$$L \otimes_R R[x]/f \cong L \times \cdots \times L$$

Thus $\mathbb{G}_m[p^\infty]$ is étale.

On the contrary, suppose $p = 0$ in R, we have f is inseparable and $\mathbb{G}_m[p^\infty]$ is connected.

Suppose $\mathbb{G}$ is a formal group of height h over a complete local Noetherian ring R. We can reconstruct an associated p-divisible grop of the same height as follow.

Let G_n be the p^nth kernel $G[p^n]$, which is represented by $\operatorname{Spf} R[[x]]/[p^n](x)$. By Weierstrass preparation theorem, $[p^n](x)$ can be written as

$$[p^n](x) = f(x)g(x)$$

where f is a monic polynomial of degree p^h and g is a unit in $R[[x]]$. One can check that the canonical inclusion i_n fits in the sequence

$$0 \to \mathbb{G}[p^n] \xrightarrow{i_n} \mathbb{G}_{n+1} \xrightarrow{p^n} \mathbb{G}_{n+1}$$

Note that this p-divisble group is connected. Because over the residue field $R/\mathfrak{m}$, the polynomial f can be written as $\tilde{f}(x^{p^h})$ 2.5, which is inseparable.

The converse is also true.

Proposition 4.4: Let R be a complete local ring. The above construction

$$\mathbb{G} \longmapsto (\mathbb{G}_n, i_n)$$

is an equivalence from the category of the *divisible* formal groups over R to the category of connected p-divisible groups over R. A formal group $\mathbb{G}$ is divisible if the multiplication by p map

$$[p] : \mathbb{G} \to \mathbb{G}$$

makes $\mathscr{O}_{\mathbb{G}}$ a free module over itself.

Hence we can freely change our dictionary between formal groups and p-divisible groups.

Example 4.6: Let C be an elliptic curve over a complete local Noetherian ring R, with residue field K of characteristic p. From Proposition 4.1, we know that if $N \neq p$, then the N-divisible group $C[N^\infty]$ is purely étale. For $N = p$, there are two cases. If $C[p^e] = \mathbb{Z}/p^e\mathbb{Z}$, the p-divisible group $C[p^\infty]$ falls into two parts, namely, the formal part and étale part. Each is of height 1. If $C[p] = 0$, the corresponding p-divisible group $C[p^\infty]$ is connected and hence formal. In this case, we say C is supersingular and

$$C[p^\infty] = \widehat{C}$$

where $\widehat{C}$ is the formal group associated to C obtaining via completing along the identity section $[0]$ of C.

Proposition 4.5: For each p, there are only finite supersingular elliptic curves. Moreover, any supersingular elliptic curve can be defined over $\mathbb{F}_p^2$.

4.2 Power Operations on E_2

Now back to topology. From the previous section 4.1, we can construct a height 2 Morava E-theory, which is also denoted by E from elliptic curves. Namely, one just picks a supersingular elliptic curve C_0 over some perfect field k, which carries a height 2 formal group $\widehat{C_0}$. By considering the deformation of $\widehat{C_0}$, one obtian such an E-theory with

$$\pi_0 E = W(k)[[h]]$$

From works of Strickland[73,87], Ando and Hopkins[23,71,72,98] and Rezk[74], we know that to compute the total power operation on E, it is sufficient to find where the universal degree p subgroup of $\widehat{C}$ is defined, where $\widehat{C}$ is the universal deformation of $\widehat{C_0}$. Since we have

$$\mathcal{O}_{\mathrm{Sub}(\widehat{C})} = E^0 B\Sigma_p / I$$

Definition 4.6: Let G_0 be a p-divisible group over k. A deformation of G_0 to a complete local Noetherian ring R consists of a triple (G, i, α)

- G is a p-divisible group over R
- $i : k \to R/\mathfrak{m}$ is an inclusion of fields
- $\alpha : G_0 \otimes_k R/\mathfrak{m} \to G \otimes_R R/\mathfrak{m}$ is an isomorphism of p-divisible groups over $R/\mathfrak{m}$

Clearly, if G_0 is connected, i.e. formal, then the above definition is just the deformation of G_0 as formal groups. Hence the deformation theory of $\widehat{C_0}$ is the same as the deformation theory of $C_0[p^\infty]$. Thanks to the theorem below, the deformation theory of $C_0[p^\infty]$ is equivalent to deformation of C_0.

Theorem 4.2 (Serre-Tate): Let R be a complete local ring and k be its residue field of characteristic p. The following two categories are equivalent.

$\mathscr{A}$: Objects are elliptic curves C over R. Morphisms are homomorphisms over R.

$\mathscr{B}$: Objects consist of triples $(C_0/k, G, i)$

- C_0 is an elliptic curve over k.
- G is a p-divisible group over R.
- $i : C_0[p^\infty] \to G \otimes_R k$ is an isomorphism.

In another words, the objects are deformations of $C_0[p^\infty]$.

Morphism in $\mathscr{B}$ are pairs (f_0,f), where f_0 is a k-homomorphism of elliptic curves and f is a homomorphism between p-divisible groups over R. They are compatible with the isomorphism i, i.e.

$$\begin{array}{ccc} C_0[p^\infty] & \xrightarrow{i} & G \otimes_R k \\ \Big\downarrow f_0 & & \Big\downarrow f \\ C_0'[p^\infty] & \xrightarrow{i'} & G' \otimes_R k \end{array}$$

The functor is given by

$$\begin{aligned} \nu : \mathscr{A} &\longrightarrow \mathscr{B} \\ C/R &\longmapsto (C_0/k, C[p^\infty], id) \end{aligned}$$

Now let C_0 be a supersingualr elliptic curve over k. Let $R = W(k)[[h]]$. From the Lubin-Tate theorem 2.6 we know that there exsists a universal deformation of $C_0[p^\infty] = \widehat{C_0}$ over R, denoted by $\widehat{C}$. By the Serre-Tate theorem, there is an elliptic curve C over R, such that $C[p^\infty] = \widehat{C}$.

Let R' be the complete local ring where the universal degree p isogeny is defined, or equivalently, the place where the universal degree p subgroup of $\widehat{C}$ lives. This ring exists, and indeed

$$R' = \mathscr{O}_{\mathrm{Sub}(\widehat{C})}$$

Since a degree p subgroup of $\widehat{C} = C[p^\infty]$ is contained in $C[p]$, a degree p subgroup of $\widehat{C}$ is the same as a degree p subgroup of C. Therefore the total power operation ring of E is R', which is the universal degree p isogeny starting from C, or equivalently, degree p subgroup of C being defined.

We can summarize the above discussion in the following picture.

$$\begin{array}{c}
\text{(Universal) deformation of } \widehat{C_0} \\
\Big\downarrow {\scriptstyle C_0 \text{ is of height } 2} \\
\text{(Universal) deformation of } C_0[p^\infty] \\
\Big\downarrow {\scriptstyle \text{Serre-Tate}} \\
\text{(Universal) deformation of } C_0
\end{array} \tag{4.2.1}$$

To make things more accessible, we will use the representability of certain moduli problems.

Definition 4.7: Let C be a scheme over S. An effective Cartier divisor D is a closed subscheme of C, which is flat over S, and its associated ideal shaef $I(D)$ is an invertible $\mathcal{O}_X$-module. The rank of $\mathcal{O}_D$ over $\mathcal{O}_S$ is called the degree of D.

Example 4.7: It is clear that for any section $s \in C(S)$, there is an associated effective Cartier divisor, denoted by $[s]$, of degree 1.

Definition 4.8: Let D, D' be two effective Cartier divisors over S. Their sum $D + D'$ is defiend to be an another effective Cartier divisor. Locally over R, let A be $\mathcal{O}_X$, we have $\mathcal{O}_D = A/f$ for some $f \in A$. Similarly $\mathcal{O}_{D'} = A/f'$. Then $D + D'$ is defined by $f \cdot f'$.

Clearly, we have $\deg(D + D') = \deg D + \deg(D')$. We say $D' \leq D$ if there is D'' with $D' + D'' = D$. This means the defining equation f' of D' divides f of D.

From the definition, $D = D'$ iff $D' \leq D$ and $\deg D' = \deg D$.

These are enough for defining what so called *level structures*.

Definition 4.9: Let A be an finite abelian group and C/S be an elliptic curve over base scheme S. We say

$$\phi : A \longrightarrow C(S)$$

from A to the group of sections of C over S is an A-*structure* on C/S if the effective Cartier divisor

$$[\phi(A)] = \sum_{a \in A} [\phi(a)]$$

is a subgroup of C/S.

Remark 4.5: From the definition, it is clear that $\deg[\phi(A)] = |A|$. There is an S-scheme which carries the universal A-structure on C. First, there is an S-scheme $\mathrm{Hom}(A, C)$ where a universal homomorphism

$$\phi_{\text{univ}} : A \longrightarrow C(\text{Hom}(A, C))$$

lives. Hence it is sufficient to find a $\text{Hom}(A, C)$-scheme T with $[\phi_{\text{univ}}(A)]$ being a subgroup of C_T/T. This can be accomplished thanks to the existance of intermediate shceme. It asserts that for any two distinct divisor D, D' over a base scheme S, there is a closed subscheme Z of S, such that $D'_Z \leq D_Z$ over Z. Moerover, Z is universal in the sense that, if there is a scheme T with $D'_T \leq D_T$ over T, then the map $T \to S$ factors through Z.

The condition that D being a subgroup of C is equivalent to the following data

- $[e] \leq D$.
- Let $i : C \to C$ stand for taking inverse. Then $D = i^*D$.
- Let $m : C \times_S C \to C$ be the multiplication on C. Let $P = D \times_S D$, denote the two projection maps by P_1, P_2. There are two sections in $C(P)$, which are composition of projections and inclusion from D_P to C_P. These we have a section in $C_P \times_P C_P$, denoted by (P_1, P_2). We require that

$$[m(P_1, P_2)] \leq D_W$$

This means the multiplication on C restricts to D.

Example 4.8: One of the most important level structure is the $\mathbb{Z}/N\mathbb{Z}-$structure, which also denoted by $\Gamma_1(N)-$structure. From the definition, one can immediately figure that a $\Gamma_1(N)$-sturcture on C is equivalent to an N-torsion point on C, given by $[\phi(1)]$.

Example 4.9: Another example is $\Gamma(N)$-structure, which is associated to group $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. This is equivalent to a basis of $C[N]$, given by $([\phi(1, 0), \phi(0, 1)])$.

Definition 4.10: Let *Ell* be the category of all elliptic curves, with morphism

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

being Cartesian product.

A *moduli problem* is a contravariant functor

$$\mathscr{P} : Ell \longrightarrow \text{Sets}$$

Remark 4.6: In the language of stacks, a moduli problem can be viewed as a presheaf of sets over the moduli stack $\mathscr{M}_{ell}$.

Example 4.10: The assignment

$$C/S \longmapsto \{A - \text{structure on } C\}$$

defines a moduli problem.

Let $A = \mathbb{Z}/N\mathbb{Z}$, the corresponding moduli problem is denoted by $[\Gamma_1(N)]$, which assigns each elliptic curve with the set of its N-torsion points. Let $A = (\mathbb{Z}/N\mathbb{Z})^2$, we have similarly $[\Gamma(N)]$ moduli problem, which assigns C with the set of basis of $C[N]$.

Example 4.11: The moduli problem

$$C/S \longmapsto \{\text{degree } N \text{ subgroup of } C\}$$

is denoted by $[\Gamma_0(N)]$.

Definition 4.11: We say a moduli problem $\mathscr{P}$ is relative representable if for any elliptic C/S, the functor

$$T \longmapsto \mathscr{P}(C_T/T)$$

from Sch/S to Sets is representable.

Example 4.12: Clearly, $[\Gamma(N)]$ and $[Gamma_1(N)]$ is relative representable.

In fact, $[\Gamma_0(N)]$ is also relative representable.

In light of relative representability, to calculate the total power operation ring 4.2.1, one could take the universal deformation $C/W(k)[[h]]$ then find the representable object $\mathscr{P}_{C/W(k)[[h]]}$ for $[\Gamma_0(p)]$.

However, to keep track of where the element h goes, we shall use the representability of $[\Gamma_1(N)]$.

Definition 4.12: We say a moduli problem $\mathscr{P}$ is representable if $\mathscr{P}$ is represented by an object in Ell.

Theorem 4.3: Over $\mathbb{Z}[1/N]$, $[\Gamma_1(N)]$ is representable, denote the representing object by $C_N/\mathscr{M}_N$.

Proposition 4.6: Let $\mathscr{P}$ be a representable moduli problem and $\mathscr{P}'$ relative representable. Then the simultaneou moduli problem $\mathscr{P} \times \mathscr{P}'$ is representable.

Proof: Let $\mathscr{M}(\mathscr{P})$ be the representaing object of $\mathscr{P}$. Then $\mathscr{P} \times \mathscr{P}'$ is represented by $\mathscr{P}'_{\mathscr{M}(\mathscr{P})}$. ■

Corollary 4.1: Over $\mathbb{Z}[1/N]$, the simultaneous problem $[\Gamma_1(N)] \times [\Gamma_0(p)]$ is representable, which is denoted by $C_N/\mathscr{M}_{N,p}$.

Now we can finially set up our computation framework.

Recall from previous discussion, let C_0 be a supersingular elliptic curve, and C be its universal deformation. To calculate the total power operation in E, we proceed as follows.

Step 1: Choose N prime to p and find $C_N/\mathscr{M}_N$. Find a supersingular locus on $\mathscr{M}_N$, thus we obtain a supersingular elliptic curve C_0 and its universal deformation C and an associated Morava E-theory.

Step 2: According to the representability of $[\Gamma_1(N)] \times [\Gamma_0(p)]$, find the scheme $\mathscr{M}_{N,p}$. The scheme $\mathscr{M}_{N,p}$ should be finite flat of rank $p+1$ over $\mathscr{M}_N$ for there are $p+1$ subgroups ($C[p]$ has rank p^2). Base change $\mathscr{M}_{N,p}$ via $\operatorname{Spec} W(k)[[h]] \to \mathscr{M}_N$, where $W(k)[[h]] = E^0$ is the place universal deformation of C_0 defined. Then we have

$$\mathscr{O}_{\mathscr{M}_{N,p}\times_{\mathscr{M}_N}\operatorname{Spec} W(k)[[h]]} = E^0 B\Sigma_p/I$$

Step 3: Find the expression of the universal target curve, and see how h changes through the universal degree p. This will give the expression of $\psi^p(h)$. See the following example 4.13, 4.14.

Example 4.13: When $p = 2$, $[\Gamma_0(2)] = [\Gamma_1(2)]$. Therefore the above procedure can be simplified to find $\mathscr{M}_2$. Let C be such a curve:

$$C : Y^2Z + aXYZ + YZ^2 = X^3$$

over the ring $\mathbb{Z}[a]$. The discriminant is $\Delta = a^3 - 27$. The Hasse invariant is a, which means mod $(2,h)$ the curve

$$C : Y^2Z + YZ^2 = X^3$$

is supersingualr over $\mathbb{F}_2$. We thus produce a universal deformation which is C is self over

$$\mathbb{Z}[a][\Delta^{-1}]^{\wedge}_{(2,a)} = \mathbb{Z}_2[[a]]$$

Now we calculate the 2-torsion point on C. Let $u = X/Y$ and $v = Z/Y$, over the $Y \neq 0$ chart, C can be interpreted as

$$C : v^2 + auv + v = u^3$$

with the identity $(0,0)$. To calculate a torsion 2 point Q, observe that the line through the origin and Q must tangent to C. Let $Q = (d,e)$, we have

$$\frac{3d^2 - ae}{2e + ad + 1}\frac{dv}{du} = \frac{e}{d}\frac{dv}{du}$$
$$e^2 + ade + e = d^3$$

which yields

$$e + d^3 = 0$$

$$d^3 - ad - 2 = 0$$

Thus C with the universal 2-torsion point lives on $\mathbb{Z}[a, d]/d^3 - ad - 2$. And hence

$$E^0 B\Sigma_2/I = \mathbb{Z}_2[[a]][d]/d^3 - ad - 2$$

To calculate the total power operation, it is sufficient to find $\psi^2(a)$. Note that let C' be the target curve of Ψ starting from C with the kernel generated by Q, then we must have

$$C' : v'^2 + \psi^2(a)u'v' + v' = u'^3$$

The coordinate u', v' can be written as

$$u'(\Psi(P)) = u(P) \cdot u(P - Q)$$
$$v'(\Psi(P)) = v(P) \cdot v(P - Q)$$

Using the group law on C and expanding everything on the right hand side as power series in terms of u, we find

$$\psi^2(a) = a^2 + 3d - ad^2$$

Example 4.14: When $p = 3$. The $[\Gamma_1(4)]$ problem over $\mathbb{Z}[1/4]$ is representable, which is represented by

$$C : y^2 + axy + aby = x^3 + bx^2$$

over the graded ring

$$S^\bullet = \mathbb{Z}[1/4][a, b, \Delta^{-1}]$$

with $|a| = 1, |b| = 2$ and $\Delta = a^2 b^4 (a^2 - 16b)$. The Hasse invariant in $H = a^2 + b$ over $\mathbb{F}_3$.

Introduce elements u and c, and let $a = uc$, $b = u^2$. Consider the degree 0 part S of $S^\bullet[u^[- 1]]$, which is an affine open chart. We have

$$S = \mathbb{Z}[1/4][c, \delta^{-1}]$$

where $\delta = u^{-12}\Delta = c^2(c^2 - 16)$. The curve becomes

$$C : y^2 + cxy + cy = x^3 + x^2$$

over S, and the Hasse invariant is $h = u^{-2}H = c^2 + 1$.

By computing the universal point of exact order 3, and applying the same procedure above, we have the universal isogeny of degree 3 is defined over

$$S[\alpha]/w(h, \alpha)$$

with

$$w(h,\alpha) = \alpha^4 - 6\alpha^2 + (c^2 - 8)\alpha - 3$$

Base change it to

$$S^{\wedge}_{3,h} = \mathbb{Z}_9[[h]]$$

we have

$$E^0 B\Sigma_3/I = \mathbb{Z}_9[[h]][\alpha]/w(h,\alpha)$$

with

$$w(h,\alpha) = \alpha^4 - 6\alpha^2 + (h - 9)\alpha - 3$$

The target curve C' is

$$C' : y^2 + c'xy + c'y = x^3 + x^2$$

with

$$c' = \frac{1}{c}\left((c^2 - 4)\alpha^3 + 4\alpha^2 + (-6c^2 + 20)\alpha + c^4 - 12c^2 + 12\right)$$

Therefore we have

$$\begin{aligned}
\psi^3(h) &= \psi^3(c^2 + 1) \\
&= c'^2 + 1 \\
&= h^3 - 27h^2 + 201h - 342 + (-6h^2 + 108h - 334)\alpha \\
&+ (3h - 27)\alpha^2 + (h^2 - 18h + 57)\alpha^3
\end{aligned}$$

4.3 Modular forms and Parameters

In example 4.13 and 4.14, we see that there are two variables, namely h, α, to describe the total power operation ring

$$E^0 B\Sigma_p/I = \mathcal{O}_{\mathrm{Sub}(\widehat{C})} = W(k)[[h]] \otimes \mathcal{O}_{M_{N,p}}$$

These parameters are indeed examples of *modular forms*.

Let C/S be an elliptic curve, with the structure map

$$p : C \to S$$

The Kähler differential $\Omega_{C/S}$ is an invertible sheaf of $\mathcal{O}_C$-module. Let $\underline{\omega}_{C/S}$ be the push-forward sheaf $p_*\Omega_{C/S}$ over S.

Definition 4.13: A modular form of weight k is a section of $\omega^{\otimes k}$ over the compactified moduli stack $\overline{\mathscr{M}_{ell}}$ of elliptic curves, where ω is the pushforward of universal sheaf of differentials.

To be precise, a modular form f is a rule, which assigns each elliptic curve C/R, where R is a ring, a section $f(C/S)$ of $\underline{\omega}_{C/S}^{\otimes k}$. This assignment is natural in the sense that for each $g : C/R \to C'/R'$, we have

$$g^*(f(C'/R')) = f(C/R)$$

To be more precise, f assigns each pair $(C/R, \omega)$, where ω here is a basis of $\underline{\omega}_{C/R}$, an element $f(C/R, \omega)$ in R, such that

(1) Natural: $g^*(f(C'/R', \omega')) = f(C/R, g^*\omega')$, as above.

(2) Change of basis: $f(C/R, \lambda\omega) = \lambda^{-k} f(C/R, \omega)$

Example 4.15: Over the complex field $\mathbb{C}$, an elliptic curve can be identified with $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where τ belongs to the upper half plane $\mathbb{H}$. A canonical differential is $d\tau$. Since for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, τ and $\gamma\tau$ yield isomorphic elliptic curve, and

$$d\gamma\tau = d(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{-2} d\tau, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$$

We recover the classical modular forms, i.e. holomorphic function $f(z)$ with

$$(cz + d)^{-2k} f(\gamma z) = f(z)$$

Example 4.16: Over an $\mathbb{F}_p$-algebra R, an elliptic curve C adimits the absolute Frobenius map $F_{abs} : C \to C$, which induces an endomorphism on $H^1(C, \mathscr{O}_C)$. By Serre duality, we have a basis ω on $\underline{\omega}_{C/S}$ determines a basis η on $H^1(C, \mathscr{O}_C)$. We define a modular form A, such that

$$F_{abs}^* \eta = A(C/S, \omega)\eta$$

If we let $\lambda\omega$ take place of ω, the corresponding basis is $\lambda^{-1}\eta$. Hence we have

$$\begin{aligned} F_{abs}^*(\lambda^{-1}\eta) &= \lambda^{-p} F_{abs}^*(\eta) \\ &= \lambda^{-p} A(C/S, \omega)\eta \\ F_{abs}^*(\lambda^{-1}\eta) &= A(C/S, \lambda\omega)\lambda^{-1}\eta \end{aligned}$$

Hence we have

$$A(C/S, \lambda\omega) = \lambda^{1-p} A(C/S, \omega)$$

Thus A is a modular form over $\mathbb{F}_p$ of weight $p - 1$, called the *Hasse invariant*.

Similarly, we have modular forms of $\Gamma_0(p)$.

Definition 4.14: A weight k modular form f of $\Gamma_0(p)$ is a rule which assigns each triple $(C/R, \omega, H)$, where H is a degree p subgroup of C, an element $f(C/R, \omega, H)$, which satisfies the analogue conditions as presented in definition 4.13.

Example 4.17: Let C be the universal elliptic curve carrying a 4-torsion point over $\mathscr{M}_4$ as defined in Example 4.14. By the representability of $\Gamma_1(N) \times \Gamma_0(p)$, there is a scheme $\mathscr{M}_{4,3}$, such that $C/\mathscr{M}_{4,3}$ carries a universal 4-torsion point and a universal degree 3 subgroup $[0] + [Q] + [-Q]$. Choose a coordinate u on C, and define

$$\alpha := \prod u(Q) \cdot u(-Q)$$

as an element in $\mathscr{O}_{\mathscr{M}_{4,3}}$. This is a modular form of $\Gamma_1(N) \times \Gamma_0(p)$, as one can see for each elliptic curve over $\operatorname{Spec} R/\mathbb{Z}[1/N]$ carrying these two information, there is a unique map $f : \operatorname{Spec} R \to \mathscr{M}_{4,3}$ which classifies such a curve. We assign such curve an element $f^\#(\alpha)$ in R, where

$$f^\# : \mathscr{O}_{\mathscr{M}_{4,3}} \to R$$

is the induced map on global sections.

This modular form which we also called α is a norm parameter. In general, over $\mathbb{Z}[1/N]$, we have a sequence of moduli schemes

$$\mathscr{M}_{N,p} \to \mathscr{M}_N \to \operatorname{Spec} \mathbb{Z}[1/N]$$

Let $C/\mathscr{M}_{N,p}$ be the universal curve with degree p subgroup H. We can similarly construct a norm

$$\alpha = N(u(Q)) := \prod_{Q \in H - 0} u(Q)$$

which is a modular form as explained previously.

Note that α is also the multiples of the cotangent map of the universal degree p isogeny over the moduli scheme $\mathscr{M}_{N,p}$. To be explicit, let $\Psi : C \to C/H$ be the universal isogeny, then we have

$$\psi^*(d\tilde{u}) = \alpha du$$

where du and $d\tilde{u}$ are the invariant differentials over C and C/H respectively.

Let C_0/k be a supersingular elliptic curve, where k is a perfect field of characteristic p. Let $C/W(k)[[T]]$ be its universal deformation. Let A be the complete local ring with

$$\mathscr{P}_{C/W(k)[[T]]} = \operatorname{Spec} A$$

In[22] (Section 5.4, 7.7), the parameters for describing A is listed below. Thus we are

Table 4-1 Parameters for Moduli Problems near supersingular locus

Moduli Problems	Data contained	Parameters for A
$\Gamma_1(N)$	N-torsion point P	$T, u(P)$
$\Gamma_0(p)$	Degree p subgroup	$T, N(u(P))$

guarenteed that the ring $E^0B\Sigma_p/I = \mathscr{O}_{\mathscr{M}_{N,p}}$ is of the form

$$E^0B\Sigma_p/I = W(k)[[h]][\alpha]/w(h, \alpha)$$

as shown in Example 4.13 and 4.14.

The compactified modular curve $\overline{\mathscr{M}}_{N,p}$ has cusps $\overline{\mathscr{M}}_{N,p} - \mathscr{M}_{N,p}$. For any cusp on $\overline{\mathscr{M}}_N$, there are exactly two cusps lying over it, corresponding to étale subgroup and formal subgroup on the corresponding Tate curve. Katz[99] (Section 1.11) and Zhu[24] (Lemma 2.15) showed that the modular form α is 1 at unramified/étale cusps and is p at ramified/formal cusps.

Thus, at least at cusps, we have the equation

$$(\alpha - p)(\alpha - 1)^p = 0$$

Since near a supersingular locus, we have the equation

$$w(h, \alpha) = \alpha^{p+1} + \sum_{i=0}^{p} \omega_i \alpha^i = 0$$

In[24], it is shown that ω_i is constant for all i but $\omega_1 = -h$. Since the Hasse invariant takes value 1 near each cusp, by comparing these two equations we have the following.

Theorem 4.4 ([24], Theorem A): After choosing a preferred model for E[24] (Definition 2,23), the ring $E^0B\Sigma_p/I$ can be interpreted as

$$E^0B\Sigma_p/I = W(\overline{\mathbb{F}}_p)[[h, \alpha]]/w(h, \alpha)$$

with

$$w(h, \alpha) = (\alpha - p)\left(\alpha + (-1)^p\right)^p - \left(h - p^2 + (-1)^p\right)\alpha. \tag{4.3.1}$$

From the modular forms point of view, the total power operation can be explained as follow.

The map, which is called an *exotic morphism on moduli problems*[22] (Chapter 11)

$$(C, N, H, du) \mapsto (C/H, \Psi(N), C[p]/H, d\tilde{u})$$

gives an endomorphism of $\mathscr{M}_{N,p}$, denoted by a. This operation, which is called the *Atkin-*

Lehner involution, is indeed an involution since the twice composite $a \circ a$ sends C to $C/C[p] \cong C$. It hence induces an involution on $\Gamma_1(N) \times \Gamma_0(p)$ modular forms, which is exactly the Atkin-Lehner theory on modular forms.

Hence under this interpretation, the total power operation can be understood as

$$\begin{tikzcd} \mathscr{M}_{N,p} \arrow[rr, "a"] \arrow[dr] & & \mathscr{M}_{N,p} \arrow[dl] \\ & \mathscr{M}_N & \end{tikzcd}$$

over the global moduli schemes. Near a supersingular locus, it can be interpreted as

$$\begin{tikzcd} E^0 B\Sigma_p/I & & E^0 B\Sigma_p/I \arrow[ll, "a"'] \\ & E^0 \arrow[ul, "\psi^p"] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

For the convenience of notation, we usually denote the image of the Atkin-Lehner involution of a modular form f by $\tilde{f}$. Therefore the total power operation is actually determined by

$$\psi^p(h) = \tilde{h}$$

the Hasse invariant of the target curve, as computed in Example 4.13 and 4.14.

4.4 Calculations on height 2 case

Let E be a Morava E-theory of height 2 over the field $\overline{\mathbb{F}}_p$, with

$$E^* = W(\overline{\mathbb{F}}_p)[[u_1]][u^\pm].$$

Let F be the $K(1)$ localization of E, whose coefficients ring is

$$F^* = W(\overline{\mathbb{F}}_p)((u_1))_p^\wedge[u^\pm].$$

Let $\mathbb{G}_E$ and $\mathbb{G}_F$ be the formal groups over E^0 and F^0 respectively.

In this section, we give an explicit calculation of the additive total power operation ψ_F^p in terms of the expression of ψ_E^p for the $n = 2$ case. The naturality of the total power operations gives a diagram:

$$\begin{tikzcd} E^0 \arrow[r, "\psi_E^p"] \arrow[d] & E^0 B\Sigma_p/I \arrow[d, "t"] \\ F^0 \arrow[r, "\psi_F^p"] & F^0 B\Sigma_p/J = F^0 \end{tikzcd} \tag{4.4.1}$$

where I and J are the corresponding transfer ideals. The equality on the right corner is because the formal group $\mathbb{G}_F$ is of height 1, hence $F^0B\Sigma_p/J$ is free of rank $\bar{d}(1,1) = 1$ over F^0.

Remark 4.7: From now on, we will use h instead of u_1 in E^* and F^*. This is because when height is 2, the ring E^0 can be viewed as the place where the universal deformation of a certain supersingular elliptic curve is defined. The letter h here stands for the Hasse invariant for it being a lift of Hasse invariant.

The map t in the middle is E^0 linear. To see this, consider the diagram

$$\begin{tikzcd}
E^0(\bigvee_{i=1}^{p-1} B\Sigma_i \times B\Sigma_{p-i}) \arrow[r, "tr_E"] \arrow[d] & E^0B\Sigma_p \arrow[r] \arrow[d] & E^0B\Sigma_p/I \arrow[d, dashed, "t"] \\
F^0(\bigvee_{i=1}^{p-1} B\Sigma_i \times B\Sigma_{p-i}) \arrow[r, "tr_F"] & F^0B\Sigma_p \arrow[r] & F^0B\Sigma_p/J
\end{tikzcd}$$

The maps in the top row are between E^0 modules and maps in the bottom can also be viewed as E^0 linear maps via $E^0 \to F^0$. Then one can check that the left two vertical maps are E^0 linear, which implies t is E^0 linear as well.

Now we can deduce the explicit expression of ψ_F^p via the calculation of ψ_E^p, which is summarized in the theorem below.

Theorem 4.5 ([24], Theorem B): The image of h under ψ_E^p is

$$\psi_E^p(h) = \alpha + \sum_{i=0}^{p} \alpha^i \sum_{\tau=1}^{p} w_{\tau+1} d_{i,\tau}, \tag{4.4.2}$$

where w_i's are defined to be

$$w_i = (-1)^{p(p-i+1)} \left[\binom{p}{i-1} + (-1)^{p+1} p \binom{p}{i} \right]$$

and

$$d_{i,\tau} = \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\cdots m_{\tau-n}=\tau+i \\ 1\le m_s \le p+1 \\ m_{\tau-n}\ge i+1}} w_{m_1} \cdots w_{m_{\tau-n}}.$$

Proof: This theorem is deduced from the following observations. The modular form α associated to C can be identified with the multiplicity of the cotangent map associated to the isogeny $C \to C/H$, for specific degree p subgroup H of C. Hence the image $\widetilde{\alpha}$ of α under the Atkin-Lehner involution is the cotangent for $C/H \to C/C[p]$. While we have

$$\begin{tikzcd}
C \arrow[r, "\Psi^p"] \arrow[rr, bend right, "p"'] & C/H \arrow[r, "\widetilde{\Psi}^p"] & C/C[p] \cong C
\end{tikzcd}$$

Hence

$$pdu = \Psi^*\widetilde{\Psi}^*du = \Psi^*\widetilde{\alpha}d\tilde{u} = \widetilde{\alpha}\alpha du$$

and

$$\widetilde{\alpha}\alpha = (-1)^{p-1}p \tag{4.4.3}$$

Since h and α satisfies

$$w(h,\alpha) = \alpha^{p+1} + \sum_{i=2}^{p} \omega_i \alpha^i - h\alpha + (-1)^{p-1}p = 0 \tag{4.4.4}$$

where the middle coefficients ω_i are all constants.

Applying Atkin-Lehner involution, we see that

$$\widetilde{\alpha}^{p+1} + \sum_{i=2}^{p} \omega_i \widetilde{\alpha}^i - \tilde{h}\widetilde{\alpha} + p = 0 \tag{4.4.5}$$

Therefore from equations 4.4.3 and 4.4.5, we have

$$\tilde{h} = \widetilde{\alpha}^p + \omega_p \widetilde{\alpha}^{p-1} + \cdots + \omega_2\widetilde{\alpha} + \alpha \tag{4.4.6}$$

Dually, from equations 4.4.3 and 4.4.4, we have

$$\widetilde{\alpha} = -\alpha^p - \omega_p \alpha^{p-1} - \cdots - \omega_2\alpha + h \tag{4.4.7}$$

Substituting the equation 4.4.7 into 4.4.6, we have the desired formula of $\tilde{h}$ in terms of h and α. ∎

Example 4.18: In Example 4.13, we calculate

$$w(h,\alpha) = \alpha^3 - h\alpha - 2$$

where d stands for α and a stands for h.

By the method above, we have

$$\alpha^2 - h + \widetilde{\alpha} = 0$$

and

$$\begin{aligned}
\tilde{h} &= \widetilde{\alpha}^2 + \alpha \\
&= (h-\alpha^2)^2 + \alpha \\
&= h^2 - 2h\alpha^2 + \alpha^4 + \alpha \\
&= h^2 - 2h\alpha^2 + h\alpha^2 + 2\alpha + \alpha \\
&= h^2 - h\alpha^2 + 3\alpha
\end{aligned}$$

which is just as what has been calculated in Example 4.13.

Example 4.19: In $p = 3$ case, Example 4.14 we calculate

$$w(h, \alpha) = \alpha^4 - 6\alpha^2 + (h - 9)\alpha - 3$$

which is incompatible with Theorem 4.4 in which we have

$$\begin{aligned} w(h, \alpha) &= (\alpha - 3)(\alpha - 1)^3 - (h - 10)\alpha \\ &= \alpha^4 - 6\alpha^3 + 12\alpha^2 - h\alpha + 3 \end{aligned}$$

This incompatability comes from the model for Morava E theory in Example 4.14 is not the preferred model. In the preferred model, we have $\alpha\widetilde{\alpha} = 3$, while in this case, we have -3.

Example 4.20: In $p = 5$ case, we have

$$w(h, \alpha) = \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - h\alpha + 5 = 0 \tag{4.4.8}$$

Applying Atkin-Lehner involution, we have

$$\widetilde{\alpha}^6 - 10\widetilde{\alpha}^5 + 35\widetilde{\alpha}^4 - 60\widetilde{\alpha}^3 + 55\widetilde{\alpha}^2 - \tilde{h}\widetilde{\alpha} + 5 = 0$$

with

$$\alpha\widetilde{\alpha} = 5$$

Hence we have

$$\begin{aligned} \psi^5(h) = \tilde{h} &= \widetilde{\alpha}^5 - 10\widetilde{\alpha}^4 + 35\widetilde{\alpha}^3 - 60\widetilde{\alpha}^2 + 55\widetilde{\alpha} + \alpha \\ &= (-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^5 \\ &- 10(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^4 \\ &+ 35(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^3 \\ &- 60(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^2 \\ &+ 55(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h) \\ &+ \alpha \\ &= h^5 - 10h^4 - 1065h^3 + 12690h^2 + 168930h \\ &- 1462250 + (-55h^4 + 850h^3 + 39575h^2 \\ &- 608700h - 1113524)\alpha + 60(h^4 - 775h^3 \\ &- 45400h^2 + 593900h + 2008800)\alpha^2 \\ &+ (-35h^4 - 400h^3 - 27125h^2 - 320900h \\ &- 1418300)\alpha^3 + (10h^4 - 105h^3 - 7850h^2 \end{aligned}$$

$$+ 86975h + 445850)\alpha^4 + (-h^4 + 10h^3$$
$$+ 790h^2 - 8440h - 46680)\alpha^5$$

Remark 4.8: As shown in above examples, different models for Morava E theories may yield different formulas for the interpretation of $E^0B\Sigma_p/I$ and the total power operation ψ^p. To be explicit, in[24], Zhu defined a preferred model for Morava E theory, which consists of the following data:

Mod.1 a supersingular elliptic curve C_0 over the algebraically closed field $\overline{\mathbb{F}}_p$

Mod.2 C_N is the universal deformation of C_0 over $\mathscr{M}_N$.

Mod.3 a coordinate u on $\widehat{C_N}$

Mod.4 an isomorphism between $\operatorname{Spf} E^0$ and the formal completion of $\mathscr{M}_N$ around the supersingular locus corresonding to C_0.

Mod.5 an isomorphism of formal groups between $\operatorname{Spf} E^0(\mathbb{C}P^\infty)$ and $\widehat{C_N}$, which sends $x_E \cdot u$ to u.

Mod.6 a universal degree p isogeny $C_N \to C_N^{(p)}$ over $\mathscr{M}_{N,p}$

Mod.7 an isomorphism between $\operatorname{Spf} E^0B\Sigma_p/I$ and the formal completion of $\mathscr{M}_{N,p}$ around the supersingular locus.

Note that the first u is the periodic element in π_*E, and $x_E \cdot u$ is a coordinate on the formal group $\operatorname{Spf} E^0(\mathbb{C}P^\infty)$. The second u means the coordinate on $\widehat{C_N}$ as defined in Mod.3.

To get the desired formulas, we need to modify this model.

Mod.1_+: The twice Frobenius on C_0 is $(-1)^{p-1}p$, i.e.

$$\mathrm{Frob}^2 = (-1)^{p-1}[p]$$

This guarantees $\alpha\widetilde{\alpha} = (-1)^{p-1}p$.

Mod.3_+: The coordinate u should be choosen properly, such that the induced modular form α takes value p on ramified cusps and $(-1)^{p-1}$ on unramified cusps. This guarantees the modular equation $w(h,\alpha)$ is of the desired form. By[24] (Lemma 2.15), this could be done.

The desired equation and formalas depend on different choices of Mod.1 and Mod.3. As explained, Mod.3 effects on the modular equation $w(h,\alpha)$ and Mod.1 effects the total power operation formula.

Back to the streamline, to determine the image of $h \in F^0 = W(\overline{\mathbb{F}}_p)((h))_p^\wedge$ under ψ_F^p, it suffices to determine the image of α in Theorem 3.2 under the map t. We have

$$\psi_F^p(h) = t \circ \psi_E^p(h)$$

by the diagram 4.4.1. Since t is an E^0 linear map, this requires us to find the solutions of $w(h, \alpha)$ in F^0.

Proposition 4.7: There is a unique solution α^* of $w(h, \alpha)$ in $W(\overline{\mathbb{F}}_p)((h))_p^\wedge$ with

$$\alpha^* = (-1)^{p+1}p \cdot h^{-1} + \left(1 + (-1)^{p+1}\frac{p(p-1)}{2}\right)p^3 \cdot h^{-3} + \textit{lower terms} \tag{4.4.9}$$

satisfies

$$w(h, \alpha) = (\alpha - p)(\alpha + (-1)^p)^p - (h - p^2 + (-1)^p)\alpha = 0.$$

Moreover, we have $\alpha^* = 0 \bmod p$.

Proof: We write $w(h, \alpha)$ as

$$w_{p+1}\alpha^{p+1} + w_p\alpha^p + \cdots + w_1\alpha + w_0$$

where $w_{p+1} = 1$, $w_1 = -h$, $w_0 = (-1)^{p+1}p$, and

$$w_i = (-1)^{p(p-i+1)}\left[\binom{p}{i-1} + (-1)^{p+1}p\binom{p}{i}\right]$$

for other coefficients.

Since h is invertible in $W(\overline{\mathbb{F}}_p)((h))_p^\wedge$, the equation $w(h, \alpha) = 0$ implies

$$\begin{aligned}\alpha &= h^{-1}(\alpha^{p+1} + w_p\alpha^p + \cdots w_2\alpha^2 + w_0)\\ &= h^{-1}w_0 + \alpha^2(\alpha^{p-1} + w_p\alpha^{p-2} + \cdots + w_2)h^{-1}\\ &= h^{-1}w_0 + h^{-3}w_0^2w_2 + \textit{lower terms}\end{aligned}$$

Substituting the second equation into itself recursively gives the desired formula for α^* as described in 4.4.9.

This iteration makes sense because the highest term of α^* is $h^{-1}w_0$ and $p|w_0$. Hence each substitution only create a lower terms, which is divided by a higher power of p, than current stage. Thus

$$\alpha^* = \Sigma_k a_k h^{-k}$$

and the coefficient a_k satisfies

$$\lim_{k\to\infty} |a_k| = 0$$

which implies α^* is indeed an element in $W(\overline{\mathbb{F}}_p)((h))_p^\wedge$.

The uniqueness comes from the following observation. Note that

$$w(h,\alpha) = \alpha(\alpha^p - h) \bmod p$$

This implies $w(h,\alpha)$ has only one solution 0 in the residue field of $W(\overline{\mathbb{F}}_p)((h))^\wedge_p$. Therefore it also has a unique solution in $W(\overline{\mathbb{F}}_p)((h))^\wedge_p$, which is α^*. ■

Remark 4.9: This uniqueness can not be explained as there is a unique degree p subgroup of a height 1 formal group, beacause the map $t : E^0 B\Sigma_p/I \to F^0$ is not continuous. Hence we cannot using Strickland's theorem directly.

Combining our analysis on the naturality of total power operations and Proposition 4.7, we have the following theorem.

Theorem 4.6: Let F be a $K(1)$-local Morava E-theory at height 2. The total power operation ψ^p_F on F^0 is determined by

$$\psi^p_F(h) = \alpha^* + \sum_{i=0}^{p} (\alpha^*)^i \sum_{\tau=1}^{p} w_{\tau+1} d_{i,\tau}, \tag{4.4.10}$$

where

$$\alpha^* = (-1)^{p+1} p \cdot h^{-1} + \left(1 + (-1)^{p+1}\frac{p(p-1)}{2}\right) p^3 \cdot h^{-3} + \textit{lower terms}$$

is the unique solution of

$$w(h,\alpha) = (\alpha - p)(\alpha + (-1)^p)^p - (h - p^2 + (-1)^p)\alpha$$

in $W(\overline{\mathbb{F}}_p)((h))^\wedge_p \cong F^0$.

The other coefficients w_i and $d_{i,\tau}$ are defined in Theorem 4.5.

In particular, ψ^p_F satisfies the Frobenius congruence, i.e.

$$\psi^p_F(h) \equiv h^p \bmod p$$

.

Proof: The formula 4.4.10 is obtained by assembling Theorem 3.3 and Proposition 3.7. The last sentence comes from $\psi^p_F \equiv \sum_{\tau=1}^{p} w_{\tau+1} d_{0,\tau} \bmod p$, for α^* being zero after modulo p. Also notice that

$$w_i \equiv 0 \bmod p,\ i = 0, 2, \cdots, p.$$

Therefore

$$\psi^p_F(h) \equiv \sum_{\tau=1}^{p} w_{\tau+1} d_{0,\tau} \equiv d_{0,p}$$

$$\equiv \sum_{n=0}^{p-1} (-1)^{p-n} w_0^n \sum_{\substack{m_1+\cdots m_{p-n}=p \\ 1\le m_s\le p+1 \\ m_{p-n}\ge 1}} w_{m_1} \cdots w_{m_{p-n}}$$

$$\equiv (-1)^p \sum_{\substack{m_1+\cdots m_p=p \\ 1\le m_s\le p+1 \\ m_p\ge 1}} w_{m_1} \cdots w_{m_p}.$$

The only possibility in the last summation is $m_s = 1$, hence

$$\psi_F^p(h) \equiv (-1)^p w_1^p = (-1)^p(-h)^p = h^p \bmod p$$

. ■

Example 4.21: We calculate these formulas for small p.

When $p = 2$, we have

$$\alpha^* = \frac{-2}{h} + \frac{-8}{h^4} + \frac{96}{h^7} + O(h^{-10})$$

and

$$\begin{aligned}\psi_F^2(h) &= h^2 + \alpha^* - h\cdot(\alpha^*)^2 \\ &= h^2 - \frac{6}{h} - \frac{40}{h^4} - \frac{544}{h^7} + O(h^{-10}).\end{aligned}$$

When $p = 3$, we have

$$\alpha^* = \frac{3}{h} + \frac{108}{h^3} - \frac{162}{h^4} + \frac{7857}{h^5} + O(h^{-6})$$

and

$$\psi_F^3(h) = h^3 - 6h^2 - 96h + 594 - \frac{1158}{h} + \frac{14580}{h^2} + \textit{ lower terms.}$$

Remark 4.10: In the $p = 3$ case, this power operation formula is different from which in[100] (Section 5.4). This is because the equation for α in[100] is not of the form as 4.3.1, but these two equations are equivalent[24] (Remark 2.25). In the semi-stable model of Morava E-theory[24] (Definition 2.23, Mod.1$^+$), it is required that $\mathrm{Frob}^2 = (-1)^{p-1}[p]$, for instance, $[3]$ in this case. While in[100], the model used is $\mathrm{Frob}^2 = [-3]$, as explained in Example 4.19

Remark 4.11: The formula 4.4.10 relies on the E_∞ structure on F. In our analysis, we equipped F with the E_∞ structure induced from E via localization. However, F itself may admit a different E_∞ structure. See[18] (Section 6). Not as what happened in Morava E theory, different models for F do yield *inequivalent* interpretation and formulas.

Recall that the total power operation $\psi_E^p : E^0 \to E^0 B\Sigma_p/I$ stands for taking the target of the universal deformation of Frobenius. It can also be viewed as taking the target curve of the universal degree p isogeny as explained previously.

Over F^0, the p-divisible group $\mathbb{G}_E$ becomes an extension

$$0 \to \mathbb{G}_F = \mathbb{G}_E^0 \to \mathbb{G}_E \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

where $\mathbb{G}_E^0$ is the connected component of $\mathbb{G}_E$ over F^0, which is $\mathbb{G}_F$.

Or equivalently

$$0 \to \widehat{C_u} \to C_u[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

over F^0, where C_u is the universal elliptic curve over $\mathscr{M}_{N,p}$ and F^0 corresponds to a punctured formal neighborhood of a supersingualr locus.

The map

$$t : E^0 B\Sigma_p/I \to F^0$$

in the diagram 4.4.1 classifies a degree p cyclic subgroup of C_u over F^0. However, in this case, C_u has only one cyclic subgroup of degree p, which coincides with the solution of $w(h, \alpha)$ in F^0 being unique, or equivalently, the map t being the unique map from $E^0 B\Sigma_p/I$ to F^0, as stated in Proposition 4.7.

Moreover, this subgroup is also the unique subgroup of degree p of $\widehat{C_u} = \mathbb{G}_F$ over F^0.

Therefore, in the interpretation of elliptic curves, we can explain the diagram 4.4.1 as follow.

$$\begin{array}{ccc} C_u & \xmapsto{\psi_E^p} & C_u/K \\ \downarrow & & \downarrow t \\ C'_u & \xmapsto{\psi_F^p} & C'_u/H \end{array}$$

where C'_u is the base change of C_u over F^0, and H is the degree p cyclic subgroup of C'_u as explained above. The maps ψ_E^p and ψ_F^p take the target curves of degree p isogenies starting from C_u over $E^0 B\Sigma_p/I$ and F^0 respectively. And the map t transform C_u to C'_u and K to H, hence it takes the curve C_u/K to C'_u/H. The element $\psi_F^p(h)$ can be viewed as the Atkin Lehner involution $\tilde{h}$ restricted over F^0.

In the interpretation of formal groups, we have

$$\begin{array}{ccc} \mathbb{G}_E & \xmapsto{\psi_E^p} & (\psi_E^p)^*\mathbb{G}_E = \mathbb{G}_E/K \\ \downarrow & & \downarrow t \\ \mathbb{G}_F & \xmapsto{\psi_F^p} & (\psi_F^p)^*\mathbb{G}_F = \mathbb{G}_F/H \end{array}$$

where K is the universal degree p subgroup of the formal group $\mathbb{G}_E$ and H is the unique degree p subgroup of $\mathbb{G}_F$. The groups K and H are the same thing as which appear in the interpretation of elliptic curves.

CHAPTER 5 AUGMENTED DEFORMATION SPECTRA

Let $\mathbb{H}$ be any height $n-1$ formal group over $k((u_{n-1}))$. In section 2.3, we have constructed the universal deformation $\mathbb{H}^u$ over the ring

$$F^0 = W(k)((u_{n-1}))^{\wedge}_p[[u_1, \dots, u_{n-2}]],$$

and let

$$F^* = F^0[\beta^{\pm}]$$

with $|\beta| = -2$. The ring F^* is Landweber exact, via the map

$$\begin{aligned} MU^* &\to F^* \\ x_{2p^i-2} &\mapsto u_i u^{n+1}. \end{aligned}$$

Hence we can construct a homotopy ring spectrum, called augmented deformation spectrum, denoted by $L_{\mathbb{H}} \in \mathrm{CAlg}(h\mathrm{Sp})$, which is complex oriented and carries the formal group $\mathbb{H}^u$.

5.1 The underlying spectra are equivalent

In this section we will show that the underlying spectra of $L_{\mathbb{H}}$ are independent of the choice of formal groups $\mathbb{H}$, which means they are all equivalent.

Let us recall what happens in Morava E-theories. Suppose the field k is perfect, $\mathbb{F}_1$ and $\mathbb{F}_2$ are two formal groups over k. Then we have $\mathbb{F}_1$ and $\mathbb{F}_2$ are isomorphic over the algebraic closure $\overline{k}$ of k, and in fact, they are isomorphic over the separable closure of k.

Let $E(k, \mathbb{F}_1)$ and $E(k, \mathbb{F}_2)$ be the corresponding Morava E-theories. It has been shown in[101] that the underlying spectra of them are equivalent, but not as homotopy commutative ring spectra. Hence one can at least take k to be an algebraically closed field, and talk about *the* Morava E-theory of height n over it.

While things are slightly different when consider formal groups over $k((u))$ even if k is algebraically closed.

Example 5.1: Let H be the Honda formal group over $\overline{\mathbb{F}}_p((u))$ with its p series given by

$$[p]_H(x) = x^p,$$

and let G be the special fiber of the base change of the universal deformation of the height

2 Honda formal group, which is defined over $\overline{\mathbb{F}}_p((u))$ with p series

$$[p]_G = ux^p +_G x^{p^2}.$$

We claim that H and G are not isomorphic over $\overline{\mathbb{F}}_p((u))$.

Suppose $\phi(t) = b_1 t + \cdots$ is an isomorphism from H to G, hence $b_1 \neq 0$. We have

$$\begin{aligned} \phi([p]_H(x)) &= [p]_G(\phi(x)) \\ \phi(x^p) &= u\phi(x)^p +_G \phi(x)^{p^2}. \end{aligned}$$

Calculating with mod x^{p^2}, we have

$$\begin{aligned} \sum_{i=1}^{p-1} b_i x^{ip} = b_1 x^p + \cdots + b_{p-1}x^{(p-1)p} &\equiv u\phi(x)^p \\ &\equiv ub_1^p x^p + ub_2^p x^{2p} + \cdots + ub_{p-1}^p x^{(p-1)p} \\ &\equiv \sum_{i=1}^{p-1} ub_i^p x^{ip} \mod x^{p^2}. \end{aligned}$$

This implies that $b_i = ub_i^p$ in $\overline{\mathbb{F}}_p((u))$. But the equation $ux^p = x$ does not have a non-zero solution in $\overline{\mathbb{F}}_p((u))$ and $b_1 \neq 0$ by the assumption.

Example 5.2: Following the above notation, we know that the Honda formal group H processes all its automorphism over $\mathbb{F}_p$, and in particular, over $\overline{\mathbb{F}}_p((u))$. On the other hand, by the same calculation, one can see that G does not have all its automorphism over $\overline{\mathbb{F}}_p((u))$.

These examples illustrate that one can not talk about *the* augmented deformation spectrum of a given height without specifying the formal group associated to it.

To show, despite of their ring structures, their underlying spectra are all equivalent, we need the following lemma.

Lemma 5.1 ([101], Lemma 7): Let R be a commutative ring with two Landweber exact formal group laws $e, f : MU_* \to R$ and let E and F be the corresponding spectra. If there is a ring extension $u : R \to S$ which is split as an R-module map and over S, the formal groups $u \circ e$ and $u \circ f$ are isomorphic and Landweber exact, then E and F has the same homotopy type.

Proof: We may assume $E = H$ is the height $n-1$ Honda formal group law and F is p-typical. We have

$$L_H^* = L_F^* = W(k)((u_{n-1}))_p^\wedge[[u_1, \ldots, u_{n-2}]][\beta^\pm].$$

Fix a separable closure of $k((u_{n-1}))$, and an isomorphism

$$\Phi(X) = \sum_{i \geq 0} \Phi_1 X +_H \Phi_2 X^{p^2} +_H + \cdots$$

from F to H. Let L denote the field $k((u_{n-1}))(\Phi_0, \Phi_1, \dots)$, and

$$L_n = k((u_{n-1}))(\Phi_0, \Phi_1, \dots, \Phi_n).$$

Therefore $L = \varinjlim L_i$ and F and H are isomorphic over L.

Each L_i is a finite Galois extension of $k((u_{n-1}))$ [16] (Section 2.3). By [102] (Proposition 4.4) or [17], there is a sequence of finite étale $W(k)((u_{n-1}))^\wedge_p$ algebras

$$W(k)((u_{n-1}))^\wedge_p = B(-1) \to B(0) \to \cdots.$$

Note that each $B(i)$ is a free $W(k)((u_{n-1}))^\wedge_p$ module, so does their limit $B(\infty)$. We have the map $W(k)((u_{n-1}))^\wedge_p \to B(\infty)$ splits as $W(k)((u_{n-1}))^\wedge_p$-modules. Since this is a $\mathbb{Z}_p$ module map, this splitting property extends to the p-adic completion, $B(\infty)^\wedge_p$, denoted by B. By [17] (Lemma 4.2), the ring B is a Cohen ring of its residue field L.

Now Let

$$R = L^*_H = L^*_F = W(k)((u_{n-1}))^\wedge_p[[u_1, \dots, u_{n-2}]], [\beta^\pm]$$
$$S = B[[u_1, \dots, u_{n-2}]][\beta^\pm].$$

The map $u : R \to S$ sends u_i to u_i and β to β, which splits as an R-module morphism as explained above. Let $\widetilde{F}$ and $\widetilde{H}$ be the universal augmented deformations of F and H respectively, over R. It is clear that $u^*\widetilde{F}$ is isomorphic to $u^*\widetilde{H}$ over S, because they are deformations of two isomorphic formal group laws. The conclusion follows from the Lemma 5.1. ■

CONCLUSION

In this dissertation, we calculated the $K(n-1)$-local E-theory of symmetric groups, obtained the corresponding modular interpretation of total power operation on the $K(n-1)$-localized Morava E-theory at height n and deduced the Dyer-Lashof algebra structure of $K(n-1)$-local E_n-algebras, using the duality result of $K(n-1)$-localized E_n.

In the height $n = 2$ case, we calculated explicitly the formula of total power operation on $K(1)$-localized E_2, and explained our works in the setting of elliptic curves and modular forms. We also proved all spectra constructed from augmented deformations are homotopy equivalent.

Our results

(1) extend results from $K(n)$-local E-algebras to $K(n-1)$-local E-algebras, which gives a more comprehensive perspective of E-algebras.

(2) is an attempt to get a full understanding of transchromatic phenomenon, which gives an insight of what happening on the moduli stack of formal groups when Bousfield localization changes chromatic heights.

(3) establish the modular interpretation of the total power operation on the overlap of the ordinary part and supersingular locus over the moduli stack of elliptic curve, which may have its use on getting a modular interpretation of the total power operation on the global moduli, i.e. TMF.

5.2 Future works

In future works, we could consider the total power operation on $K(t)$-localized Morava E-theory of height n, with $n - t \geq 2$. This may have its roots in the Lurie's refined Landweber exact functor theorem.

Recall the theorem from Lurie.

Theorem 5.1 ([55]Theorem 3.0.11): Let R_0 be a Noetherian $\mathbb{F}_p$ algebra, which is F-finite (i.e. Frobenius is a finite morphism), and let $\mathbf{G}_0$ be a *nonstationary* p-divisible group over R_0. Then there exists a morphism of connective E_∞ rings $R^{un}_{\mathbf{G}_0}$, which classifies (unoriented) deformation of $\mathbf{G}_0$ over R_0, i.e.

$$\mathrm{Maps}_{\mathrm{CAlg}/R_0}(R^{un}_{\mathbf{G}_0}, A) \xrightarrow{\sim} \mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$$

where A is a Noetherian E_∞ ring with a map $\rho_A : A \to R_0$ and a surjection $\epsilon_A : \pi_0 A \to R_0$, which is complete with respect to ker ϵ_A.

Since every classical p-divisible group is (canonically) oriented, we have an E_∞ ring (usually not connective) $R^{or}_{\mathbf{G}_0}$, which is called the orientation classifier of $R^{un}_{\mathbf{G}_0}$.

Lurie took R_0 to be a perfect field, and $\mathbf{G}_0$ be a height n formal group over R_0, and showed that

$$L_{K(n)} R^{or}_{\mathbf{G}_0} = R^{or}_{\mathbf{G}_0} = E_n$$

We could follow his approach, to be precise

- Let $R_0 = k((u_t))$, where k is a perfect field. This ring R_0 is F-finite.
- Let $\mathbf{G}_0$ be the base change of the formal group $\mathbf{G}_E$ over $\pi_0(E_n)$ along the map

$$\pi_0(E_n) \to \pi_0(L_{K(t)} E_n) \to k((u_t))$$

which is a height n p-divisible group with height k formal part. I don't know whether it is nonstationary or not.

- Show that $L_{K(t)} R^{or}_{\mathbf{G}_0}$ is equivalent to $L_{K(t)} E_n$

If this could work, then we have showed that $L_{K(t)} E_n$ classifies oriented deformations of $\mathbf{G}_0$ in the spectral setting.

We could also study the total power operation on $L_{K(1)} TMF$, which corresponds to the ordinary part of moduli stack of elliptic curves. Since the total power operation on supersingular loci can be identified with a map of E_∞ rings, we could patch them and which over the ordinary locus along their intersection, to obtain an expression of the total power operation on TMF.

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

Wu Yifan was born in 1997, in Bengbu, Anhui, China.

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Academic Achievements during the Study for an Academic Degree

学术论文

[1] Power Operations on $K(n-1)$-Localized Morava E-theory at Height n, (submitted to Journal of Homotopy and Related Structures)