Resurfaced discriminant surfaces

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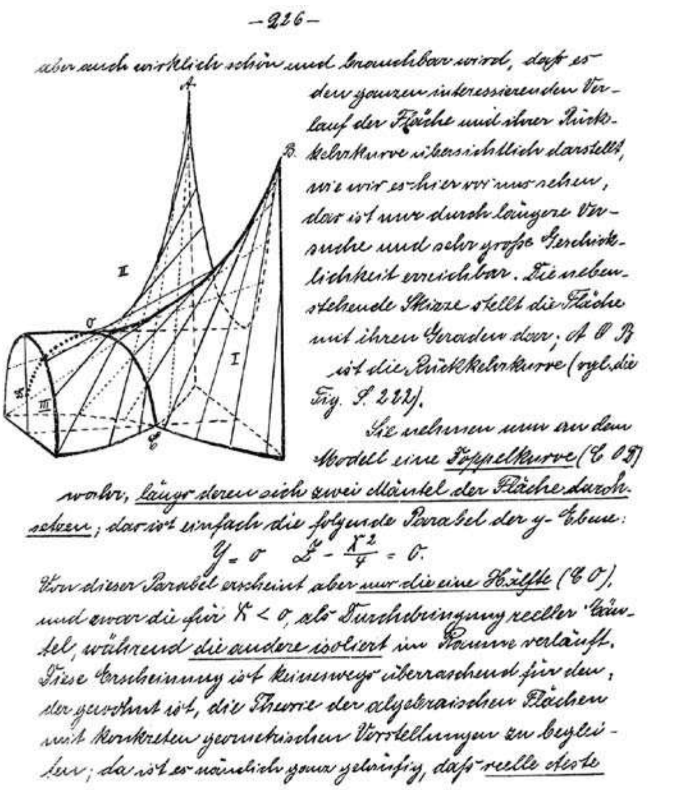

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aber auch wirklich schön und brauchbar wird, daß es den ganzen interessierenden Verlauf der Fläche und ihrer Rückkehrkurve übersichtlich darstellt, wie wir es hier vor uns sehen, das ist nur durch längere Versuche und sehr große Geschicklichkeit erreichbar. Die nebenstehende Skizze stellt die Fläche mit ihren Geraden dar; A O B ist die Rückkehrkurve (vgl. die Fig. S. 222).

Sie nehmen nun an dem Modell eine Doppelkurve (C O D) wahr, längs deren sich zwei Mäntel der Fläche durchsetzen; das ist einfach die folgende Parabel der y-Ebene:

$$y = 0 \quad z - \frac{x^2}{4} = 0.$$

Von dieser Parabel erscheint aber nur die eine Hälfte (C O), und zwar die für $x < 0$, als Durchdringung reeller Mäntel, während die andere isoliert im Raume verläuft. Diese Erscheinung ist keineswegs überraschend für den, der gewohnt ist, die Theorie der algebraischen Flächen mit konkreten geometrischen Vorstellungen zu begleiten; da ist es nämlich ganz geläufig, daß reelle Äste

The text shown here is from lectures by Felix Klein (1849–1925). The beautiful handwriting belongs to Ernst David Hellinger (1883–1950), who was asked to write down Klein's course while he worked in Göttingen as an assistant between 1907 and 1909. As Klein writes in the preface,

> Man wolle dabei von der Arbeit, die Herr Dr. Hellinger zu erledigen hatte, nicht gering denken. Denn es ist auch so noch ein weiter Weg von der durch allerlei zufällige Umstände bedingten mündlichen Darlegung des Dozenten zu der schriftlichen, hinterher noch wesentlich abgeglichenen, lesbaren Darstellung.
> [One should not underestimate the work that Dr. Hellinger had to accomplish. It is a long way from the by various coincidences influenced oral exposition by the lecturer, to the written and afterwards substantially adjusted, readable exposition.]

Hellinger's notes resulted in the textbook [Kl08] (1908). An English translation [Kl32] based on the third German edition was made by E. R. Hedrick and C. A. Noble and appeared in 1932. Although the original is more than a century old, the text is surprisingly modern and readable. Reprints and new editions have appeared quite regularly, one as recent as 2007.

The page shown here is taken from Klein's exposition concerning equations of degree 4:

$$t^4 + xt^2 + yt + z = 0.$$

Here x, y, z are real parameters, and Klein discusses the (simple) problem: how many real roots does the equation have? His approach may strike us as very geometric: every equation as given here corresponds to a point (x, y, z) in $\mathbb{R}^3$. In fact this geometric approach is not new; it is already present in the paper [Sy64] by J. J. Sylvester published in 1864.

The picture shows points (x, y, z) for which the corresponding equation has a root with multiplicity at least 2, i.e. the discriminant surface. This is by definition the set of points where the discriminant of the quartic polynomial vanishes. Klein proves in detail that the real points of this surface subdivide $\mathbb{R}^3$ into three connected parts, depending on the number of real roots of the equation being 0, 2 or 4. The picture shows more, namely that the discriminant locus contains a lot of straight lines. Indeed, it is a special kind of ruled surface. This is not a new discovery, nor was it new when Klein presented his lectures just over a century ago. For example, the same example with the same illustration also appears in 1892 in a text [Ke92] by G. Kerschensteiner, and again as §78 in H. Weber's *Lehrbuch der Algebra* (1895); see [We95]. An even older publication [MBCZ] from 1877 by the civil engineers V. Malthe Bruun and C. Crone (with an appendix by the geometer H. G. Zeuthen) includes cardboard models of the type of surface considered here. Its geometric properties are discussed in Volume 1 of É. Picard's *Traité d'analyse* (1895; pp. 290 ff.); see [Pi91].

In this note we discuss some of the fascinating history and classical theory of discriminant surfaces. In particular we focus on models of such surfaces that were designed in the period 1892–1906. Although it will not be treated here, we note that surfaces like these and in particular their singularities are still studied in algebraic geometry, differential geometry and dynamical systems (catastrophe theory, bifurcation theory, singularity theory, …).

1 String models

The mathematics department of the Johann Bernoulli Institute in Groningen (the Netherlands) owns a large collection of models of geometrical figures. Most of these models originate in Germany, where first L. Brill in Darmstadt between 1880 and 1899 and after him M. Schilling (at first in Halle a.d. Saale, later in Leipzig) between 1899 and 1935 produced hundreds of different models. In fact, the obituary [Yo28] mentions that at the end of his life, Klein proudly asserted that

> thanks to his efforts and those of his friends, the Brills and Dyck, among others, no German university was without a proper collection of mathematical models.

To some extent, the truth of this statement can still be verified: an Internet search for the phrase 'Sammlung mathematischer Modelle' finds websites of such collections in Dresden, Freiberg, Göttingen, Halle-Wittenberg, Regensburg and Vienna (Wien); moreover it mentions similar such collections at a number of other universities. The books [Fi86] and the

PhD thesis [P07] contain information about some of these models. Groningen owes its large collection mainly to the geometer Pieter Hendrik Schoute (1846–1913).

In identifying the models, catalogues such as M. Schilling's [S11] play a crucial role. We will discuss here three string models from the Groningen collection. It turns out that these models are not contained in the Schilling Catalogue, although they are closely related to Schilling's Series XXXIII. Fifteen years before the appearance of this series, they were designed by Schoute in Groningen and presented by him in Amsterdam on 27 May 1893 at the monthly physics meeting of the royal Dutch academy of sciences KNAW.

Series XXXIII of Schilling's Catalogue also consists of three string models. Two of these were designed by Roderich Hartenstein, a student of Klein, as part of his "Staatsexamen" in Göttingen in 1905/06. In fact, Klein's discussion shown above reviews Hartenstein's work. The third model in Series XXXIII was designed by Mary Emily Sinclair, a student of Oskar Bolza (who himself was also a student of Klein) as part of her Master's thesis at the University of Chicago in 1903. We will elaborate on the relation between Schoute's models and the models in Schilling's Series XXXIII below.

Schoute

Biographical notes concerning the Groningen geometer Pieter Hendrik Schoute (1846–1913) are presented at [Sch1]. Schoute's explanation of his models appeared in the minutes, written in Dutch, of the KNAW meetings [KNAW, p. 8–12] in 1894 and also, written by him in German, in the supplement to Dyck's Catalogue [Dy93] published in 1893. The text in the KNAW minutes has as title *Drie draadmodellen van ontwikkelbare oppervlakken, die met hoogere-machtsvergelijkingen in verband staan.* [Three string models of developable surfaces related to equations of higher degree.]

He considers three families of equations, namely

$$t^3 + xt^2 + yt + z = 0$$

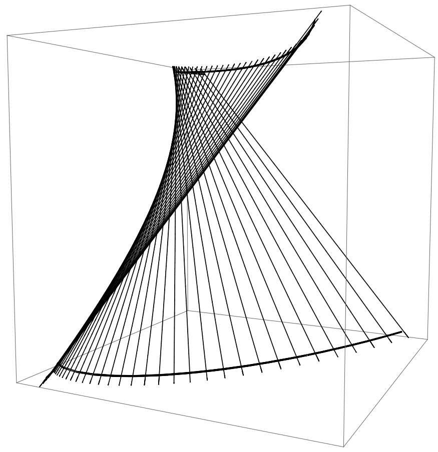
and

$$t^4 + xt^2 + yt + z = 0$$

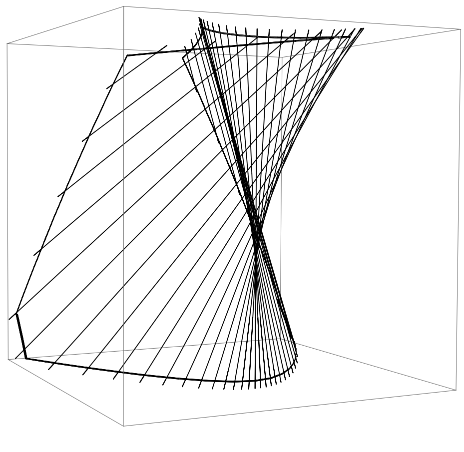
and

$$t^6 - 15t^4 + xt^2 + yt + z = 0.$$

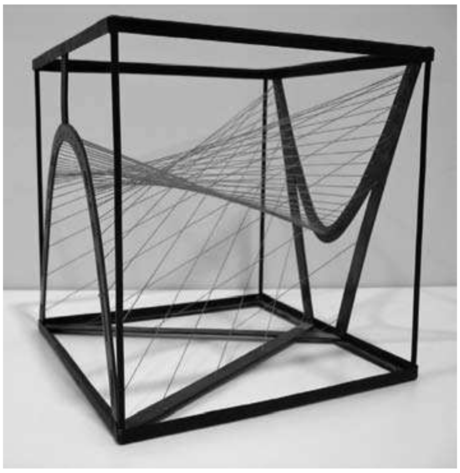
In each case, any real point $(x, y, z) \in \mathbb{R}^3$ corresponds to a polynomial of the given shape. The models show the real points for which the corresponding polynomial has a multiple zero; these points form a (ruled) surface. Moreover, in his description Schoute exhibits the singular points of this surface, for all three cases, and he explains in which intervals he takes x, y, z and how he scales these parameters, in order to obtain the actual models. The precise information given in this way makes it possible to verify that indeed the three models that are still present in the mathematics institute of the university of Groningen are Schoute's models from 1893. Below we show a picture of each of the models together with a mathematica plot based on Schoute's description.

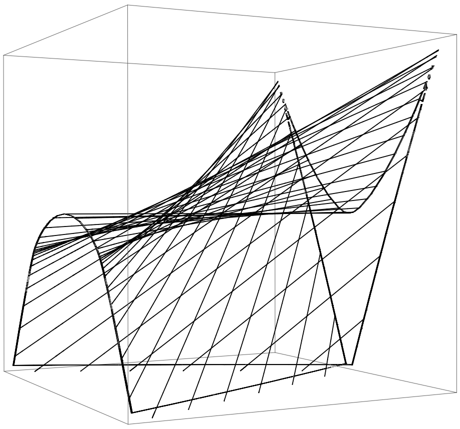


Cubic polynomials: Schoute's model and a plot



Bi-quadratic polynomials: Schoute's model and a plot





Sextic polynomials: Schoute's model and a plot

Schoute also explains why he considers polynomials of degree 3, 4 and 6: the points in space not on the surface correspond to real polynomials with only simple zeros. For degree 3, there are two possibilities for the number of real zeros, so the surface partitions space in two parts depending on this number of zeros. In the case of degree 4, there are three possibilities for the number of real zeros (which obviously all occur in the given family of polynomials). Schoute does not consider degree 5 since that also gives three possibilities hence a somewhat similar partitioning of space. In degree 6, however, there are four possibilities for the number of real zeros. Schoute chooses the particular family of sextics since all four possibilities occur in this family.

Finally, we find in the minutes of the 1893 KNAW meeting motivation for why Schoute designed such models. He explains that it was inspired by the last sentence of Klein's text [Dy93, p. 3–15] in Dyck's Catalogue. Here Klein writes concerning the visualisation of discriminant surfaces

> Es wird sehr dankenswert sein, wenn jemand die Herstellung solcher Modelle in die Hand nehmen wollte.
> [We will be very indebted if someone takes up the construction of such models.]

This is exactly what Schoute did. He writes in the minutes that after his models were ready he noted that pp. 168–173 of Dyck's Catalogue contain a discussion of discriminant surfaces, written by 'Gymnasiallehrer' (teacher at a grammar school) G. Kerschensteiner. This account, published in 1892, even contains sketches of such surfaces:

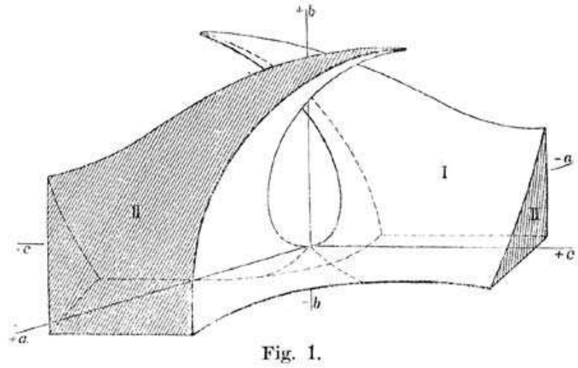


Fig. 1.

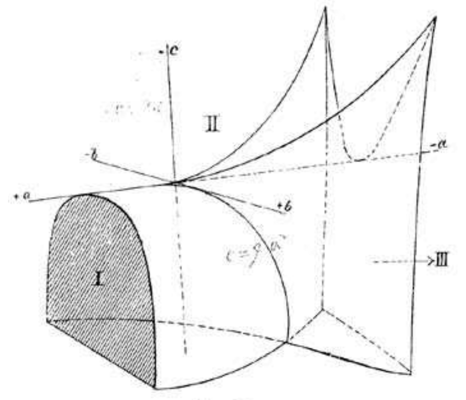


Fig. 2.

G. Kerschensteiner's drawings of discriminant surfaces (1892)

Schoute finds no indication in Kerschensteiner's text that, apart from drawing, he also constructed models. So Schoute continues and presents his models. However, during the next KNAW meeting (Saturday 24 June 1893), Schoute informs his colleagues that

> Dr. Kerschensteiner in Schweinfurt indeed constructed models of discriminant surfaces, but not string models. Rather than content himself therefore with only drawings, Dr. Kerschensteiner made tinplate cross sections which he moulded together using a kneadable substance. (See [KNAW, p. 44] for the original text in Dutch.)

The evident correspondence between Schoute and Kerschensteiner probably inspired the latter to continue his research on discriminant surfaces. This resulted in a text [Ke93] published in the supplement of Dyck's Catalogue [Dy93]. Here, knowing that Schoute had already discussed polynomials of degree 3, 4 and 6, Kerschensteiner treats the family of degree 5 polynomials

$$t^5 + xt^2 + yt + z.$$

One of his results is that if x, y, z are real numbers such that the discriminant of the polynomial is nonzero then the number of real zeros is either 1 or 3. The discriminant surface in this case partitions real 3-space into two parts, (probably disappointingly) much like what also happens for polynomials of degree 3. Kerschensteiner finishes his text by noting that Schoute had shown that discriminant surfaces are unions of straight lines (in fact, developable surfaces) and that this allows one to construct string models of such surfaces. He did this for his family of quintic polynomials.

According to Kerschensteiner, Schoute's presentation at the KNAW on Saturday 27 May 1893 took place in the "Accademie in Groningen". This is very unlikely: KNAW meetings took (and still take) place in Amsterdam. One may expect that on the particular Saturday, Schoute, carrying his three models each of size $8.5 \times 8.5 \times 8.5$ inches, spent several hours on the steam train between Groningen and Amsterdam.

The KNAW minutes record that Schoute's presentation "is followed by a short discussion with Grinwis and Korteweg". The latter is the Amsterdam mathematician famous for the partial differential equation having his name (KdV = Korteweg–de Vries equation). C. H. C. Grinwis (1831–1899) was a mathematical physicist who worked in Delft and in Utrecht.

Dr G. Kerschensteiner (1854–1932) later became well-known as a school reformer and pedagogist working in Munich. In particular his ideas concerning vocational schools have had a large influence on the German school system and have brought him worldwide recognition. Although he taught in his early years mathematics and physics in grammar schools (in Schweinfurt from 1890 till 1893), and he also published some mathematical texts such as the one described here, in mathematics Kerschensteiner seems to be mostly forgotten. In physics however, Germany still has a 'Kerschensteiner Medal' honouring particularly good physics teachers.

Hartenstein

The fact that Roderich Hartenstein did his 'Staatsexamen' with Klein in Göttingen in 1905/06 indicates that he probably became a high school teacher. His thesis work on the discriminant surface corresponding to

$$t^4 + xt^2 + yt + z$$

was not just recalled by Klein in his celebrated lecture notes [Kl08], [Kl32]; two of the string models in Schilling's Series XXXIII were based on it, and as a supplement to these models Schilling published the thesis in 1909 [Ha09]. This was noticed on several occasions. For example, in 1910 the Bulletin of the AMS mentions it [BAMS10, p. 503]. It is also referred to in a paper [Em35] of Arnold Emch in 1935.

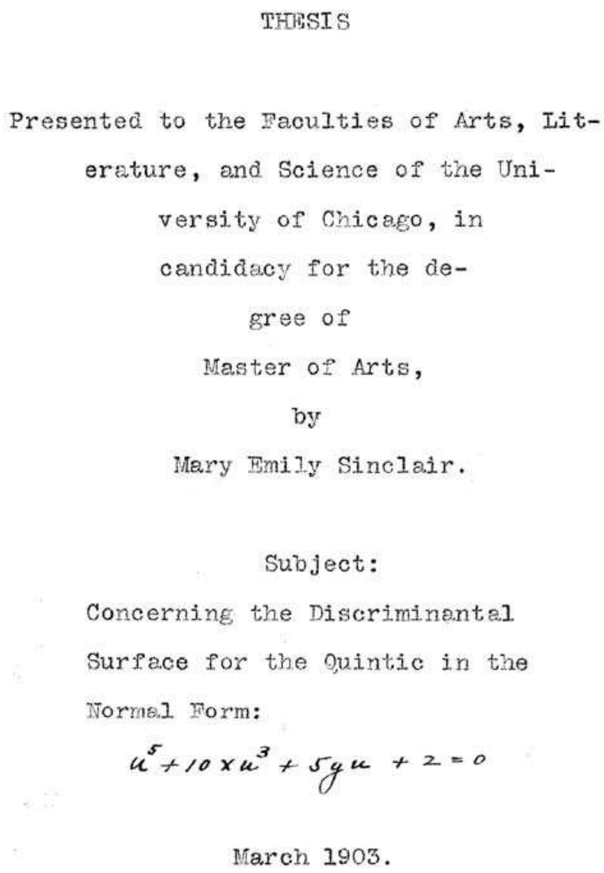
To a large extent, the results of Hartenstein coincide with what Schoute already wrote concerning the polynomials of degree 4. Hartenstein discusses one novel issue. Remarkably, this is not reviewed by Klein in his treatment in [Kl08], [Kl32]. The problem is, given two real bounds L, B with $L < B$, how to find geometrically using the discriminant surface all polynomials $t^4 + xt^2 + yt + z$ having a zero τ such that $L \leq \tau \leq B$. One of the two Hartenstein models in Series XXXIII simply shows the discriminant surface corresponding to these biquadratic polynomials. The other model illustrates his solution to the problem with the two bounds.

In a 'Vorbemerkung' (preceding remark) [Ha09, p. 5], Hartenstein writes that he had already created his models in 1905/06, inspired by Klein. From footnotes on p. 10 and p. 12 one can see that he is well aware of the fact that, in the meantime, Klein used his results in his lectures and text [Kl08]. Hartenstein also gives credit to Schoute and to Kerschensteiner (footnote on p. 15) by observing that they also represented discriminant surfaces. He includes the precise places [Ke92], [Sch93] in Dyck's Catalogue [Dy93] where they describe their work.

His two models in Schilling's Series XXXIII can be admired on the web.[1] Note that M. Schilling's firm was located in the same town Halle for a while before it moved to nearby Leipzig. So it is not surprising that here one finds a large collection of models.

Sinclair

Some biographical notes concerning Mary Emily Sinclair (1878–1955) can be found in [GL08]. Her Master's thesis [Si03] completed at the University of Chicago in 1903 (supervised by Klein's former student Bolza, who returned to Germany (Freiburg) in 1910) is closely related to the work of Schoute and Kerschensteiner discussed above.



THESIS

Presented to the Faculties of Arts, Literature, and Science of the University of Chicago, in candidacy for the degree of Master of Arts, by Mary Emily Sinclair.

Subject: Concerning the Discriminantal Surface for the Quintic in the Normal Form:

$$u^5 + 10xu^3 + 5yu + z = 0$$

March 1903.

Sinclair considers the family

$$t^5 + xt^3 + yt + z.$$

This appears only slightly different from the quintics considered ten years earlier by Kerschensteiner in [Ke93] but there are two differences. The first and important one is that in Sinclair's family real x, y, z exist such that the discriminant is nonzero and the number of real zeros of the polynomial is any of the possibilities 1, 3, 5. So her discriminant surface partitions real 3-space into three parts, which was not the case for Kerschensteiner's surface. The second difference is a computational one: Sinclair's family is chosen in such a way that the zeros of the derivative are quite easily expressible in the coefficients x, y, z. This is an advantage exploited throughout her thesis.

In the last sentence of the Introduction, Sinclair states "A model of the surface accompanies this investigation." This model seems to have disappeared. A sketch of the surface can be found in her thesis (p. 36). Five years after this Master's thesis was completed, Schilling reproduced Sinclair's model as Nr. 1 in the Series XXXIII. Accompanying this new model, an eight page summary Sinclair wrote about her Master's work was published [Si08]. Searching through the collections available at various German universities, one finds Schilling's reproduction of Sinclair's model (Series XXXIII Nr. 1) in, for example, the Mathematics Department of the Martin Luther Universität of Halle-Wittenberg.[2]

In 2003, the American sculptor Helaman Ferguson made a stone model [Fe03] based on Sinclair's thesis.

2 Discriminant surfaces

Here we discuss some of the classical geometry of discriminant surfaces. We do this using the language of algebraic geometry. Of course, for the actual classical models one was interested in the surfaces over the real numbers $\mathbb{R}$ (and to some extent, over the complex numbers $\mathbb{C}$). Here we work over an arbitrary field k. We fix a separable closure of k, which will be denoted $\overline{k}$. All classical examples fit in the following framework.

Consider two polynomials $f, g \in k[t]$ such that f is monic and $\deg(f) > \deg(g) > 1$. Put

$$P(x, y, z, t) := f(t) + xg(t) + yt + z \in k[x, y, z, t].$$

Replacing z by $z' = f(0) + xg(0) + z$ and y by $y' = \frac{df}{dt}(0) + x\frac{dg}{dt}(0) + y$ and x by $x' = \frac{d^2f}{dt^2}(0) + x$, one can moreover assume that $g(t) \equiv 0 \bmod t^2$ and $f(t) \equiv 0 \bmod t^3$. Also, multiplying x by the leading coefficient of g allows one to assume that g is monic.

The examples treated above are indeed of this kind:

Designer	f	g
Schoute	t^3	t^2
	t^4	t^2
	$t^6 - 15t^4$	t^2
Kerschensteiner	t^4	t^2
	t^5	t^2
Sinclair	t^5	t^3
Hartenstein	t^4	t^2

There is one more condition, which we assume only in Proposition 2.4 below: the polynomials f, g should be chosen in such a way that the derivative of f and the second derivative of g are not identically zero. Since we also assume $\deg(f) > \deg(g) > 1$, this is automatic in the case $\mathrm{char}(k) = 0$. However, in positive characteristic it poses an extra condition. For example, the cases treated by Sinclair satisfy $\frac{d^2g}{dt^2} \neq 0$ and $\frac{df}{dt} \neq 0$ precisely when $\mathrm{char}(k) \neq 2, \neq 3, \neq 5$.

We now review some geometric properties of the set

$$S := \left\{(\alpha, \beta, \gamma) \in \overline{k}^3 \mid P(\alpha, \beta, \gamma, t) \text{ has a zero with multiplicity } \geq 2\right\}.$$

Obviously S is an algebraic set: it consists of all points where the discriminant $\Delta(x, y, z) \in k[x, y, z]$ of P (regarded as polynomial in the variable t) vanishes. In more detail, put $K := k(x, y, z)$ the field of fractions of $k[x, y, z]$. Then $P \in K[t]$. Write $P' = \frac{dP}{dt}$.

By definition, S consists of all (α, β, γ) such that $p := P(\alpha, \beta, \gamma, t)$ and $p' := P'(\alpha, \beta, \gamma, t)$ have a zero in common. This condition implies in particular that no polynomials $r, s \in \overline{k}[t]$ exist such that $rp + sp' = 1$. The converse also holds: if 1 is not a linear combination of p and p' then p, p' have a common factor, so p has a multiple zero. This simple observation leads to the following.

For $d > 0$, consider

$$\mathcal{P}_d := \{r \in K[t] \mid \deg(r) < d\}.$$

This is a finite dimensional vector space over K. Put $d := \deg(P) = \deg(f)$ and write $P = z + yt + (\sum_{m=2}^{d-1} a_m t^m) + t^d$. The K-linear map

$$\varphi : \mathcal{P}_{d-1} \oplus \mathcal{P}_d \to \mathcal{P}_{2d-1} : (r, s) \mapsto rP + sP'$$

is invertible precisely when 1 is in the image, which is the case precisely when $\gcd(P, P') = 1$. Choose the basis

$$\left\{t^{2d-2}, t^{2d-3}, \ldots, t^2, t, 1\right\}$$

for $\mathcal{P}_{2d-1}$ and

$$\left\{(t^{d-2}, 0), (t^{d-3}, 0), \ldots, (1, 0), (0, t^{d-1}), (0, t^{d-2}), \ldots, (0, t), (0, 1)\right\}$$

for $\mathcal{P}_{d-1} \oplus \mathcal{P}_d$. With respect to these bases, φ is given by the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & d & 0 & \ldots & 0 \\ a_{d-1} & 1 & & 0 & (d-1)a_{d-1} & d & \ldots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ a_2 & a_3 & & 1 & 2a_2 & 3a_3 & & 0 \\ y & a_2 & & a_{d-1} & y & 2a_2 & & d \\ z & y & & & 0 & y & & (d-1)a_{d-1} \\ 0 & z & & & 0 & 0 & & \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & z & 0 & 0 & \cdots & y \end{pmatrix}.$$

The determinant of this matrix is called the resultant of P and P'; this is (up to a sign) the discriminant $\Delta(x, y, z)$ of P. This is a polynomial in x, y, z; from the description as a determinant given here one concludes that its degree with respect to z equals $d - 1$, unless $\mathrm{char}(K)$ divides d, in which case the degree is strictly less than $d - 1$.

So as an algebraic set,

$$S := \left\{(\alpha, \beta, \gamma) \in \overline{k}^3 \mid \Delta(\alpha, \beta, \gamma) = 0\right\}.$$

An immediate consequence of this is that S is an (affine) algebraic surface. To prove more properties of S, observe that S consists of all points (α, β, γ) such that $\tau \in \overline{k}$ exists with $P(\alpha, \beta, \gamma, \tau) = P'(\alpha, \beta, \gamma, \tau) = 0$. In other words, S is the image under the projection map $(\alpha, \beta, \gamma, \tau) \mapsto (\alpha, \beta, \gamma)$ of the algebraic set $\tilde{S} \subset \overline{k}^4$ defined by $P = P' = 0$.

Proposition 2.1. *S is an irreducible algebraic set.*

Proof. This assertion means that S is not the union of two proper nonempty algebraic subsets. Equivalently, the ideal $I(S) \subset \overline{k}[x, y, z]$ consisting of all polynomials that vanish on every point of S, should be a prime ideal. To show that this is indeed true, consider the ideal $I(\tilde{S}) \subset \overline{k}[x,y,z,t]$ consisting of all polynomials vanishing on every point of $\tilde{S}$. Since S is the projection of $\tilde{S}$ it follows that $I(S) = I(\tilde{S}) \cap \overline{k}[x,y,z]$. Hence it suffices to prove that $I(\tilde{S})$ is a prime ideal.

Consider $I := \overline{k}[x,y,z,t]\cdot P + \overline{k}[x,y,z,t]\cdot P'$. By definition, $\tilde{S}$ is the algebraic set defined by the vanishing of all polynomials in I. By Hilbert's Nullstellensatz, this implies that $I(\tilde{S})$ is the radical of the ideal I. However, I is a prime ideal because

$$\overline{k}[x,y,z,t]/I \cong \overline{k}[x,y,t]/\overline{k}[x,y,t]\cdot P' \cong \overline{k}[x,t]$$

is a domain. Hence $I = \mathrm{rad}(I) = I(\tilde{S})$, showing that $I(\tilde{S})$ and therefore also $I(S)$ is a prime ideal. □

Since $I(S)$ and Δ both define the irreducible algebraic set S, it follows that $I(S) = \mathrm{rad}(\overline{k}[x,y,z]\cdot\Delta)$. This means that $I(S)$ is a principal ideal generated by an irreducible $Q \in \overline{k}[x,y,z]$ such that $\Delta = uQ^m$ for some nonzero u in $\overline{k}$ and an integer $m > 0$. Because $I(S)$ is generated by polynomials defined over k (namely, those in $(k[x,y,z,t]\cdot P + k[x,y,z,t]\cdot P') \cap k[x,y,z]$), we can even take $Q \in k[x,y,z]$ (and therefore also $u \in k$) here. In the classical examples one has $m = 1$, so Δ is irreducible. However, this is not true in general, as the case $t^{2p} + xt^p + yt + z$ in characteristic $p > 0$ shows (see Examples 2.5 below).

Proposition 2.2. *S is a ruled surface, i.e. it is a union of straight lines.*

Proof. Indeed, $(\alpha,\beta,\gamma) \in S$ precisely when τ exists such that $P(\alpha,\beta,\gamma,\tau) = P'(\alpha,\beta,\gamma,\tau) = 0$. This can be written as

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ -f'(\tau) \\ \tau f'(\tau) - f(\tau) \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -g'(\tau) \\ \tau g'(\tau) - g(\tau) \end{pmatrix}.$$

Clearly this defines a union of straight lines. □

Note that the above result shows that the straight lines are described by the following property: the points (α,β,γ) on any such line correspond to the polynomials $P(\alpha,\beta,\gamma,t)$ that have a common multiple zero.

The next property shows that in fact S is a developable surface: not only is it a union of straight lines; the points on any such line all have a common tangent plane, which in fact is provided with an easy equation.

Proposition 2.3. *Suppose char$(k) \neq 2$. Let $(\alpha,\beta,\gamma) \in S$ correspond to a zero τ of $P(\alpha,\beta,\gamma,t)$ with multiplicity 2. Then the plane with equation*

$$f(\tau) + xg(\tau) + y\tau + z = 0$$

is tangent to S in (α,β,γ).

Proof. First, observe that by assumption indeed (α,β,γ) is a point on the given plane. The multiplicity of τ being 2 implies, since char$(k) \neq 2$, that

$$f''(\tau) + \alpha g''(\tau) \neq 0.$$

Our strategy will be to determine the tangent plane Π to S in (α,β,γ) by first computing the tangent plane $\tilde{\Pi}$ to $\tilde{S}$ in $v := (\alpha,\beta,\gamma,\tau)$ and then project this to the x,y,z-space.

By definition, $\tilde{\Pi}$ consists of all points $v + w$ (for $w = (w_1,w_2,w_3,w_4)$) such that

$$q(v + \lambda w) \equiv 0 \bmod \lambda^2,$$

for all $q \in I(\tilde{S})$. We know that the ideal $I(\tilde{S})$ is generated by P and P' hence, instead of using all q, it suffices to consider only these two polynomials. Using that $(\alpha,\beta,\gamma,\tau) \in \tilde{S}$, the condition $v + w \in \tilde{\Pi}$ then reads

$$\begin{cases} g(\tau)w_1 + \tau w_2 + w_3 = 0; \\ g'(\tau)w_1 + w_2 + (f''(\tau) + \alpha g''(\tau))w_4 = 0. \end{cases}$$

Note that since $f''(\tau) + \alpha g''(\tau) \neq 0$, the second equation uniquely determines w_4 in terms of the other coordinates. This implies that the projection Π of $\tilde{\Pi}$ to the x,y,z-space consists of all $(\alpha,\beta,\gamma) + (w_1,w_2,w_3)$ satisfying the equation

$$g(\tau)w_1 + \tau w_2 + w_3 = 0.$$

Alternatively, this can be written as $P(w_1,w_2,w_3,\tau) = f(\tau)$. Since $P(\alpha,\beta,\gamma,\tau) + P(w_1,w_2,w_3,\tau) = f(\tau) + P(\alpha + w_1, \beta + w_2, \gamma + w_3, \tau)$, one concludes that the tangent plane consists of all (x,y,z) satisfying $P(x,y,z,\tau) = 0$, which is what we wanted to prove. □

From now on we will exploit the additional assumption that $g''(t)$ is nonzero. Let s be a new variable and put

$$w := \begin{pmatrix} -f(s) \\ -f'(s) \\ -f''(s) \end{pmatrix}.$$

The condition says that

$$A := \begin{pmatrix} g(s) & s & 1 \\ g'(s) & 1 & 0 \\ g''(s) & 0 & 0 \end{pmatrix}$$

is invertible over $k(s)$. In particular, there exists a unique solution

$$v := \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix}$$

to the linear equation $Av = w$. This system says that $t = s$ is a triple zero of the polynomial

$$p(t) := f(t) + x(s)g(t) + y(s)t + z(s) \in k(s)[t].$$

Hence $p(t)$ can be written as

$$p(t) = (t - s)^3 q(t)$$

for some $q(t) \in k(s)[t]$. Taking the derivative with respect to s, one concludes that $t = s$ is a zero of $\frac{\partial p}{\partial s} = x'(s)g(t) + y'(s)t + z'(s)$ of multiplicity at least 2. So for fixed $s_0 \in \overline{k}$ such that $g''(s_0) \neq 0$, and for every $\lambda \in \overline{k}$, the polynomial

$$f(t) + (x(s_0) + \lambda x'(s_0))g(t) + (y(s_0) + \lambda y'(s_0))t + z(s_0) + \lambda z'(s_0)$$

in $\overline{k}[t]$ has a zero $t = s_0$ of multiplicity ≥ 2. In other words,

$$(x(s_0), y(s_0), z(s_0)) + \lambda\,(x'(s_0), y'(s_0), z'(s_0))$$

is in the discriminant surface S.

Proposition 2.4. *Assume that g'' and f' are not identically zero.*

Then $s \mapsto (x(s), y(s), z(s))$ parametrizes a curve in S corresponding to the polynomials $f(t) + xg(t) + yt + z$ with a triple zero.

For general $s_0 \in \overline{k}$, the tangent line to this curve in the point corresponding to s_0 is contained in S, and S is the Zariski closure of the union of all these tangent lines.

Proof. As before, let s be a variable over k. Define $(x(s), y(s), z(s))$ as above. We claim that the point $(x'(s), y'(s), z'(s)) \in S$ is not the point $(0, 0, 0)$: if it were, then $(0, 0, 0)$ is a point on the line in S corresponding to all polynomials with a double zero at $t \quad s$. The parametrization of this line given in Proposition 2.2 now implies $f'(s) \quad 0$ (and also $f(s) - sf'(s) \quad 0$). This contradicts the assumption that $f' \neq 0$.

Since $(x'(s), y'(s), z'(s)) \neq (0, 0, 0)$, it follows that $(x(s), y(s), z(s))$ indeed parametrizes a curve in S. The Zariski closure of the set of tangent lines to this curve is an irreducible subvariety V of S. By Proposition 2.1, S is irreducible, so V equals S unless it has dimension strictly smaller than 2. This can only happen if the closure of the image of $s \mapsto (x(s), y(s), z(s))$ is a line ℓ in S. However, that is impossible: if $s_0 \in \overline{k}$ is taken such that the derivative $(x'(s_0), y'(s_0), z'(s_0))$ is nonzero then we have seen that the tangent line to the parametrized curve corresponding to s_0 is the line in S coming from all polynomials having s_0 as a multiple zero. On the other hand, this tangent line is obviously equal to ℓ. This is absurd, since it would imply that the polynomials corresponding to the points of ℓ would have infinitely many multiple zeros. This finishes the proof. □

Examples 2.5.

1. Suppose p is a prime number. Consider $f(t) \quad t^{2p}$ and $g(t) \quad t^p$ over a field k of characteristic p. Obviously, the conditions $g''(t) \neq 0$ and $f'(t) \neq 0$ used in Proposition 2.4 are not satisfied.
   The discriminant surface S is given by the equation $y \quad 0$. Every point in S corresponds to a polynomial of which the roots have multiplicity a multiple of p. This implies that also the conditions used in Proposition 2.3 are not satisfied. Clearly, S has no singular points. The discriminant is a constant times $y^{p \quad 1}$. The tangent planes to S in points of S are not of the form described in Proposition 2.3 in this case.
2. Suppose $p > 3$ is a prime number. Take $f(t) \quad t^{p \quad 3}$ and $g(t) \quad t^{p \quad 1}$ over a field k of characteristic p. Then $f' \neq 0$ but $g'' \quad 0$. In this example, $f(t) \quad xg(t) \quad yt \quad z$ has a zero of multiplicity > 2 if and only if $z \quad y \quad 0$. These equations define a line in the surface S, namely the one corresponding to the polynomials with $t \quad 0$ as a multiple zero.
3. Suppose $p > 2$ is a prime number. We now consider an example in which $g''(t)$ is nonzero but $f'(t)$ is identically zero: put $f(t) \quad t^p$ and $g(t) \quad t^2$ over a field k of characteristic p. Here the curve in S corresponding to polynomials with a zero of multiplicity > 2 is parametrized by $s \mapsto (0, 0, -s^p)$. It is a line in (the singular locus of) S but not a line obtained by demanding that the polynomial has some fixed multiple zero.

3 An "appendix"

One may wonder why M. Schilling decided to reproduce the models by Sinclair and Hartenstein, and not the older ones by Schoute. According to the catalogue [S11, p. 158], the models in Series XXXIII were inspired by Bolza and by Klein. Since Bolza was a former graduate student of Klein, and Klein's interest in models was well-known, probably Bolza informed him of the work of his Master's student Sinclair. It is only natural that in some contact with Schilling, Klein mentioned the models of his student Hartenstein and the related model of Sinclair, leading to Series XXXIII.

Nevertheless, Klein was quite likely aware of the existence of Schoute's models. They are described in the well-known 'Nachtrag' (supplement) of Dyck's Catalogue [Dy93] and Hartenstein even refers to this. Moreover, mathematically the results of Schoute and Sinclair nicely complement each other: together they provide 3-parameter families of polynomials of degree 3, 4, 5 and 6, such that in each family all possible numbers of real zeros occur. Incidentally, there is no evidence that Sinclair was aware of Schoute's work: she very carefully mentions in her thesis where to find relevant results she uses and this does not include a reference to Schoute.

One mathematical reason to take a model by Hartenstein is that he describes the set of polynomials with a given number of (real) zeros in a prescribed interval. This topic is not considered by Schoute or Sinclair. The model XXXIII Nr. 3 illustrates Hartenstein's description, providing an argument for using this model in the series.

However, there may be a di erent mathematical reason. As we have seen, the lines (strings) in the models correspond to real polynomials with a fixed (real) multiple zero. Now it is possible that a real polynomial does not have a real multiple zero but instead it has a pair of complex conjugate ones. In the example $t^4 \quad xt^2 \quad yt \quad z$ this happens precisely for two double zeros $t \quad \pm\sqrt{-b}$ and $b > 0$, so for the polynomials $(t^2 \quad b)^2$. It defines one half of a parabola $(2b, 0, b^2)$ with $b > 0$, which is part of the discriminant surface but no real lines (strings) pass through this part. Although Schoute describes this phenomenon, his actual models consist of strings only and ignore this "appendix". Hartenstein did include it in his models by means of a brass wire. And exactly at the place in his text [Ha09] where he describes this wire ("Messingdraht"), he places a footnote referring to older models by Schoute and by Kerschensteiner. So this makes a case in favour of Hartenstein's models over one by Schoute. There is one problem with this reasoning: the same phenomenon also appears in Sinclair's family $t^5 \quad xt^3 \quad yt \quad z$, namely, the polynomials $t(t^2 \quad b)^2$ with $b > 0$ give the points $(2b, b^2, 0)$ in the corresponding discriminant surface. For $b > 0$, there are no real lines in the surface passing through such a point. Sinclair describes this in detail; however, when she discusses how to construct her model, no mention of these special points is made. Contrary to Hartenstein's examples, it also appears that this "appendix" is missing in Schilling Model XXXIII Nr. 1. So maybe the choice for Hartenstein and Sinclair was not motivated by the "appendix".

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Notes

1. See, for example, http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dj-002 and http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dj-003 (both at the Martin Luther Universität in Halle-Wittenberg, Germany)
2. See http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dj-001

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