博士学位论文

色展同伦论中的谱代数几何方法

METHODS OF SPECTRAL ALGEBRAIC GEOMETRY IN CHROMATIC HOMOTOPY THEORY

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METHODS OF SPECTRAL ALGEBRAIC GEOMETRY IN CHROMATIC HOMOTOPY THEORY

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摘 要

谱是拓扑空间的稳定化，是阿贝尔群的链复形的类比，由一列拓扑空间和其中的 suspension 映射所构成。环谱是一类重要的谱，是经典代数中环的高阶范畴化类比，它们所代表的是具有乘性的广义上同调理论。代数拓扑中一个基本的问题是如何构造具有结构乘法的环谱，比如乘法在同伦范畴具有交换性的环谱和乘法在所有高阶同伦中交换的环谱。在一些经典的方法中，我们需要使用复杂的阻碍理论去得到交换环谱的结构，比如 Goerss-Hopkins-Miller 定理。但是 Lurie 在他的一系列书籍和文章中使用了谱代数几何的方法给了这个定理一个新的证明。但另一方面，Morava E-理论在色展同伦论中扮演了重要角色，它们和形式群的形变的表示对象有关，但是当我们考虑带水平结构的形式群的形变的表示对象，我们却不能直接从 Lurie 的机制中得到一个谱。这是因为这些对象到一维 p 可除群的模叠的映射不是平展的。

在这篇文章中，我们在谱代数几何中定义并研究了所谓的导出水平结构。我们证明了谱椭圆曲线的同源诱导它下面的经典椭圆曲线之间的同源。这个结果说明我们导出版本的水平结构必须诱导经典的水平结构。我们定义并研究了谱代数几何中的相对 Cartier 除子，并且我们证明了一些可表性结果。基于这些结果，我们定义了谱代几何中的谱椭圆曲线的导出水平结构，我们证明了导出水平结构所结合的函子是被一些谱代数空间所表示的。除此之外，我们还考虑了谱 p 可除群的导出水平结构，我们证明了谱 p 可除群的水平结构的所结合的函子是可表的。

导出水平结构在代数拓扑中有很多应用。使用 Lurie 发展的表示定理，我们证明了附带导出水平结构的谱椭圆曲线可以形成一个谱 Deligne-Mumford 叠。我们证明了 p-可除群的带有导出水平结构的谱形变的模问题是仿射可表的。这些仿射可表对象所对应的谱使我们可以把 Morava E-理论提升到带水平结构的形变上，虽然这些提升是不平展的。对于附带全水平结构的形变，我们可以得到经典的 Lubin-Tate 塔的一个高阶范畴提升。而对于附带一个选定子群的形变, 由 Strickland 的工作，可以看做是 Frobenius 的形变的模问题。它们所对应的导出水平结构可以给我们 Morava E-理论的幂运算环的拓扑实现。

关键词：代数拓扑；色展同伦论；Morava E-理论；谱代数几何

ABSTRACT

Spectra are stabilizations of topological spaces, analogous to chain complexes of abelian groups. And ring spectra are higher categorical refinements of rings from classical algebra. They represent multiplicative generalized cohomology theories. A fundamental question is how to construct ring spectra with structured multiplication, such as ring spectra whose multiplications are commutative in homotopy categories and ring spectra whose multiplications are commutative in all higher homotopy. Classical methods use complicated obstruction theory to obtain commutative ring structures, such as Goerss-Hopkins-Miller theorem. But Lurie uses methods of spectral algebraic geometry give this theorem a new proof. On the other hand, Morava E-theories play an important role in chromatic homotopy theory, they correspond to universal deformations of formal groups. But moduli problems concerning deformations with level structures do not have immediate topological realizations readily from Lurie's framework. This is because the representable objects are not étale over the moduli stack of one dimensional p-divisible groups of height n.

In this thesis, we define and study moduli problems called derived level structures in Lurie's spectral algebraic geometry. We prove that isogenies of spectral elliptic curves must induce isogenies of their underlying classical elliptic curves. This provides evidence that the derived version of level structures must induce classical level structures. We define relative Cartier divisors in spectral algebraic geometry and prove those associated functors are representable by certain spectral Deligne-Mumford stacks. Analogous to Drinfeld, we define derived level structures for spectral elliptic curves. We prove that for spectral elliptic curves, moduli problems of derived level structures are representable, similar to the classical case. We also consider derived level structures of spectral p-divisible groups. We prove that those problems associated with them are representable in certain cases.

The study of derived level structures has many applications in algebraic topology. Using the spectral Artin representability theorem, we prove that the moduli stack of spectral elliptic curves with derived level structures has the structure of spectral Deligne-Mumford stacks. When we consider spectral deformations with derived level structures of p-divisible groups, those affine representable objects can provide us with many interesting spectra. We can lift Morava E-theories to deformations with level structures, although these lifts are not étale over Morava E-theories. For deformations with full-level structures, we can obtain higher categorical analogs of Lubin-Tate towers. And for deformations involving the selection of subgroups, which can be interpreted as moduli problems of deformations of Frobenius based on Stickland's work. We can obtain spectra whose π_0 are power operation rings of Morava E-theories.

Keywords: Algebraic topology; Chromatic homotopy theory; Morava E-theory; Spectral algebraic geometry

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LIST OF SYMBOLS AND ACRONYMS

$\mathrm{CAlg}^{\heartsuit}$	The category of commutative rings
Sch	The category of schemes
$\mathrm{Sch}_{ét}$	The étale site of schemes
Stk	The 2-category of stacks over $\mathrm{Sch}_{ét}$
$\mathcal{S}$	The ∞-category of spaces (∞-groupoids)
Sp	The ∞-category of spectra
CAlg	The ∞-category of $\mathbb{E}_\infty$-rings
CAlg^{ad}_{cpl}	The ∞-category of complete adic $\mathbb{E}_\infty$-rings
Mod_R	The ∞-category of R-modules for an $\mathbb{E}_\infty$-ring R
∞-Top	The ∞-category of ∞-topoi
$\mathrm{Sp}étR$	The étale spectrum of an $\mathbb{E}_\infty$-ring R, which is a spectrally ringed ∞-topos
SpDM	The ∞-category of spectral Deligne-Mumford stacks
$\mathrm{Spf}R$	The formal spectrum of an adic complete $\mathbb{E}_\infty$-ring R,which is a spectrally ringed ∞-topos
$\mathrm{Var}(R)$	The ∞-category of spectral varieties over an $\mathbb{E}_\infty$-ring R
$\mathrm{AVar}(R)$	The ∞-category of spectral abelian varieties over an $\mathbb{E}_\infty$-ring R
$\mathrm{Ell}(R)$	The ∞-category of spectral elliptic curves over an $\mathbb{E}_\infty$-ring R
FFG(A)	The ∞-category of commutative finite flat groups schemes over an $\mathbb{E}_\infty$-ring R
$\mathrm{BT}^p(A)$	The ∞-category of spectral p-divisible groups over an $\mathbb{E}_\infty$-ring A
$\mathcal{M}_{ell}$	The moduli stack of spectral elliptic curves
$\mathcal{M}^{cl}_{ell}$	The moduli stack of classical elliptic curves
$\mathrm{CDiv}(X/R)$	The space of relative Cartier divisors of a spectral Deligne-Mumford stack X over an $\mathbb{E}_\infty$-ring R
$\mathrm{Level}(\mathcal{A}, X/R)$	The space of derived A-level structures of a spectral elliptic curve X over an $\mathbb{E}_\infty$-ring R
$\mathrm{Level}(k, G/R)$	The space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of a height h spectral p-divisible group G/R
$\mathcal{M}_{ell}(\mathcal{A})$	The moduli stack of spectral elliptic curves with derived level structures

CHAPTER 1 INTRODUCTION

1.1 Background

By Brown's representability theorem, a general cohomology theory of topological spaces corresponds to a spectrum. All spectra form a closed symmetric monoidal category, called the stable homotopy category. Studying the stable homotopy category is a central topic in algebraic topology. There are many models of spectra, making it become a closed symmetric monoidal category. See[1] for an early discussion of the stable homotopy category,[2] for the S-module approach, and[3] for the ∞-category approach.

Chromatic homotopy theory uses chromatic localizations and the chromatic filtration to study the stable homotopy category. The heart of chromatic homotopy theory is the study of spectra, which represent general cohomology theories that are complex oriented. We can associate each complex oriented cohomology theory with a one-dimensional formal group. Studying those associated formal groups can help us understand complex oriented cohomology theories. The heights of formal groups can distinguish certain complex oriented cohomology theories. Choosing a coordinate of a formal group can yield a formal group law. Quillen[4] proved that the complex cobordism MU is the universal complex oriented cohomology theory, and its associated formal group law is the universal formal group law over the Lazard ring. Using the Landweber exact functor theorem[5], one can construct many complex oriented cohomology theories. Morava E-theories are constructed by using this theorem. Morava K-theories are another important complex oriented cohomology theories in chromatic homotopy theory, which are constructed by tensoring certain spectra together. Localizing with respect to Morava E-theories and Morava K-theories is the most common method in chromatic homotopy theory when working with spectra. Another very important example in chromatic homotopy theory is elliptic cohomology theories and their global section, the topological modular forms, which are useful in quantum field theory.

Homotopical algebraic geometry was founded in[6-7], which replaces commutative rings with simplicial rings, E_∞-ring spectra, and so on. One version of homotopical algebraic geometry is derived algebraic geometry, which replaces commutative rings with simplicial rings. One can refer to[8-10] for the foundation of derived algebraic geometry.

Derived algebraic geometry is useful in intersection problems, deformation problems, mathematical physics (homological mirror symmetry, BRST or BV quantization), p-adic Hodge theory, the geometric version of Langlands correspondences, and many other fields in mathematics. Spectral algebraic geometry is another version of homotopical algebraic geometry, which replaces commutative rings with E_∞-rings. It was founded by Lurie[11], and has increasingly more applications in algebraic topology, such as elliptic cohomology and equivariant topological modular forms.

As we mentioned, the stable homotopy category is a central topic in algebraic topology. Structured ring spectra are the most common examples studied, such as H_∞ spectra and E_∞ spectra. In[12] and[13], Lurie uses spectral algebraic methods give a proof of the Goerss-Hopkins-Miller theorem for topological modular forms. Except for the application of elliptic cohomology, Lurie also proved the E_∞ structures of Morava E-theories[13], which use the spectral version of deformation theory of certain p-divisible groups. The earliest proof of E_∞ structures of Morava E-theories is due to Goerss, Hopkins and Miller[14]. They turned the problem into a moduli problem and developed an obstruction theory. One can finish the proof by computing the Andre-Quillen groups. Comparing with their method, Lurie's proof is more conceptual. There are more and more applications of spectral algebraic geometry in algebraic topology. Such as topological automorphic forms[15], Morava E-theories over any F_p-algebra[13], not only just for a perfect field k. The construction of equivariant topological modular forms[16], elliptic Hochschild homology[17] and more.

On the other hand, moduli problems concerning deformations of formal groups with level structures are also representable, and the moduli spaces of different levels form a Lubin-Tate tower[18-19]. We know that the universal objects of deformations of formal groups have higher algebra analogues, which are the Morava E-theories. A natural question is what are higher categorical analogues of moduli problems of deformations with level structures? And can we find higher categorical analogues of Lubin-Tate towers. Although the $\mathbb{E}_\infty$-structure of topological modular forms with level structures can be obtained from[20], we still hope that there exists a derived stack of spectral elliptic curves with level structures which provide us with a more moduli interpretation. Except this, in the computation of unstable homotopy groups of sphere, after applying the EHP spectral sequences and the Bousfield-Kuhn functor, we observe that some terms on the E_2-page also arise from the universal deformation of isogenies of formal groups. They are computed by the Morava E-theories on the classifying spaces of symmetric groups[21-22]. They can be viewed as sheaves on the Lubin-Tate tower. We hope to provide a more conceptual perspective on this fact within the higher categorical Lubin-Tate tower.

In this paper, we give an attempt to address this problem by studying specific moduli problems in spectral algebraic geometry. The main ingredient of our work is the derived version of Artin's representability theorem established in[7,23]. We will use the spectral algebraic geometry version[11] in this paper. We study relative Cartier divisors in the context of spectral algebraic geometry. By imposing certain conditions, we define derived level structures of certain geometric objects in spectral algebraic geometry. Using Artin representability theorem, we prove some representable results of moduli problems that arise from our derived level structures. We give some examples of applications involving derived level structures. We consider the moduli problem of spectral deformations with derived level structures of p-divisible groups. We prove that these moduli problems are representable by certain formal affine spectral Deligne-Mumford stacks and the corresponding spectra can provide us many interesting generalized cohomology theories.

1.2 Statement of Main Results

We work on spectral algebraic geometry in this thesis. For a spectral Deligne-Mumford stack X over a spectral Deligne-Mumford stack S, a relative Cartier divisor is a morphism $D \to S$ of spectral Deligne-Mumford stacks such that $D \to X$ is a closed immersion, the ideal sheaf of D is a line bundle over X, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation. We use Lurie's representability theorem prove that the relative Cartier divisor is representable in certain cases. Our first main result is:

Theorem A. (Theorem 3.2.7) Suppose that E is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \;:\; & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

We define derived level structures of spectral elliptic curves. Roughly speaking, for a finite abstract abelian group A, usually equals $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, a derived A-level structure of a spectral elliptic curve E over an $\mathbb{E}_\infty$-ring R is just a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the space derived A-level structures of a spectral elliptic curve E/R. We prove that moduli problems associated with derived level structures are representable. Our second main result is:

Theorem B. (Theorem 3.3.5) Suppose that E is a spectral elliptic curve over a connective $\mathbb{E}_\infty$-ring R, then the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne-Mumford stack which is locally almost of finite presentation over the $\mathbb{E}_\infty$-ring R.

In classical algebraic geometry, except one-dimensional group curves, we also care level structures of p-divisible groups, it comes the full sections of commutative finite flat group schemes. In chapter three, we also consider derived level structures of spectral p-divisible groups. Let $\mathrm{Level}(k, G_R/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of a height h spectral p-divisible group G/R. Out third main result is:

Theorem C. (Theorem 3.4.11) Suppose G is a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\mathrm{Level}_{G/R}^k : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\mathcal{P}_{G/R}^k$.

For applications of derived level structures. We first prove that the moduli of spectral elliptic curves with derived level structures is representable by a spectral Deligne-Mumford stack. Our fourth main result is:

Theorem D. (Theorem 4.1.7) Let $\mathrm{Ell}(\mathcal{A})(R)$ denote the space of spectral elliptic curves with derived A-level structures over the $\mathbb{E}_\infty$-ring R. The functor

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne-Mumford stack and moreover this stack is locally almost of finite presentation over the sphere spectrum $\mathbb{S}$.

In[13], Lurie consider the spectral deformations of a classical formal group. As we have the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures. Suppose G_0 is a p-divisible group of height h over a perfect F_p-algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{or} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the ∞-category spanned by those quaternions (G, ρ, e, η)

（1）G is a spectral p-divisible group over R.

（2）ρ is a equivalence class of G_0-taggings of R.

（3）e is an orientation of the identity component of G.

（4）$\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G/R.

Our last main result is:

Theorem E. (Theorem 4.2.2) The functor $\mathcal{M}_k^{or}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is a finite $R_{G_0}^{or}$-algebra, $R_{G_0}^{or}$ is the orientation deformation ring of G_0 defined in[13].

1.3 Outline

The second chapter of this paper is an introduction of spectral algebraic geometry. We review main definitions and propositions of Lurie's book[11] and his series paper on elliptic cohomology[13,24-25]. We review spectral stacks, and morphisms between spectral Deligne-Mumford stacks, such as flat, étale, proper and finite conditions. These conditions will be useful in our future discussions. Spectral abelian varieties and spectral p-divisible groups are our main objects of study in this paper, we will review their basic properties in this chapter. The spectral Artin representability theorem is the main ingredient of this paper, we will use it to prove some representability results later. We will introduce the main conditions of this theorem. Deformations and orientations are the main tools for applying spectral algebraic geometry to algebraic topology. We present some useful concepts and theorems in the final section of this chapter.

The third chapter is the heart of this paper. We define derived isogenies and prove that the kernel of a derived isogeny in some cases have the same phenomenon as in the classical case. This provides evidence that our derived versions of level structures must induce classical level structures. For representability reasons, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures. We define relative Cartier divisors in the context of spectral algebraic geometry. We then use Lurie's representability theorem to prove that functors associated with relative Cartier divisors are representable by certain spectral Deligne-Mumford stacks. The main part of our proof involves computing of cotangent complex. We define derived level structures of spectral elliptic curves. Roughly speaking, a derived A-level structure of a spectral elliptic curve E over an $\mathbb{E}_\infty$-ring R is just a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure. We prove that moduli problems associated with derived level structures are representable. We also explore derived level structures of spectral p-divisible groups in this chapter and prove that the corresponding moduli problems are representable in certain cases.

In the last chapter, we give some applications of derived level structures. We first prove that the moduli problem of spectral elliptic curves with derived A-level structures is representable by a spectral Deligne-Mumford stack. In[13], Lurie consider the spectral deformations of a classical formal group. As we have the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures of certain p-divisible groups. We prove that these moduli problems are representable by certain spectral Deligne-Mumford stacks. And by choose different level structures, we obtain some interesting spectra. We will give examples of spectra constructed by consider moduli of spectral deformations with various level structures, such as higher categorical analogues of Lubin-Tate towers and topological realizations of representable objects of Strickland's deformations of Frobenius. In the second section of this chapter, we propose some idea about representation theory in spectral algebraic geometry.

We give an introduction to chromatic homotopy theory in Appendix A. We review formal groups, complex-oriented cohomology theory, Morava E-theories and Morava K-theories. We state some great achievements in chromatic homotopic theory, including nilpotence theorem, periodicity theorem and thick subcategories theorem. In the last part of appendix A, we review something about power operations.

We also give some necessary introduction about ∞-categories and higher algebra in Appendix B, including ∞-categories, homotopy limits and colimits, ∞-operads, modules and algebras in $\mathbb{E}_\infty$-ring context, finite, perfect, flat and étale morphism in $\mathbb{E}_\infty$-algebras.

CHAPTER 2 SPECTRAL ALGEBRAIC GEOMETRY

Spectral algebraic geometry was founded by Lurie in[11], it replaces commutative ring by $\mathbb{E}_\infty$-spectra in algebraic. Since there are homotopy coherence in the category of spectra, for convenience, we will work on ∞-categories. There are many references for ∞-categories, such as[3] and[26]. We assume that reader are familiar with the basic knowledge of ∞-categories and higher algebra. If not, the appendix B will give you a quick review. We will review some base knowledge of spectral algebraic geometry, most of materials comes forms[11]. I recommand readers to find more details in Lurie's book.

2.1 Spectral Deligne-Mumford Stacks

In the context of classical algebraic geometry, a stack is a functor from schemes to groupoid and satisfying some descending conditions, we recommend readers[27] and[28] for more discussion about stacks. We let Stk denote the 2-category of stacks. We recall that a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ in Stk is representable by schemes if for any $S \in$ Sch and $S \to \mathcal{X}_2$, the Cartesian product

$$S \times_{\mathcal{X}_2} \mathcal{X}_1$$

is representated by a scheme.

Definition 2.1.1: Suppose $\mathcal{X}$ is a sheaf of sets on $\mathrm{Sch}_{ét}$, we will say $\mathcal{X}$ is an algebraic space if there exists a scheme U and a surjective étale morphism $U \to \mathcal{X}$ is representable by schemes. The map $U \to X$ is called an étale presentation.

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of prestacks (or presheaves) over Sch, we will say f is representable if for every morphism $T \to \mathcal{Y}$ from a scheme T, the fiber product $\mathcal{X} \otimes_{\mathcal{Y}} T$ is an algebraic space.

Definition 2.1.2: Suppose X is a stack $\mathcal{X}$ over $\mathrm{Sch}_{ét}$, we will say that $\mathcal{X}$ is an algebraic stack if there exits a scheme U and a surjective, smooth, and representable morphism $U \to \mathcal{X}$. We will call this morphism $U \to \mathcal{X}$ a smooth presentation.

Definition 2.1.3: Let $\mathcal{X}$ be a stack over $\mathrm{Sch}_{ét}$, we will say that $\mathcal{X}$ is a Deligne-Mumford stack if there exits a scheme U and a surjective, étale, and representable morphism $U \to \mathcal{X}$. We will call this morphism $U \to \mathcal{X}$ an étale presentation.

Our definition of spectral Deligne-Mumford stacks will follow[11], which are ringed ∞-topoi satisfying certain conditions. Let's first say something about classical topoi. When we say a category X is a topos (Grothendieck topos), we always mean that X is equivalent to a category which has the form Shv($\mathcal{C}$), which is the category sheaves on a site $\mathcal{C}$. And when we say ringed topos, we mean a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathcal{X}$ is a topos and $\mathcal{O}_{\mathcal{X}}$ is a commutative ring object in the category $\mathcal{X}$.

For a certain commutative ring R, we let $\mathrm{CAlg}_R^{\acute{e}t}$ denote the 1-category of étale-R algebra. By the properties of étale motphism, we can equip a Grothendieck topology on the opposite category of $\mathrm{CAlg}_R^{\acute{e}t}$. It is defined by setting the family of étale maps generate a cover sieve if there exists some finite collection of morphism which is indicated by $\alpha_1, \alpha_2, \cdots, \alpha_n$, satisfying the map $A \rightarrow \prod_{1 \le i \le n} A_{\alpha_i}$ is faithfully flat. We let $\mathcal{O} : \mathrm{CAlg}_R^{\acute{e}t} \rightarrow \mathrm{Set}$ be the forgetful functor defined by $\mathcal{O}(R) = R$. Then it can be prove that $\mathcal{O}$ is sheaf for the étale topology, and moreover it is a commutative ring object of the topos $\mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_R^{\acute{e}t})$. We refer $(\mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_R^{\acute{e}t}), \mathcal{O})$ as the étale spectrum of this commutative R and denote it as $\mathrm{Sp\acute{e}t}R$.

We know that a Deligne-Mumford stack X can be view as a functor from the category of schemes to the category of groupoids satisfying certain conditions. It is an étale sheaf $X : \mathrm{CAlg}^{\heartsuit} \rightarrow \tau_{\le 1}\mathcal{S}$.

Theorem 2.1.4: Let $X : \mathrm{CAlg}_R^{\heartsuit} \rightarrow \tau_{\le 1}\mathcal{S}$ be a functor, X is representable by a classical Deligne-Mumford stack if there exits a collection of objects U_α which is indicated by $\alpha \in I$ in the category $\mathrm{CAlg}_R^{\heartsuit}$, and it satisfies the following two conditions.

(1) These objects $\{U_\alpha,\}_{\alpha \in I}$ cover $\mathrm{CAlg}_R^{\heartsuit}$. That is, the canonical map $\coprod_\alpha U_\alpha \rightarrow 1$ is an epimorphism in $\mathrm{CAlg}_R^{\heartsuit}$.

(2) For each $\alpha \in I$, the ringed topos $(X_{/U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ is equivalent to a ringed topos which has the form $\mathrm{Sp\acute{e}t}R_\alpha$, such that R_α is an ordinary commutative ring.

Proof: See[11]Remark 1.2.5.5 and[11]Theorem 1.2.5.9. ■

∞-Topoi

We now turn to spectral algebraic geometry. The main ingredients of spectral algebraic geometry are spectral Deligne-Mumford stacks, they are spectrally ringed ∞-topoi satisfying certain conditions.

Definition 2.1.5: Suppose we have an ∞-category $\mathcal{X}$, we will say that $\mathcal{X}$ is an ∞-topos, if we have an accessible left exact localization functor $\mathcal{P}(\mathcal{C}) \rightarrow \mathcal{X}$, where $\mathrm{P}(\mathcal{C})$ is the ∞-category of presheaves on small ∞-category $\mathcal{C}$. This condition means that there is an adjoint pair

$$a : \mathcal{P}(\mathcal{C}) \leftrightarrows \mathcal{X} : i$$

where a is left exact, and i is acessible.

Theorem 2.1.6: [29]Theorem 6.1.0.6 Suppose $\mathcal{X}$ is an ∞-category, then we have the following equivalent conditions:

（1）$\mathcal{X}$ is an ∞-topos.

（2）$\mathcal{X}$ is presentable, if we have a small simplicial set K and a natural transformation $\bar{\alpha} : \bar{p} \to \bar{q}$ of diagrams in Fun$(K^{\triangleright} \to \mathcal{X})$, $\mathcal{X}$ satisfies the following conditions:

If $\bar{q}$ is a colimit diagram and $\alpha = \bar{\alpha}|K$ is a Cartesian transformation, then we have $\bar{p}$ is a colimit diagram if and only if $\bar{\alpha}$ is a Cartesian transformation.

（3）$\mathcal{X}$ satisfying the Giraud's axioms:

① $\mathcal{X}$ is a presentable ∞-category.

② Colimits in the ∞-category $\mathcal{X}$ are universal.

③ Coproducts in the ∞-category $\mathcal{X}$ are disjoint.

④ Every groupoid object of $\mathcal{X}$ is an effective object.

Definition 2.1.7: Suppose we have two ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$. A geometric morphism form $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f_* : \mathcal{X} \to \mathcal{Y}$ of ∞-categories, such that f_* have a left exact adjoint (denote by $f^* : \mathcal{Y} \to \mathcal{X}$).

It is obvious that a classical topos is an ∞-topos whose morphism spaces are all discrete. Generally, we have the definition of n-topos.

Definition 2.1.8: Suppose $\mathcal{X}$ is an ∞-category, for $0 \le n \le \infty$, we will say that $\mathcal{X}$ is a n-topos if there exists an accessible left exact localization

$$L : \mathcal{P}_{\le n-1}(\mathcal{C}) \to \mathcal{X}$$

such that $\mathcal{C}$ is a small ∞-category, and $\mathcal{P}_{\le n-1}(\mathcal{C})$ denote the full ∞-subcategory of $\mathcal{P}(\mathcal{C})$ spanned by those $(n-1)$-truncated objects of the presheaves category $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$.

Example 2.1.9: Suppose $\mathcal{X}$ is an ∞-category, $\mathcal{X}$ is a 0-topos if and only if there is an equivalence of ∞-categories $\mathcal{X} \simeq N(\mathcal{U})$, here $\mathcal{U}$ is a locale. Let $\mathcal{U}$ be a partially ordered set, we say $\mathcal{U}$ is a locale if it satisfies the following two conditions:

（1）Let $\{U_\alpha\}$ be a subset of $\mathcal{U}$, which consists of elements of $\mathcal{U}$, then $\{U_\alpha\}$ has a least upper bound in $\mathcal{U}$, which we denoted it by $\cup_\alpha U_\alpha$ in $\mathcal{U}$.

（2）The least upper bounds commutes with meets, that is we have an equality

$$\bigcup(U_\alpha \cap V) = (\bigcup U_\alpha) \cap V.$$

where $(U \cup V)$ is the greatest lower bound of the two elements U and V.

Spectrally Ringed ∞-Topoi

Definition 2.1.10: Suppose that $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is an ∞-category. We will say a functor $F : \mathcal{X}^{op} \to \mathcal{C}$ is a $\mathcal{C}$ valued sheaf if it preserves small limits in ∞-categories. We let $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ denote the∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

Remark 2.1.11: In general, the definition above is not equal to the definition of $\mathcal{C}$-valued sheaves with respect to a certain Grothendieck topology on the ∞-category $\mathcal{X}$. But there is still a connection between them. Suppose that $\mathcal{T}$ is a small ∞-category equipped with a certain Grothendieck topology. We let $j : \mathcal{T} \to \mathcal{P}(\mathcal{T})$ denote the ∞-categorical Yoneda embedding. We have an inclusion functor $i : \mathrm{Shv}(\mathcal{T}) \hookrightarrow P(\mathcal{T})$, since it preserves small limits, so by the ∞-categorical adjoint functor theorem, it admits a left adjoint. We let $L : P(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T})$ denote the left adjoint to inclusion functor. Suppose we have an ∞-category $\mathcal{C}$ which admits all small limits. Then we have an equivalence of ∞-categories $\mathrm{Shv}_{\mathcal{C}}(\mathrm{Shv}(\mathcal{T})) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{T})$ which is induced by composition with $L \circ j$.

Definition 2.1.12: A spectrally ringed ∞-topos X is a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ is a sheaf of E_∞-rings on $\mathcal{X}$.

Spectral Deligne-Mumford Stacks

For an ∞-ring A, we consider the ∞-category of $\mathrm{CAlg}_A^{ét}$, it is equipped with the étale-topology. The sheaf category $\mathrm{Shv}_{\mathcal{S}}(\mathrm{CAlg}_A^{ét})$ is an ∞-topos, we let $\mathcal{O} : \mathrm{Shv}_{\mathcal{S}}(\mathrm{CAlg}_A^{ét}) \to$ CAlg denote the forget functor (since its value on represent objects are spectra), then it can be proved that $(\mathrm{Shv}_R^{ét}, \mathcal{O})$ is a spectrally ringed topoi, we call this ∞-topoi the étale spectrum of A.

Definition 2.1.13: Suppose we have a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we will say that X is a nonconnective spectral Deligne-Mumford stack if there exists a collection of objects $U_\alpha \in \mathcal{X}$ satisfying the following two conditions:

（1）Those object $\{U_\alpha\}$ is a cover of the ∞-topos $\mathcal{X}$.

（2）For each index α, the restriction ∞-topoi $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ of $(\mathcal{X}, \mathcal{O}_X)$ to U_α is equivalent to an étale spectrum $\mathrm{Sp}\acute{e}tA_\alpha$ for an $\mathbb{E}_\infty$-ring A_α.

We will say $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack if in addition, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is connective.

Example 2.1.14: For a connective $\mathbb{E}_\infty$-ring A, $\mathrm{Sp}\acute{e}tA = (\mathrm{Shv}_R^{ét}, \mathcal{O})$ is a spectral Deligne-Mumford stack.

Proposition 2.1.15: Suppose we have a nonconnective spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then the connective cover construction $\tau_{\geq 0}\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ determine a spectral Deligne-Mumford stack $(\mathcal{X}, \tau_{\geq 0}\mathcal{O}_{\mathcal{X}})$. And it has the following universal property: for every $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \infty\mathrm{Top}_{\mathrm{CAlg}}^{sHen}$, if we have $\mathcal{O}_{\mathcal{Y}}$ is connective, then the canonical map

$$\mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{sHen}}((\mathcal{X}, \tau_{\geq 0}\mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) \to \mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{sHen}}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is a homotopy equivalence. Moreover, the inclusion functor SpDM $\hookrightarrow$ SpDM^{nc} has a left adjoint. And its left adjoint is given by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$.

Proof: See[11]Proposition 1.4.5.1 and[11]Corollary 1.4.5.2. ■

Truncated spectral Deligne-Mumford stacks

Definition 2.1.16: Suppose $n \geq 0$ is an integer, and $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumfor stack. We will say that X is n-truncated if its structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated. We let $\mathrm{SpDM}^{\leq n}$ denote the full subcategory of SpDM, which is spanned by those spectral Deligne-Mumford stacks which are n-truncated.

Example 2.1.17: Suppose A is a connective $\mathbb{E}_\infty$-ring, then $\mathrm{Sp\acute{e}t}A$ is an affine spectral Deligne-Mumford stack. And $\mathrm{Sp\acute{e}t}A$ is n-truncated if and only if A is an n-truncated $\mathbb{E}_\infty$-ring.

Proposition 2.1.18: Suppose $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then the truncated construction $\tau_{\leq n}$ of structural sheaves determines a spectral Deligne-Mumford stack $(\mathcal{X}, \tau_{\leq 0}\mathcal{O}_{\mathcal{X}})$. And it has following universal property: for each $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}$, if we have $\mathcal{O}_{\mathcal{Y}}$ is connective and n-truncated. Then the canonical map

$$\mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{sHen}}((\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) \to \mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{sHen}}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is a homotopy equivalence. Moreover, the construction $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathcal{X}})$ determines a left adjoint of the inclusion functor $\mathrm{SpDM}^{\leq n} \hookrightarrow \mathrm{SpDM}$.

Proof: See[11]Proposition 1.4.6.3 and[11]Corollary 1.4.6.4. ■

For an ∞-topos $\mathcal{X}$, it can be prove that its heart $\mathcal{X}^\heartsuit$ is an ordinary topos. What is the relations between $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{X}^\heartsuit, \pi_0\mathcal{O}_{\mathcal{X}})$? The following recognition criterion give a relation between spectral Deligne-Mumford stacks and classical Deligne-Mumford stacks.

Theorem 2.1.19: [11]Theorem 1.4.8.1 Suppose $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectrally ringed ∞-topos, then it is a nonconnective spectral Deligne-Mumford stack if and only it satisfying the following four conditions:

（1）The underlying ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$ is a classical Deligne-Mumford stack.

（2）The canoncial geometric morphism $\phi_* : \mathcal{X} \to \mathrm{Shv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is étale.

（3）The homotopy groups sheaves $\pi_n\mathcal{O}_{\mathcal{X}}$ are all quasi-coherent sheaf on the classical stack $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$.

（4）The sheaf $\mathcal{O}_{\mathcal{X}}$ is hypercomplete.

Proposition 2.1.20: [11]Proposition 1.4.9.1 Let SpDM be the ∞-category spectral Deligne-Mumford stacks, it is the homotopy limit of following tower

$$\cdots \to \mathrm{SpDM}^{\leq 3} \overset{\tau_{\leq 2}}{\to} \mathrm{SpDM}^{\leq 2} \overset{\tau_{\leq 1}}{\to} \mathrm{SpDM}^{\leq 1} \overset{\tau_{\leq 0}}{\to} \mathrm{SpDM}^{\leq 0}.$$

Functor of Points

Assume we have a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we define functors

$$\begin{aligned} h_X^{nc} \; : \; & \mathrm{CAlg} \to \hat{\mathcal{S}} \\ & R \longmapsto \mathrm{Map}_{\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}}(\mathrm{Sp}\acute{e}tR, X) \end{aligned}$$

$$\begin{aligned} h_X \; : \; & \mathrm{CAlg}^{cn} \to \hat{\mathcal{S}} \\ & R \longmapsto \mathrm{Map}_{\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}}(\mathrm{Sp}\acute{e}tR, X). \end{aligned}$$

It can be prove that for a nonconnective spectral Deligne-Mumford stack X and for every $\mathbb{E}_\infty$-ring R, the mapping space $h_X^{nc}\mathrm{Map}_{\infty\mathrm{Top}^{loc}_{\mathrm{CAlg}}}(\mathrm{Sp}\acute{e}tR, X)$ is essentially small[11]Proposition 1.6.4.2.

Proposition 2.1.21: [11]Proposition 1.6.4.2 Let h_X^{nc} and h_X be the two functors defined above, we have

（1）$X \mapsto h_X^{nc}$ determines a fully faithful embedding $\mathrm{SpDM}^{nc} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$.

（2）$Y \mapsto h_Y$ determines a fully faithful embedding $\mathrm{SpDM} \to \mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$.

We will refer these two functors h_X^{nc} and h_Y as the functor of points of nonconnective spectral Deligne-Mumford stack X and spectral Deligne-Mumford stack Y respectively.

Definition 2.1.22: Suppose X is a spectral Deligne-Mumford stack, we will say X is a spectral Deligne-Mumford n-stack if for every commutative ring R, the mapping space $\mathrm{Map}_{\mathrm{Sp}DM}(\mathrm{Sp}\acute{e}tR, X)$ is n-truncated. And a spectral algebraic space is a spectral Deligne-Mumford 0-stack.

Example 2.1.23: Suppose that we have a connective $\mathbb{E}_\infty$-ring A, then $\mathrm{Sp}\acute{e}tA$ is a spectral algebraic space.

Geometric Points

Suppose that $\mathcal{X}$ is an ∞-topos, the ∞-category of points of X are the full subcategory of the functor ∞-category $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$ spanned by those geometric morphism $x^* : \mathcal{X} \to \mathcal{S}$.

Definition 2.1.24: Suppose that we have a spectral Deligne-Mumford stack X. A geometric point is a morphism of spectral Deligne-Mumford stacks $\eta : \mathrm{Sp}\acute{e}tk \to X$, such that k is a separably closed filed. And moreover, we say such a geometric point η is minimal if η can be written as a composition

$$\mathrm{Sp}\acute{e}tk \xrightarrow{\eta'} \mathrm{Sp}\acute{e}tA \xrightarrow{\eta''} X$$

and it satisfies the following conditions:

（1）η'' is étale.

（2）The map of commutative rings $\phi : \pi_0 A \to k$ which is induced by η'' exhibits k as a separable extension of a certain residue field of the ring $\pi_0 A$.

We let $Pt_g(X)$ denote the full subcategory of $\mathrm{SpDM}_{/X}$ which is spanned by those minimal geometric points $\eta : X_0 \to X$.

The following theorem gives an relation between geometric points and points of the underlying ∞-topos of a spectral Deligne-Mumford stack.

Proposition 2.1.25: [11]Proposition 3.5.4.2 Suppose $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack. Then we have a equivalence of ∞-categories between the ∞-category $Pt_g(X)$ and ∞-category $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$. Where $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ is the functor ∞-category spanned by those functors which preserves small colimit and finite limits. This equivalence is given by

$$(\eta : X_0 \to X) \mapsto (\eta^* \in \mathrm{Fun}(\mathcal{X}, \mathcal{S})).$$

Proposition 2.1.26: Suppose that $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are two spectral Deligne-Mumford stacks, and $f : X \to Y$ is a morphism between them. Then we have the following equivalent conditions:

（1）$f_* : \mathcal{X} \to \mathcal{Y}$ is a sujective morphism between their underlying ∞-topoi.

（2）Suppose that k be is a filed and $\eta : \mathrm{Sp}\acute{e}tk \to Y$ is a morphism in SpDM. Then we have an field extension of k' of k, it satisfies the composite $\mathrm{Sp}\acute{e}tk' \to \mathrm{Sp}\acute{e}tk \to Y$ factor through f.

（3）Suppose k is a filed and $\eta : \mathrm{Sp}\acute{e}tk \to Y$ is a morphism in SpDM, the fiber product $\mathrm{Sp}\acute{e}t_k \times_Y X$ is nonempty.

Proof: See[11]Proposition 3.5.5.4. ■

Definition 2.1.27: Suppose X and Y are two spectral Deligne-Mumford stacks and $f : X \to Y$ is a morphism between X and Y, we will say f is surjective if it satisfies those equivalent conditions in the above proposition.

2.2 Properties of Morphisms

We first recall something about local properties of geometric objects and morphisms between them. Let $\mathcal{T}$ be a Grothendieck topology on the ∞-category of spectral Deligne-Mumford stacks, like open, étale, flat, fpqc and so on.

Suppose P is a property of spectral Deligne-Mumford stacks, we will say that the property P is local for the $\mathcal{T}$-topology, if $\mathcal{P}$ satisfies the following conditions:

（1）For a morphism in $f : X \to Y$ belongs to $\mathcal{T}$, if once we know Y has the property P, then we can get X also has the property P.

（2）For cover morphisms $\{X_\alpha \to Y\}$ in $\mathcal{T}$, if every X_α has the property P, then we can get Y also has the property P.

Let Q be a property of morphisms in the ∞-category SpDM, Q is said to be local on the source with respect to the $\mathcal{T}$-topology, if the following conditions hold:

（1）Suppose we have a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$, if f belongs to $\mathcal{T}$, and g is a morphism which has property Q, then we can get $g \circ f$ also has the property Q.

（2）Suppose $g : X \to Y$ be a morphism in SpDM, for a collection of cover morphisms $\{f_\alpha : X_\alpha \to X\}$ in $\mathcal{T}$, if each of the composition $g \circ f_\alpha$ is a morphism the property Q, then we get g also has the property Q.

Let Q be a property of morphisms in SpDM, we will say that the property Q is local on the target with respect to the $\mathcal{T}$-topology, if it satisfying the following conditions:

（1）For every pullback square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

such that g belongs to $\mathcal{T}$, if f has the property Q, we get f' has the property Q.

（2）Let $g : X \to Y$ be a morphism in SpDM, for a collection of cover morphisms $\{f_\alpha : Y_\alpha \to Y\}$ in $\mathcal{T}$, if each of induced morphism $Y_\alpha \times_Y X \to Y_\alpha$ has the property Q, we can get g has the property Q.

Étale Morphisms

By the definition of spectral Deligne-Mumford stacks, étale locally, they are étale spectrum of $\mathbb{E}_\infty$-rings. The étale morphisms play the role of local in the word of spectral Deligne-Mumford stacks,just like open subscheme in classical algebraic geometry. We recall that a morphism $f : A \to B$ of $\mathbb{E}_\infty$-ring is called étale if it satisfies the following conditions:

（1）$\pi_0 A \to \pi_0 B$ is a étale morphism in the sense of classical algebraic geometry (flat and unramified).

（2）There are isomorphism $\pi_n A \otimes_{\pi_0 A} \pi_0 B \cong \pi_0 B$ of groups.

Definition 2.2.1: Let X and Y be two nonconnective spectral Deligne-Mumford stacks, We say a morphism $f : X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ between them is étale if it satisfies the following conditions:

（1）The morphism of the underlying ∞-topos $f_* : \mathcal{X} \to \mathcal{Y}$ is étale, i.e., it induces an equivalence of ∞-topos, $\mathcal{X} \simeq \mathcal{Y}_{/U}$ for a certain object $U \in \mathcal{Y}$.

（2）We have an equivalence

$$f^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$$

of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Proposition 2.2.2: [11]Corollary 1.4.10.3 Suppose that we have two nonconnective spectral Deligne-Mumford stacks $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ and $f : X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a morphism between them, then f is étale if and only if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t} B & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t} A & \longrightarrow & Y \end{array}$$

where the horizontal maps are étale, the underying map of $\mathbb{E}_\infty$-rings $A \to B$ is étale.

Definition 2.2.3: Suppose that we have two nonconnective spectral Deligne-Mumford stacks $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ and $f : X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is morphism between them, we will say f is flat if for every commutative square

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t} B & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t} A & \longrightarrow & Y \end{array}$$

where the horizontal maps are étale, the underying map of $\mathbb{E}_\infty$-rings $A \to B$ is flat.

Remark 2.2.4: By[11]Example 2.8.1.8 and[11]Proposition 2.8.2.4, being a étale (flat) morphism is a property which is local on source for the étale topology.

Closed Immersion

In classical algebraic geometry, assume that we have two schemes X and Y a morphism $f : X \to Y$ between them. We say f is a closed immersion if it induce a homeomorphism of the underlying topological space of X to a closed subset of Y, and induced morphism of structure sheaves $f^{-1} : \mathcal{O}_Y \to \mathcal{O}_X$ is an epimorphism. We say a geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$ between ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$ is a closed immersion if we have a composition

$$\mathcal{Y} \overset{g_*}{\to} \mathcal{X}/U \overset{i_*}{\to} \mathcal{X},$$

and satisfying U is an object of $\mathcal{X}$ which is (-1)-truncated and g_* is an equivalence of ∞-topoi.

Definition 2.2.5: Suppose that we have two spectrally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$, we say a morphism $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ between them, we say f is a closed immersion if it satisfies the following conditions:

（1）$f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion of ∞-topoi.

（2）Both $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ are connective.

（3）The induce morphism $\pi_0 f^{-1}\mathcal{O}_\mathcal{Y} \to \pi_0\mathcal{O}_\mathcal{X}$ is an epimorphism.

Proposition 2.2.6: [11]Proposition 3.1.1.1 Suppose that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a locally spectrally ringed ∞-topos, if we have $\mathcal{O}_\mathcal{X}$ is connective and we have a morphism $\alpha : \mathcal{O}_\mathcal{X} \to \mathcal{O}'$ of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ such that the induced morphism $\pi_0\mathcal{O}_\mathcal{X} \to \mathcal{O}'$ is surjective. Then there exists a closed immersion locally spectrally ringed ∞-topoi $f : (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and an equivalence $\beta : \mathcal{O}' \simeq f_*\mathcal{O}_\mathcal{Y}$.

Proposition 2.2.7: [11]Corollary 3.1.2.3 Suppose that we have a pullback square in SpDM

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

If we know that f is a closed immersion, we get f' is also a closed immersion.

Proposition 2.2.8: [11]Corollary 3.1.2.4 Suppose that we have a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

of spectral Deligne-Mumford stacks. If we already know that g is a closed immersion. Then the condition that f is a closed immersion is equivalent to h is a closed immersion.

Separated Morphisms

Definition 2.2.9: Suppose $X, Y \in$ SpDM, and $f : X \to Y$ is a morphism between. We will say f is separated if the diagonal morphism $X \to X \times_Y X$ is a closed immersion. Since Sp$\acute{e}t$ S is final object of SpDM, we say that a $X \in$ SpDM is separated if the morphism $X \to$ Sp$\acute{e}tS$ is separated.

It can be prove that for a separated morphism $f : X \to Y$ between spectral Deligne-Mumford stacks, the map $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp}\acute{e}tR, X) \to \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp}\acute{e}tR, Y)$ is 0-truncated. By this result, if we know Y is a spectral algebraic space, we get X is a spectral algebraic space.

Remark 2.2.10: By the base change of closed immersion, suppose that we have a pull-back square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of spectral Deligne-Mumford stacks. If f is separated, then basechange f' is also separated. And if g is an étale surjection, then the converse is also true.

By the definition of closed immersion, we find that a morphism $f : X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in SpDM to be separated only depends on the morphism of their underlying 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0\mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0\mathcal{O}_{\mathcal{Y}})$.

Finiteness Conditions on spectral Deligne-Mumford stacks

Let us first review some finiteness conditions in higher categorical algebra .

Suppose A is an $\mathbb{E}_\infty$-ring, M is an A-module. We say M is

（1）perfect, if it is an compact object of $L\mathrm{Mod}_R$.

（2）almost perfect, if there exits a integer k satisfying $M \in (L\mathrm{Mod}_R)_{\geq k}$ and M is an almost perfect object of $(L\mathrm{Mod}_R)_{\geq k}$.

（3）perfect to order n, if it satisfying the following conditions:

Suppose that we have a filtered diagram $\{N_\alpha\}$ in $(L\mathrm{Mod}_A)_{\geq 0}$, then the canonical map $\underset{\to\alpha}{\lim}\mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \underset{\to\alpha}{\lim} N_\alpha)$ is injective for $i = n$ and bijective for $i \leq n$.

（4）finitely n-presented if M is n-truncated and perfect to order (n+1).

（5）finite generated, if it is perfect to order 0.

And when we consider the finiteness conditions on algebra. We say a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is

（1）finite presentation if B belongs to the smallest full subcategory of $\mathrm{CAlg}_A^{\mathrm{free}}$ which is closed under finite colimits.

（2）locally of finite presentation if B is a compact object of ∞-category CAlg_A.

（3）almost of finite presentation if A is an almost compact object of the ∞-category CAlg_A, that is, $\tau_{\leq n}B$ is a compact object of $\tau_{\leq n}\mathrm{CAlg}_A$ for all $n \geq 0$.

（4）finite generation to order n if it satisfying the following conditions:

Suppose that we have a filtered diagram of connective $\mathbb{E}_\infty$-rings over A, $\{C_\alpha\}$, it has colimit C. If we know that each C_α is n-truncated and that those transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then there is a homotopy equivalence

$$\lim_{\alpha} \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$

（5）finite type if B is an A-algebra of finite generation to order 0.

（6）finite if B as an A-module is finitely generated.

Proposition 2.2.11: [11]Proposition 2.7.2.1, Proposition 4.1.1.3 Suppose that we have two connective $\mathbb{E}_\infty$-rings A and B, and $\phi : A \to B$ be a morphism between them. Then the following conditions are equivalent.

（1）ϕ is finite (finite type).

（2）The commutative ring $\pi_0 B$ is finite (finite type) over $\pi_0 A$.

Definition 2.2.12: [11]Definition 4.2.0.1 Suppose that we have $X, Y \in \mathrm{SpDM}$, and $f : X \to Y$ is a morphism between them. We say that f is locally of finite type, (locally of finite genreration to order n, locally almost of finite presentation, locally of finite presentation) if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}B & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t}A & \longrightarrow & Y \end{array}$$

in SpDM, such that the horizontal morphisms are étale, we always have the map of $\mathbb{E}_\infty$-rings $A \to B$ is finite type (finite generation to order n, almost of finite presentation, locally of finite presentation).

Definition 2.2.13: [11]Definition 5.2.0.1 Suppose that we have $X, Y \in \mathrm{SpDM}$, and $g : X \to Y$ is a morphism between them, we say f is finite, if f satisfying the following conditions:

（1）f is affine.

（2）The push-forward sheaves $f_*\mathcal{O}_X$ is perfect to order 0.

Remark 2.2.14: By[11]Example 4.2.0.2, a morphism $f : X \to Y$ in SpDM is locally of finite type if the underlying map of spectral Deligne-Mumford 0-stacks is locally of finite type.

And by[11]Remark 5.2.0.2, a morphism of $f : X \to Y$ is finite if the underlying map of spectral Deligne-Mumford 0-stacks is finite. If X and Y are spectral algebraic spaces, then f is finite is equivalent to $f^\heartsuit$ is finite is the sense of classical algebraic geometry.

Proposition 2.2.15: Suppose we have a pullback diagram

$$\begin{CD} X' @>>> X \\ @V{f'}VV @VV{f}V \\ Y' @>>> Y \end{CD}$$

in SpDM. If we know f is locally of finite generation to order n (locally of finite type, locally almost of finite presentation), we get f' also satisfies the same condition.

Proof: This is easy to see by the pullback property. ■

Proper Morphisms

Definition 2.2.16: Suppose that we have $X, Y \in$ SpDM, and $f : X \to Y$ is a morphism between them. We say f is universally closed if we have a pullback square in SpDM

$$\begin{CD} X' @>>> X \\ @VVV @VVV \\ Y' @>>> Y \end{CD}$$

such that Y' is a quasi-separated spectral algebraic space, we alwasy get the map $|X'| \to |Y'|$ between topological spaces is closed.

Definition 2.2.17: Suppose that we have $X, Y \in$ SpDM, and $f : X \to Y$ is a morphism between them. We call f a proper morphism if f is quasi-compact, separated, locally of finite type and universally closed.

Proposition 2.2.18: Proper morphism is stable under base change. Suppose we have a pull-back diagram

$$\begin{CD} X' @>>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD}$$

in SpDM. Then we have

（1）if f is proper, then so is f' .

（2）if f' is proper, and we know f is separated and g is a flat cover, we can get f

is proper.

Proof: This just follows from the base change property of separated, universally closed and locally of finite type. ■

Corollary 2.2.19: The condition that a morphism $f : X \to Y$ be proper is local on the target for the étale topology. This means that, if we get a étale surjection such that the projection map $X \times_Y Y' \to Y'$ is proper, then f is proper. And moreover, if we have a collection morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}$ such that each of them is proper. Then the we get the induced map $\coprod X_\alpha \to \coprod Y_\alpha$ is proper.

2.3 Quasi-Coherent Sheaves

Definition 2.3.1: Suppose that we have a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathcal{F}$ is a sheaves of spectra on $\mathcal{X}$ which is a $\mathcal{O}_\mathcal{X}$-module. We say that $\mathcal{F}$ is a quasi-coherent sheaves if we can find a collection of objects $U_\alpha \in \mathcal{X}$ such that they cover $\mathcal{X}$ (i.e.,the map $\coprod_\alpha U_\alpha$ is an effective epimorphism) and they satisfies:

For every α, there exits an E_∞-ring A_α, an A_α-module M_α, and an equivalence

$$(\mathcal{X}_{/U_\alpha}, \mathcal{O}|_{U_\alpha}, \mathcal{F}|_{U_\alpha}) \simeq \mathrm{Sp}\acute{e}t_{\mathrm{Mod}}(A_\alpha, M_\alpha)$$

in the ∞-category $\infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{Mod}}$.

For a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, we let $\mathrm{QCoh}(X)$ denote the ∞-category of quasi-coherent sheaves of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}$.

Let $f : X \to Y$ be a morphism of functors $X, Y : \mathrm{CAlg}^{cn} \to \mathcal{S}$ which is locally of finite presentation, representable, proper, locally of finite Tor-amplitude. We define

$$f_+\mathcal{F} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y), \quad \mathcal{F} \mapsto f_*(\omega_{X/Y}, \otimes \mathcal{F}).$$

Proposition 2.3.2: Suppose that we have two functors $X, Y : \mathrm{CAlg}^{cn} \to \mathcal{S}$ and $f : X \to Y$ be a morphism between such that f is representable, locally of finite presentation, proper and locally of finite Tor-amplitude. Then there exits an adjunction

$$f_+ : \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}(Y) : f^*$$

2.4 Formal Spectral Algebraic Geometry

Suppose that A is an E_∞-ring, we say A is an adic $\mathbb{E}_\infty$-ring if $\pi_0 A$ is an I adic ordinary ring for an ideal $I \subseteq \pi_0 A$. In classical commutative algebra, for a $M \in \mathrm{Mod}_R$ and an ideal of R, we can talk about the I -adic completion of M. There is a similar story in spectral algebraic geometry, reader can find more details in[11]Section 7. For any finitely generated ideals $I \subset \pi_0 A$, we have the following the I-completion functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_A^I : M \to \hat{M}_I$$

We consider a functor $\mathcal{O}_{\mathrm{Spf}A} : \mathrm{CAlg}_A^{\acute{e}t} \to \mathrm{CAlg}_A$ defined by $B \mapsto B_I^\wedge$. Let Shv_A^{ad} denote the closed subtopos of $\mathrm{Shv}_A^{\acute{e}t}$ corresponds to vanishing locus of an ideal of definition of $\pi_0 A$. It can be prove that $\mathcal{O}_{\mathrm{Spf}A} : \mathrm{Shv}_A^{ad} \to \mathrm{CAlg}_A$ is connective and strictly Henselian, so $(\mathrm{Shv}_A^{ad}, \mathcal{O}_{\mathrm{Spf}A})$ is an spectrally ringed ∞-topos. One can see chapter 8 of Lurie's book for more details.

Definition 2.4.1: Let A be an adic E_∞-ring, the formal spectrum $\mathrm{Sp}Af$ is the spectrally ringed-topoi $\mathrm{Spf}(A) := (\mathrm{Shv}_A^{ad}, \mathcal{O}_{\mathrm{Spf}A})$.

Definition 2.4.2: Suppose that we have an spectrally ∞-topos $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, we say f is a formal spectral Deligne-Mumford stack if there is a cover $\{U_i\}$ of $\mathcal{X}$, such that each $(\mathcal{X}_{|U_i}, \mathcal{O}_\mathcal{X}|U_i)$ is equivalent to $\mathrm{Spf}A_i$ for an adic E_∞-ring A_i.

Example 2.4.3: By[11]Proposition 8.1.6.6, suppose that $X \in \mathrm{SpDM}$ and $K \subset |X|$ is a co-compact closed subset of the underlying topological space of X. Then we can get a map in $i : \mathfrak{X} \to \mathcal{X}$ in SpDM, this $\mathfrak{X}$ can be viewed a formal completion of X along the closed subset K.

Formal GAGA Theorem

Theorem 2.4.4: [11]Theorem 8.5.3.1 Suppose that we have an I-adic complete $\mathbb{E}_\infty$-ring R, where I is an ideal $\pi_0 R$. If X is a spectral algebraic space over$R$and $X^\wedge$ is the formal completion along I, that is $X^\wedge = \mathrm{Spf}R \times_{\mathrm{Sp\acute{e}t}R} X$. Then we have a homotopy equivalence

$$\mathrm{Map}_{\mathrm{SpDM}}(X, Y) \to \mathrm{Map}_{\mathrm{fSpDM}}(X^\wedge, Y)$$

for any quasi-separated spectral algebraic space Y.

2.5 Spectral Artin Representability Theorem

Suppose that we have a spectral Deligne-Mumford stack X, its functor of points determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. A fundamental question in spectral algebraic geometry is what kinds of functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ are representable by spectral Deligne-Mumford stacks? We will review spectral representability theorem in this section. Let us first recall the classical Artin representability theorem

Theorem 2.5.1: Let R be a Grothendieck ring and $X : \mathrm{CAlg}_R^\heartsuit \to \mathrm{Set}$ be a functor. If X

satisfying the following conditions:

（1）$X \to X \times_{\mathrm{Spec}R} X$ is representable by a classical algebraic space.

（2）X is an étale sheaf on the category of R-algebra.

（3）We have an equivalence of sets

$$X(B) \to \varprojlim X(B/m^n)$$

for any complete local Noetherian R-algebra B with maximal ideal m.

（4）X admits a cotangent complex, and satisfying Schlessinger's criteria for formal representability.

（5）X commutes with filtered colimits.

Then X is representable by an algebraic space which is locally of finite presentation over R.

In derived algebraic geometry, there is a similar theorem developed by[7] and[23]. But we will focus on following spectral algebraic geometry version[11].

Spectral Artin Representability Theorem

Theorem 2.5.2: [11]Theorem 16.0.1 Suppose that we have a functor $M : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ between ∞-categories and R is a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring. If $f : M \to \mathrm{Spec}R$ is a natural transformation. If there exits a non-negative integer n , and X satisfying the following conditions:

（1）The space $M(R_0)$ is n-truncated for any discrete commutative ring R_0.

（2）The presheaf M is an étale sheaf.

（3）M admits a connective cotangent complex L_M.

（4）M is nilcomplete, integrable and infinitesimally cohesive.

（5）f is locally almost of finite presentation as a natural transformations between functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Then M is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

We will explain these conditions in the left of this section.

Cotangent Complex

Definition 2.5.3: Suppose that we have a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{A})$ and $\mathcal{M}$ is an $\mathcal{A}$-modules we let $\mathcal{A} \oplus M$ denote the trivial square extension $\mathcal{A}$ by $\mathcal{M}$, see[3]Theorem 7.3.4.7 for more details. A derivation is a map $\mathcal{A} \to \mathcal{A} \oplus \mathcal{M}$ satisfying it is a section of the canonical map $\mathcal{A} \oplus \mathcal{M} \to \mathcal{A}$. We let $\mathrm{Der}(\mathcal{A}, \mathcal{M}) = \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})/\mathcal{A}}(\mathcal{A}, \mathcal{A} \oplus M)$ denote ∞-category of derivations of $\mathcal{A}$ into $\mathcal{M}$.

Definition 2.5.4: Suppose $\mathcal{X}$ is an ∞-topos. We let

$$L : \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathrm{Mod}(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})), \quad \mathcal{A} \mapsto L_{\mathcal{A}}$$

denote the absolute cotangent complex functor defined in[3]Subsection 7.3.2. And for a morphism $\phi : \mathcal{A} \to \mathcal{B}$ of $\mathbb{E}_\infty$-ring sheaves on $\mathcal{X}$, the relative cotangent complex $L_{\mathcal{B}/\mathcal{A}}$ is given by the cofiber of the map $\mathcal{B} \otimes_{\mathcal{A}} L_{\mathcal{A}} \to L_{\mathcal{B}}$ determined by ϕ.

By[11]Subsection 7.3.2, the absolute cotangent complex is characterized by the following properties: There exists a universal derivation $d \in \mathrm{Der}(\mathcal{A}, L_{\mathcal{A}})$ for which composition with d induces an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathcal{A}}}(L_{\mathcal{A}}, M) \to \mathrm{Der}(\mathcal{A}, M).$$

of ∞-categories.

The cotangent complex of a spectral Deligne-Mumford stack X is the cotangent of X as a spectrally ringed topos. Assume that we have two functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a natural transformation $f : X \to Y$, they are determined by spectral Deligne-Mumford stacks X, Y and a morphism $f : X \to Y$ between them. Then for any $A \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta \in X(A)$, there exists a connective A-module M_η which corepresents the functor

$$\mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}, \quad N \mapsto \mathrm{fib}(X(A \times N) \to X(A) \otimes_{Y(A)} Y(A \oplus N)).$$

Definition 2.5.5: Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, we define a functor $F : \mathrm{Mod}_{\mathrm{cn}}^X \to \mathcal{S}$ by

$$F(A, \eta, M) = \mathrm{fib}(X(A \times M) \to X(A) \otimes_{Y(A)} Y(A \oplus M))$$

We will say f admits a cotangent complex if the functor F is locally almost corepresentable, see[11]Subsection 17.2.4 for more details. We say a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a cotangent complex if the natural transformation $X \to *$ admits a cotangent complex.

It can be prove that a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ admits a cotangent complex in the sense of above definition if it satisfies:

（1）For every $A \in \mathrm{CAlg}^{cn}$ and any point $\eta \in X(A)$, the functor

$$F_\eta : \mathrm{Mod}_A^{cn} \to \mathcal{S}, \quad F_\eta(N) = X(A \oplus N) \times_{X(A)} \{\eta\}$$

is corepresented by an A-module M_η by which is almost connective .

（2）Let $A \to B$ be a morphism between two connective $\mathbb{E}_\infty$-rings A and B, then for every B-module M which is connective, the diagram

$$\begin{array}{ccc} X(A \oplus M) & \longrightarrow & X(B \oplus M) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is a pullback square.

Remark 2.5.6: Suppose that we have a diagram in SpDM

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in Fun(CAlg$^{\mathrm{cn}}$, $\mathcal{S}$), if g and h admits a cotangent complexes.We can get f also admits a cotangent complex, and we have a fiber sequence

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$$

in the stable ∞-category QCoh(X).

Cohesive, Nilcomplete, and Integrable Functors

Definition 2.5.7: Let X : CAlg$^{cn} \to \mathcal{S}$ be a functor. We say that this functor X is

（1）Cohesive if X satisfying the condition: for every pull-back diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

in CAlg$^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced square

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow X(f) \\ X(B') & \xrightarrow{X(g)} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

（2）Infinitesimally cohesive if X satisfying the condition: for every pull-back square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective whose kernel are nilpotent ideals in $\pi_0 A$ and $\pi_0 B'$, the induced square diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow X(f) \\ X(B') & \xrightarrow{X(g)} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Remark 2.5.8: （1）Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a cohesive functor, then X is infinitessimally cohesive.

（2）If X is representable by a spectral Deligne-Mumford stack, then X is infinitessimally cohesive.

（3）Let $\{X_\alpha\}_{\alpha \in I}$ be a filtered diagram in Fun($\mathrm{CAlg}^{cn} \to \mathcal{S}$), and the colimit of this diagram is X, if we know that each X_α is cohesive(infintiesimally cohesive), then X is cohesive (infinitessimally cohesive).

Definition 2.5.9: We say a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is nilcomplete if for every $R \in \mathrm{CAlg}^{\mathrm{cn}}$, the natural map $X(R) \to \varprojlim X(\tau_{\leq n})$ is a homotopy equivalence.

Definition 2.5.10: We say a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is integrable if for every complete local Noetherian E_∞-ring A, we have an equivalence

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\mathrm{Spec}A, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\mathrm{Spf}A, X).$$

which is induced by $\mathrm{Spf}A \to \mathrm{Spec}A$.

Proposition 2.5.11: [11]Proposition 17.3.5.1 A functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is integrable if and only if for a local Noetherian ring A which is complete with respect to the maximal ideal m_A, we have an equivalence

$$X(A) \to \varprojlim_{n} X(A/m^n).$$

Relative Version of Cohesive, Nilcomplete, and Integrable

Definition 2.5.12: Let $g : X \to Y$ be a natural transformation between two functors, $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that g is:

（1）**cohesive** if g satisfies the condition: for every pullback square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

in CAlg^{cn} such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are all surjective, the diagram

$$\begin{array}{ccccc}
X(A') & \longrightarrow & X(A) & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
 & & Y(A') & \longrightarrow & Y(A) \\
X(B') & \longrightarrow & X(B) & & \downarrow \\
 & \searrow & \downarrow & \searrow & \\
 & & Y(B') & \longrightarrow & Y(B)
\end{array}$$

in $\mathcal{S}$ is a limit digram.

（2）**infinitesimally cohesive**, if g satisfies the condition: for any pullback square

$$\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}$$

of $\mathrm{CAlg}^{\mathrm{cn}}$, such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjections with nilpotent kernel, we get diagram of spaces

$$\begin{array}{ccccc}
X(A') & \longrightarrow & X(A) & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
 & & Y(A') & \longrightarrow & Y(A) \\
X(B') & \longrightarrow & X(B) & & \downarrow \\
 & \searrow & \downarrow & \searrow & \\
 & & Y(B') & \longrightarrow & Y(B)
\end{array}$$

is a limit diagram.

（3）**nilcomplete** if it satisfyies the condtion: for every $A \in \mathrm{CAlg}^{\mathrm{cn}}$, the diagram

$$\begin{array}{ccc}
X(A) & \longrightarrow & \varprojlim X(\tau_{\leq n} A) \\
\downarrow & & \downarrow \\
Y(A) & \longrightarrow & \varprojlim Y(\tau_{\leq n} A)
\end{array}$$

is a pullback square.

（4）**integrable** if it satisfies the condition: for every complete local Noetherian

E_∞-ring A, the induced diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}A, X) \\ \downarrow & & \downarrow \\ Y(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}A, Y) \end{array}$$

is a pullback square.

Remark 2.5.13: Suppose we are given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Y & \end{array}$$

in Fun(CAlg$^{\mathrm{cn}}$, $\mathcal{S}$), where g is cohesive. Then f is cohesive if and only if h is cohesive. The statement is also holds for conditions: infinitesimally cohesive, nilcomplete and integrable.

Take Z to be the final object of Fun(CAlg^{cn}, $\mathcal{S}$), we can find that if Y : CAlg $\to \mathcal{S}$ is cohesive, then a morphism $f : X \to Y$ is cohesive if and only if X is cohesive. The statement is also holds for conditions: infinitesimally cohesive, integrable and nilcomplete.

Locally of Finite Presentation

Definition 2.5.14: Suppose $X, Y \in$ Fun(CAlg$^{\mathrm{cn}} \to \mathcal{S}$), let $f : X \to Y$ be a natural transformation. We will say f is

(1) **locally of finite presentation** if it satisfies the condition: for every filtered diagram of connective E_∞-rings $\{A_\alpha\}$ whose colimit is A, the canonical map we have an equivalence

$$\theta : \varinjlim \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$$

(2) **locally almost of finite presentation** if it satisfies the condition: for $m \geq 0$ and for any filtered diagram $\{A_\alpha\}$ in CAlg$^{\mathrm{cn},\tau_{\leq n}}$, we have an equivalence

$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha).$$

(3) **locally of finite generation to order n** if it satisfies the condition: for any filtered diagram $\{A_\alpha\}$ in CAlg$^{\mathrm{cn}}$ such that A_α is n-truncated and the transition map $\pi_n A_\alpha \to \pi_n A_\beta$ are monomorphism, we have an equivalence

$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} \varinjlim Y(A_\alpha).$$

Proposition 2.5.15: Suppose X, Y : Fun(CAlg$^{\mathrm{cn}} \to \mathcal{S}$), let $f : X \to Y$ be a natural transformation between them. Then we have the following statements which are equivalent.

（1）f is locally of finite presentation .

（2）For every pull-back square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in Fun(CAlg^{cn}, $\mathcal{S}$), f' is locally of finite presentation.

（3）For every pull-back square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in Fun(CAlg^{cn}, $\mathcal{S}$) where Y' is a corepresent functor, the map f' is locally of finite presentation.

Moreover, these statements holds for the conditions: locally almost of finite presentation and locally of finite generation to order n.

Proposition 2.5.16: [11]Proposition 17.4.2.1 Suppose X, Y : Fun(CAlg^{cn} → $\mathcal{S}$), let $f : X \to Y$ be a natural transformation between them and suppose that f admits a cotangent complex $L_{X/Y}$. Then:

（1）If f is locally of finite generation to order n, then $L_{X/Y} \in$ QCoh(X) is perfect to order n.

（2）Assume that f is infinitessimally cohesive and satisfies the following addition condition

(*) For every filtered diagram $\{A_\alpha\}$ of commutative rings have colimits A, the diagram of spaces

$$\begin{array}{ccc} \varinjlim X(A_\alpha) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ \varinjlim Y(A_\alpha) & \longrightarrow & Y(A) \end{array}$$

is a pull-back diagram. Moreover, if $L_{X/Y}$ is perfect to order n, then f is locally of finite generation to order n.

Étale sheaves in Spectral Algebraic Geometry

Suppose that $\mathcal{C}$ is an ∞-category and $\mathcal{C}$ been equipped with a Grothendieck topology $\mathcal{T}$ (See[29]Definition 6.2.2.1 for the details of Grothendieck topology on ∞-categories). Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$ be a presheaf, we say $\mathcal{F}$ is an $\mathcal{T}$-sheaf if for any object $C \in \mathcal{C}$, and a $\mathcal{T}$ cover sieve $\{U_i \to C\}$, $\mathcal{F}(C)$ is the limit of the simplicial diagram

$$\mathrm{Tot} : \Delta^{\mathrm{op}} \to \mathcal{S}, \quad [n] \mapsto \coprod \mathcal{F}(U_{i_1,i_n})$$

This definition is similar with the classical definition, while $\mathcal{F} : C^{op} \to \tau_{\leq 0}\mathcal{S} \simeq \mathrm{Set}$ is a classical sheaf from a 1-category to Set if for any object $C \in \mathcal{C}$, and an $\mathcal{T}$ cover $\{U_i \to C\}$, $\mathcal{F}(C)$ is the limit of the diagram

$$\coprod \mathcal{F}(U_i) \to \coprod \mathcal{F}(U_{ij})$$

The following theorem gives a relation between an étale sheaf and its restriction to discrete case.

Proposition 2.5.17: [11]Proposition 18.1.1.1 Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a nilcomplete, infinitesimally cohesive functor and admits a cotangent complex. Then the following conditions are equivalent:

(1) The functor X is an étale sheaf in higher categorical word.

(2) The restriction of X restricts to discrete is an étale sheaf, that is $X|_{\mathrm{CAlg}^{\heartsuit}}$ is an étale sheaf .

Proof: The direction $(1) \Rightarrow (2)$ is obvious, we will prove the other direction. Suppose that we already know that $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology. We wish to prove that $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an étale sheaf, but étale sheaf is a local condition, so we only need to prove that $X|_{\mathrm{CAlg}_R^{\acute{e}t}}$ is an étale sheaf.

We know that X is a nilcomplete sheaf, so we only need to prove that $X_{\tau_{\leq n}R} : \mathrm{CAlg}^{et}_{\tau_{\leq n}R} \to \mathcal{S}, A \mapsto X(\tau_{\leq A})$ is an étale sheaf. We will use the induction to prove this statement. The case $n = 0$ follows from the assumption, now we assume it is true for $n-1$. We know that R is a square-zero extension of $\tau_{\leq n-1}R$ by $M = \Sigma^n(\pi_n R)$, we then have a pullback diagram

$$\begin{array}{ccc} \tau_{\leq n}R & \longrightarrow & \tau_{\leq n-1}R \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}R & \longrightarrow & \tau_{\leq n-1}R \oplus \Sigma M \end{array}$$

We define two functors $Y_{\tau_{\leq n-1}R}, Z_{\tau_{\leq n-1}R} : \mathrm{CAlg}^{\acute{e}t}_{\tau_{\leq n}R} \to \mathcal{S}$ by the formula

$$Y_{\tau_{\leq n-1}R}(A) = X(A \otimes_{\tau_{\leq n}R} \tau_{\leq n-1}R) = X(\tau_{\leq n-1}A)$$

$$Z_{\tau_{\leq n-1}R}(A) = X(A \otimes_{\tau_{\leq n}R} (\tau_{\leq n-1}R \oplus \Sigma M)) = X(\tau_{\leq n-1}A \oplus (A \otimes_{\tau_{\leq n}R} M)).$$

By the infinitessimally cohesiveness of X, we then have a pullback diagram of functors

$$\begin{array}{ccc} X_{\tau_{\leq n}R} & \longrightarrow & Y_{\tau_{\leq n-1}R} \\ \downarrow & & \downarrow \\ Y_{\tau_{\leq n-1}R} & \longrightarrow & Z_{\tau_{\leq n-1}R} \end{array}$$

By the assumption, we have $Y_{\tau_{\leq n-1}R}$ is an étale sheaf, so it is enough to prove that $Z_{\tau_{\leq n-1}R}$ is an étale sheaf. We consider the nature projection $Z_{\tau_{\leq n-1}R} \to Y_{\tau_{\leq n-1}R}$, by the fiber principle[11]Lemma D.4.3.2, it is enough to prove that each fiber of this functor is an étale sheaf. This is equivalent to say that:

(*) For every étale $\tau_{\leq n}R$-algebra A, and every point $\eta \in X(\tau_{\leq n-1}A)$, the functor $\mathcal{F} : \mathrm{CAlg}^{\acute{e}t}_{A} \to \mathcal{S}$ defined by

$$B \mapsto \mathrm{fib}(X(\tau_{\leq n-1}B \oplus (A \otimes_{\tau_{\leq n}R} M)) \to X(\tau_{\leq n-1}B))$$

is an étale sheaf. But by the definition of cotangent complex of L_X, we find that $\mathcal{F}(B) = \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n-1}A}}(\eta^* L_X, B \otimes_R M)$. It then follows from that Hom and $\otimes$ [11]Corollary 6.3.4 satisfying étale descent[11]Propositon 5.2.7. ■

And the spectral Artin representability can deduced from the following version.

Theorem 2.5.18: [11]Theorem 18.1.0.2 Suppose that we have a functor $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, then Z is representable by a spectral Deligne-Mumford stack if and only if it satisfying the following conditions:

(1) There exists a $Y \in \mathrm{SpDM}$ representing a functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a equivalence of functors $Z_{\mathrm{CAlg}^{\heartsuit}} \simeq Y|_{\mathrm{CAlg}^{\heartsuit}}$.

(2) Z have a cotangent complex.

(3) Z is nilcomplete .

(4) Z is infinitesimally cohesive.

2.6 Spectral Varieties

Algebraic varieties are the earliest objects people studied in classical algebraic geometry. They are the common zeros of a collection of polynomials. Then Grothendieck give these objects a more general description, they are schemes satisfies certain conditions: integral separated scheme of finite type over an algebraically closed field k. In spectral algebraic geometry, spectral varieties are also comes from some restrictions on more general objects.

Definition 2.6.1: A spectral variety X over an E_∞-ring R is a morphism in SpDM^{nc} which is flat, and satisfying the induced map $\tau_{\geq 0}X \to \mathrm{Spet}\tau_{\geq 0}R$ of spectral Deligne-Mumford stacks is proper, locally almost of finite presentation, geometrically reduced and geometrically connected. We let $\mathrm{Var}(R)$ denote the ∞-category of spectral varieties over R.

Suppose that $\mathcal{X}$ is an ∞-category and it has all finite products. We let Lat denote the ∞-category of free abelian group of finite rank. A functor $A : \mathrm{Lat}^{op} \to \mathcal{X}$ is called an abelian group object if it preserves finite products. We let $\mathrm{Ab}(\mathcal{X})$ denote the ∞-category of abelian group objects of $\mathcal{X}$.

Suppose that $\mathcal{X}$ is an ∞-category and it has all finite products. We recall that a commutative monoid object of $\mathcal{X}$ is a functor $M : \mathrm{Fin}_* \to \mathcal{X}$ which satisfies: For each $n \geq 0$, the maps $\{M(\rho^i) : M(\langle n\rangle) \to M\langle 1\rangle\}_{1\leq i\leq n}$ determines an equivalence $M(\langle n\rangle) \to M(\langle 1\rangle)^n$ in $\mathcal{X}$. And we denote $\mathrm{CMon}(\mathcal{X})$ the ∞-category of commutative monoid objects of $\mathcal{X}$.

Definition 2.6.2: Let R be an E_∞-ring. A spectral abelian variety over R is a commutative monoid object of the ∞-category $\mathrm{Var}(R)$. We let $\mathrm{AVar}(R)$ denote the category of spectral abelian varieties over R.

Definition 2.6.3: Suppose that we have an $R \in \mathrm{CAlg}$. A strict spectral abelian variety over R is an abelian group object of the ∞-category $\mathrm{Var}(R)$. We let $\mathrm{AVar}^s(R)$ denote the ∞-category of strict abelian varieties over R.

Remark 2.6.4: We have the functor of points construction $\mathrm{Var}(R) \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$, which induce a fully faithful embedding

$$\begin{aligned}\mathrm{AVAr}(R) &= \mathrm{CMon}(\mathrm{Var}(R)) \\ &= \mathrm{CMon}^{gp}(\mathrm{Var}(R)) \\ &\hookrightarrow \mathrm{CMon}^{gp}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})) \\ &= \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon}^{gp}(\mathcal{S}))\end{aligned}$$

So for an abelian variety X, its value $X(R)$ on an $\mathbb{E}_\infty$-ring R is an group like $\mathbb{E}_\infty$-space. We also have the functors of strict abelian varieties their values on $\mathbb{E}_\infty$-rings are topological abelian groups.

Spectral Elliptic Curves

Definition 2.6.5: Suppose that we have $R \in \mathrm{CAlg}$. A spectral elliptic curve over R is an spectral abelian variety of dimension 1 over R. We let $\mathrm{Ell}(R) = \mathrm{AVar}_1(R)$ denote the ∞-category of spectral elliptic curves over R.

A strict spectral elliptic curve is a strict spectral ableian variety of dimensional 1 over R. We let $\mathrm{Ell}^s(R) = \mathrm{AVar}_1^s(R)$ denote the ∞-category of strict spectral elliptic curves over R.

By the definition of spectral elliptic curves and strict spectral elliptic curves, we can define functors

$$\begin{aligned} \mathcal{M}_{ell} \ &: \ \mathrm{CAlg} \to \mathcal{S} \\ & \quad R \longmapsto \mathcal{M}_{ell}(R) = \mathrm{Ell}(R)^{\simeq} \end{aligned}$$

$$\begin{aligned} \mathcal{M}^s_{ell} \ &: \ \mathrm{CAlg} \to \mathcal{S} \\ & \quad R \longmapsto \mathcal{M}^s_{ell}(R) = \mathrm{Ell}(R)^{\simeq} \end{aligned}$$

Theorem 2.6.6: [24]Theorem 2.4.1 The two functors $\mathcal{M}_{ell}$ and $\mathcal{M}^s_{ell}$ are representable spectral Deligne-Mumford stacks. Moreover, these two representable stacks are locally almost of finite presentation over the sphere spectrum.

2.7 Spectral p-Divisible Groups

Definition 2.7.1: Suppose that we have A is a $\mathbb{E}_\infty$-ring and $M \in \mathrm{Mod}_A$. We will say that M is finite flat it satisfies the following conditions:

（1）Every homotopy group $\pi_n M$ as a $\pi_0 A$-module is locally free of finite rank over the commutative ring $\pi_0 A$.

（2）For each integer n, we have an isomorphism $\pi_0 M \otimes_{\pi_0 A} \pi_n A \to \pi_n M$ of homotopy groups.

Definition 2.7.2: Let $f : X \to Y$ be a map in SpDM. We say that f is a finite flat morphism of degree d, if for every map $\mathrm{Sp}\acute{e}tA \to Y$, the fiber product $X \times_Y \mathrm{Sp}\acute{e}tA$ has the form $\mathrm{Sp}\acute{e}tB$, where B is a finite flat rank d A-module. We let $\mathrm{FF}(A)$ denote the full subcategory of SpDM^{nc}_A spanned by finite flat morphisms $X \to \mathrm{Sp}\acute{e}tA$.

It is easy to see that if $f : X \to \mathrm{Sp}\acute{e}tA$ is finite flat, then $X = \mathrm{Sp}\acute{e}tB$ for some finite flat A-algerba B. And one can also define spectral commutative finite flat schemes over

A. They are just grouplike commutative monoid objects in $\mathrm{FF}(A)$. We let FFG(A) denote the ∞-category of spectral commutative finite flat group schemes over A.

Definition 2.7.3: Suppose $A \in \mathrm{CAlg}$ and S be a set of prime numbers. When we say a S-divisible group over A, we mean a functor $X : (\mathrm{Ab}^S_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{FFG}(A)$ satisfies the following conditions:

（1）The spectral commutative finite flat scheme $X(0)$ is trivial.

（2）For every short exact sequence $M'' \to M \to M'$ of finite abelian S-groups, we have a pullback square of spectral commutative finite flat group schemes over A as follows

$$\begin{array}{ccc} X(M'') & \longrightarrow & X(M) \\ \downarrow & & \downarrow \\ X(0) & \longrightarrow & X(M'). \end{array}$$

（3）The S-divisible group has height h, if for a M which is a finite abelian S group, $X(M)$ is a degree $|M|^h$ spectral commutative finite flat group scheme over A.

when S consists of only one prime p, then we call it a p-divisible group over A, we write $\mathrm{BT}^p_h(A)$ for the ∞-category of height h spectral p-divisible group.

Theorem 2.7.4: [24]Theorem 7.0.1 Assume that we have a connective $\mathbb{E}_\infty$-ring $A \in \mathrm{CAlg}^{\mathrm{cn}}$ and M be a connective A-module, let $\bar{R}$ be a square-zero extension of R by M. For every integer $g \geq 0$, the p^∞-torsion construction determines a pullback square

$$\begin{array}{ccc} \mathrm{AVar}^s_g(\bar{R}) & \longrightarrow & \mathrm{AVar}^s_g(R) \\ \downarrow & & \downarrow \\ \mathrm{BT}^s_{2g}(\bar{R}) & \longrightarrow & \mathrm{BT}^s_{2g}(R) \end{array}$$

By this theorem, we can find that just like the classical case, the deformations of spectral abelian varieties are controled by deformations of their associated spectral p-divisible groups. One can see[24]Section 6, 7 for more details about spectral p-divisible groups.

It is know that for a classical simple p-divisible group G over a perfect field k of characteristic p, there is a short exact sequence,

$$0 \to G^\circ \to G \to G^{\acute{e}t} \to 0$$

such that G^0 is formal and $G^{\acute{e}t}$ is étale. This is also a similar theorem in spectral algebraic geometry.

Definition 2.7.5: [13]Definition 1.6.1 Suppose that we have $R \in \mathrm{CAlg}^{\mathrm{cn}}$, A spectral formal group over R is a functor $\hat{G} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ such that the composition

$$\mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{\hat{G}} \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^\infty} \mathcal{S}$$

is a formal hyperplane over R, i.e., this functor is representable by a formal spectrum of the dual of a smooth coalgebra, see[13]Section 1 for more details about spectral formal groups.

Theorem 2.7.6: Suppose that we have a p-complete E_∞-ring R, and G is a spectral p-divisible over R. Then there exits an essentially unique spectral formal group $G^\circ \in \mathrm{FGroup}(R)$ satisfying that G^o restrict to those connective $\tau_{\leq 0}R$-algebras which are truncated and p-nilpotent is given by

$$A \mapsto \mathrm{fib}(G(A) \to G(A^{red})).$$

We call G° the identity component of G. Moreover, if the connective component G° is a spectral p-divisible formal group, then we can get a short exact sequence

$$0 \to G^\circ \to G \to G^{ét} \to 0,$$

satisfying G° is formally connected and $G^{ét}$ is étale.

Deformations of Spectral p-Divisible Groups

In this subsection, suppose that we have a commutative ring R_0 and G_0 is a p-divisible group over R_0. Let $A \in \mathrm{CAlg}^{\mathrm{cn}}$ and we have a map $\rho_A : A \to R_0$

Definition 2.7.7: A spectral deformation of G_0 along the ring map ρ_A consists of a pair (G, α), where G is a spectral p-divisible group over A and $\alpha : G_0 \simeq \rho_A^* G$ is an equivalence of spectral p-divisible groups over R_0. We let $\mathrm{Def}_{G_0}(A, \rho_A)$ denote the ∞-category of all spectral deformations fo G_0 along the map ρ_A.

The following theorem due to Lurie establish the universal spectral deformation theory of p-divisible groups. Suppose that R_0 is Noetherian F_p-algebra such that the Frobenius morphism is finite and G_0 is a p-divisible group over R_0.

Theorem 2.7.8: [13]Theorem 3.0.11 There exists a E_∞-ring $R_{G_0}^{\mathrm{un}} \in \mathrm{CAlg}^{\mathrm{cn}}$ with a morphism of E_∞-rings $\rho : R_{G_0}^{\mathrm{un}} \to R_0$ satisfying following properties:

- The E_∞-ring $R_{G_0}^{\mathrm{un}}$ is Noetherian, and the map $\pi_0(\rho) : \pi_0(R_{G_0}^{un}) \to R_0$ is surjective, and $R_{G_0}^{\mathrm{un}}$ is complete with respect to the ideal $\ker(\pi_0(\rho))$.
- For any complete Noetherian E_∞-ring A with a map $\rho_A : A \to R_0$, such that

$\epsilon_A : \pi_0(A) \to R_0$ is surjective, we have an equivalence of ∞-categories

$$\mathrm{Map}_{\mathrm{CAlg}_{/R_0}}(R^{\mathrm{un}}_{G_0}, A) \to \mathrm{Def}_{G_0}(A, \rho_A).$$

The proof of existence of universal deformations along a map follows from the follow definition of G_0-taggings.

Definition 2.7.9: Suppose that A is an adic E_∞-ring and $G \in \mathrm{BT}^P(A)$. A G_0-tagging of G consists of a triple (I, μ, α), where $I \subset \pi_0 A$ is an ideal of definition, $\mu : R_0 \to \pi_0(A)/I$ is a ring homomorphism, and $\alpha : (G_0)_{\pi_0 A/I} \simeq G_{\pi_0 A/I}$ is an isomorphism of p-divisible groups over $\pi_0 A/I$.

We then define a spectral deformation of G_0 over the $\mathbb{E}_\infty$-ring A consists of a spectral p-divisible group G over A together with an equivalence class of G_0-tagging of G. We let $\mathrm{Def}_{G_0}(A)$ denote the collection of all deformations of G_0 over A, i.e., it is the filtered colimit

$$\varinjlim_I \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)$$

where I ranges over all ideals of defintion $I \subset \pi_0(A)$ which are finitely generated. What is the relation between $\mathrm{Def}_{G_0}(A, \rho_A)$ and $\mathrm{Def}_{G_0}(A)$?. It can be proved that there is a fiber sequence

$$\mathrm{Def}_{G_0}(A, \rho) \to \mathrm{Def}_{G_0}(A) \xrightarrow{\rho} \mathrm{Def}_{G_0}(R_0).$$

Lemma 2.7.10: [13]lemma 3.1.10 Suppose that R_0 is a commutative ring and G_0 is a p-divisible group. If R is a complete adic $\mathbb{E}_\infty$-ring, the ∞-category $\mathrm{Def}_{G_0}(R)$ is an ∞-groupoids.

By this lemma, we have a functor

$$\mathrm{Def}_{G_0} : \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}.$$

Theorem 2.7.11: [13]Theorem 3.1.15 If R_0 is Noetherian F_p algebra such that the Frobenius morphism is finite, and G_0 is a p-divisible group over R_0. Then we have the following statements:

（1）There exists an universal deformation of G_0. i.e., there exists a complete adic $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{G_0}$, and a morphism $\rho : R^{\mathrm{un}}_{G_0} \to R_0$ such that the functor Def_{G_0} is corepresentable by $R^{\mathrm{un}}_{G_0}$. i.e. , for any complete adic $\mathbb{E}_\infty$-ring R，there is a equivalence

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{\mathrm{un}}_{G_0}, R) \to \mathrm{Def}_{G_0}(R).$$

（2）$\mathbb{E}_\infty$ ring $R^{\mathrm{un}}_{G_0}$ is a connective and Noetherian $\mathbb{E}_\infty$-ring.

（3）The induced map $\pi_0(\rho) : \pi_0(R^{\mathrm{un}}_{G_0}) \to R_0$ is surjective, and $R^{\mathrm{un}}_{G_0}$ is complete with respect to the ideal $\ker(\pi_0(\rho))$.

How do we get univesal deformations along a map from universal deformations consists of G_0-taggings. For $\rho_A : A \to R_0$ which induces a surjection of commutative rings $\epsilon : \pi_0 A \to R_0$. We have a commutative digram σ

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{\mathrm{un}}_{G_0}, A) & \xrightarrow{\rho_A \circ} & \mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{\mathrm{un}}_{G_0}, R_0) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{G_0}(A) & \longrightarrow & \mathrm{Def}_{G_0}(R_0) \end{array}$$

for any $u : R^{\mathrm{un}}_{G_0} \to A$, it fits into a commutative diagram

$$\begin{array}{ccc} R^{\mathrm{un}}_{G_0} & \xrightarrow{u} & A \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho_A} & \\ & R_0 & \end{array}$$

Passing to the homtopy fiber of the lower horiznetal map, we get a map

$$\theta : \mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{\mathrm{un}}_{G_0}, A) \to \mathrm{Def}_{G_0}(A, \rho)$$

If A is complete with respect to $\ker \epsilon$, vertical maps in σ are all equivalence, so we find that θ is a equivalecne.

2.8 Orientations

Suppose that we have an $R \in \mathrm{CAlg}$ and $X : \mathrm{CAlg}^{cn}_{\tau_{\geq 0}(R)} \to \mathcal{S}_*$ is a pointed formal hyperplane over this $\mathbb{E}_\infty$-ring R. We call a map of pointed spaces

$$e : S^2 \to X(\tau_{\geq 0}(R))$$

is a preorientation of X.

Definition 2.8.1: A preorientation of an 1-dimensional spectral formal group $\hat{G}$ over an E_∞-ring R is a map

$$e : S^2 \to \Omega^\infty \hat{G}(\tau_{\geq 0} R)$$

of based spaces, where the based points goes to the identity of the group structure. We let $\mathrm{Pre}(X)$ denote the space preorientation of X.

Fro every 1-dimensional spectral formal group $\hat{G}$, the dualizing line of $\hat{G}$ is an R-module defined by

$$\omega_{\hat{G}} := R \otimes_{\mathcal{O}_{\hat{G}}} \mathcal{O}_{\hat{G}}(-\eta)$$

where $\mathcal{O}_{\hat{G}}(-\eta)$ is the fiber of $\mathcal{O}_{\hat{G}} \to \tau_{\geq 0}R \to R$, $\eta \in \hat{G}(\tau_{\geq 0}R)$ is the connective element of the group. For every preorientation $e : S^2 \to \hat{G}(\tau_{\geq 0}R)$, there is an associated map

$$\beta_e : \omega_{\hat{G}} \to \Sigma^{-2}R$$

called the Bott map. See[13]Section 4.2 for more details about preotientations and orientations.

Definition 2.8.2: Fro a one dimensional spectral formal group G, an orientation is a preorientation whose Bott map is an equivalence.

The reason why we require that the Bott map is an equivalence is because, for a complex periodic $\mathbb{E}_\infty$-ring, we can define a spectral formal group G_Q^A, called the Quillen formal group over A. And the preorientation of of a spectral formal group $\hat{G}$ is classified by the mapping space of $\hat{G}_A^Q$ to $\hat{G}$. And the Bott map of a preorientation of Quillen formal groups is an equivalence. So if we want a preorientation e of $\hat{G}$ to be an orientation, then the image of this proentation under the map $\phi : \Omega^\infty \hat{G}(\tau_{\geq 0}R) \to \Omega^\infty \hat{G}_Q^A(\tau_{\geq 0}R)$ must be an orientation, i.e. the Bott map of $\phi(e)$ is an equivalence, then we get the Bott map of e is an equivalence.

Proposition 2.8.3: [13]Proposition 4.3.21 Let R be E_∞-ring which is complex periodic. Then for any spectral formal group $\hat{G}$ over R, there is canonical equivalence

$$\mathrm{Map}_{\mathrm{FGroup}}(G_A^Q, G) = \mathrm{Pre}(G)$$

Proposition 2.8.4: [13]Proposition 4.3.13 Suppose that we have $R \in \mathrm{CAlg}$ and X is a formal hyperplane over R which is dimension one. Then there exists an $\mathbb{E}_\infty$-ring $\mathfrak{D}_X$ and a orientation $e \in \mathrm{OrDat}(X_{\mathfrak{D}_X})$ satisfying for any $R' \in \mathrm{CAlg}_R$, evaluation on e induces an homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathfrak{D}_X, R') \to \mathrm{OrDat}(X_{R'}).$$

The representability of orientation comes from the following representability of preorientation, we notice that $\mathrm{Pre}(Y) = \Omega^2 Y(\tau_{\geq 0}R)$ for a pointed formal hyperplane Y .

Lemma 2.8.5: Suppose that we have $R \in \mathrm{CAlg}$ and X is a pointed formal hyperplane over R. Then the functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \quad R' \mapsto \mathrm{Pre}(X_{R'})$$

is corepresentable by an $\mathbb{E}_\infty$-ring A over R.

Applications

Definition 2.8.6: Suppose that we have an $\mathbb{E}_\infty$-ring R, and E is a strict elliptic curve over R. A presentation of E is a map $e : S^2 \to \Omega^{\infty+2}E(\tau_{\geq 0}R)$ of pointed spaces. An orientation is a preorientation such that its image under the equivalence $\mathrm{Pre}(E) = \mathrm{Pre}(\hat{E})$ is an orientation of the formal group $\hat{E}$.

We let $\mathrm{Ell}^{or}(R)$ denote the ∞-category of pairs (E, e), such that E is a strict elliptic curve over R, and e is an orientation of E.

Theorem 2.8.7: The functor

$$\mathcal{M}_{ell}^{or} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$$
$$R \mapsto \mathrm{Ell}^{or}(R)^{\simeq}$$

is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over $\mathbb{S}$.

Remark 2.8.8: It follows that[13]Remark 7.3.2 that the étale topos $\mathcal{U}$ of the classical Deligne-Mumford stack of classical elliptic curves is the full subcategory of the underlying topos $\mathcal{X}$ of $\mathcal{M}_{ell}^{s}$ spectral Deligne-Mumford stack of spectral elliptic curves. We have a map $\phi : \mathcal{M}_{ell}^{or} \to \mathcal{M}_{ell}^{s}$ of nonconnective spectral Deligne-Mumford stacks, we consider the direct image sheaf $\phi_* \mathcal{O}_{\mathcal{M}_{ell}^{or}}$, which is a sheaf of $\mathbb{E}_\infty$-rings over $\mathcal{X}$. So we get a functor $\mathcal{O}_{\mathcal{M}_{ell}}^{Top} : \mathcal{U}^{op} \to \mathrm{CAlg}$. This construction can be viewed as a construction of elliptic cohomology theories. It follows that[13]Remark 7.3.2 and[30], those ∞-structure determined by this construction are actually homotopy equivalent to the $\mathbb{E}_\infty$-structure in Goerss-Hopkins-Miller's proof[14].

Let G_0 be a nonstationary p-divisible group over a Noetherian $\mathbb{F}_p$-algebra. Let G be the universal deformation of G_0, and $R_{G_0}^{or}$ denote the orientation classifier for the identity component G°, we refer $R_{G_0}^{or}$ as the orientation deformation ring.

Theorem 2.8.9: Let R_0 be a Noetherian $\mathbb{F}_p$-algebra and G_0 be a one dimensional nonstationary p-divisible over R_0 with a classical universal deformation ring $R_{G_0}^{cl}$. Then we have:

（1）The odd degree homotopy groups of $R_{G_0}^{or}$ equals to zero, and $R_{G_0}^{cl} \cong \pi_0(R_{G_0}^{or})$.

（2）Suppose that we have an adic $\mathbb{E}_\infty$-ring A, the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}_{cpl}^{ad}}(R_{G_0}^{or}, A) = \mathrm{Def}_{G_0}^{or}$$

classifying triples (G, α, e), where

① G is a spectral deformation of G_0 to A.

② α is an equivalence class of G_0-taggings of A.

③ e is an orientation G° of the connective component of G.

Proof: See[13]Theorem 6.0.3 and[13]Remark 6.0.7. ■

By the deformation construction and orientation construction, we get the following celebrated theorem due to Lurie[13].

Theorem 2.8.10: Let M_{BT}^n denote the moduli stack of one dimensional height n p-divisible group, then there is a sheaf of E_∞-ring space, $\mathcal{O}^{\mathrm{Top}}$ on the étale site. such that for any

$$E := \mathcal{O}^{\mathrm{Top}}(\mathrm{Spec}R \overset{G_0}{\to} M_{\mathrm{BT}}^n)$$

we have

$$\mathrm{Spf}E^0(\mathbb{C}P^\infty) = G_0$$

where G_0 is the formal part of the p-divisible group G.

The construction this sheaf of $\mathbb{E}_\infty$-rings: $\mathcal{O}^{\mathrm{Top}}$ is as follows: when we have a one-dimensional height n p-divisible group G over a commutative ring R, which is classified by a map $G_0 : R \to M_{\mathrm{BT}}^n$. We consider its unorientated deformation ring R_G^{un}, and its universal deformation $G_{0,\mathrm{univ}}$.The orientation classifier $R_{G_0}^{or}$ of $G_{0,\mathrm{univ}}^\circ$ is an even periodic spectrum E. And it satisfies conditions in this theorem.

We recall that the Goerss-Hopkins-Miller theorem[14]. For any formal groups over a perfect field of characteristic $p > 0$. We can get a even periodic ring spectrum E, such that $\pi_0 E$ is the Lubin-Tate ring and the universal deformation was obtained from $G_E^{Q_0}$ by base change of scalars.

Now let us give a strategy of Lurie's proof of Goerss-Hopkins-Miller theorem. If $\hat{G}_0$ is a formal group over k, then it can be viewed as a identity component of a connected classical p-divisible group G_0 over k. Then there exists a universal deformation G over the spectral deformation ring $R_{G_0}^{un}$. Let G^0 be the identity component of G, and $R_{G_0}^{or}$ be the orientation classifier of the identity component G^0. Lurie proved that $E_{G_0} = L_{K_n} R_{G_0}^{or}$ is even periodic. We refer to this as the Lubin-Tate spectrum. We then prove that the spectrum E_{G_0} satisfying the same property with Morava E-theories. And then using the uniquess of Morava E-theories.

Theorem 2.8.11: [13]Theorem 5.1.5 For every complex periodic $K(n)$-local E_∞-ring A. We have a homototpy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}}(E_{G_0}, A) \to \mathrm{Hom}_{\mathcal{FG}}((R_0, G_0), (\pi_0(A)/J_n^A, G_A^{Q_n})).$$

And there are some new cohomology theories which are constructed by this theorem, like topological automorphic forms, we recommand readers find more details in[15].

CHAPTER 3 DERIVED LEVEL STRUCTURES

3.1 Isogenies of Spectral Elliptic Curves

This chapter is heart of this paper, Our main innovation is derived level structures defined in this chapter. The start is derived version of isogenies. We prove that the kernel of a derived isogeny in some cases have the same phenomenon as the classical case. This gives us an evidence that over derived version of level structures must induce classical level structures. In section 2, we define relative Cartier divisors in the setting of spectral algebraic geometry. We then use Lurie's representability theorem prove that functors associated with relative Cartier divisors are representable by certain spectral Deligne-Mumford stacks. In the third and fourth section, we study derived level structures of spectral elliptic curves and spectral p-divisible groups. The main content of last two sections are the proof of representability of derived level structures.

Definition 3.1.1: Assume that we have a connective $\mathbb{E}_\infty$ ring R. Let $f : X \to Y$ be a morphism of spectral abelian varieties over R, we say f is an isogeny if it is flat, finite and surjective.

Lemma 3.1.2: Let $f : X \to Y$ be a morphism of spectral abelian varieties, then $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ is an isogeny in the classical sense.

Proof: In classical abelian varieties, $f^\heartsuit$ is an isogeny means $f^\heartsuit$ is surjective and $\ker f^\heartsuit$ is finite. But it is equivalent to $f^\heartsuit$ is finite, flat and surjective[31]Proposition 7.1. And it is easy to see that $f^\heartsuit$ is finite, flat. We only need to prove that $f^\heartsuit$ is surjective.

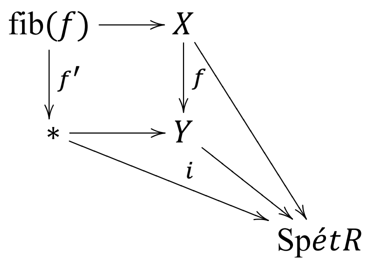
For every morphism $|\mathrm{Spec}k| \to |Y^\heartsuit|$, this correspond to a morphism $\mathrm{Sp\acute{e}t}k \to Y^\heartsuit$, by the inclusion-truncation adjunction[11]Proposition 1.4.6.3, this corresponds to a morphism $\mathrm{Sp\acute{e}t}k \to Y$. By the definition of surjective, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}k' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}k & \longrightarrow & Y \end{array}$$

The upper horizontal morphism corresponds to a morphism $\mathrm{Sp\acute{e}t}k' \to X^\heartsuit$ by inclusion-truncation adjunction. On the underlying topological space level, this corresponds to a point $|\mathrm{Sp\acute{e}t}k| \to |Y^\heartsuit|$. It is clear that this point in $|Y^\heartsuit|$ is a preimage of $|\mathrm{Sp\acute{e}t}k|$ in $X^\heartsuit$. So $f^\heartsuit$ is surjective. ■

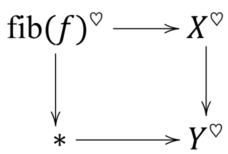
Lemma 3.1.3: Let $f : X \to Y$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring R, then $\mathrm{fib}(f)$ exists and is a finite and flat nonconnective spectral Deligne-Mumford stack over R.

Proof: By[11]Proposition 1.14.1.1, the finite limits of nonconnective spectral Deligne-Mumford stacks exists, so we can define fib(f). We consider the following diagram



where the square is a pullback diagram. We find that $\mathrm{fib}(f)$ is over $\mathrm{Sp\acute{e}t}R$. By[11]Remark 2.8.2.6, $f' : \mathrm{fib}(f) \to *$ is flat because it is a pull-back of a flat morphism. Obviously $i : * \to \mathrm{Sp\acute{e}t}R$ is flat, so by[11]Example 2.8.3.12 (flat morphism is local on the source for the flat topology), $i \circ f' : \mathrm{fib}(f) \to \mathrm{Sp\acute{e}t}R$ is flat.

Next, we show $\ker f$ is finite over R. Since $*$, X and Y are all spectral algebraic spaces, so we have $\mathrm{fib}f$ is also a spectral algebraic space. And $\mathrm{Sp\acute{e}t}R$ is an algebraic space[11]Example 1.6.8.2. By the above remark 2.2.14, we only need to prove that the underlying morphism is finite. The truncation functor is a right adjoint , so preserve limits. So we get a pull-back diagram



So we are reduced to prove that for an isogeny $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ of ordinary abelian varieties over a commutative ring R. $\ker f$ is finite over R. But this is true in classical algebraic geometry[31]Proposition 7.1.

■

Lemma 3.1.4: Let $f_N : E \to E$ be an isogeny of spectral elliptic curves over R, such that the underline map of ordinary elliptic curve is the multiplication N map, $N : E^\heartsuit \to E^\heartsuit$. Then $\mathrm{fib}f$ is finite locally free of rank N in the sense of[11]Definition 5.2.3.1. And moreover if N is invertible in $\pi_0 R$, then $\mathrm{fib}f$ is a locally constant étale sheaf.

Proof: By[32]Theorem 2.3.1, we know that $N : E^\heartsuit \to E^\heartsuit$ is locally free of rank N in the classical sense. When N is invertible in $\pi_0 R$, then $\ker N$ is locally constant étale sheaf. $\mathrm{fib}(f_N)$ is a spectral algebraic space which is finite and flat and its underlying map $\mathrm{fib}(f_N)^\heartsuit = \ker N$ is locally free of rank N. We need to prove that $\mathrm{fib} f_N \to \mathrm{Sp\acute{e}t} R$ is locally free of rank N in spectral algebraic geometry. But $\mathrm{fib} f_N$ is finite and flat, so is affine. We are reduce to prove this in local affine, i.e., we need ot prove that $f_N|_{\mathrm{Sp\acute{e}t} S} : \mathrm{Sp\acute{e}t} S \to \mathrm{Sp\acute{e}t} R$ is locally free, for $\mathrm{Sp\acute{e}t} S$ is an affine substack of $\mathrm{fib} f_N$. This is equivalent to prove that $R \to S$ is locally free of rank N in the sense of[11]Definition 2.9.2.1. So we need to prove

（1）S is locally free of finite rank over R.(By[3]Proposition 7.2.4.20, this is equivalent to say S is a flat and almost perfect R-module.)

（2）For every $\mathbb{E}_\infty$-ring maps $R \to k$, the vector space $\pi_0(M \otimes_R k)$ is a N-dimensional k-vector space.

For (1), we know that $\pi_0 S$ is projective $\pi_0 R$-module, and S is a flat R-module, so by[29]Proposition 7.2.2.18, S is a projective R-module. And since $\pi_0 S$ is a finitely generate R-module, so by[3]Corollary 7.2.2.9, S is a retract of a finitely generated free R-module M, so is locally free of finite rank.

For (2), $\pi_0(k \otimes_R M)$, since R and M are connective, by[3]Corollary 7.2.1.23, we get $\pi_0(k \otimes_R M) \simeq k \otimes_{\pi_0 R} \pi_0 M$ is a rank N k-vector space ($\pi_0 M$ is rank N free $\pi_0 R$ module).

We next show that if N is invertible in $\pi_0 R$, then $\mathrm{fib} f$ is a locally constant sheaf. By the above discussion, $\mathrm{fib} f$ is a spectral Deligne-Mumford stack, so the associated functor points $\mathrm{fib} f : \mathrm{CAlg}_R \to \mathcal{S}$ is nilcomplete and locally of almost finite presentation. By[32]Theorem 2.3.1, $\mathrm{fib} f|_{\mathrm{CAlg}^\heartsuit_{\pi_0 R}}$ is a locally constant sheaf, the desired results follows form the following lemma. ■

Lemma 3.1.5: Let $\mathcal{F} \in \mathrm{Shv}^{\acute{e}t}(\mathrm{CAlg}^{\mathrm{cn}}_R)$, and is nilcomplete, locally of almost finite presentation and $\mathcal{F}|_{(\mathrm{CAlg}_R^{cn})^\heartsuit}$ is the associated sheaf of constant presheaf valued on A. Then $\mathcal{F}$ is a homotopy locally constant sheaf (i.e., sheafification of a homotopy constant presheaf).

Proof: We choose a étale cover U_i^0 of $\pi_0 R$, such that $\mathcal{F}|_{U_i^0}$ is a constant sheaf for each i. By[3]Theorem 7.5.1.11, this corresponds to an étale cover $U_i \to R$ such that $\pi_0 U_i = U_i^0$. We consider the following diagram

$$\begin{array}{ccc} \tau_{\leq 0} R & \longrightarrow & \tau_{\leq 0} U \\ \downarrow & & \downarrow \\ \tau_{\leq n} R & \longrightarrow & \tau_{\leq n} U \end{array}$$

which is push-out diagram, since U_i is an étale R algebra. This is a colimit diagram in

$\tau_{\leq n}\mathrm{CAlg}_R$. $\mathcal{F}$ is a sheaf of locally of almost finite prsentation, so we get push-out diagram

$$\begin{array}{ccc} \mathcal{F}(\tau_{\leq 0}R) & \longrightarrow & \mathcal{F}(\tau_{\leq 0}U_i) \\ \downarrow & & \downarrow \\ \mathcal{F}(\tau_{\leq n}R) & \longrightarrow & \mathcal{F}(\tau_{\leq n}U_i) \end{array}$$

For each i, we have such diagram. Without loss of generality, we can assume each U_i is connective. So $\mathcal{F}(\tau_{\leq 0}U_i)$ are always same for all i. That means we have $\mathcal{F}(\tau_{\leq n}U_i)$ are all equivalence. But we have $\mathcal{F}$ is nicomplete, this means $\mathcal{F}(U_i) \simeq \mathrm{colim}\mathcal{F}(\tau_{\leq n}U_i)$. So we get all $\mathcal{F}(U_i)$ are homotopy equivalence. ■

3.2 Relative Cartier Divisors

In this section, we will define relative Cartier divisors in the context of spectral algebraic geometry. And we use Lurie's spectral Artin's representability theorem to prove that functors associated relative Cartier divisors are representable in certain cases.

For a locally spectrally topoi $X = (\mathcal{X}, \mathcal{O}_x)$, we can consider its functor of points

$$h_X : \infty\mathrm{Top}^{loc}_{\mathrm{CAlg}} \to \mathcal{S}, \quad Y \mapsto \mathrm{Map}_{\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(Y, X)$$

By[11]Remark 3.1.1.2, the closed immersion of locally spectrally ringed topos $f : X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to Y = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ corresponds to morphism of sheaves of connective $\mathbb{E}_\infty$-rings $\mathcal{O}_\mathcal{X} \to f_*\mathcal{O}_\mathcal{Y}$ over $\mathcal{X}$ such that $\pi_0\mathcal{O}_\mathcal{X} \to \pi_0 f_*\mathcal{O}_\mathcal{Y}$ is surjective. We consider the the fiber of this map fib$f$. For a closed immersion $f : D \to X$ of spectral Deligne-Mumford stack, we let $I(D)$ denote fib(f), called the ideal sheaf of D.

To prove the relative representability, we need the representability of the Picard functor. If we have a map $f : X \to \mathrm{Sp\acute{e}t}R$ of spectral Deligne-Mumford stack, we can define a functor

$$\mathcal{P}ic_{X/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{P}ic(\mathrm{Sp\acute{e}t}R' \times_{\mathrm{Sp\acute{e}t}R} X)$$

If f admits a section $x : \mathrm{Sp\acute{e}t}R \to X$ then there exists a natural transformation of functors $\mathcal{P}ic(X/R) \to \mathcal{P}ic_{R/R}$. We let

$$\mathcal{P}ic^x_{X/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}$$

denote the fiber of this map.

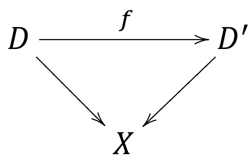
Theorem 3.2.1: [11]Theorem 19.2.0.5 Let X be a map spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected over an $\mathbb{E}_\infty$-ring R. And suppose that $x : \mathrm{Sp\acute{e}t}R \to X$ is a section, the functor

$\mathcal{P}ic^{x}_{X/R}$ is representable by a spectral algebraic space which is locally of finite presentation over R.

In the classical case, relative Cartier divisors schemes are open subschemes of Hilbert schemes[33]. But in the derived case, the Hilbert functor is representable by a spectral algebraic space[23]Theorem 8.3.3, it is hard to say relation to say the relation between them. We will directly study relative Cartier divisors in derived world.

Definition 3.2.2: Suppose that X is a spectral Deligne-Mumford stack over a spectral Deligne-Mumford stack S. We let $\mathrm{CDiv}(X/S)$ denote the ∞-category of closed immersions $D \to X$, such that D is flat, proper, locally almost of finite presentation over S and the associated ideal sheaf of D is locally free of rank one over X.

Remark 3.2.3: It is easy to say that for any spectral Deligne-Mumford stack X over S, $\mathrm{CDiv}(X/S)$ is a kan complex, since all objects are closed immersions of X, let $D \to D'$ be morphism, then we have a diagram



by the definition of closed immersions, they all equivalent to the same substack of X, so f is a equivalence.

Lemma 3.2.4: Let X/S be a spectral Deligne-Mumford stack, and $T \to S$ be a map of spectral Deligne-Mumford stacks. If we have a relative Cartier divisor $i : D \to X$, then D_T is a relative Cartier divisor of X_T.

Proof: This is easy to see, we just notice that D_T is still closed immersion of X_T[11]Corollary 3.1.2.3. And after base change, D_T is flat, proper, locally almost of finite presentation over T. The only thing we need to worry is that whether $I(D_T)$ is a line bundle over X_T? But this is also true. Since we have a fiber sequence

$$I(D) \to \mathcal{O}_X \to \mathcal{O}_D$$

after applying the morphism $f^* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X_T}}$, due to the flatness of D. We get fiber sequence

$$f^*(I(D)) \to \mathcal{O}_{X_T} \to \mathcal{O}_{D_T}$$

So we get $I(D_T)$ is just $f^*I(D)$, so is invertible. ■

By the construction of relative Cartier divisors, suppose that X is a spectral Deligne-Mumford stack over an affine spectral Deligne-Mumford stack $S = \mathrm{Sp\acute{e}t}R$. We then have a functor

$$\begin{aligned} \mathrm{CDiv}_{X/R} \ : \ & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

Our main target in this section is to prove this functor is representable when E/R is a spectral algebraic space satisfying certain conditions. Before we start the prove of represenability of relative Cartier divisor, we need some preparations for computing the cotangent complex of a relative Cartier divisor functor. The main issiue is square extension. We need following truth about pushout of two closed immersions.

By[11]Theorem 16.2.0.1, Proposition 16.2.3.1, suppose we have a pushout square of spectral Deligne-Mumford stacks:

$$\begin{array}{ccc} X_{01} & \xrightarrow{i} & X_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ X_1 & \xrightarrow{i'} & X, \end{array}$$

such that i and j are closed immersions. Then the induced square of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X_{01}) & \longleftarrow & \mathrm{QCoh}(X_0) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(X_1) & \longleftarrow & \mathrm{QCoh}(X) \end{array}$$

determines emdbedding $\theta : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_0) \times_{\mathrm{QCoh}(X_{01})} \mathrm{QCoh}(X_1)$ and restricts to an equivalence

$$\mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{QCoh}(X_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(X_{01})^{\mathrm{cn}}} \mathrm{QCoh}(X_1)^{\mathrm{cn}}$$

Let $\mathcal{F} \in \mathrm{QCoh}(X)$, and set

$$\mathcal{F}_0 = j'^{*} \in \mathrm{QCoh}(X_0) \quad \mathcal{F}_1 = i'^{*}\mathcal{F} \in \mathrm{QCoh}(X_1).$$

Then the quasi-coherent sheaf $\mathcal{F}$ is n-connective is equivalent $\mathcal{F}_0$ and $\mathcal{F}_1$ are n-connective, and this statement is also true for the condition, almost connective, Tor-amplitude $\leq n$ flat, perfect to order n, almost perfec, perfect, locally free of finite rank.

And by[11]Theorem 16.3.0.1, we the have a pullback square

$$\begin{array}{ccc} \mathrm{SpDM}_{/X} & \longrightarrow & \mathrm{SpDM}_{/X_0} \\ \downarrow & & \downarrow \\ \mathrm{SpDM}_{/X_1} & \longrightarrow & \mathrm{SpDM}_{/X_{01}} \end{array}$$

of ∞-categories Let $f : Y \to X$ be a map of spectral Deligne-Mumford stacks. Let

$Y_0 = X_0 \times_X Y$, $Y_1 = X_1 \times_X Y$ and let $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ be the projections maps. Then we have[11]Proposition 16.3.2.1 f is locally almost of finite presentation if and only if both f_0 and f_1 are locally almost of finite presentation. And the statement is also trur for conditions: locally of finite generation to order n, locally of finite presentation, étale, equivalence, open immersion, closed immersion, flat, affine, separated and proper.

Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a spectral Deligne-Mumford stack, and $\mathcal{E} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ is a quasi-coherent sheaf, and $\eta \in \mathrm{Der}(\mathcal{O}_X, \Sigma\mathcal{E})$, that is map $\eta : \mathcal{O}_X \to \mathcal{O}_X \oplus \Sigma\mathcal{E}$. We let $\mathcal{O}_X^\eta$ denote the square-zero extension of $\mathcal{O}_X$ by $\mathcal{E}$ determined by η, then we have a pull-back diagram

$$\begin{array}{ccc} \mathcal{O}_X^\eta & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \eta \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \oplus \Sigma\mathcal{E} \end{array}$$

By[11]Proposition 17.1.3.4, $(\mathcal{X}, \mathcal{O}_X^\eta)$ is a spectral Deligne-Mumford stack, which we will denote it by $\mathcal{X}^\eta$. In the case of $\eta = 0$, we denote it by $X^{\mathcal{E}} = (\mathcal{X}, \mathcal{O}_X \oplus \mathcal{E})$. We then have a pullback square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} X^{\Sigma\mathcal{E}} & \xrightarrow{g} & X \\ \downarrow f & & \downarrow \\ X & \longrightarrow & X^\eta \end{array}$$

such that f and g are closed immersions.

We have a pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X^\eta)^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(X)^{\mathrm{acn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X)^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(X^{\Sigma\mathcal{E}})^{\mathrm{acn}}. \end{array}$$

by[11]Theorem 16.2.0.1, Proposition 16.2.3.1. Taking $\eta = 0$ and passing ti homotopy fiber over some $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{acn}}$, we can get

$$\mathrm{QCoh}(X^{\mathcal{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(X)} \{\mathcal{F}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{F}, \Sigma(\mathcal{E} \otimes \mathcal{F}))$$

by[11]Proposition 19.2.2.2.

Taking $\eta = 0$ and passing to the homotopy fibers over some $Z \in \mathrm{SpDM}_{/X}$, we can get classification of the first order deformations

$$\mathrm{SpDM}_{/X^{\mathcal{E}}} \times_{\mathrm{SpDM}_{/X}} \{Z\} \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(L_{Z/X}, \Sigma f^* \mathcal{E}),$$

see details in[11]Porposition 19.4.3.1.

Lemma 3.2.5: Let $f : X \to \mathrm{Sp}\acute{e}tR$ be a morphism of spectral Deligne-Mumford stacks. For a connective R-module M, then the ∞-categories of Deigne-Mumford stacks X' with a morphism $X \to \mathrm{Sp}\acute{e}t(R \oplus M)$ such that fitting into the following pull back diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Sp}\acute{e}tR & \longrightarrow & \mathrm{Sp}\acute{e}tR \oplus M \end{array}$$

is a Kan complex, which is canonically equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}}(L_{X/Y}, \Sigma f^{*}M)$, and moreover if f is flat, proper and locally of almost finite presnetation, then any such $f' : X' \to S[M]$ is flat, proper and locally almost of finite presentation.

Proof: We have a pullback square in $\mathbb{E}_{\infty}$-rings

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R \\ \downarrow & & \downarrow {\scriptstyle (Id,0)} \\ R & \longrightarrow & R \oplus \Sigma M, \end{array}$$

this corresponds a pushout square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathrm{Sp}\acute{e}tR \oplus \Sigma M & \longrightarrow & \mathrm{Sp}\acute{e}tR \\ \downarrow & & \downarrow \\ \mathrm{Sp}\acute{e}tR & \longrightarrow & \mathrm{Sp}\acute{e}tR \oplus M \end{array}$$

such that $\mathrm{Sp}\acute{e}tR \oplus \Sigma M \to \mathrm{Sp}\acute{e}tR$ are closed immersion. That makes $\mathrm{Sp}\acute{e}tR \oplus M$ be an infinitesimal thickening of $\mathrm{Sp}\acute{e}tR$ determined by $R \overset{(id,0)}{\longrightarrow} R \oplus \Sigma M$.

The first part of this lemma is just the formula of first order deformations[11]Proposition 19.4.3.1, and the second part is properties of pushout of two closed immersions[11]Corollary 19.4.3.3. ∎

Lemma 3.2.6: Suppose that we are given a pushout diagram of spectral Deligne-Mumford stacks σ:

$$\begin{array}{ccc} X_{01} & \xrightarrow{i} & X_0 \\ \downarrow {\scriptstyle j} & & \downarrow \\ X_1 & \longrightarrow & X, \end{array}$$

where i and j are closed immersions. Let $f : Y \to X$ be a map of spectral Deligne-Mumford stacks. Let $Y_0 = X_0 \times_X Y$, $Y_1 = X_1 \times_X Y$ and let $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ be the projections maps.

If both f_0 and f_1 are closed immersions and determine line bundles over Y_0 and Y_1, then f is a closed immersion and determines a line bundle.

Proof: The closed immersion part is just Lurie's theorem. And for the line bundle part, we notice that by[11]Theorem 16.2.0.1, Proposition 16.2.3.1, f determine a sheaf of locally free of finite rank. To prove it is a line bundle, we can do it locally. By[11]Theorem 16.2.0.2, for a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

of E_∞-rings such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, then there is an equivalence $F : \mathrm{Mod}_A^{cn} \to \mathrm{Mod}_{A_0}^{cn} \times_{\mathrm{Mod}_{A_{01}}^{cn}} \mathrm{Mod}_{A_1}^{cn}$. Actually this a symmetric monoidal equivalence. Sice we have $F(M) = (A_0 \otimes_A M, A_{01} \otimes_A M, A_1 \otimes_A M)$. They satisfying $F(M \otimes N) \simeq F(M) \otimes F(N)$. But by[11]Propsition 2.9.4.2, line bundles of $A_1, A_{0,1}$ and A_0 determines invertible objects of $\mathrm{Mod}_{A_1}^{cn}$, $\mathrm{Mod}_{A_{01}}^{cn}$ and $\mathrm{Mod}_{A_1}^{cn}$, so determine a invertible object of Mod_A^{cn}, hence a line bundle over A by[11]Propsition 2.9.4.2. ■

Theorem 3.2.7: Let E/R be a spectral algebraic space which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \ : \ & \mathrm{CAlg}_R \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

Proof: We use Lurie's spectral Artin's represnetability theorem to prove this theorem.

（1）For every discrete commutative R_0, the space $\mathrm{CDiv}_{E/R}(R_0)$ is 0-truncated.

We just notice that $\mathrm{CDiv}_{E/R}(R_0)$, consists of closed immersions $D \to E \times_R R_0$, such that D is flat proper over R_0, so all D are discrete object, so $\mathrm{CDiv}_{E/R}(R_0)$ is 1-truncated.

（2）The functor $\mathrm{CDiv}_{E/R}$ is a sheaf for the étale topology.

Let $\{R' \to U_i\}_{i \in I}$ be an étale cover of R', and $U_\bullet$ be the associate check simplicial object. We need to prove that the map

$$\mathrm{CDiv}_{E/R}(R') \to \lim_{\Delta} \mathrm{CDiv}_{E/R}(U_\bullet)$$

is an equivalence. Unwinding the definitions, we only need to prove following general result: for a spectral Deligne-Mumford stack $X \to S$ and we have a étale cover $T_i \to S$, then

$$\mathrm{CDiv}(X/S) \to \lim_{\Delta} \mathrm{CDiv}(X \times_S T_\bullet)$$

is a homotopy equivalence. But this obvious, since our conditions of relative Cartier divisor is local for the étale topology.

（3）The functor $\mathrm{CDiv}_{E/R}$ is nilcomplete.

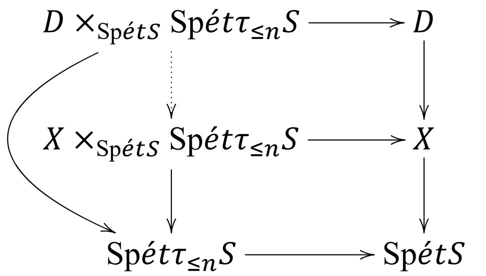
This is equivalent to say that the canonical map

$$\mathrm{CDiv}_{E/R}(R') \to \varprojlim \mathrm{CDiv}_{E/R}(\tau_{\leq n} R')$$

This can be deduced form the following results: for a flat, proper, locally almost of finite presentation spectral spectral algebraic space X over a connective E_∞-ring S, we have a equivalence

$$\mathrm{CDiv}(X/\mathrm{Sp\acute{e}t}S) \to \varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n} S).$$

Let us prove this equivalence now. For a relative Cartier divisor $D \to X$, we have the following commutative diagram



We then get a induce map $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. It is easy to prove that this map is a closed immersion[11]Corollary 3.1.2.3, and $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}S$ is flat, proper and locally almost of finite presentation, since $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is the base change of D along $\mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}S$, and the associated ideal sheaf of $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is still a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. So $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is a relative Cartier divisor of $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. Thus we have define a functor

$$\theta : \mathrm{CDiv}(X/S) \to \varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S), \quad D \mapsto \{D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S\}$$

This functor is fully faithful, since we have equivalence $\mathrm{SpDM}_{/S} \to \varprojlim \mathrm{SpDM}_{/\tau_{\leq n}S}$ defined by $X \mapsto X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$[11]Proposition 19.4.1.2. To prove the functor θ is an equivalence, we need to show it is essentially surjective. Suppose $\{D_n\} \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is an object in $\varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S)$. It is a morphism in $\varprojlim \mathrm{SpDM}_{/\tau_{\leq n}S}$, by[11]Proposition 19.4.1.2, there is a morphism $D \to X$ in $\mathrm{SpDM}_{/S}$, satisfying $D \times_{\mathrm{Sp\acute{e}t}S}$

$\mathrm{Sp\acute{e}t}\tau_{\leq n}S \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ are just $D_n \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$.

Next, we need to show that such $D \to X$ is relative Cartier divisor. The condition that $D \to S$ is flat, proper and locally almost of finite presentation follows immediately from[11]Proposition 19.4.2.1. We need to prove that $D \to X$ is a closed immersion and determine a line bundle over X. Without loss of generality, we may assume that $X = \mathrm{Sp\acute{e}t}B$ is affine, so we have closed immersion $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}B \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \simeq \mathrm{Sp\acute{e}t}(B \otimes_S \tau_{\leq n}S)$, the second equivalence comes from[11]Proposition 1.4.11.1(3). And by[11]Theorem 3.1.2.1, $D \times_{SpetS} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ equals $\mathrm{Sp\acute{e}t}B'_n$ for each n, such that $\pi_0(B \times_S \tau_{\leq n}S) \to \pi_0 B'_n$ is surjective. Since we have $\tau_{\leq n}S \to B'_n$ is flat, we get $\mathrm{Sp\acute{e}t}B'_n = \mathrm{Sp\acute{e}t}B'_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S = \mathrm{Sp\acute{e}t}(B'_{n+1} \times_{\tau_{\leq n+1}S} \tau_{\leq n}S) \simeq \mathrm{Sp\acute{e}t}\tau_{\leq n}B'_{n+1}$. So we get a spectrum B' such that $\tau_{\leq n}B' \simeq \mathrm{Sp\acute{e}t}B'_n = D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. Consequently $D = \mathrm{Sp\acute{e}t}B'$, and $\pi_0 B \to \pi_0 B'$ is surjective, so $D = \mathrm{Sp\acute{e}t}B' \to \mathrm{Sp\acute{e}t}B = X$ is a closed immersion. To prove that the associated ideal sheaf of D is a line bundle, we notice that there is a pullback diagram.

$$\begin{array}{ccc} I_n & \longrightarrow & B \times_S \tau_{\leq n}S \\ \downarrow & & \downarrow \\ * & \longrightarrow & B' \times_S \tau_{\leq n}S, \end{array}$$

each I_n is an invertible $B \times_S \tau_{\leq n}S = \tau_{\leq n}B$ module. Passing to the inverse limit, we get

$$\begin{array}{ccc} \lim\limits_{\longleftarrow} I_n & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & B'. \end{array}$$

Consequently, we have $I(D) \simeq \lim\limits_{\longleftarrow} I_n$. So by the nilcompleteness of Picard functor[11]Corollary 19.2.4.6, Propostion 19.2.4.7, We get I is a invertible B-module. So the associated ideal sheaf of D is a line bundle of X.

（4）The functor $\mathrm{CDiv}_{E/R}$ is cohesive.

This statement follows from Proposition 3.2.6 and[11]Proposition 16.3.2.1.

（5）The functor $\mathrm{CDiv}_{E/R}$ is integrable. We need to prove that for R' a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$. Then the inclusion functor induces a homotopy equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Sp\acute{e}t}R', \mathrm{CDiv}_{E/R}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}R', \mathrm{CDiv}_{E/R}).$$

But this follows from the following result: for a flat proper, locally almost of finite presentation and separated spectral spectral algebraic space X over a connective E_∞-ring S, we have equivalence

$$\mathrm{CDiv}(X/S) \simeq \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)$$

Let $\mathrm{Hilb}(X/S)$ denote the full subcategory of $\mathrm{SpDM}_{/X}$ consists of those $D \to X$, such that $D \to X$ is a closed immersion and $D \to S$ is flat, proper and locally almost of finite presentation. Then by the formal GAGA theorem[11]Theorem 8.5.3.4 and base change properties of flat, proper and locally almost of finite presentation, we have $\mathrm{Hilb}(X/S) \simeq \mathrm{Hilb}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)$. To prove the equivalence of relative Cartier divisors, we need to check that $D \to X$ associated a line bundle over X if and only if $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ associated a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$. We notice that since $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ is flat over X, we have $I(D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S) = I(f^*D) \simeq f^*I(D)$

$$\begin{array}{ccc} D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S & \longrightarrow & D \\ \downarrow & & \downarrow \\ X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S & \xrightarrow{f} & X. \end{array}$$

By[11]Proposition 19.2.4.7, we have an equivalence

$$\mathrm{QCoh}(X/S)^{\mathrm{aperf,cn}} \simeq \mathrm{QCoh}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)^{\mathrm{aperf,cn}}$$

By restricting to subcategories spanned by invertible object and using[11]Proposition 2.9.4.2, we get D associated a line bundle over X if and only if $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ associated a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$.

（6）$\mathrm{CDiv}_{E/R}$ is locally almost of finite presentation.

We need to prove that $\mathrm{CDiv}_{E/R} : \mathrm{CAlg}_R \to \mathcal{S}, R' \mapsto \mathrm{CDiv}(E_{R'}/R')$ commutate with filtered colimits when restrict to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. But we notice that $\mathrm{CDiv}(E_{R'}/R')$ are full categories of $\mathrm{SpDM}_{/E_{R'} \to R'}$, we consider the functor

$$R' \mapsto \mathrm{Var}^+_{/E_{R'} \to R}$$

where $\mathrm{Var}^+_{/E_{R'} \to R}$ consists of the diagram

$$\begin{array}{ccc} D & \longrightarrow & E_{R'} \\ & \searrow & \downarrow \\ & & \mathrm{Sp\acute{e}t}R' \end{array}$$

such that $D \to R'$ is flat, proper, and locally almost of finite presentation. Then by[11]Proposition 19.4.2.1. This functor commutates with filtered colimits when restrict to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. Then we just need to prove that when $\{D_i \to E^i_{R'}\}_{i \in I}$ are closed immersions and determine line bundles in $\{E^i_{R'}\}$, then $\mathrm{colim}D_i$ are closed immersion of $\mathrm{colim}E^i_{R'}$ and determine line bundle in colim$E^i_{R'}$. But this fact follows from the locally almost of finite presentationnes of Picard functor and properties of closed immersions.

Consider the functor $\mathrm{CDiv}_{E/R} \to *$, it is infitesimally cohesive and admits a cotangent complex which is almost perfect, so by[11]17.4.2.2, it is locally almost of finite presentation. So $\mathrm{CDiv}_{E/R}$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

（7）The functor $\mathrm{CDiv}_{E/R}$ admits a complex L which is connective and almost perfect.

For a connective E_∞-ring S, and every $\eta \in \mathrm{CDiv}_{E/R}(S)$, and a connective S-module M. We have a pullback diagram

$$\begin{array}{ccc} F_\eta(M) & \longrightarrow & \mathrm{CDiv}_{E/R}(S \oplus M) \\ \downarrow & & \downarrow \\ \eta & \longrightarrow & \mathrm{CDiv}_{E/R}(S) \end{array}$$

Then we have a functor

$$F_\eta : \mathrm{Mod}_S \to \mathcal{S}, \quad M \mapsto F_\eta(M)$$

We need to prove that this functor is corepresentable. η corresponds a morphism $D \to E \times_R S$, and $E \times_R (S \oplus M)$ is a square zero extension of $E \times_R S$. So by the classification of first order deformation theory[11]Propostion 19.4.3.1, the space of D', which satisfying the pullback diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \downarrow f & & \downarrow \\ E \times_R S & \longrightarrow & E \times_R (S \oplus M) \\ \downarrow p & & \downarrow \\ \mathrm{Sp\acute{e}t} S & \longrightarrow & \mathrm{Sp\acute{e}t}(S \oplus M) \end{array}$$

is equivalent to

$$\mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E \times_R S}, \Sigma f^* \mathcal{E}) = \mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E \times_R S}, \Sigma f^* \circ p^* M)$$

Push forward along $p \circ f$, and by[11]Proposition 6.4.5.3 we have

$$\mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E \times_R S}, \Sigma f^* \circ p^* M) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t} S)}(\Sigma^{-1} p_+ \circ f_+ L_{D/E \times_{\mathrm{Sp\acute{e}t} R} \mathrm{Sp\acute{e}t} S}, M).$$

And by[11]Proposition 16.3.2.1 and Lemma 3.2.6, any such D' is a closed immersion of

$\mathrm{CDiv}_{E/R}(S \oplus M)$ and determine a line bundle of $\mathrm{CDiv}_{E/R}(S \oplus M)$. Since the diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}S & \longrightarrow & \mathrm{Sp\acute{e}t}S \oplus M \end{array}$$

is a pullback diagram, so D' is a square zero extension of D. By[11]Proposition 16.3.2.1, we get $D' \to \mathrm{Sp\acute{e}t}(S \oplus M)$ is flat, proper and locally almost of finite presentation. Combining these facts, we find that

$$F_\eta(M) = Map_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}S)}(\Sigma^{-1}p_+ \circ f_+ L_{D/E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S}, M).$$

Consequently, the functor $\mathrm{CDiv}_{E/R}$ satisfies condition (a) of[11]Example 17.2.4.4 and condition (b) follows form the compatibility of f_+ with base change. It then follows that $\mathrm{CDiv}_{E/R}$ admits a cotangent complex $L_{\mathrm{CDiv}_{E/R}}$ satisfying $\eta^* L_{\mathrm{CDiv}_{E/R}} = \Sigma^{-1}p_+ \circ f_+ L_{D/E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S}$. Since the quasi-coherent sheaf $L_{D/E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S}$ is connective and almost perfect. The R-module $\Sigma^{-1}p_+ \circ f_+ L_{D/E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S}$ is (-1) connective.

$L_{\mathrm{CDiv}_{E/R}}$ is almost perfect, since we have $\mathrm{CDiv}_{E/R}$ it is infitesimally cohesive and admits a cotangent complex. And it is locally almost of finite presentation, so by[11]17.4.2.2, its cotangent complex is almost perfect.

We next show that it is connective. Let R' be an $\mathbb{E}_\infty$-ring, and $\eta \in \mathrm{CDiv}(E_{R'}/R)$, we wish to prove that $M = \eta^* L_{\mathrm{CDiv}_{E/R}} \in \mathrm{Mod}_R'$ is connective. We already know that M is is (-1)-connective and almost perfect, the homotopy group $\pi_{-1}M$ is a finitely generated $\pi_0 R'$ module. To prove that π_{-1} vanishes. By the Nakayama's lemma, this is equivalent to prove that

$$\pi_{-1}M(k \otimes_{R'} M) \simeq \mathrm{Tor}_0^{\pi_0 R'}(k, \pi_{-1}M)$$

equals to 0 for every residue filed of R. Then we may replace R' by k and assume k is a algebraically closed filed.

Let $A = k[t]/(t^2)$, unwinding the definitions, we find that the dual space $\mathrm{Hom}_k(\pi_{-1}M, k)$ can be identify with the set of automorphism of η_A such that it restrict identity of η. we wish to prove this set is trivial. But this follow from the fact : Let X/k be scheme, L is an line bundle on X, if L_A is also a line bundle of X_A. If we have f is an automorphism of L_A such that $f|L$ is identity on L, then f is the identity. (This fact follows from the connectiveness of cotangent complexes of Picard functors.)

■

3.3 Derived Level Structures of Spectral Elliptic Curves

Let C be a one dimensional smooth commutative group scheme over a base scheme S, and A be an abstract finite abelian group. A homomorphism of abstract groups

$$\phi : A \to C(S)$$

is said to be an A-Level structure on C/S if the effective Cartier divisor D in C/S defined by

$$D = \Sigma_{a\in A}[\phi(a)]$$

is a subgroup of C/S.

The following result due to Katz-Mazur[32] give the representability of level structures moduli problems.

Proposition 3.3.1: [32]Proposition 1.6.2 Let C/S be an one dimensional smooth commutative group scheme over S. Then the functor

$$\mathrm{Level}_{C/S} : \mathrm{Sch}_S \to \mathrm{Set}$$

$$T \mapsto \text{the set of A-level structures on } C_T/T$$

is representable by a closed subscheme of $\mathrm{Hom}(A, C) \cong C[N_1] \times_S \cdots \times_S C[N_r]$.

Definition 3.3.2: Let E/R be a spectral elliptic curve. In the level of objects, a derived A-level structure is a relative Cartier divisor $\phi : D \to E$ of E, such that the underlying morphism $D^\heartsuit \to E^\heartsuit$ is the inclusion of the associated relative Cartier divisor $\Sigma_{a\in A}[\phi_0(a)]$ into $E^\heartsuit$, where $\phi_0 : A \to E^\heartsuit(R^\heartsuit)$ is any classical level structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the ∞-category of derived A-level structures of E/R, whose objects can be viewed as pairs $\phi = (D, \phi)$.

It is easy to see that for a spectral elliptic curve E/R, the ∞-category $\mathrm{Level}(\mathcal{A}, E/R)$ is a ∞-groupoid, since it is a full subcategory of $\mathrm{CDiv}(E/R)$, which is a ∞-groupoid.

Lemma 3.3.3: Let E/R be a spectral elliptic curve and $\phi_S : D \to E$ be a derived level structure. Suppose that $T \to S$ be a morphism of nonconnective spectral Deligne-Mumford stacks, then the induce morphism $\phi_S : D_T \to E_T$ is a derived level structure of E_T/T.

Proof: We notice that derived level structure is stable under base change. So $\phi_S^\heartsuit : A \to (E \times_S T)^\heartsuit(T_0) = E^\heartsuit(T_0)$ is classical level structure, so $D_T^\heartsuit$ is the associated classical relative Cartier divisor of a classical level structure. And $D_T \to E_T$ is a relative Cartier divisor in spectral algebraic geometry, this is just the base change of relative Cartier divisor (Lemma

3.2.4). ■

We first recall a proposition in Katz and Mazur's book[32]Corollarly 1.3.7: Suppose that C/S is a smooth group curve, and D is a relative Cartier divisor of C, then exists a closed subscheme Z of S, satisfying for any $T \to S$, D_T is a subgroup of C_T if and only if T passing through Z.

Lemma 3.3.4: Let E/R be a spectral elliptic curve, and $D \to E$ be a relative Cartier divisor. There exists a closed spectral Deligne-Mumford substack $\mathrm{Sp\acute{e}t}Z \subset \mathrm{Sp\acute{e}t}R$, satisfying the following universal property:

For any $S \in \mathrm{CAlg}_R^{cn}$, such that the associated sheaf of D_S is a relative Cartier divisor of X_S and $(D_S)^\heartsuit$ is a subgroup of $(E_S)^\heartsuit$ if and only if $R \to S$ factor through Z.

Proof: For a map $R \to S$, it is obvious that D_S is a relative Cartier divisor of X_S. By[32]Corollarly 1.3.7, we just notice that if $(D_S)^\heartsuit/\pi_0 S$ is a subgroup of $(E_S)^\heartsuit/\pi_0 S$, we have $\mathrm{Spec}\pi_0 S$ must passing through a closed subscheme $\mathrm{Spec}Z_0$ of $\mathrm{Spec}\pi_0 R$. This corresponds a closed spectral subscheme $\mathrm{Spec}Z$ of $\mathrm{Spec}R$, sice we have the map $R \to S$ such that $\pi_0 R \to \pi_0 S$ pass through $\pi_0 R/I$ for some ideal I of $\pi_0 R$, so we have $R \to S$ passing through $R^{Nil(I)}$, see[11]Chapter 7 for details about nilpotent R-module. Conversely, suppose that $R \to S$ passing through Z, then we have $S = \mathcal{O}_{\mathrm{Sp}}\acute{e}tS$ is vanishing on I. That is we have $\pi_0 R \to \pi_0 S$ passing through $\pi_0 R/\sqrt{I}$, but this is equivalent to say $\mathrm{Spec}\pi_0 S \to \mathrm{Spec}\pi_0 R$ passing through $\mathrm{Spec}\pi_0 R/I = \mathrm{Spec}Z_0$, and so $(D_S)^\heartsuit$ is a subgroup of $(E_S)^\heartsuit$. ■

Theorem 3.3.5: Let E/R be a spectral elliptic curve, then the functor

$$\begin{aligned}\mathrm{Level}_{E/R} \ :\ & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R')\end{aligned}$$

is representable by a closed substack $S(A)$ of $\mathrm{CDiv}_{X/R}$. Moreover, $S(A) = \mathrm{Sp\acute{e}t}\mathcal{P}_{E/R}$ for an $\mathbb{E}_\infty$-ring $\mathrm{Sp\acute{e}t}\mathcal{P}_{E/R}$, which is locally almost of finite presentation over R, .

Proof: By definition, the functor $\mathrm{Level}_{E/R}$ is a subfunctor of the representable functor $\mathrm{CDiv}_{X/R}$. We consider a spectral Deligne-Mumford stack GroupCDiv defined by the pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc}\mathrm{GroupCDiv}_{E/R} & \longrightarrow & \mathrm{CDiv}_{E/R} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}Z & \longrightarrow & \mathrm{Sp\acute{e}t}R.\end{array}$$

It is easy to say that $\mathrm{GroupCDiv}_{E/R}$ valued on a R-algebra R' is the space of relative Cartier divisors D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^\heartsuit$ is a subgroup of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^\heartsuit$.

It is cleared that

$$\mathrm{GroupCDiv}_{E/R} = \coprod_{A_0 \in \mathrm{FinAb}} A_0 - \mathrm{CDiv}_{E/R}$$

where $A_0 - \mathrm{CDiv}_{E/R}$ valued on a R-algebra R' is the space of relative Cartier divisors D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^{\heartsuit}$ is an algebric subgroup of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^{\heartsuit}$ and $D^{\heartsuit}(R') = A_0$. It is cleared that $\mathrm{Level}_{E/R} = A - \mathrm{CDiv}_{E/R}$, so we have $\mathrm{Level}_{E/R}$ is representable by a open substack of $\mathrm{GroupCDiv}_{E/R}$.

To prove the second part, we consider the map $S(A) \to \mathrm{Sp\acute{e}t}R$, they are all spectral algebraic spaces. By[11]Remark 5.2.0.2, a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. So we only need to prove $S(A)^{\heartsuit}$ is finite over $\mathrm{Spec}\pi_0 R$, but this is just the classical case since $S(A)^{\heartsuit}$ is the representable object of the classical level structure, which is finite over R_0 by[32]Corollary 1.6.3. ■

3.4 Derived Level Structures of Spectral p-Divisible Groups

Before we talk about derived level structures of spectral p-divisible groups, let us first review something about the classical level structures of commutative finite flat group schemes. Let X/S be a finite flat S-scheme of finite presentation of rank N, it can be prove that X/S is finite locally free of rank N. This means that for every affine scheme $\mathrm{Spec}R \to S$, the pullback scheme $X \times_S \mathrm{Spec}R$ over $\mathrm{Spec}R$ have the form $\mathrm{Spec}R'$, where R' is an R-algebra which is locally free of rank N. For an element $f \in R'$ which can acts on R' by multiplication, define an R-linear endmorphism of B'. Because R' is a locally free of rank N. Then multiplication of f can be representable by a $N \times N$ matrix M_f. Then we can define the characteristic polynomial of f to be the characteristic polynomial of M_f, i.e.,

$$\det(T - f) = \det(T - M_f) = T^N - \mathrm{trace}(M_f) + \cdots + (-1)N\mathrm{Norm}(f).$$

Let $\{P_1, \cdots, P_N\}$ be a set of N points in $X(S)$, we say this set is a full set of sections of X/S if one of the following two conditions are satisfied:

(1) For any $\mathrm{Spec}R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have the equality

$$\det(T - f) = \prod_{i=1}^{N} (T - f(p_i)).$$

（2）For every Spec$R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have

$$\mathrm{Norm}(f) = \prod_{i=1}^{N} f(p_i).$$

Actually, these conditions are equivalent.

If we have N not-necessarily-distinct points $\{P_1, \cdots, P_N\}$ in $X(S)$, then we have a morphism

$$\mathcal{O}_Z \to \bigotimes_i (P_i)_*(\mathcal{O}_S)$$

of sheave over X. It is easy to see that this map is surjective, and it defines a closed subscheme D of X, which is flat, proper over S. So by the construction, for a $\phi : A \to X(S)$, we can define closed subscheme D of X which corresponds to the sheave $\otimes_{a\in A}\phi(a)_*\mathcal{O}_S$.

Lemma 3.4.1: For a finite flat and finite presentation S-scheme Z, $\mathrm{Hom}(A, Z)$ is an open subscheme of $\mathrm{Hilb}_{Z/S}$.

Proof: Let $T \to S$ be a S-scheme, for any $D \to Y = T \times_S Z$ in $\mathrm{Hilb}(Y) = \mathrm{Hilb}(T \times_S Z)$, we need to prove that the set of points $t \in T$ which satisfying $D_t \to Y_t$ is coming from the closed subscheme associated with a map $\phi : A \to Z(T) = Y(T)$ is an open subset of T. Since D is the closed subscheme defined by $\mathcal{O}_Y \to \mathcal{O}_D$, if D_t comes form $\mathcal{O}_Y|_t \to \otimes(P_i)_*(\mathcal{O}_T)|_t$. Then by the definition of stalks of sheaves, there exists an open subset U of D such that $t \in U$, and D_U is defined by $\mathcal{O}_Y|_U \to \otimes(P_i)_*(\mathcal{O}_T)|_U$. ■

Definition 3.4.2: Suppose that G/S be a rank N commutative finite flat S-group scheme of finite presentation and A is a finite abelian group of order N. A group homomorphism

$$\phi : A \to G(S)$$

is called an A-generator of G/S, if the N points $\{\phi(a)\}_{a\in A}$ are a full subset of sections of $G(S)$. In these cases, we say ϕ is a Drinfeld level structure.

Proposition 3.4.3: [32]Proposition 1.10.13 Suppose that G is a rank N finite flat commutative group scheme of finite presentation over S and A is a finite abelian group of order N. Then we have the following two propositions:

（1）The functor $A-\mathrm{Gen}(G/S)$ on S-schemes defined by

$$T \mapsto \{\phi|\phi : A \to G(T) \text{ is a Drinfeld level structure}\}$$

is representable by a finite S-scheme of finite presentation. Actually, it is the closed subscheme of $\mathrm{Hom}_{\mathrm{Sch}_S}(A, G)$ over which the image of sections $\{\phi_{univ}(a)\}_{a\in A}$ of the universal homomorphism $\phi_{univ} : A \to G$ form a full set of sections.

（2）If G/S is finite étale over S of rank N, we have

$$A-\mathrm{Gen}(G/S) \simeq \mathrm{Isom}_{\mathrm{Sch}_S}(A, G),$$

such that each connected component of S, $A-\mathrm{Gen}(S)$ is either empty or is a finite étale $\mathrm{Aut}(A)$-torsor.

Derived Level Structures of Spectral Finite Flat Group Schemes

For a spectral commutative finite flat group scheme G over R. By the definition of finite flat, we have $G = \mathrm{Sp\acute{e}t}B$ for a finite flat R-algebra B. We let $\mathrm{Hilb}(G/R)$ denote the full subcategory of $\mathrm{SpDM}_{/G}$ spanned by those $D \to G$ such that $D \to G$ is a closed immersion of spectral Deligne-Mumford stacks, and the composition $D \to G \to R$ is flat, proper and locally almost of finite presentation. Then we find $\mathrm{Hilb}(G/R)$ is actually equivalent to the ∞-category of diagrams which have the form

$$\begin{array}{ccc} R & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & R' & \end{array}$$

such that R' is flat, proper and locally almost of finite presentation over R and satisfies certain conditions. It is easy to see that $\mathrm{Hilb}(G/R)$ is a Kan complex. Then we can define a functor

$$\mathrm{Hilb}_{G/R} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$$
$$R' \to \mathrm{Hilb}(G_{R'})$$

Theorem 3.4.4: Suppose that G is a commutative finite flat group scheme over an $\mathbb{E}_\infty$-ring R, then $\mathrm{Hilb}_{G/R}$ is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

Proof: This is just a special case of spectral algebraic geometry version of Lurie's theorem[23]Theorem 8.3.3. ■

Remark 3.4.5: We can proof this theorem by the same argument of the proof of representability of relative Cartier divisors.

Definition 3.4.6: Let G be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R, and A be an abstract finite abelian group of order N, an A-level structure of G is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $\pi_0\phi_*\mathcal{O}_D \simeq \otimes\phi(a)_*\mathcal{O}_{\mathrm{Spec}\pi_0 R}$, where $\phi(a)_*\mathcal{O}_{\mathrm{Spec}\pi_0 R}$ comes from a map $\phi : A \to G^\heartsuit(\pi_0 R)$.

Lemma 3.4.7: Let G/R be a spectral commutative finite flat group scheme of rank N

over an $\mathbb{E}_\infty$-ring R and let D be a Hilbert closed subscheme of G. Then there exists a $\mathbb{E}_\infty$-ring Z, satisfying the following universal property:

For any $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, $(D_{R'})^\heartsuit$ is a derived A-level structures of $(G_{R'})^\heartsuit$ if and only if $R \to R'$ factor through Z.

Proof: For $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, it is obvious that $D_{R'}$ is in $\mathrm{Hilb}(G_{R'}/R')$. This means that $(D_{R'})^\heartsuit$ is a Hilbert closed subscheme of $(G_{R'})^\heartsuit$. For $D_{R'}$ to be a derived level structure, we have $D_{R'}^\heartsuit$ must lie in $\mathrm{Hom}(A, G^\heartsuit)(\pi_0 R')$, this means that $\mathrm{Spec}\pi_0 R' \to \mathrm{Spec}\pi_0 R$ must passing through an open of $\mathrm{Spec}\pi_0 R$, since $\mathrm{Hom}(A, G^\heartsuit)$ can be viewed as a open sub scheme of $\mathrm{Hilb}(G^\heartsuit/R^\heartsuit)$. Then we have $\pi_0 R \to \pi_0 R'$ passing through W_0, where W_0 is a localization of $\pi_0 R$, so we have $R \to R'$ must passing through W, where W is an $\mathbb{E}_\infty$-ring, which is a localization of R. As for now, we already have a map $\mathrm{Sp\acute{e}t}R' \to \mathrm{Sp\acute{e}t}W$, such that $D_{R'}$ is a Hilbert closed subscheme of $G_{R'}$, and $\pi_0 i_* \mathcal{O}_{D_{R'}}$ comes from a map $\phi : A \to G^\heartsuit(\pi_0 R')$. For $D_{R'}$ want to be a derived level structure, $\mathcal{O}_{G^\heartsuit} \to \phi(a)_*(\mathcal{O}_{\mathrm{Spec}\pi_0 R'})$ needs to be an isomorphism, i.e., these N points $\phi(a)_{a\in A}$ must be a full section of $G^\heartsuit(\pi_0 R')$. By[32]Proposition 1.9.1, for a set of N points of $(G^\heartsuit(\pi_0 R'))$ to be a full section of $G^\heartsuit(\pi_0 R')$, $\mathrm{Spec}\pi_0 R' \to \mathrm{Spec}\pi_0 W$ must passing through a closed subscheme of $\mathrm{Spec}W_0$. Then $\pi_0 W \to \pi_0 R'$ must passing through Z_0, where Z_0 is equals $\pi_0 W/I$ for some ideal I of $\pi_0 W$. This means that we have $W \to R'$ pass through $Z = W^{\mathrm{Nil}(I)}$. By the discussion above, we have Z is the desired $\mathbb{E}_\infty$-ring. And the converse is also true by the same discussion in the derived level structures of curves.

■

Proposition 3.4.8: Suppose that G is a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R and A is an abstract finite abelian group of order N. Then the following functor

$$\mathrm{Level}_{H/R}^{\mathcal{A}} : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(\mathcal{A}, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(A) = \mathrm{Sp\acute{e}t}\mathcal{P}_{G/R}$.

Proof: We first prove the representability. By definition, the functor $\mathrm{Level}_{G/R}^{\mathcal{A}}$ is a subfunctor of the representable functor $\mathrm{Hilb}_{G/R}$. We consider a spectral Deligne-Mumford stack $S(A)$ defined by the pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} S(A) & \longrightarrow & \mathrm{Hilb}_{G/R} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}Z & \longrightarrow & \mathrm{Sp\acute{e}t}R. \end{array}$$

It is easy to say that $S(A)$ valued on a R-algebra R' is the Hilbert closed subscheme D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^{\heartsuit}$ is a derived level A-structure of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^{\heartsuit}$. Then $S(A)$ is the desried stack.

For the affine condition, we need to prove that $S(A)$ is finite in spectral algebraic geometry. By[11]Remark 5.2.0.2, a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. We have $S(A)$ and $\mathrm{Sp\acute{e}t}R$ are spectral spaces. So we only need to prove $S(A)^{\heartsuit}$ is finite over R_0, but this is just the classical case, which is finite by[32]Proposition 1.10.13. ■

Derived Level Structures of Spectral p-Divisible Groups

Remark 3.4.9: We let $\mathrm{FFG}(R)$ denote the ∞-category of spectral commutative finite flat group schemes over an $\mathbb{E}_\infty$-ring R. By[24]Proposition 6.5.8, there is another equivalent definition of spectral p-divisible group[13]Definition 6.0.2. A spectral p-divisible group over a connective $\mathbb{E}_\infty$-ring R is just a functor

$$G : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$$

which satisfies the following conditions:

(1) Suppose that $S \in \mathrm{CAlg}_R^{cn}$, the spectrum $G(S)$ is p-nilpotent, i.e., $G(S)[1/p] \simeq 0$.

(2) For M be a finite ableian p-group, the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad S \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}}(M, G(S))$$

is copresentable by a finite flat R-algebra.

Let X be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R, that is a functor

$$X : \mathrm{Ab}_{\mathrm{fin}}^{p} \to \mathrm{FFG(R)}.$$

For every $p^k \in \mathrm{Ab}_{\mathrm{fin}}^{p}$, we let $X[p^k]$ denote the image of p^k of X. We find that $X[p^k]$ is a rank $(p^k)^h$ spectral commutative finite flat group schemes over R.

Definition 3.4.10: Let G be a spectral p-divisible group of height h over a connective E_∞-ring R . For A a finite abelian group, an derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure

$$\phi : D \to G[p^k]$$

of $G[p^k]$, which is a spectral commutative finite flat scheme over R. We let $\mathrm{Level}(k, G/R)$

denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of G/R.

Theorem 3.4.11: Let G be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R. Then the following functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^k_{G/R}$.

Proof: We just notice that by the definition of spectral p-divisible group, $G[p^k]$ is a spectral commutative finite flat scheme. Then the theorem follows form the above result of general spectral commutative finite flat group scheme. ■

Non-Full Level Structures

The above cases only cares full level structures of commutative finite flat schemes, actually we can define general level structures of finite flat group schemes. Let G be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R, and A be an abstract finite abelian group, an derived A-level structure of G is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $D^\heartsuit$ is a subgroup of G and $G^\heartsuit(\pi_0 R)$ is isomorphic to A. We let $\mathrm{Level}_1(\mathcal{A}, G/R)$ denote th space of derived A-level structure. And $\mathrm{Level}_0(\mathcal{A}, G/R)$ denote the space of equivalence class $D \to G$ in $\mathrm{Hilb}(G/R)$ such that $G^\heartsuit(\pi_0 R)$ is isomorphic to A, two object D, D' are equivalent if the image of $D^\heartsuit \to G^\heartsuit$ and $D'^\heartsuit \to G^\heartsuit$ are same.

Proposition 3.4.12: Suppose that G is a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R and A is an abstract finite abelian group of order not necessarily equal to N. Then the following functor

$$\mathrm{Level}^{1,\mathcal{A}}_{G/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(\mathcal{A}, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack.

Proof: We just notice that the classical level structure functor $\mathrm{Level}(A, G^\heartsuit/\pi_0 R)$ is representable by a closed subscheme $\mathrm{Hom}(A, G)$, the using the same discussion of full level case, we get the desired result. ■

Remark 3.4.13: The above proposition also true for $\mathrm{Level}^{0,\mathcal{A}}$. By the spectral commutative finite flat scheme cases, we can get the representability results of spectral p-divisible group case.

We let $\mathrm{Level}_1(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level structures of G/R. Then the following functor

$$\mathrm{Level}^{1,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S_1(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^{1,k}_{G/R}$.

We let $\mathrm{Level}_0(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level generators of G/R. Then the following functor

$$\mathrm{Level}^{0,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_0(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S_0(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^{0,k}_{G/R}$.

CHAPTER 4 APPLICATIONS TO CHROMATIC HOMOTOPY THEORY

4.1 Spectral Elliptic Curves with Derived Level Structures

In the second chapter, we have introduced that there exists a spectral Deligne-Mumford stack $\mathcal{M}_{ell}$ whose functor of points is

$$\begin{aligned} \mathcal{M}_{ell} \ &: \ \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ &R \longmapsto \mathcal{M}_{ell}(R), \end{aligned}$$

where $\mathcal{M}_{ell}(R) = \mathrm{Ell}(R)^{\simeq}$ is the underline ∞-groupoid of the ∞-category of spectral elliptic curves over R.

And we have the classical Deligne-Mumford stack of classical elliptic curves, which can be viewed as a spectral Deligne-Mumford stack

$$\begin{aligned} \mathcal{M}_{ell}^{cl} \ &: \ \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ &R \longmapsto \mathcal{M}_{ell}^{cl}(\pi_0 R) \end{aligned}$$

where $\mathcal{M}_{ell}^{cl}(\pi_0 R)$ is the groupoid of classical elliptic curves over the commutative ring $\pi_0 R$.

And for A denote $\mathbb{Z}/N\mathbb{Z}$, or $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, we have the classical Deligne-Mumford stack of classical elliptic curves with level-A structures, which can also be viewed as a spectral Deligne-Mumford stack.

$$\begin{aligned} \mathcal{M}_{ell}^{cl}(A) \ &: \ \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ &R \longmapsto \mathcal{M}_{ell}^{cl}(A)(\pi_0 R) \end{aligned}$$

where $\mathcal{M}_{ell}^{cl}(A)(\pi_0 R)$ is the groupoid of classical elliptic curves with level A-structures over the commutative ring $\pi_0 R$.

In last chapter, we define and study derived level structures. The construction $X \mapsto \mathrm{Level}(\mathcal{A}, X/R)$ determines a functor $\mathrm{Ell}(R) \to \mathcal{S}$ which is classified by a left fibration $\mathrm{Ell}(\mathcal{A})(R) \to \mathrm{Ell}(R)$. Objects of $\mathrm{Ell}(\mathcal{A})(R)$ are pairs (E, ϕ), where E is a spectral elliptic curve and ϕ is a derived level structures of E.

For every $R \in \mathrm{CAlg}^{\mathrm{cn}}$, we can consider all spectral elliptic curves over R with derived level structures. This moduli problem can be thought as a functor

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \ : \ & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

where $\mathrm{Ell}(\mathcal{A})(R)$ is the space of spectral elliptic curves E with a derived level structure $\phi : \mathcal{A} \to E$.

Proposition 4.1.1: The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ is an étale sheaf.

Proof: Let $\{R \to U_i\}$ be an étale cover of R, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{array}{ccc} \mathrm{Ell}(\mathcal{A})(R)^{\simeq} & \xrightarrow{f} & \lim_\Delta \mathrm{Ell}(\mathcal{A})(U_\bullet)^{\simeq} \\ \downarrow p & & \downarrow q \\ \mathrm{Ell}(R)^{\simeq} & \xrightarrow{g} & \lim_\Delta \mathrm{Ell}(U_\bullet)^{\simeq}. \end{array}$$

The left map p is a left fibration between Kan complex, so is a Kan fibration[29]Lemma 2.1.3.3. And the right vertical map is pointwise Kan fibration. By picking a suit model for the homotopy limit we may assume that q is a Kan fibration as well. We have g is an equivalence by[24]Lemma 2.4.1. To prove that f is a equivalence. We only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence. We have the $\mathrm{Level}(\mathcal{A}, E)$ as full ∞-subcategory of $\mathrm{CDiv}(E/R)$ and $\lim_\Delta \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet)$ as a full subcategory of

$$\lim_{\Delta} \mathrm{CDiv}(E \times_R U_\bullet(U_\bullet))$$

But CDiv is an étale sheaf. So the functor

$$\mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet).$$

is fully faithful. To prove it is a equivalence, we only need to prove it is essentially surjective.

For any $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$ in $\lim_\Delta \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet)$. Clearly, we can find a morphism $\phi_R : D \to E$ in $\mathrm{CDiv}(E/R)$ whose image under the equivalence $\mathrm{CDiv}(E/R) \simeq \lim_\Delta \mathrm{CDiv}(E \times_R U_\bullet/U_\bullet)$ is $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$. We just need to prove this $\phi_R : D \to E$ is a derived level structure. This is true since in the classic case, $\mathrm{Level}(A, E^{\heartsuit}(R_0)) \simeq \lim_\Delta \mathrm{Level}(A, E^{\heartsuit}(\tau_{\leq 0} U_\bullet))$ and $\phi_R : D \to E$ is already a relative Cartier divisor. ■

Lemma 4.1.2: $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is a nilcomplete functor, i.e., $\mathcal{M}_{ell}(\mathcal{A})(R)$ is the homotopy limit of the following diagram

$$\cdots \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m}R) \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m-1}R) \to \cdots \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq 0}R)$$

Proof: For a spectral elliptic curve R, there is an obvious functor

$$\theta : \mathcal{M}_{ell}(\mathcal{A})(R) \to \lim_{\leftarrow n} \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq n}R)$$

define by $(E, \phi : D \to E) \mapsto \{(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R \to E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R)\}_n$. Here we notice that $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R \to E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R)$ is in $\mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq n}R)$.

First, we prove that θ is essentially surjective. An object in $\lim_{\leftarrow m} \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m}R)$ can be written as a diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots & \longrightarrow & D_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & E_{n+1} & \longrightarrow & E_n & \longrightarrow & E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_0
\end{array}$$

where each E_n is spectral elliptic curve over $\tau_{\leq n}R$ and $D_n \to E_n$ is a derived level structure, and satisfying $D_n = D_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, E_n = E_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R$. By the nilcompletness of $\mathcal{M}_{ell}$, we get a spectral elliptic curves E, such that $E \times_R \tau_{\leq n}R \simeq E_n$, and by the nilcompletness of Var_+ [11]Proposition 19.4.2.1, we get a spectral Deligne-Mumford stack D, such that $D_n = D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R$. We need to prove the induce map $D \to E$ is a derived level structure, but this follows form nilcompletness of $\mathrm{Level}_{E/R}$.

Second, we need to prove that this functor is fully faithful. Unwinding the definitions, we need to prove that for every $(X, D_1 \to X), (Y, D_2 \to Y) \in \mathcal{M}_{ell}(\mathcal{A})(R)$, the following map is a homotopy equivalence.

$$\mathrm{Map}_{\mathcal{M}_{ell}(\mathcal{A})(R)}((X, D_X), (Y, D_Y)) \to \mathrm{Map}_{\mathcal{M}_{ell}(\mathcal{A})(R)}(\lim_{\leftarrow n}(X_n, D_{X,n}), \lim_{\leftarrow n}(Y_m, D_{Y,m})).$$

where X_n is $\tau_{\leq n}X = X \times_R \tau_{\leq n}R$, and Y, $D_{X,n}, D_{Y,n}$ similarly.

But we notice that this is equivalent to following equivalence

$$\mathrm{Map}_{\mathrm{SpDM}_{/R}}((X, D_X), (Y, D_Y)) \to \lim_{\longleftarrow n} \mathrm{Map}_{\mathrm{SpDM}_{\tau_{\leq n}}}((X_n, D_{X,n}), (Y_n, D_{Y,n})).$$

And this equivalence follows from [11]Proposition 19.4.1.2 ■

Lemma 4.1.3: $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a cohesive functor.

Proof: For every pullback diagram

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ such that the underlying homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjective. We need to prove that

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A})(D) & \longrightarrow & \mathcal{M}_{ell}(\mathcal{A})(A) \\ \downarrow & & \downarrow \\ \mathcal{M}_{ell}(\mathcal{A})(C) & \longrightarrow & \mathcal{M}_{ell}(\mathcal{A})(B) \end{array}$$

is a pullback diagram.

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & \searrow^{f} & \downarrow h \\ & & * \end{array}$$

By[11]Remark 17.3.7.3, $\mathcal{M}_{ell} * (\mathcal{A})$ is a cohesive fucntor if and only if f is cohesive. Since we have $\mathcal{M}_{ell}$ is cohesive functor, h is a cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$. And again by[11]Remark 17.3.7.3, f is cohesive if and only if g is cohesive. So we only need to prove that g is a cohesive morphism. But by[11]Proposition 17.3.8.4 g is cohesive if and only if each fiber of g is cohesive, i.e., for $R \in \mathrm{CAlg}^{cn}$ and a point $\eta_E \in \mathcal{M}_{ell}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

is cohesive. But we have $R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The cohesive of $\mathcal{M}_{ell}(\mathcal{A})$ then follows from the cohesive of $\mathrm{Level}_{E/R}$.

■

Lemma 4.1.4: The fucntor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable

Proof: We need to prove that for R a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$, then there is an equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Sp\acute{e}t} R', \mathcal{M}_{ell}(\mathcal{A})) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Spf} R', \mathcal{M}_{ell}(\mathcal{A})).$$

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} \\ & & * \end{array}$$

By[11]Remark 17.3.7.3, $\mathcal{M}_{ell}(\mathcal{A}) \to *$ is a integrable fucntor if and only if f is integrable. Since we have $\mathcal{M}_{ell}$ is integrable functor, h is a integrable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. And again by[11]Remark 17.3.7.3, f is integrable if and only if g is integrable. So we only need to prove that g is a integrable morphism. But by[11]Proposition 17.3.8.4 g is integrable if and only if each fiber of g is integrable, i.e., for $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta_E \in \mathcal{M}_{ell}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}_R^{cn} \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

is integrable. But we have $R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The integrable of $\mathcal{M}_{ell}(\mathcal{A})$ then follows from the integrable of $\mathrm{Level}_{E/R}$.

■

Lemma 4.1.5: The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ admits a cotangent complex $L_{\mathcal{M}_{ell}^{de}}$, and moreover $L_{\mathcal{M}_{ell}^{de}}$ is connective and almost perfect.

Proof: We have a commutative diagram in $\mathrm{CAlg}^{cn} \to \mathcal{S}$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} \\ & & * \end{array}$$

Since we have h is infitessimally coheisve and admits a connective cotangent complex, and f,g is infitessimally cohesive. By[11]Proposition 17.3.9.1, to prove that f admits a cotangent complex. We only need to prove g admits a relative cotangent complex. By[11]Proposition 17.2.5.7, a morphism $j : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$ admits a relative cotangent complex if and only if, for any corepresentbale $Y' = \mathrm{Map}(R, -) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ and any natural transformation $Y' \to U$, j' in the following pullback diagram admit a cotangent complex.

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ Y' & \longrightarrow & Y \end{array}$$

To prove that $\mathcal{M}_{ell}(\mathcal{A}) \to \mathcal{M}_{ell}$ admits a cotangent a cotangent complex, we just need to prove that for any $R \in \mathrm{CAlg}^{cn}$, and a spectral elliptic curve E which represents a natural transformations Spec$R \to \mathcal{M}_{ell}$. The functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

admits a connective cotangent complex. But we have $\mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} = \mathrm{Level}(E \times_R R') = \mathrm{Level}_{E/R}(R')$. So the results of $f : \mathcal{M}_{ell}(\mathcal{A}) \to *$ admits a cotangent complex follows from $\mathrm{Level}_{E/R}$ admits a cotangent complex. And the properties of connective and almost perfect also follows from the property of the cotangent complex of $\mathrm{Level}_{E/R}$. ■

Lemma 4.1.6: The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ is locally almost of finite presentation.

Proof: Consider the functor $\mathcal{M}_{ell}(\mathcal{A}) \to *$, it is infitesimally cohesive and admits a cotangent complex which is almost perfect, so by[11]17.4.2.2, it is locally almost of finite presentation. So $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation, since $*$ is a final object of Fun($\mathrm{CAlg}^{cn}, \mathcal{S}$).

■

Theorem 4.1.7: The functor

$$\begin{aligned} \mathcal{M}_{ell}(A) \ : \ & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is representable by a spectral Deligne-Mumford stack.

Proof: By the spectral Artin representability theorem, we need to prove that the functor $\mathcal{M}_{ell}(\mathcal{A})$ satisfying the following condition

(1) The space $\mathcal{M}_{ell}(\mathcal{A})(R_0)$ is n-truncated for every discrete commutative ring R_0.

(2) $\mathcal{M}_{ell}(\mathcal{A})$ is a sheaf for the étale topology.

(3) $\mathcal{M}_{ell}(\mathcal{A})$ is a nilcomplete, infinitesimally cohesive, and integrable functor.

(4) $\mathcal{M}_{ell}(\mathcal{A})$ admits a cotangent complex $L_{\mathcal{M}_{ell}(A)}$ which is connective.

(5) $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation.

But these follows form the above series of lemmas. ■

4.2 Higher Categorical Lubin-Tate Towers

We recall that for a height h p-divisible group G_0 over a commutative ring R_0 and suppose $A \in \mathrm{CAlg}_{cpl}^{ad}$. We recall that a deformation of G_0 over R is a spectral p-divisible group over R together with an equivalence class of G_0-tagging of G. We let $\mathrm{Level}(k, G/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of a height h spectral p-divisible group. We consider the following functor

$$\begin{aligned} \mathcal{M}_k \ : \ & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}(G_0, R, k)$ is the ∞-category whose objects are triples (G, ρ, η)

(1) G is a spectral p-divisible group over R.

(2) ρ is an equivalence of G_0 taggings of R.

(3) $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G.

Theorem 4.2.1: The functor $\mathcal{M}_k$ is corepresentable by a $\mathbb{E}_\infty$-ring which is finite over the unoriented spectral deformation ring of G_0.

Proof: We let $E_{univ}/R_{G_0}^{\mathrm{un}}$ denote the universal spectral deformation of G_0/R_0. Suppose that G is a spectral deformation G_0 to R, we get a map of $\mathbb{E}_\infty$-rings $R_{G_0}^{\mathrm{un}} \to R$, and an equivalence $E_{univ} \times_{R_{G_0}^{un}} R \simeq G$ of spectral p-divisible groups. By the universal objects of level structures. We have the following equivalence

$$\mathrm{Level}(k, G/R) \simeq \mathrm{Level}(k, E_{univ} \times_{R_{G_0}^{un}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R_{G_0}^{un}}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R),$$

where $\mathcal{P}_{E_{univ}/R_{G_0}^{un}}$ is the universal object of derived level structure functor associated with the p-divisible group $E_{univ}/R_{G_0}^{un}$.

Then we consider the following moduli problem

$$\mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R_0}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R).$$

For $R \in \mathrm{CAlg}_{R_0}^{ad,cpl}$, $\mathrm{Map}_{\mathrm{CAlg}_{R_0}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R)$ can viewed the ∞-categories of pairs (α, f), where

$$\alpha : R_{G_0}^{un} \to R$$

is the classified map of a spectral p-divisible group G, which is a deformation of G_0, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}_{R_{G_0}^{un}}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R) = \mathrm{Level}(k, E_{univ} \times_{R_{G_0}^{un}} R)$ is a derived level structure of G/R. So we get $\mathrm{Map}_{\mathrm{CAlg}_{R_0}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R)$ is just the ∞-category of pairs (G, ρ, η). By lemma 3.4.11, $\mathcal{P}_{E_{univ}/R_{G_0}^{un}}$ is finite over $R_{G_0}^{un}$. So we have $\mathcal{P}_{E_{univ}/R_{G_0}^{un}}$ is the desired spectrum. ■

Although we get spectra come from a conceptual derived moduli problems, but these spectra may be complicated, since we didn't know the homotopy groups. In algebraic topology, orientation of $\mathbb{E}_\infty$-spectra make E_2 page of Atiyah-Hirzebruch spectral sequences degenerating, and give us the information of homotopy groups.

Let G_0 be a height h p-divisible group over R_{G_0}. We consider the following functor

$$\begin{aligned}\mathcal{M}_k^{or} \ : \ & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k)\end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the space of four tuples (G, ρ, e, η), where

（1）G is a spectral p-divisible over R.

（2）ρ is an equivalence class of G_0 taggings of R.

（3）$e : S^2 \to \Omega^\infty G^\circ(R)$ is an orientation of the G°, where G° is the identity component of G.

（4）$\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G.

Theorem 4.2.2: The functor $\mathcal{M}_k^{or} : \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JK}_k$, which is finite over the orientated deformations ring $R_{G_0}^{or}$.

Proof: Let $\mathrm{Def}^{or}(G_0, R)$ denote the ∞-groupoid of triples (G, ρ, e), where G is a p-divisible of over R, ρ is an equivalence class of G_0-taggings of R, and e is an orientation of the identity conpoment of G. By[13]Theorem 6.0.3, Remark 6.0.7, the functor

$$\begin{aligned}\mathcal{M}^{or} \ : \ & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{Def}^{or}(G_0, R)\end{aligned}$$

is corepresnetable by the orientated deformation ring $R_{G_0}^{or}$, that is we have an equivalence of spaces

$$\mathrm{Map}_{\mathrm{CAlg}_{cpl}^{ad}}(R_{G_0}^{or}, R) \simeq \mathrm{Def}^{or}(G_0, R).$$

Let E_{univ}^{or} be the associated universal orientation deformation of G_0 to $R_{G_0}^{or}$, then it is obvious that $\mathcal{JL}_k = \mathcal{P}_{E_{univ}^{or}/R_{G_0}^{or}}$, the universal object of derived level structures of $E_{univ}^{or}/R_{G_0}^{or}$, is the desired spectrum similar to th unorientated case. ■

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbb{Z}/p^k\mathbb{Z}) \times \mathrm{Aut}(G_0)$. And when k varies, we have a tower

$$\begin{array}{c}\mathrm{Sp}\acute{e}t\mathcal{JL}_k\\ \downarrow\\ \mathrm{Sp}\acute{e}t\mathcal{JL}_{k-1}\\ \downarrow\\ \cdots\\ \downarrow\\ \mathrm{Sp}\acute{e}t\mathcal{JL}_0.\end{array}$$

We call this tower higher categorical Lubin-Tate tower.

Let E be a local field, G be a reductive group over E. The classical local Langlands correspondence predict that for any irreducible smooth representation π of $G(E)$, we can naturally associate an L-parameter

$$\phi_E : W_E \to \widehat{G}(\mathbb{C}).$$

The geometric Langlands correspondence actually aim to construct an equivalence of categories

$$D(\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(X)) \simeq D(\mathcal{D}(\mathrm{Bun}_G))$$

from the derived category of quasi-coherent sheaves on $G^\vee$ local systems on X and the derived categories of D-modules on the moduli stack of G-bundles over X[34]. Due to the work of Fargues-Scholze[35], the arithmetic local Langlands correspondence can also be some kinds of geometric Langlands correspondence, but in the perfectoid world.

In the classical arithmetic geometry, the Lubin-Tate tower can be used to realize the Jacquet-Langlands correspondence[36]. Is there a topological realization of the Jacquet-Langlands correspondence? Actually, in a recent paper[37], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structures such that representing object is étale over representing object of universal deformations.

We hope our higher categorical analogues of Lubin-Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra. By the construction of Jacquet-Langlands spectra above, Let $\mathbb{G}$ be a formal group over a field of characteristic p, $\mathcal{JL}$ be its ℓ-adic complete Jacquet-Langlands spectrum. Let X be a spectrum with an action of $\mathrm{Aut}(\mathbb{G}_h)$. We have the following brave conjecture.

Conjecture 4.2.3: The function spectrum $F(X, \mathcal{JL})$ admits an action of $GL_h(\mathbb{Z}_p)$ and all its homotopy groups are $\mathbb{Z}_l$-modules.

Representation Theory in Spectra Algebraic Geometry

The reason why we need spectra and spectral algebraic geometry in representation theory is due to the fact, in general the derived category of G-objects $\mathrm{Mod}(R)$ is not equal to the category of G-objects in $D(R)$. But in algebraic topology, it seems that group actions of spectra are more easy to find, like actions of Morava stabilizer groups on Morava E-theories.

It follows that[38], some topological realizations of classical cohomology rings may have a good structures, like the topological Hochschild homology of quasiregualr semiperfectoid rings. These leads to the establishment of some special p-adic cohomology theories, Breuil-Kisin-modules cohomology theory and its refinement, prismatic cohomology[39]. The heart of this topic are δ-rings and their topological realization derived δ-rings[40]. It turns out homotopy groups of these topological cohomology of perfectoid rings are crystalline Galois representations[38], But those entire spectra are not equivalent spectra.

We hope to establish representation theory in derived category, like $D(R)$, $D(\mathrm{QCoh}(X))$. But as we said, they are not the derived category of G-objects. We proposed an viewpoint that how do we use spectral algebraic geometry to solve this problem.

（1）Representations in $\mathrm{Var}_k, \mathrm{QCoh}(X)$;

（2）Explain these $\mathrm{Var}_k, \mathrm{QCoh}(X)$ as classical moduli spaces;

（3）Find associated derived moduli problems in spectral algebraic geometry ;

（4）Using repersentability theorem to get derived geometric objects;

（5）Representations in derived categories.

Now, let's see some examples of this strategy.

Example 4.2.4: (**Spherical Witt Vectors**) We consider the spherical Witt-vector functor defined in[13] and[41].

$$\mathbb{SW} : \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{Sp}_p).$$

form the category of perfect $\mathbb{F}_p$ algebras to the ∞-category of p-complete $\mathbb{E}_\infty$-rings. This functor is defined by studying a derived moduli problem, thickenings of relatively perfect morphisms. And it has many application in chromatic homotopy theory, like[41] and[42].

And it is easy to see that this functor can find some Galois representations in derived category.

Example 4.2.5: (**Spectral Deformations of p-Divisible Groups**) For a classical p-divisible group G_0 over a perfect field k, we consider the Morava stabilizer group $S = \mathrm{Aut}(G_0) \rtimes Gal(k)$. We can consider its spectral deformations over an $\mathbb{E}_\infty$-ring R, which consists of pairs (G, ρ), where G is a spectral p-divisible group over R, and ρ is an equivalence class of G_0 taggings. In[13], Lurie proved that there exits an universal deformation of G_0. i.e., there exists a complete adic $\mathbb{E}_\infty$-ring $R^{un}_{G_0}$, and a morphism $\rho : R^{un}_{G_0} \to R_0$ such that the functor Def_{G_0} is corepresentable by $R^{un}_{G_0}$. i.e., for any complete adic $\mathbb{E}_\infty$-ring R, there is an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{un}_{G_0}, R) \to \mathrm{Def}_{G_0}(R).$$

It is easy to see that this spectrum $R^{un}_{G_0}$ admits an action of S.

Example 4.2.6: (**Derived Level Structures**) Let k be a p-adic field with residue field k of characteristic p. Let LT_n denote the moduli space of deformations with level $(\mathbb{Z}/\mathbb{Z}^n)^h$-structures of a height h formal group G_0 . Passing to the direct limit over n of vanishing cycle sheaves of LT_n. This give an collection $\{\Psi^i_m\}$ of infinite-dimensional $\bar{\mathbf{Q}}_l$-vector spaces which admits admissible nature actions of the subgroup of $GL_g(K) \times D^\times_{K,g} \times W_K$. Then by our construction of derived level structures, we find these actions can lift to actions on certain ∞-spectra.

Topological Langlands Correspondence

We know actions of certain Galois groups and automorphism groups on certain objects, like Morava E-theories, THH, TC. And this means that these groups acting on their homotopy groups. By the Langlangs correspondence, we can associated certain objects which have the action of GL_n, or more generally, reductive groups. But can these objects lift to GL_n equivalent spectra. Our derived level structure give an attempt on this idea by considering the function spectrum $\mathrm{Fun}(X, \mathcal{JL})$.

Let G be an algebraic group, viewed as a 0-truncated spectral Deligne-Mumford stack, Let X be a spectral Deligne-Mumford stack admits a G-action. Then does this make R for an affine substack $\mathrm{Sp\acute{e}t}R$ to become a G-equivariant spectrum? See[43] for equivariant spectra and[44] for the equivariant $\mathbb{E}_\infty$ setting. On the other hand, what is the meaning of the action of an algebraic group on a spectrum, since spectra are topological, they don't have algebraic structures.

We want to develop a representation theory in E_∞-spectra, spectral schemes, and spectral stacks, such that it is compatible with the classical definition of actions of algebraic groups on schemes. And we want to know how does actions of Galois side on certain objects can related to actions of some algebraic groups on another certain objects. And the name topological Langlands correspondence comes from that we want certain spectral algebraic geometry objects play the roles of homotopy representations of dual reductive algebraic groups, which can be viewed as automorphic side of topological Langlands correspondence.

4.3 Topological Lifts of Power Operation Rings

We recall the deformation of formal groups. Let G_0 be a formal group over a perfect field k such that $\mathrm{char} k = p$, a deformation of G_0 to R is a triple (G, i, Φ) satisfying

- G is a formal group over R,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Phi : \pi^* G \cong i^* G_0$ of formal groups over R/m.

Suppose that we have a complete local ring R whose residue filed has characteristic p. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map. For each formal group G over R, the **Frobenius isogeny** $\mathrm{Frob} : G \to \phi^* G$ is the homomorphism of formal group over R induced by the relative Frobenius map on rings. We write $\mathrm{Frob}^r : G \to (\phi^r)^* G$ which is the composition $\phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$

Let G_0 be a formal group over k, (G, i, α) and (G', i', α') be two deformations of G_0 to R. A deformation of Frob^r is a homomorphism $f : G \to G'$ of formal groups over R which satisfying

（1）$i \circ \phi^r = i'$ and $i^*(\phi^r)^* G_0 = (i')^* G_0$.

$$\begin{array}{ccc} k & \xrightarrow{i'} & R/m \\ {\scriptstyle \phi^r}\downarrow & \nearrow{\scriptstyle i} & \\ k & & \end{array}$$

（2）the square

$$\begin{array}{ccc} i^* G_0 & \xrightarrow{i^*(\mathrm{Frob}^r)} & i^*(\phi^r)^* G_0 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ \pi^* G & \xrightarrow{\pi^*(f)} & \pi^* G' \end{array}$$

of homomorphisms of formal groups over R/m commutes.

We let Def_R denote the category whose objects are deformations fo G_0 to R, and whose morphisms are deformation of Frob^r for some $r \geq 0$. We will say that a morphism in Def_R has height r, if it is a deformation of Frob^r, and the we denote the corresponding subcategory as $\mathrm{Sub}^r R$. Let G be deformation of G_0 to R, then it can be proved that the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r.

Theorem 4.3.1: [21] Let G_0/k be a height n formal group over a perfect field k. For each $r > 0$, there exists a complete local ring A_r which carries a universal height r morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \mapsto (G_t, i_t, \alpha_t) \in \mathrm{Sub}^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set Sub^r_R. Furthermore, we have:

（1）$A_0 \approx W(k)[[v_1, \cdots, v_{n-1}]]$ is the Lubin-Tate ring.

（2）There is a map $s : A_0 \to A_r$ which classifies the source of the universal height r map, i.e. $G_s = s^* G_E$, where $G_E = G_{univ}/A_0$ be the universal deformation of G_0, and A_r is finite and free as an A_0 module.

（3）There is a map $t : A_0 \to A_r$ which classifies the target of the universal height r map, i.e. $G_t = t^* G_E$.

（4）And there is a bijection $\{g : A_r \to R\} \to \mathrm{Sub}^r(R)$ given by $g \to g^*(f^r_{univ})(g^* G_s \to g^* G_t)$.

We know that those rings $A_r, r \geq 0$ have topological meansings.

Theorem 4.3.2: [22] The ring A_r in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$, i.e,

$$A_r \cong E^0(B\Sigma_{p^r})/I$$

where I is transfer ideal.

The collections $\{A_r\}$ have the structures of graded coalgerbas, for $s = s_k, t = t_k : A_0 \to A_k$, which is induced by E^0 cohomology on $B\Sigma \to *$, we have

$$\mu = mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s \otimes_{A_0}{}^t A_l$$

which classifying the source,target, and composite of morphisms. So for the power operation $R^k(X) \to R^k(X \times B\Sigma_m)$. For $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a Γ-module, where Γ are duals of $A[r]$.

For more details about power operation in Morava E-theory, one can see[45-46] and[47]. Direct computations are in[48] for height 2 at the prime 2,[49] for height 2 at prime 3,[50] for height 2 at all primes. Cases of height > 2 is still lack of computations.

Because we have the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub_R^r with source G and the finite subgroup of G which have rank p^r. So it is easy to see that A_r corepresent the following moduli problem

$$\begin{aligned} \mathcal{M}_{0,r} \quad : \quad & \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S} \\ & R \to \mathrm{Def}(G_0, R, p^r) \end{aligned}$$

where $\mathrm{Def}(G_0, R, p^r)$ consists of pairs (G, H) where G is an defomration G_0 to R, and H is a rank p^r subgroup of G.

Proposition 4.3.3: For every integer $r \geq 1$, there exists a E_∞-ring $E_{n,r}$, such that $\pi_0 E_{n,r} = A_r$.

Proof: For the formal group G_0 over a field k of characteristic p. We just consider the functor $\mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ by sending an E_∞-ring R to quadruples (G, ρ, e, η), where (G, ρ) is spectral deformation of G_0 to R. e is an orientation of G°, the identity component G, and $\eta \in \mathrm{Level}_0(k, G/R)$ is a derived level structure. Using the same argument in full level structure and the fact $\mathrm{Level}_{G/R}^{0,k}$ is representable, see Remark 3.4.13. We get this proposition. ∎

Remark 4.3.4: Although, we obtain spectra whose π_0 are the power operation rings of Morava E-theories. But we don't know higher homotopy groups of these spectra, since these spectra are not even periodic and they are not étale over Morava E-theories. We will continue to study such spectra in the future.

CONCLUSION

We now give an conclusion of this paper. By our proves and results, it is reasonable to consider more moduli spaces in the context of spectral algebraic results, like vector bundles on a spectral curves and how this moduli space can give us interesting cohomology theory. The main contributions of this paper are

（1）Give a reasonable definition of derived versions of level structures.

（2）Prove that moduli spaces of relative Cartier divisors have the structure of spectral Deligne-Mumford stacks.

（3）Give a higher categorical analogues of moduli stack of elliptic curves with level structures.

（4）Give higher categorical analogues of Lubin-Tate towers.

（5）Give topological realizations of power operation rings of Morava E-theories (The representable objects of deformations with given finite subgroups).

But there are still many problems in this project. First is computations of homotopy groups of higher categorical Lubin-Tate towers, since we only know their π_0 correspond to moduli spaces of deformations with level structures. And as cohomology theories, we also want some results about computations on certain spaces, like $B\Sigma_n$ and so on. The relation between these cohomology theories and Morava E-theories is also interesting topic for us.

The second question is more complicated. We know that our derived level structure follows from relative Cartier divisors. But what if we choose other moduli problems, it follows that different moduli problems will generating different cohomology theories. We want find a relation between theses moduli problems and those representable spectra.

APPENDIX A CHROMATIC HOMOTOPY THEORY

We review some basic definitions and results in chromatic homotopy theory. More details can be found in[51-55].

A.1 Formal Groups

A formal scheme is a functor the category of profinite commutative rings (completion of some commutative ring) to the category of sets, which carries every profinite ring R to its R-points $X(R)$

A formal group is a formal scheme G which admits a group structure, $m : G\times G \to G$. G is a functor, so m is actually a natural transformation from the product functor $G \times G$ to functor G,i.e, for every object $R \in$ **ProCommR**, there is a binary operation

$$(G \times G)(R) = G(R) \times G(R) \to G(R)$$

In algebraic topology, we usually consider dimension one affine group schemes. One can see[56] and[57] for more discussions about formal groups.

Suppose that we have a complete local ring R and with char$R = p > 0$. Let C_R denote the category of local Noetherian R-algebras. For a functor

$$F : C_R \to \text{Set},$$

the elements of $F(A)$ will be called the A-valued points of F. And we define the formal affine line by

$$\hat{\mathbb{A}}^1(A) := C_R(R[[t]], A)$$

for any $A \in C_R$. It's easy to see that $\hat{\mathbb{A}}^1(A)$ is isomorphic to the maximal ideal of A.

Definition A.1.1: A commutative one-dimensional formal group over R is a functor

$$F : C_R \to \mathbf{Ab}$$

which is isomorphic to $\hat{\mathbb{A}}^1$.

It is known that the morphisms between affine schemes is unique determined by the morphisms of their global sections, i.e. ring of functions. If G is a group scheme over

Spec$R$ and has group multiplication $m : G \times G$, we have a ring morphism

$$\mathcal{O}_G \to \mathcal{O}_{G\times G} \cong \mathcal{O}_G \otimes \mathcal{O}_G$$

The ring of functions $\mathcal{O}_G$ is just $R[\![X]\!]$ and $\mathcal{O}_G \otimes \mathcal{O}_G$ is $R[\![X]\!] \otimes_R R[\![Y]\!] = R[\![X,Y]\!]$. So the multiplication is actually determined by

$$\begin{array}{rccc} \phi: & R[\![X]\!] & \to & R[\![X,Y]\!] \\ & X & \to & f(X,Y) \end{array}$$

So we find that the multiplication of a dimension one group scheme is actually determined by a former power series $f(X,Y)$ over R.

A coordinate X on F is a natural isomorphism $x : F \to \hat{\mathbb{A}}^1 = \hat{\mathbb{A}}^1_R$ of functors. It gives an isomorphism $\Gamma(F, \mathcal{O}_F) \cong R[[X]]$.

Formal Group Laws

Definition A.1.2: Suppose that we have a ring R and $F \in R[\![x_1, x_2]\!]$, we call f a formal group law over R if it satisfying the following conditions:

- $F(x,0) = F(0,x) = x$ (Identity)
- $F(x_1,x_2) = F(x_2,x_1)$ (Commutativity)
- $F(F(x_1,x_2),x_3) = F(x_1,F(x_2,x_3))$ (Associativity) If R is a graded ring, we require F to be homogeneous of degree 2 where $|x_1| = |x_2| = 2$.

Theorem A.1.3: There is a universal formal group law $F_{univ}(x,y) \in L[\![x,y]\!]$ over a ring L, such that for any other formal group law $F(x,y) \in R[\![x,y]\!]$ over a ring R, there is a ring morphism $f : L \to R$ such that $f^*(F_{univ}(x,y)) = F(x,y)$

Proof: We let $L = \mathbb{Z}[c_{ij}]/\sim$, where $\sim$ stands for a equivalence relation of x_{ij} given by the condition of formal group law. And we define

$$F_{univ}(x,y) = \sum c_{ij} x^i y^j$$

So for any other formal group Law $F(x,y) = \sum a_{ij} x^i y^j \in R[\![x,y]\!]$ over a ring R, we define a ring morphism

$$f : L \to R, c_{ij} \mapsto a_{ij}$$

Clearly we have $f^* F_{univ} = F$ ■

Theorem A.1.4: (**Lazard's Theorem**) $L \cong \mathbb{Z}[t_1, t_2, \cdots]$, where each t_i has degree 2i.

Proof: See[58]. ■

Hights of Formal Groups

Definition A.1.5: Let $f(x,y) \in R[\![x,y]\!]$ be a formal group law over a commutative ring R. For every non-negative integer n, we define the n-series $[n](t) \in R[\![t]\!]$ as

（1）If $n = 0$,we set $[n](t) = 0$.

（2）If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

It can be prove that the n-series $[n](t)$ of a formal group law determine a homomorphism from f to itself, i.e., we have $f([n](x), [n](y)) = [n]f(x,y)$.

Proposition A.1.6: Suppose that R is a commutative ring, $p = 0$ in R and f is a formal group law over R, then s $p[t]$ is either 0 or $\lambda t^{p^n} + O(t^{p^n+1})$ for an integer $n > 0$.

Proof: See[54]Lecture 12. ■

Definition A.1.7: Suppose we have a commutative ring R and F is a formal group law over R . Let v_n denote th coefficient of $t^P n$ in the p-series of F. We call F has height $\leq n$ if $v_i = 0$ fro $i < n$, and we call f has height exactly n if it has height $\leq n$ and the coefficient v_n is invertible.

Example A.1.8: For the formal group law $F(x,y) = x+y+xy$, its n-series is $[n](t) = (1+t)^n - 1$. If $p = 0$ in R, then $[p](t) = (1+t)^p - 1 = t^p$, so F is height 1.

Example A.1.9: For the formal group law $F(x,y) = x+y$, if $p = 0$ in R. Its p-series $[p](t) = 0$, so f has infinite height.

There is a geometric interpretation of the height of a formal group. Let $\mathcal{F} : \mathrm{Alg}_R \to$ Ab be a height n formal group. Then $\mathcal{F}[p] = \ker(\mathcal{F} \xrightarrow{p} \mathcal{F})$ is representable by a finite flat group scheme of rank p^n. And moreover, if we assume $\mathcal{F}$ is defined by a formal group law $f(x,y)$ whose p-series $[p](t) = \lambda p^{t^n} + \cdots$ where λ is invertible. Then we have $\mathcal{F}[p] = \mathrm{Spec} R[[t]]/(\lambda t^{p^n} + \cdots)$.

Example A.1.10: We consider the formal multiplicative group $\mathcal{F}$, then $\mathcal{F}[p]$ is exactly the group scheme μ_p, defined by $\mu_p(A) = a \in A, a^p = 1$, and we have $\mu_p = \mathrm{Spec} R[a]/(a^p - 1)$ which has rank p.

A.2 Complex Oriented Cohomology Theories

Suppose that E is a general cohomology theory, we say E is multiplicative if there is a map $E^p(X) \otimes E^q(Y) \to E^{p+q}(X)$ for every topological space and every integers p, q.

Definition A.2.1: A multiplicative cohomology theory E is even periodic if $E^i(pt) = 0$ whenever i is odd and there exists $\beta \in E^{-2}(pt)$ such that multiplication with β induces an isomorphism $E^n(-) \cong E^{n-2}(-)$ for all n.

Definition A.2.2: A complex orientation of E is a natural, multiplicative, collection of Thom classes $\mathcal{U}_V \in \widetilde{E}^{2n}(Th(V))$ for all complex vector bundles $V \to X$, where $\dim_{\mathbb{C}} V = n$, and satisfying the following condition

- $f^*(\mathcal{U}_V) = \mathcal{U}_{f^*V}$ for $f : Y \to X$
- $\mathcal{U}_{V_1 \otimes V_2} = \mathcal{U}_{V_1} \circ \mathcal{U}_{V_2}$.
- For any $x \in X$, the class $\mathcal{U}_V$ maps to 1 under the composition

$$\widetilde{E}^{2n}(Th(V)) \to \widetilde{E}^{2n}(Th(V_x)) \cong \widetilde{E}^{2n}(S^{2n}) \cong E^0(pt).$$

We know that $E^2(\mathbb{C}P^\infty)$ is set of morphisms of spectrum $e : \Sigma^{\infty-2}(\mathbb{C}P^\infty) \to E$. If there is a unit map $e : \mathbb{S} \to E$, then E is complex orientable if the map e factor as a composition

$$\mathbb{S} \simeq \Sigma^{\infty-2}\mathbb{C}P^1 \to \Sigma^{\infty-2}\mathbb{C}P^\infty \to E$$

By using the Atiyah-Hirzebruch spectral sequence $H^p(X, E^q(*)) \Rightarrow E^{p+q}(X)$. The complex orientation of E determines an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong E^*(*)[\![t]\!] = (\pi_* E)[[t]]$$

for some generator $t \in E^2(*)$. Furthermore given such an isomorphism, is equivalent to a complex orientation. In particularly, any even periodic theory is complex orientable.

We know that there is a multiplication map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

(We can view $\mathbb{C}^\infty$ as function space $\mathbb{C}[x]$, then we get a commutative multiplication on $\mathbb{C}P^\infty$). Still using the Atiyah-Hirzebruch spectrtal sequence, we can get $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (\pi_* E)[[x, y]]$. We then get a map

$$(\pi_* E)[[t]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (\pi_* E)[[x, y]]$$

We let $f(x, y) \in (\pi_* E)[[x, y]]$ denote the image of t under this map. It is easy to prove that $f(x, y)$ is a formal group law.

Complex Cobordism Spectrum MU

Let $EU(n) \to BU(n)$ be the universal bundle over the classifying space $BU(n)$, then we define spectrum $MU(n)$ to be $\Sigma^{\infty-2n} BU(n)/BU(n-1)$ which $BU(n)/BU(n-1)$actually is $Th(EU(n))$, the Thom space of $EU(n)$.

And we define a new spectrum $MU = \lim MU(n)$. This spectrum MU is called the complex cobordism. The n-th homotopy group is just the bordsim group of n-dimensional complex manifold. MU admits a E_∞ structure since there is a diagram commutes up to homotopy for any complex oriented spectrum.

$$\begin{array}{ccc} MU(m) \times MU(n) & \longrightarrow & MU(m+n) \\ \downarrow & & \downarrow \\ E \otimes E & \longrightarrow & E \end{array}$$

Theorem A.2.3: (**Quillen's theorem**) MU is the universal complex oriented cohomology theory, i.e., $L \cong \pi_* \mathrm{MU}$

Proof: See[4]. ■

Construction of Even Periodic Cohomology Theories

Suppose that E is a complex oriented cohomology theory. Then $\pi_* E$ is an algebra over the Lazard ring $L = \pi_* MU$. So it is natural to ask a question: if we have a ring map $L \to R$, how can we construct a general cohomology theory E which is complex oriented such that $R = \pi_* E$. There is a natural way to construct such cohomology theory by defining

$$E_*(X) = MU_*(X) \otimes_{\pi_* MU} R = MU_*(X) \otimes_L R$$

However the axiom of cohomology theory require some exactness of a certain sequence, but the functor $- \otimes_L R$ general doesn't preserve exact sequence. If R is flat over L, then there is no problem. But this condition is too restrictive，because the Lazard ring is too big. there is a weaker condition proved by Landweber.

Theorem A.2.4: (**The Landweber Exact Functor Theorem**) Let M be a module over the Lazard ring L. Then M is flat over $\mathcal{M}_{FG}$ if and only if for every prime number p, the elements $v_0 = p, v_1, v_2, \cdots \in L$ form a regular sequence for M.

Proof: See[5]. ■

Example A.2.5: Let $R = \oplus_n L^{\otimes n} = \oplus_n L \cong L[\beta^{\pm 1}]$, and $L \to R = L[\beta^{\pm 1}]$ be the obvious map. We can define a cohomology theory E_R

$$(E_L)_*(X) = MU_*(X) \otimes_L L[\beta^{\pm}] \cong MU_*(X)[\beta^{\pm 1}].$$

This spectrum is called the **periodic complex bordism spectra** and is denoted by MP.

Example A.2.6: Suppose that R is a commutative ring over L and R is an invertible L module, and Let f be a formal group law over the graded commutative ring $\oplus_n R^{\otimes n}$ such that associated ring morphism $L \to \oplus_n R^{\otimes n}$ satisfying Landweber's criterion. Then we get a homology theory

$$(E_R)_*(X) = MU_*(X) \otimes_L R[\beta^{\pm 1}] \simeq MP_*(X) \otimes_L R.$$

In particular, we have $(E_R)_0(X) \otimes_L R = MU_{\text{even}}(X) \otimes_L R$.

A.3 Morava E-theories and Morava K-theories

Lubin-Tate Theory

Definition A.3.1: Suppose that k is filed , an infinitesimal thickening of k is a surjective map $\phi : A \to k$ of commutative rings and its kernel $m_A = \ker(\phi)$ satisfying: $m_A^n = 0$ for $n \gg 0$ and m_A^n / m_A^{n+1} is a finite dimensional k-vector space .

Definition A.3.2: (**Deformation of formal groups**) Suppose that G_0 be a formal group over a perfect field k and $\text{char}(k) = p$, a deformation of G_0 to R is a triple (G, i, Ψ) such that $G \in \text{FG}(R)$, $i : k \to R/m$ is an isomorphism and $\Psi : \pi^* G \cong i^* G_0$ is an isomorphism of formal groups over R/m.

Theorem A.3.3: (**Lubin-Tate**) There is a universal formal group G over $R = W(k)[[v_1, \cdots, v_{n-1}]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijective map

$$\text{Hom}_{/k}(R, A) \to \text{Def}(A).$$

Proof: See[59]. ■

Morava E-Theories

Let k be a perfect field and $\text{char} k = p$, f is a height n formal group law. By Lubin -Tate's theorem, the deformation of by is classified by the ring $R = W(k)[\![v_1, \cdots, v_{n-1}]\!]$. Notice that this universial deformation over R is Landweber-exact: the sequence $v_0 = p, v_1, \cdots, v_{n-1}$ is regular, and v_n has invertible image in $R/(v_1, \cdots, v_n)$. So using the construction in last section, there is a even periodic spectrum $E(n)$ with

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$$

where β has degree 2. It's called **Morava E-theory**. The cohomology theory $E(n)$ not only depends on n, but also a choice of k and f .

Theorem A.3.4: (**Goerss-Hopkins-Miller**[14]) Those spectra E(n) are E_∞ ring spectra.

Morava K-Theories

Suppose that p is a prime number, we can consider the p-local complex cobordism spectrum $MU_{(p)}$ whose homotopy groups are $\pi_* MU_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, \cdots,]$, and we may assume that $v_i = t_{p^i - 1}$ for each $i > 0$.

For $k \in \mathbb{Z}$, write $M(k)$ for the cofiber of the map $\Sigma^{2k} MU_{(p)} \to MU_{(p)}$ given by the multiplication by t_k. One can prove that each $M(k)$ admits a unital and homotopy associative multiplication .

Let $K(n)$ denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum $K(n)$ is called **Morava K-theory**. It is obvious that the homotopy groups of $K(n)$are

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n - 2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

where v_n has degree $2(p^n - 1)$.

Elliptic Cohomology

The elliptic curve is an very important object in arithmetic geometry. It is the most nontrival example in algebraic geometry. But it still can gives us some interesting things. One can see[60] for information of elliptic curves and[32] for the moudli stack and level structures of the elliptic curves. If we do completetion for an elliptic curve, then we get an one dimensional formal group. Does this formal group can give us a good cohomology theory.

Definition A.3.5: An elliptic cohomology theory is a generalized cohomology theory E, which is representated by a spectrum E which satisfies.

（1）E is an even periodic spectrum.

（2）There exists a elliptic curve C over $\pi_0 E$.

（3）There is an isomorphism of formal groups, $\phi : \mathrm{Spf}\pi_0(E^{\mathbb{C}P^\infty}) \cong \hat{C}$.

We denote this data as $(E, C\phi)$

Theorem A.3.6: (**Goerss-Hopkins-Miller Theorem** [14]) There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}$ for the etale topology. For any étale morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}$ there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0 \mathcal{O}_{tmf}(f) = R$, and C_f is the generalized elliptic curve over R classified by f.

Let $Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}} \to \overline{\mathcal{M}})$, the spectrum topological modular forms.

Let $TMF = \mathcal{O}_{tmf}(\mathcal{M} \to \overline{\mathcal{M}})$, the periodic spectrum of topological modular forms

Let $tmf = \tau_{\geq 0}\mathcal{O}_{tmf}(\bar{\mathcal{M}}_{ell})$ be the connect cover of Tmf.

We know that the modular forms can be viewed as global sections of the moduli stack of elliptic curve over complex plane $\mathbb{C}$. And it is easy to see that if we take homotopy group of the topological modular forms , then we can get the classical modular forms.

The construction of topological modular forms is complicated, one can see[61] for more details.

A.4 Chromatic Localizations

Suppose that we have a spectrum E , a spectrum F is called E-acyclic if $F \otimes E$ is 0, we denote G_E the collection of E-acyclic spectra. And we say spectrum is E-local if every map for each $Y \in G_E$, the map $Y \to X$ is nullhomotopic. For each $X \in$ Sp, we have a cofiber sequence

$$G_E(X) \to X \to L_E(X).$$

where $L_E(X)$ is E-local, and $G_E(X)$ is E-acyclic. So we have define a functor

$$L_E : \mathrm{Sp} \to L_E\mathrm{Sp},$$

this functor is called **Bousfield localization**functor . And the map $X \to L_E(X)$ is determined by following two properties.

（1）The spectrum $L_E(X)$ is E-local.

（2）The map $X \to L_E(X)$ is an E-equivalence.

Example A.4.1: Bousfield Localization with respect to Morava E-theories $E(n)$, $L_{E(n)}$. And one can prove that $L_{E(n)}$ behaves like restriction to the open substack $\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.

Example A.4.2: Bousfield Localization with $K(n)$, $L_{K(n)}$. One can prove that $L_{K(n)}$ is the completion along $\mathcal{M}_{FG}^{n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.

Suppose that we have two homology theory E and E', we say they are Bousfield equivalent, if for every spectrum, the homology group $E_*(X)$ vanishes if and only if $E'_*(X)$ vanishes. It can be prove that the spectrum E(n) is Bousfield equivalent to $E(n-1)\times K(n)$. Here by convention that $E(0) \simeq H\mathbf{Q}[\beta^{\pm}]$, which is Bousfield equivalent to $H\mathbf{Q}$. This is also equivalent to say that $L_{E(n)} = L_{K(n)\times E(n-1)}$.

Definition A.4.3: Suppose that G is commutative group, then the Moore spectrum MG of G is the spectrum characterized by having the following homotopy groups:

（1）$\pi_{<0}MG = 0$;

（2）$\pi_0(MG) = G$;

（3）$H_{>0}(MG,Z) = \pi_{>0}(MG \wedge HZ) = 0$.

A basic special case of E-Bousfield localization of spectra is given by $E = MA$ the Moore spectrum of an abelian group A. For $A = \mathbb{Z}_{(p)}$, this is p-localization, for $A = \mathbb{F}_p$, this is p-completion, for $A = \mathbf{Q}$, is the rationalization of X.

Example A.4.4: The p-localization of a spectrum X:

$$L_{M\mathbb{Z}_{(p)}}X \simeq M\mathbb{Z}_{(p)} \wedge X.$$

We denote this as $L_{M\mathbb{Z}_{(p)}}X \simeq X_{(p)}$.

Example A.4.5: The p-completion of a spectrum X:

$$L_{M\mathbb{F}_p}X \simeq [\Omega M\mathbb{Z}/p^\infty, X].$$

where $\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$. We denote this spectrum as $X_p^\wedge$.

Example A.4.6: The rationalization of a spectrum X:

$$L_{M\mathbf{Q}}X = X \wedge L_{\mathbf{Q}}S^0 = X \wedge M\mathbf{Q} = X \wedge H\mathbf{Q}$$

We denote this spectrum as $X_{\mathbf{Q}}$.

Periodicity Theorem and Thick Subcategories

Definition A.4.7: Suppose that we have a p-local finite spectrum X, we say X has type n if $K(n)_*(X) \neq 0$ and $K(m)_*(X) = 0$ for $m < n$. And we let $\mathcal{C}_{\geq n}$ be the category type $\geq n$ p-local spectra which

Suppose that we have a p-local finite spectrum, and let $n \geq 1$. A v_n-self map of X is a map $f : \Sigma^k X \to X$ which satisfying:

（1）$K(n)_*X \to K(n)_*X$ is an isomorphism induced by f.

（2）For $m \neq n$, $K(m)_*X \to K(m)_*X$ which is induced by f is nilpotent.

Theorem A.4.8: (Devinatz-Hopkins-Smith[62]) For a type $\leq n$ finite p-local spectrum X, it admits a v_n self map.

Suppose that $\mathcal{C}$ is a full subcategory of finite p-local spectra. We call $\mathcal{C}$ is **thick** subcategory if it contains the final object, closed under fiber and cofiber, and is stable under retract.

Theorem A.4.9: (**Thick Subcategory Theorem**[62]) Suppose that $\mathcal{T}$ is a thick subcategory of $\mathrm{Sp}_{(p)}$. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.

The Chromatic Filtration

Let $L_n(X) = L_{E(n)(X)}$, then we have the following chromatic tower.

$$\begin{array}{ccccccccc} & & M_n(X) & & & & M_2(X) & & M_1(X) & & M_0(X) = H\mathbb{Q} \wedge X \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & L_n(X) & \longrightarrow \cdots \longrightarrow & & & L_2(X) & \longrightarrow & L_1(X) & \longrightarrow & L_0(X) = H\mathbb{Q} \wedge X \end{array}$$

where $M_n(X)$ are defined by the fiber .

$$M_n(X) \to L_n(X) \to L_{n-1}(X)$$

The following chromatic convergence theorem is proved by Hopkins-Ravenel.

Theorem A.4.10: (**Chromatic Convergence Theorem**[63]) Suppose that X is a finite spectra, then the map $X \to \lim_n L_n X$ is an equivalence.

Suppose that X is a spectrum, we say X monochromatic of height n if it is $E(n)$-local and $E(n-1)$-acyclic. We let $\mathcal{M}_n$ denote the category monochromatic of height n spectra. There is an equivalence

$$L_{K(n)} : \mathcal{M}_n \rightleftharpoons \text{K(n) local spectra} : M_n.$$

See[54]Lecture 34 for details.

A.5 Power Operations

Suppose that we have $R \in \text{CAlg}$, and $M \in \text{Mod}_R$, then we can define a free commutative R-algebra on M:

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m(M) \cong \bigvee_{m \geq 0} (M \wedge_R \cdots \wedge_R M)_{h\Sigma m}.$$

And if A is commutative R-algebra , then we have a unit map

$$\mu : \mathbb{P}_R A \to A.$$

So the question is how to build a power operation? Let us study the general case.

If A is a commutative R -algebra.

(1) We can choose a $\alpha : R \to \mathbb{P}_R^m(R) \cong R \wedge B\Sigma_m^+$

(2) For any element $x \in \pi_0 A$ which is represented by $f_x : R \to A$.

(3) We define a element $Q_\alpha(x) \in \pi_0 A$ which is represented by the following composite

$$R \xrightarrow{\alpha} \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A) \subset \mathbb{P}_R(A) \xrightarrow{\mu} A$$

So we have define a map $Q_\alpha : \pi_0 A \to \pi_0 A$. And we can also define $Q_\alpha : \pi_q A \to \pi_{q+r} A$ if

$$\alpha : \Sigma^{q+r} R \to \mathbb{P}_R^m(\Sigma^q R) \cong R \wedge B\Sigma_m^{qV_m}.$$

Example A.5.1: (Steenrod Operations) Let $H = H\mathbb{F}_2$ is the mod 2 Maclane spectrum, if A is a H-algebra, then $\pi_* A$ is a graded commutative $\mathbb{F}_2$-algebra generated by $Q^r : \pi_q A \to \pi_{q+r} A$ and satisfying relations

- $Q^r(x+y) = Q^r(x) + Q^r(y)$.
- $Q^r(xy) = \sum Q^i(x) Q^{r-i}(y)$.
- $Q^r Q^s(x) = \epsilon_{r,s}^{i,j} Q^i Q^j(x)$ if $r > 2s$, where $i \leq 2j$.

Example A.5.2: (Power Operations in K-theory) If K is the complex K-theory spectrum, and A is a p-complete K-algebra, we have Adams operations $\psi^p : \pi_0 A \to \pi_0 A$, they satisfying relations:

- $\psi^p(x+y) = \psi^p(x) + \psi^p(y)$.
- $\psi^p(x) \equiv x^p \mod p$.
- $\psi(xy) = \psi(x)\psi(y)$.

APPENDIX B HOMOTOPY COHERENT MATHEMATICS

We will review basic setting of homotopy coherent mathematics, including ∞-categories, homotopy limits and homotopy colimits. Then we give an introduction of higher algebra to help readers being familiar with the $\mathbb{E}_\infty$-ring context.

B.1 Fundamental Language of ∞-Categories

Definition B.1.1: A category $\mathcal{C}$ is called a simplicial category if mapping spaces of any pairs of objects are simplicial sets.

If $\mathcal{C}$ is a simplicial category, we can define new category $|\mathcal{C}|$as

（1）Objects of $|\mathcal{C}|$ are objects of $\mathcal{C}$.

（2）$\mathrm{Map}_{|\mathcal{C}|}(X,Y) = |\mathrm{Map}_{\mathcal{C}}(X,Y)|$.

Definition B.1.2: Suppose that $\mathcal{C}$ be a simplicial categories, its homotopy categories $h\mathcal{C}$ is defined by

（1）Objects $h\mathcal{C}$ are objects of $\mathcal{C}$

（2）For $X, Y \in \mathcal{C}$, then we define $\mathrm{Map}_{h\mathcal{C}}(X,Y) = \pi_0|\mathrm{Hom}(X,Y)|$

Let

$$P_{i,j} = \{I \subseteq [i,j] | i,j \in I\}$$

We now define a category $C[\Delta^n]$ as follows:

- objects: the numbers $0, q, \cdots, n$
- morphisms

$$\mathrm{Map}_{C[\Delta_n]}(i,j) = \begin{cases} NP_{i,j}, & \text{if} \quad i \leq j \\ \emptyset, & \text{if} \quad i > j \end{cases}$$

so there is a functor $C[\Delta^\bullet] : \Delta \to s\mathrm{Cat}$

Definition B.1.3: The **homotopy coherent nerve** $N_\Delta(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is the simplicial set

$$N_\Delta(\mathcal{C})_\bullet = \hom_{s\mathrm{Cat}}(C[\Delta^\bullet], \mathcal{C}).$$

So N_Δ is actually a functor form simplicial categories to simplicial sets

$$N_\Delta : s\mathrm{Cat} \to s\mathrm{Set}.$$

On the other side, we can extend the cosimplicial object $\Delta \to s\mathrm{Cat} : [n] \mapsto C[\Delta^n]$ to a colimit-preserving functor $C[-] : s\mathrm{Set} \to s\mathrm{Cat}$. For a simplicial set, we define

$$C[X] = \mathrm{colim}_{\Delta/X} C[-] \circ p$$

where p is the canonical functor.

Theorem B.1.4: There is an adjunction

$$C[-] : s\mathrm{Set} \rightleftharpoons s\mathrm{Cat} : N_{\Delta}$$

Proposition B.1.5: Suppose that $\mathcal{C}$ be a simplicial category such that for any two objects $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex. We have the simplical nerve $N(\mathcal{C})$ is an ∞-category.

The counit of this adjunction can be described by the following theorem.

Theorem B.1.6: If $\mathcal{C}$ is a topological category. Then the counit map

$$|\mathrm{Map}_{C|N(\mathcal{C})|}(X, Y)| \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence of topological spaces.

∞-Categories

We recall that a kan complex is a simplical set which satisfies for $0 \leq k \leq n$ and any morphism $f : \Lambda^n_k \to X$, there exists a morphism $f' : \triangle^n \to X$, such that the composition of $i : \Lambda^n_k \to \triangle^n$ and f' is equal to f, this means that there exists a commutative triangle

$$\begin{array}{ccc} \Lambda^n_l & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{f'}{\nearrow} & \\ \triangle^n & & \end{array}$$

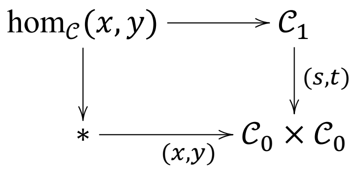
Definition B.1.7: An ∞-catgory is a simpicial set X which satisfies for any $0 < k < n$ and any morphism $f : \Lambda^n_k \to X$, there exists a morphism $f' : \triangle^n \to X$, such that the composition of $i : \Lambda^n_k \to \triangle^n$ and f' is equal to f, this means that there is a commutative triangle

$$\begin{array}{ccc} \Lambda^n_l & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{f'}{\nearrow} & \\ \triangle^n & & \end{array}$$

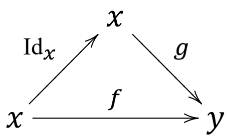
And this is also been called a weak Kan complex.

Given an ∞-category $\mathcal{C}$, objects are the vertices $x \in \mathcal{C}_0$, and the morphism are the 1-simplicies $f \in \mathcal{C}_1$. The face map $s = d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ is the source map, and $t = d_0$:

$\mathcal{C}_1 \to \mathcal{C}_0$ is the target map. We often write $f : x \to y$, if $s(f) = x$ and $t(f) = y$. We define mapping space of $\hom_{\mathcal{C}}(x, y)$ from x to y to be the fiber



Definition B.1.8: Suppose that we have $f, g : x \to y$ in an ∞-category $\mathcal{C}$, we say f and g are homotopic ($f \simeq g$) if there is a 2 simplex $\sigma : \Delta \to \mathcal{C}$ whose boundary $\partial\sigma = (d_0\sigma, d_1\sigma, d_2\sigma)$ is given by (g, f, id_x), i.e., we have the following diagram



Suppose that we have a ∞-category $\mathcal{C}$, then we can define a new category $h\mathcal{C}$ whose objects are the same as $\mathcal{C}$, and whose morphism are the homotopy class of morphisms in $\mathcal{C}$. Compositions and identities are given by

$$[g] \circ [f] := [g \circ f] \quad \text{and} \quad id_x := [id_x] = [s_0 x].$$

Construction of ∞-categories

Definition B.1.9: Suppose that we have two simplicial sets K and L, the join $K \star L$ of K and L is the simplicial set defined by the formula

$$(K \star L)_n = K_n \cup L_n \bigcup_{i+1+j=n} K_i \times L_j.$$

We have the following properties of joins:

（1）The partial join functors $K \star (-) : s\mathrm{Set} \to s\mathrm{Set}_{K/}$ and $(-) \star L : s\mathrm{Set} \to s\mathrm{Set}_{L/}$ preserves colimits.

（2）$\Delta^i \star \Delta^j \cong \Delta^{i+j+1}$

Example B.1.10: And it is not hard to prove that the nerve functor is compatible with the join constructions, i.e.,we have a natural isomorphism

$$N(A) \star N(B) \cong N(A \star B), A, B \in \mathrm{Cat}$$

If K is an arbitrary simplicial set and $L = \Delta^0$, then we define the right cone (or called cocone) on K to be $K^{\rhd} = K \star \Delta^0$. And the left cone (or called cone) is defined as $L^{\lhd} = \Delta^0 \star L$

Proposition B.1.11: [29]Proposition 1.2.8.3 Suppose that we have two ∞-categories $\mathcal{C}$ and

$\mathcal{D}$ ar, then the join $\mathcal{C} \star \mathcal{D}$ is also an ∞-category.

Proposition B.1.12: ([29]Proposition 1.2.9.2) Suppose that we have two simplicial sets A and B, let $p : A \to B$ be a functor, then there exists a simplicial set $B_{/p}$ such that there is a natural bijection

$$\mathrm{Fun}(C, B_{/p}) \cong \mathrm{Fun}_p(C \star A, B)$$

where the right-hand side denote those $C \star A \to B$, making the triangle

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow^{p} \\ C \star A & \longrightarrow & B \end{array}$$

commute.

B.1.1 Straightening and Unstraightening

We know that the Grothendieck Construction establish an equivalence between $\mathcal{C}\mathrm{at}(Set)$-valued functor on $\mathcal{C}^{op}$ and categories which are fibered over $\mathcal{C}$. The St^{+}_{ϕ} functor establish an ∞-version of this equivalence but replace $\mathcal{C}$ by a simplicial set S and replace $\mathcal{C}$ by $\mathrm{Cat}^{\triangle}_{\infty}$

Suppose that we have a simplicial set S and $\mathcal{C}$ is simplicial category, let $C[S]$ denote the coherent nerve of S. Suppose that $\phi : C[S] \to \mathcal{C}^{op}$ is functor between these two simplicial categories. Given an object $X \in (\mathrm{Set}_{\triangle})_{/S}$. Let v denote the cone point of $X^{\triangleright}$. We can view the simplicial category

$$\mathcal{M} = \mathfrak{C}[X^{\triangleright}] \coprod_{\mathfrak{C}[X]} \mathcal{C}^{op}$$

as a correspondence from $\mathcal{C}^{op}$ to v. Then we can define a simplicial functor

$$\begin{array}{rccc} St_{\phi}X : & \mathcal{C} & \to & \mathrm{Set}_{\triangle} \\ & C & \mapsto & \mathrm{Map}_{\mathcal{M}}(C, v) \end{array}$$

We can regard St_{ϕ} as a functor from $(\mathrm{Set}_{\triangle})_{/S}$ to $(\mathrm{Set}_{\triangle})^{\mathcal{C}}$.We refer to St_{ϕ} as the straightening functor associated to ϕ. In the special case $\mathcal{C} = C[S]^{op}$ and ϕ is the identity map, we will write St_S instead of St_{ϕ}.

Theorem B.1.13: [29]Theorem 2.2.1.2 There is an Quillen adjunction.

$$St_{\phi} : s\mathrm{Set}_{/S} \rightleftarrows s\mathrm{Set}^{\mathcal{C}} : Un_{\phi},$$

n where $s\mathrm{Set}_{/S}$ is endowed with the contravariant model structure, and $s\mathrm{Set}^{\mathcal{C}}$ is endowed with the projective model structure. If ϕ is an equivalence, then we have (St_{ϕ}, Un_{ϕ}) is also an Quillen equivalence.

B.1.2 Marked Case

Suppose that we have a simplicial set S and $\mathcal{C}$ is a simplicial category, let $C[S]$ denote the coherent nerve of S. Suppose that $\phi : C[S] \to \mathcal{C}^{op}$ is functor between these two simplicial categories. Let $(X, \mathcal{E})$ be an object of $(\mathrm{Set}_{\triangle}^{+})_{/S}$. Then we can define

$$\begin{array}{rccc} St_{\phi}^{+}(X,\mathcal{E}): & \mathcal{C} & \to & \mathrm{Set}_{\triangle}^{+} \\ & C & \mapsto & ((St_{\phi}X)(C), \mathcal{E}_{\phi}(C)) \end{array}$$

where $\mathcal{E}_{\phi}(C)$ is the set of all edges of $(St_{\phi}X)(C)$ having the form

$$G^{*}\widetilde{f}$$

$f : d \to e$ is a marked edge of X, giving rise to an edge $\widetilde{f} : \widetilde{d} \to F^{*}\widetilde{e}$ in $(St_{\phi}X)(D)$, and G belongs to $\mathrm{Map}_{\mathcal{C}^{op}}(C, D)_1$

- $St_{\phi}^{+} : (\mathrm{Set}_{\triangle}^{+})_{/S} \to (\mathrm{Set}_{\triangle}^{+})^{\mathcal{C}}$ preserve colimits.
- St_{ϕ}^{+} has a right adjoint $Un_{\phi}^{+} : St_{\phi}^{+} \to (\mathrm{Set}_{\triangle}^{+})_{/S}$
- $(St_{\phi}^{+}, Un_{\phi}^{+})$ determine a Quillen adjuction $(\mathrm{Set}_{\triangle}^{+})_{/S} \rightleftarrows (\mathrm{Set}_{\triangle}^{+})^{\mathcal{C}}$

Theorem B.1.14: [29]Theorem 3.2.0.1 There is an Quillen adjunction

$$St_{\phi}^{+} : (\mathrm{Set}_{\Delta}^{+})/S \leftrightarrows (\mathrm{Set}_{\Delta}^{+})^{\mathcal{C}} : Un_{\phi}^{+}.$$

where $(\mathrm{Set}_{\Delta}^{+})/S$ is endowed with the Cartesian model structure and the category $(\mathrm{Set}_{\Delta}^{+})/S$ is endowed with the projective model structure). Moreover if ϕ is an equivalence, then $(St_{\phi}^{+}, Un_{\phi}^{+})$ is a Quillen equivalence.

B.2 Limits and Colimits

We recall that in a ordinary category $\mathcal{C}$, an object $X \in \mathcal{C}$ is final if the hom set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ consists of only one point for any objects $Y \in \mathcal{C}$. And an object $X \in \mathcal{C}$ is initial if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consists of only one point for any objects $Y \in \mathcal{C}$.

Definition B.2.1: Suppose that $\mathcal{C}$ is a simplicial set. An object $X \in \mathcal{C}$ is final if it is final $h\mathcal{C}$,

Definition B.2.2: Suppose that K is a simplicial set and for any ∞ category $\mathcal{C}$. A limit of a functor $p : K \to \mathcal{C}$ is a final object in $\mathcal{C}_{/p}$. A colimit of a diagram $p : K \to \mathcal{C}$ is an initial object in $\mathcal{C}_{p/}$.

A ∞-category is complete is admits all limits of all small diagrams, is cocomplete if it admits all comlimits of all small diagrams.

B.3 Presentable ∞-Category

Definition B.3.1: Let $\mathcal{C}$ be an ∞-category and κ a regular cardinal. We say $\mathcal{C}$ is κ-accessible if $\mathcal{C}$ admits small κ-filtered colimits and contains an essentially small full subcategory $\mathcal{C}" \subseteq \mathcal{C}$ which consists of κ-compact objects and generate $\mathcal{C}$ under small κ-filtered colimits.

Definition B.3.2: An ∞-category $\mathcal{C}$ is presentable if $\mathcal{C}$ is accessible and admits small colimits.

Definition B.3.3: An adjunction between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a map $q : \mathcal{M} \to \Delta^1$ which is both a Cartesian fibration and a coCartesian fibration together with equivalences $C \to \mathcal{M}_{\{0\}}$ and $D \to \mathcal{M}_{\{1\}}$.

Assume M be an adjunction between C and D and let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be functors associated to M. In this case, we will say that f is left adjoint to g and g is right adjoint to f.

Theorem B.3.4: [29]Corollary 5.5.2.9 For presentable ∞-categories, we have following criterion for adjunctions

- A functor between presentable ∞-categories has a right adjoint if and only if it preserves small colimits.
- A functor between presentable ∞-categories has a left adjoint if and only if it preserves small limits and is accessible.

B.4 Stable ∞-Categories

Definition B.4.1: Suppose that $\mathcal{C}$ is an ∞-category, we say $\mathcal{C}$ is stable if we have $\mathcal{C}$ has a zero object, and satisfying Every morphism in C have a cofiber and a fiber, a triangle in C is a fiber if and only if it is a cofiber.

We let M^{Σ} denote the full subcategory of $Fun(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

If $\mathcal{C}$ admits cofibers, the evaluation at the initial vertex $M^{\Sigma} \to \mathcal{C}$ is a trivial fibration[29]. Let $s : \mathcal{C} \to M^{\Sigma}$ be a section of it. Let $e : M^{\Sigma} \to \mathcal{C}$ be the evaluation at the final vertex. Then the composition of $e \circ s$ is a functor from $\mathcal{C}$ to itself. And we call this suspension functor and denote it by $\Sigma : \mathcal{C} \to \mathcal{C}$. Similarly, If $\mathcal{C}$ admits fibers, then the same argument show that for the evaluation at the final vertex, there is also a functor $\Omega : \mathcal{C} \to \mathcal{C}$ and we call it the loop functor.

If $\mathcal{C}$ is a stable ∞-category and $n \geq 0$, We let

$$X \mapsto X[n]$$

denote the nth power of the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$. If $n \leq 0$, we let $X \mapsto X[n]$ denote the $(-n)$th power of the loop functor Ω. Let $\mathcal{C}$ be a stable ∞-category, Then the suspension functor $X \mapsto X[1]$ and the distinguished triangle defined above endowed $h\mathcal{C}$ with a triangulated category.

Definition B.4.2: Suppose tat we have two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and F is a functor between them, we will say f is excisive if it maps pushout to pullbacks.

Definition B.4.3: Suppose that $\mathcal{C}$ is an ∞-category. A functor $F : \mathcal{S}_*^{fin} \to \mathcal{C}$ is called a spectrum object if it satisfies the following two conditions:

- F is excisive.
- F(*) is terminal.

A spectrum is a spectrum object in the ∞-category of spaces

Definition B.4.4: A stable homotopy theory is a presentable symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{I})$ and it satisfies the conditions: all tensor product commutes with all colimits.

So a stable homotopy theory $(\mathcal{C}, \otimes, \mathbb{I})$ has the following properties

（1）Ho($\mathcal{C}$) is a symmetric monoidal triangulated category.

（2）There is an equivalence

$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega.$$

（3）We can define homtopy groups

$$\pi_n E := [\Sigma^n \mathbb{I}, E].$$

（4）We can define homology groups and cohomology groups

$$E_n(F) := \pi_n(E \otimes F),$$

$$E^m(F) := \pi_n(\mathrm{Map}(F, E)).$$

Example B.4.5: The derived category $D(R)$ of a discrete ring R with the derived tensor product admits a structure of stabel homotopy theory.

Example B.4.6: The ∞-categroy Sp of spectra.

Example B.4.7: The ∞-categroy Mod_R of modules over an E_∞-ring spectrum R.

Example B.4.8: Let X be a scheme (or algebraic stack). Then the quasi-coherent shaves complexes can admits an structure of stable homotopy theory.

Example B.4.9: Let K be an ∞-category, and $\mathcal{C}$ is a stable homotopy theory. Then $\mathrm{Fun}(K, \mathcal{C})$ admits a nature structure of stable homotopy theory. If $K = BG$, then this functor category are those objects in $\mathcal{C}$ with a G-action.

B.5 Higher Categorical Algebra

Operads

For the convenience of discussion, we first recall some setting in simplicial set theory

（1）A morphism $\alpha : \langle n\rangle \to \langle k\rangle$ in Fin_* is insert if $\alpha^{-1}(i)$ is a singleton for every $1 \leq i \leq k$

（2）A morphism $\alpha : \langle m\rangle \to \langle n\rangle$ in Fin_* is active if $\alpha^{-1}(pt)$ is a singleton (necessarily the basepoint).

（3）A morphism $\alpha : [n] \to [k]$ in Δ is conves if it is injective and the image $\mathrm{im}(\alpha) \subseteq [k]$ is convex,i.e., the image is given by the interval $[\alpha(0), \alpha(n)]$.

An operad is a gadget used to describe algebraic structures in symmetric monoidal categories.

Definition B.5.1: Let V be a symmetric monoidal category. A operad in V consists of objects $F(n)$ of V, $n \in \mathbf{N}$ equipped with the following extra structure.

- Right actions of symmetric groups $\rho_n : S_n \to \mathrm{Hom}(F(n), F(n))$;
- A unit $e : I \to F(1)$
- Composition operations

$$F(k) \otimes F(n_1) \otimes F(n_2) \otimes \cdots \otimes F(n_k) \to F(n_1 + \cdots + n_k)$$

These data are subject to obvious identities such as associativity and unitality of composition, and compatibility of composition with symmetric group actions. For example, the unit laws say that the evident composite

$$F(n) \cong I \otimes F(n) \overset{e\otimes 1}{\longrightarrow} F(1) \otimes F(n) \overset{comp}{\longrightarrow} F(n)$$

and

$$F(n) \cong F(n) \otimes I^{\otimes n} \overset{1\otimes e^{\otimes n}}{\longrightarrow} F(n) \otimes F(1)^{\otimes n} \overset{comp}{\longrightarrow} F(n)$$

are the identity map. Compatibility with symmetric group actions means that for each element $\sigma \in S_n$, the composition operation

$$F(k) \otimes \bigotimes_{i=1}^{k} F(n_i) \to F(n_1 + \cdots + n_k)$$

coequalizes a pair of automorphisms

$$\rho(\sigma) \otimes 1, 1 \otimes \lambda(\sigma) : F(k) \otimes \bigotimes_{i=1}^{k} F(n_i) \rightrightarrows F(k) \otimes \bigotimes_{i=1}^{k} F(n_i)$$

where σ acts on the big tensor product on the left by permuting tensor factors in the obvious way. If V has suitable colimits, this condition could be expressed in terms of tensor products over S_n.

Definition B.5.2: An F-algebra structure on an object v in V consists of a collection of maps

$$F(n) \otimes v^{\otimes n} \to v$$

We intuitively write this map as

$$\theta \otimes x_1 \otimes \cdots \otimes x_n \mapsto \theta(x_1, \ldots, x_n)$$

so that the element of $F(n)$ are interpreted as n-ary operations on v.

Definition B.5.3: Let C be a set, called the set of colours. Then a coloured operd is

- for each $n \in N$ and each $(n+1)$-tupe $(c_1, \cdots, c_n, c)$, there is an object $P(c_1, \cdots, c_n; c) \in V$;
- for each $c \in C$ a morphism $1_c : I \to P(c : c)$ in V - the identity on c;
- for each $(n+1)$-tuple $(c_1, \cdots, c_n, c)$ and n other tuples $(d_{1,1}, \cdots, d_{1,k1}), \cdots, (d_{n,1}, \cdots, d_{n,k_n})$ a morphism$P(c_1, \cdots, c_n; c) \otimes P(d_{1,1}, \cdots, d_{1,k_1}; c_1) \otimes \cdots \otimes P(d_{n,1}, \cdots, d_{n,k_n}; c_n) \to P(d_{1,1}, \cdots, d_{n,k_n}, c)$ the composition operation;
- for all n n, all tuples, and each permutation σ in the symmetric group Σ_n a morphism

$$\sigma^* : P(c_1, \cdots, c_n; c) \to P(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c)$$

- subject to the conditions that
  - the σsform a representation of Σ_n;
  - composition operation satisfies associativity and unitality in the obvious way;
  - and is Σ_n equivariant in the evident way.

Let $\mathcal{O}$ be a colored operad. We define a category $\mathcal{O}^{\otimes}$ as follows:

- The object of $\mathcal{O}^{\otimes}$ are finite sequence of colors $X_1, \ldots, X_n \in \mathcal{O}$.
- Given two sequence of objects

$$X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathcal{O},$$

a morphism form $\{X_i\}$ to $\{Y_j\}$ is given by a map $\alpha : \langle m\rangle \to \langle n\rangle$ in Fin_*, together with a collection of morphisms:

$$\{\phi_j \in P(\{X_i\}_{i\in\alpha^{-1}(j)}, Y_j)\}$$

- Composition of morphisms in $\mathcal{O}^{\otimes}$ is determined by the composition laws on Fin_* and on $\mathcal{O}$

∞-operads

An ∞-operad is an ∞-categorical generalization of coloured operad.

Definition B.5.4: An ∞-operad is a functor $p : \mathcal{O}^{\otimes} \to N(\text{Fin}_*)$ between ∞-categories which satisfies the following conditions:

（1）For every inert morphism $f : \langle m\rangle \to \langle m\rangle$ in $N(\text{Fin}_*)$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, there exists a p-coCartesian morphism $\bar{f} : C \to C'$ lifting f, In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$

（2）Let $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ be objects, let $f : \langle m\rangle \to \langle m\rangle$ in $N(\text{Fin}_*)$ and let $Map^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the unoion of morphism which lie over f. Choose a p-coCartesion morphism $C' \to C'_i$ lying over $\rho^i : \langle n\rangle \to \langle 1\rangle$. Then the induced map

$$\text{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\leq i\leq n} \text{Map}^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}(C, C')$$

is a homotopy equivalence.

（3）For every finite collection of objects $C_1, \cdots, C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-coCartesian morphisms $C_1 \to C_i$ covering $\rho^i : \langle n\rangle \to \langle 1\rangle$.

Example B.5.5: The commutative ∞-operad $\text{Comm}^{\otimes} = N(\text{Fin}_*)$.

Example B.5.6: $N(\text{Fin}^{inj}_*)$ is an ∞-operad which denote it by $\mathbf{E}^{\otimes}_0$.

Definition B.5.7: Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be two ∞-operads. An ∞-operad map from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ and satisfying:

（1）There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{f} & \mathcal{O}'^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\mathrm{Fin}_*) & \end{array}$$

（2）The functor f carries insert morphisms in $\mathcal{O}^{\otimes}$ to insert morphisms in $\mathcal{O}'^{\otimes}$.

We say that a map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of operads if q is a categorical fibration.

Definition B.5.8: Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A map $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-categories is a coCartesian fibration of ∞-operads if

（1）$p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-categories.

（2）The composite map $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

In this cas, we say that p exhibits $\mathcal{O}^{\otimes}$ as a $\mathcal{O}$-monoidal ∞-category.

Algebras over ∞-Operads

Definition B.5.9: Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of operads, if we have a map of operads $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$. We let $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the maps of ∞-operads.

In the special case where $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and α is the identity map, we denote the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}$ by $\mathrm{Alg}_{/\mathcal{O}}$.

Symmetric Monoidal ∞-categories

Suppose that $\mathcal{M}$ is a symmetric monoidal category with monoidal product ⊗, we construct a new category $\mathcal{M}^{\otimes}$ as follows.

（1）An object in $\mathcal{M}^{\otimes}$ is a finite sequence

$$(M_1, \cdots, M_n), M_i \in \mathcal{M}, n \geq 0$$

（2）A morphism $(M_1, \cdots, M_n) \to (L_1, \cdots, L_k)$ is a pair $(\alpha, \{f_i\}_i)$ consists of a morphism $\alpha : \langle n \rangle \to \langle k \rangle$ in $\mathcal{F}in$ together with morphism

$$f_i : \bigotimes_{j \in \alpha^{-1}(i)} \to L_i, i = 1, \cdots, k.$$

and the tensor product ,unit, composition law can be recovered as before. There is an obvious projection functor $p : \mathcal{M}^{\otimes} \to \mathcal{F}in$ given by $(M_1, \cdots, M_n) \to \langle n \rangle$ and $(\alpha, \{f_i\}_i) \to \alpha$.

Proposition B.5.10: For any symmetric monoidal category $\mathcal{M}$ the functor $p : \mathcal{M}^{\otimes} \to$

$\mathcal{F}in$ is a Grothendieck opfibration. Moreover, this functor satisfies the Segal condition,i.e., the Segal maps

$$(\rho_!^1, \cdots, \rho_!^n) : \mathcal{M}^{\otimes}_{\langle n\rangle} \xrightarrow{\sim} \mathcal{M}^{\times n}, n \geq 0$$

are equivalence.

Definition B.5.11: A symmetric monoidal ∞-category is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$.

Remark B.5.12: If we don' t want to use the language of ∞-operads, then there is a equivalent definition. A symmetric monoidal ∞-category is a coCartesian fibration $p : \mathcal{M}^{\otimes} \to N(\mathcal{F}in)$ such that the Segal maps are equivalence

$$(\rho_!^1, \cdots, \rho_!^n) : \mathcal{M}^{\otimes}_{\langle n\rangle} \xrightarrow{\sim} (\mathcal{M}^{\otimes}_{\langle 1\rangle})^{\times n}, n \geq 0$$

A symmetric monoidal ∞-category $p : \mathcal{M}^{\otimes} \to N(\mathcal{F}in)$ endows the underlying ∞-category $\mathcal{M} = \mathcal{M}^{\otimes}_{\langle 1\rangle}$ with a monoidal pairing which is associative and commutative up to coherent homotopy.

Definition B.5.13: Let $p : \mathcal{M}^{\otimes} \to N(\mathrm{Fin}_*), q : \mathcal{N}^{\otimes} \to \mathcal{N}(\mathrm{Fin}_*)$ be symmetric ∞-categories and let $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ be a functor over $N(\mathrm{Fin}_*)$.

(1) The functor F is symmetric monoidal if it sends p-coCartesian arrows to q-coCartesian arrows.

(2) The functor F is lax symmetric monoidal if it sends p-coCartesian lift of insert morphism to q-coCartesian arrows.

Definition B.5.14: We define the ∞-categories of commutative algebra objects $\mathrm{Alg}_{\mathbf{E}_\infty}(\mathcal{M}^{\otimes}) = \mathrm{Fun}^{\otimes, lax}(N(Fin_*), \mathcal{M}^{\otimes})$.

B.5.1 Monoidal ∞-category

Definition B.5.15: The category $\mathrm{Assoc}^{\otimes}$ is defined as

(1) Objects: are the object of Fin_*.

(2) Morphism: a morphism form $\langle m\rangle$ to $\langle n\rangle$ consists of $(\alpha, \{\leq_i\}_{1\leq i\leq n})$, where $\alpha : \langle m\rangle \to \langle n\rangle$ is a map of pointed finite sets and $\leq_i$ is a linear ordering on the inverse image $f^{-1}(i) \subset \langle m\rangle$ for $1 \leq i \leq n$.

We let $\mathrm{Assoc} = N(\mathbf{Assoc}^{\otimes})$. It can be proved that Assoc is an ∞-operad. We know that the ordinary monoidal category can be encoded by Grothendieck opfibrations. Using this idea, we can define monoidal ∞-categories.

Definition B.5.16: Let $\mathcal{C}^{\otimes}$ be an ∞-operad with a fibration $q : \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. We let

$\mathrm{Alg}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ of ∞-operads sections of q. The ∞-categroy of associative algebra objects of $\mathcal{C}$.

Definition B.5.17: A monoidal ∞-category is a coCartesian fibration of ∞-operads $p : \mathcal{M}^{\otimes} \to \mathrm{Assoc}^{\otimes}$.

Remark B.5.18: Just like the symmetric monoidal ∞-case, if we don' t want use the language of ∞-operads. Then one can check there is a equivalent definition that is a monoidal ∞-category is a coCartesian fibration $p : \mathcal{M}^{\otimes} \to N(\Delta^{op})$ such that the Segal maps are equivalence

$$\mathcal{M}^{\otimes}_{[n]} \xrightarrow{\sim} (\mathcal{M}^{\otimes}_{[1]})^{\times n}, n \geq 0.$$

We often refer to the category $\mathcal{M} = \mathcal{M}^{\otimes}_{[1]}$ as a monoidal ∞-category.

Example B.5.19: We ha

（1）Let $\mathcal{M}$ be a monoidal category and $p : \mathcal{M}^{\otimes} \to \Delta^{op}$ be the associated Grothendieck opfibration. An application of the nerve functor yields a monoidal category

$$N(p) : N(\mathcal{M}^{\otimes}) \to N(\Delta^{op}).$$

（2）from model categorical input

We recall that a morphism $\alpha : [n] \to [k]$ in Δ is conves if it is injective and the image $\mathrm{im}(\alpha) \subseteq [k]$ is convex,i.e., the image is given by the interval $[\alpha(0), \alpha(n)]$.

Proposition B.5.20: Let $p : \mathcal{M}^{\otimes} \to \Delta^{op}$ be a monoidal structure on $\mathcal{M} = \mathcal{M}^{\otimes}_{[1]}$. Then a section $A : \Delta^{op} \to \mathcal{M}^{\otimes}$ of p that sends convex arrows to p-coCartesian arrows encodes an algebra structure on $A_{[1]} \in \mathcal{M}$. Conversely, any algebra object in $\mathcal{M}$ determines such a section of $p : \mathcal{M}^{\otimes} \to \Delta^{op}$.

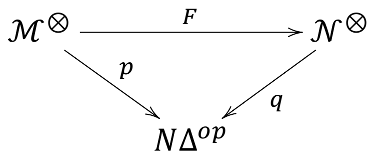
So in the ∞-category language setting, we have

Definition B.5.21: Let $p : \mathcal{M}^{\otimes} \to N(\Delta^{op})$ be a monoidal ∞-category. A section $A : N(\Delta^{op}) \to \mathcal{M}^{\otimes}$ of p is an associative algebra object in $\mathcal{M}^{\otimes}$ if A sends convex morphisms to p-coCartesian arrows in $\mathcal{M}^{\otimes}$.

Given an algebra object A in $\mathcal{M}$, the underlying object $A_{[1]}$ is endowed with a multiplciation map which is associative and unital up to coherent homotopy. In particular, an algebra object in a monoidal ∞-category defines an ordinary algebra object in the underlying homotopy category, but not conversely.

Algebra objects in monoidal ∞-categories are special case of lax monoidal functors between monoidal ∞-categories.

Definition B.5.22: Let $p : \mathcal{M}^{\otimes} \to N(\Delta^{op})$ and $q : \mathcal{N}^{\otimes} \to N(\Delta^{op})$ be monoidal ∞-categories. A lax monoidal functor $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ is a functor over $N(\Delta^{op})$, which is a commutative diagram



that sends p-coCartesian lifts of convex morphisms in $N(\Delta^{op})$ to q-coCartesian arrows. A monoidal functor $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ is a functor over $N(\Delta^{op})$ that sends arbitrary p-coCartesian arrows to q-coCartesian ones.

$\mathbb{E}_n$-Algebra

We begin by briefly recalling the notions of E_n-algebra is a closed symmetric monoidal $(\infty, 2)$-category δ which admits geometric relizations.

For an integer $k \geq 0$, we let $\square^k = (-1, 1)^k$ denote an open cube of dimension k. We will say that a map $f : \square^k \to \square^k$ is a rectilinear embedding if it is given by the formula

$$f(x_1, \ldots, x_k) = (a_1 x_1 + b_1, \ldots, a_k x_k + b_k)$$

for some real constant a_i and b_i, with $a_i \geq 0$

Definition B.5.23: We define a topological category ${}^t\mathbb{E}_k^{\otimes}$ as follows

(1) The objects ${}^t\mathbb{E}_k^{\otimes}$ are the objects $\langle n \rangle \in \mathrm{Fin}_*$.

(2) Given two objects $\langle m \rangle, \langle n \rangle$. A morphism from $\langle m \rangle$ to $\langle n \rangle$ consists of:

- A morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in Fin_*.
- For each $j \in \langle n \rangle^{\circ}$ a rectilineat embedding $\square^k \times \alpha^{-1}(j) \to \square^k$.

We let $\mathbb{E}_k^{\otimes}$ denote the nerve of the topological category ${}^tE_k^{\otimes}$. It can be that this functor $\mathbb{E}_k^{\otimes} \to N(\mathrm{Fin}_*)$ exhibits $E_k^{\otimes}$ as an ∞-operad. We refer to the ∞-operad $\mathbb{E}_k^{\otimes}$ as the ∞-operad of little k-cubes.

Definition B.5.24: Suppose that $\mathcal{C}$ is a symmetric monoidal ∞-category. An E_n-algebra in $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{A} : \mathbb{E}_k^{\otimes} \to \mathcal{C}$.

B.6 Brave New Algebra

Finiteness Conditions

Proposition B.6.1: Suppose that we have an E_1-ring R. Then we have $L\mathrm{Mod}_R$ is compactly generated ∞-category, and an object of $L\mathrm{Mod}_R$ is perfect if and only if it is compact.

Definition B.6.2: Suppose that we have a compactly generated ∞-category $\mathcal{C}$ and an object X in $\mathcal{C}$, we will say X is almost compact if $\tau_{\leq n}X$ is a compact object of $\tau_{\leq n}$ for all $n \leq 0$.

Definition B.6.3: Suppose that we have an E_1-ring R and $M \in L\text{Mod}_R$, we call M is

（1）perfect if it is a compact object of $L\text{Mod}_R$.

（2）almost perfect if $M \in (L\text{Mod}_R)_{\leq k}$ and is almost compact object of $(L\text{Mod}_R)_{\leq k}$ for an certain integer k.

（3）perfect to order n if for every filtered diagram $\{N_\alpha\}$ in $(L\text{Mod}_A)_{\leq 0}$, the canonical map $\lim_{\to\alpha}\text{Ext}^i_A(M, N_\alpha) \to \text{Ext}^i_A(M, \lim_{\to\alpha} N_\alpha)$ is injective for $i = n$ and bijective for $i \leq n$.

（4）finitely n-presented if M is n-truncated and perfect to order (n+1).

Localization, Nilpotent and Complete

Semi-Orthogonal Decomposition of Stable ∞-Categories

Definition B.6.4: Suppose that we have an ∞-category $\mathcal{C}$ and $\mathcal{D}$ be a subcategory of $\mathcal{C}$, we define two subcategories ${}^{\perp}\mathcal{D} \subseteq \mathcal{C} \supseteq \mathcal{D}^{\perp}$ as follows

（1）An object $X \in \mathcal{C}$ belongs to ${}^{\perp}\mathcal{D}$ is equivalent to say that for every object $Y \in \mathcal{D}$, $\text{Map}_{\mathcal{C}}(X, Y)$ is contractible.

（2）An object $Y \in \mathcal{C}$ belongs to $\mathcal{D}^{\perp}$ is equivalent to say that for every object $X \in \mathcal{D}$, $\text{Map}_{\mathcal{C}}(X, Y)$ is contractible.

Definition B.6.5: Suppose that we have a connective $\mathbb{E}_2$-ring R and an element t $x \in \pi_0 R$, and let $\mathcal{C}$ be a presentable R linear ∞-category. Suppose C is an object of $\mathcal{C}$, we let $C[X^{-1}]$ denote $R[x^{-1}] \otimes_R C$. We call C is x-nilpotent object if the localization $C[x^{-1}]$ vanishes. If wa have an ideal I of $\pi_0 R$. We will call this object $C \in \mathcal{C}$ is I-nilpotent object if it is x-nilpotent for each $x \in I$.

Example B.6.6: Suppose that we have a connective $\mathbb{E}_2$-ring R and let I is a finitely generated ideal of $\pi_0 R$. Suppose that M is a left R-module, then M I-nilpotent is equivalent to say that every element of $\pi_* M$ is annihilated by some power of I.

Definition B.6.7: Suppose that we have a connective $\mathbb{E}_2$-ring R and I is a finitely generated ideal of $\pi_0 R$. For any stable R-linear ∞-category $\mathcal{C}$ and C is an object of $\mathcal{C}$ is I-local if the mapping space $\text{Map}_{\mathcal{C}}(D, C)$ is contractible for every I-nilpotent object $D \in \mathcal{C}$. We let $\mathcal{C}^{Loc(I)}$ denote the full subcategory of $\mathcal{C}$ spanned by the I-local objects.

Flat

Definition B.6.8: Suppose that we have an $\mathbb{E}_\infty$-ring R and $M \in \mathrm{Mod}_R$. We will call M is a flat R-module if we have

(1) $\pi_0 M$ is flat over $\pi_0 A$, in the sense of ordinary algebraic geometry.

(2) For each n, the induces map

$$\pi_n A \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$$

is an isomorphism.

Locally Free Modules

Definition B.6.9: Suppose that we have an $\mathbb{E}_\infty$-ring R and $M \in \mathrm{Mod}_R$. We will say M is locally free of finite rank R-module if there exist an integer n, such that M is a direct summand of A^n.

We say M is locally free of rank N, if

(1) M is locally free of finite rank.

(2) the vector space $\pi_0 k \otimes_R M$ is dimension N over k for every field k and every map of E_∞-ring $R \to k$.

Étale

Definition B.6.10: Suppose that we have two $\mathbb{E}_\infty$-rings A and B and $f : A \to B$ is a map between them, we will say f is $\acute{e}$tale if f satisfying the following two conditions:

(1) B is a flat A-module.

(2) $\pi_0 f : \pi_0 A \to \pi_0 B$ is $\acute{e}$tale.

Theorem B.6.11: Suppose that we have an $\mathbb{E}_\infty$-ring A, then the map $\pi_0 : \mathrm{CAlg}_A \to \mathrm{CAlg}_{\pi_0 A}$ induces an equivalence $\mathrm{CAlg}_A^{et} \simeq \mathrm{CAlg}_{\pi_0 A}^{et}$.

Proof: See[3]Theorem 7.5.4.2. ■

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

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