Higher-periodic homotopy types through Lubin–Tate towers

In their recent work, Barthel, Schlank, Stapleton, and Weinstein determined the periodic stable homotopy groups of the sphere spectrum rationally. They made an essential use of the structure afforded by perfectoid spaces for computing relevant group cohomology in the framework of condensed mathematics. These spaces appear in an equivariant isomorphism between two towers: (1) the Lubin–Tate tower that parametrizes deformations of a formal group of fixed height with level structures and (2) the Drinfeld tower that parametrizes those for shtukas. I'm obliged to introduce this exciting mathematical landscape to the greater "perfection" community, and appeal for further insights and collaborations. This also includes: (a) my ongoing joint work with Guozhen Wang which computes unstable higher-periodic homotopy types integrally, (b) Xuecai Ma's spectral realization of finite levels of the Lubin–Tate tower as non-even commutative ring spectra, which generalize Morava, Hopkins, Miller, Goerss, and Lurie's spectra at the ground level, and (c) Hongxiang Zhao's work which connects Ando's norm through homotopical descent of level structures along the tower, to Coleman's norm in the context of Lubin and Tate's explicit local class field theory.

https://bicmr.pku.edu.cn/content/show/17-3377.html

Higher-periodic homotopy types through Lubin–Tate towers

Yifei Zhu

Southern University of Science and Technology

2024.6.25

Context and motivations

p-adic geometry ⇄ **topology**

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p**-adic geometry** ⇄ (homotopy theory) **topology**

Context and motivations

- In one direction, through a stratification of the moduli stack of formal groups by heights and primes

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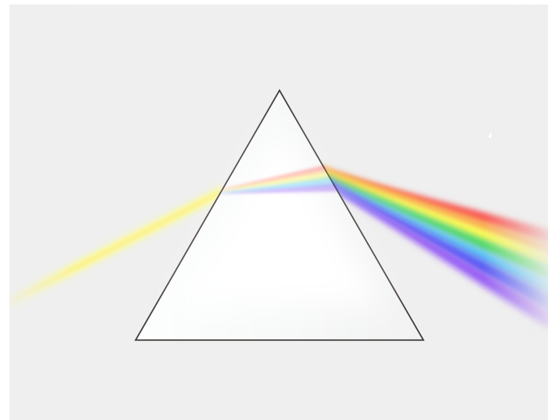

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e.g., Tate diagonal
cyclotomic spectra à la Nikolaus and Scholze
Liu–Wang, Jingbang Guo

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In this talk, we focus on some of the recent progresses in the forward direction, but the p-adic geometry involved, as well as the relevant tools, may be of independent interest beyond topology.

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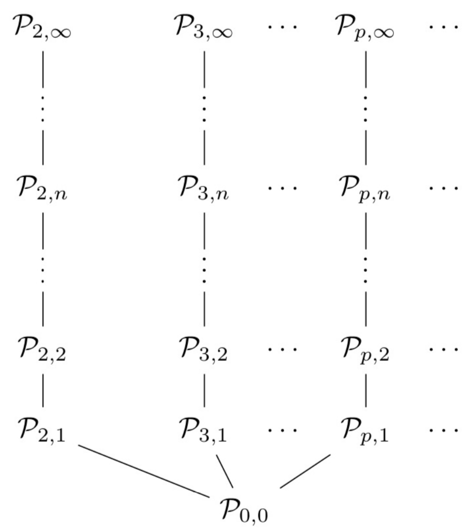
Thick subcategories of **Sp** (Hopkins–Smith '98)



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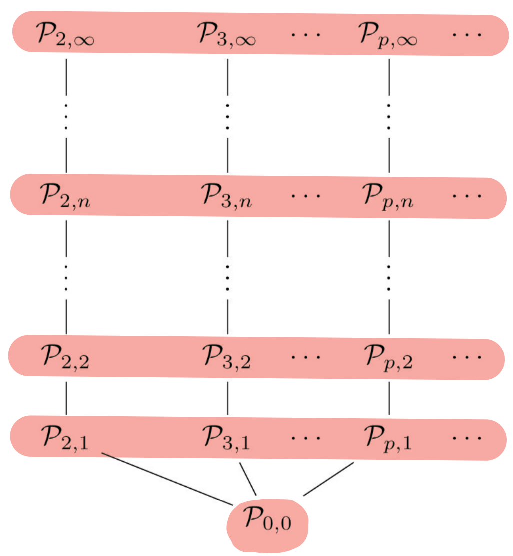

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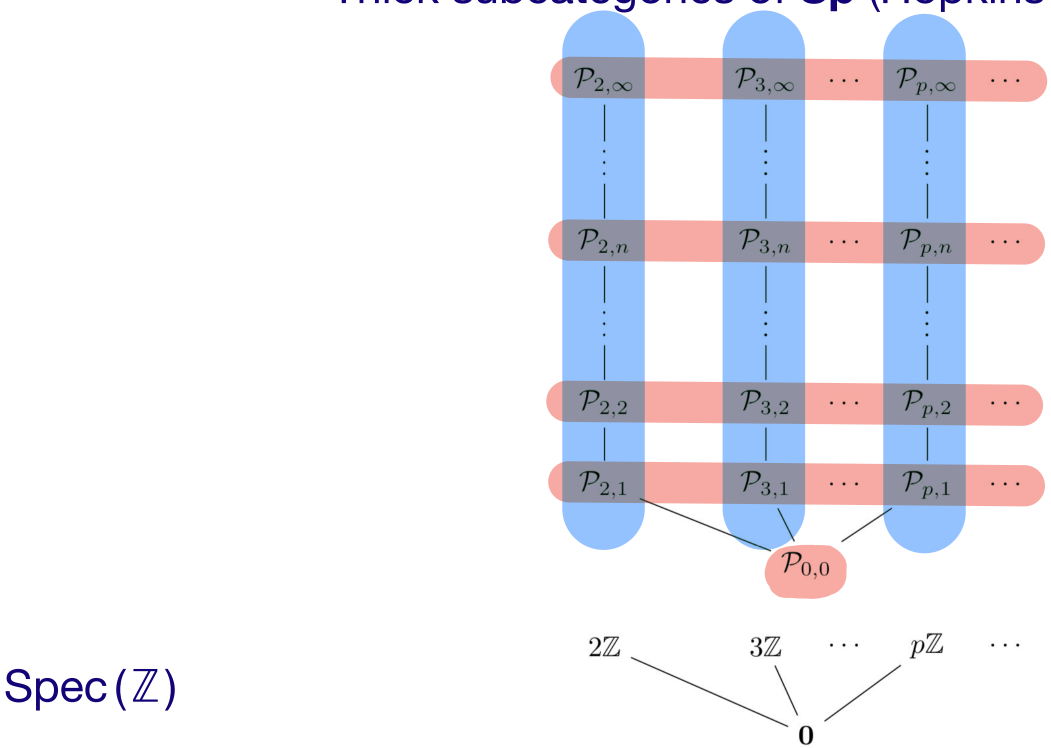

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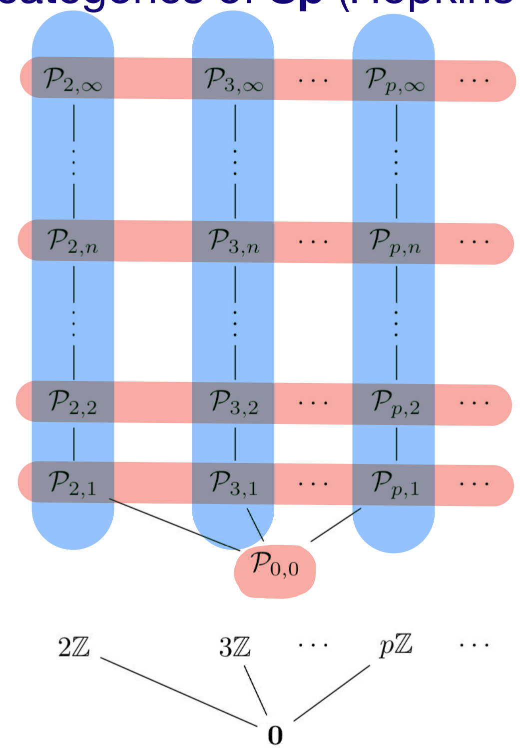

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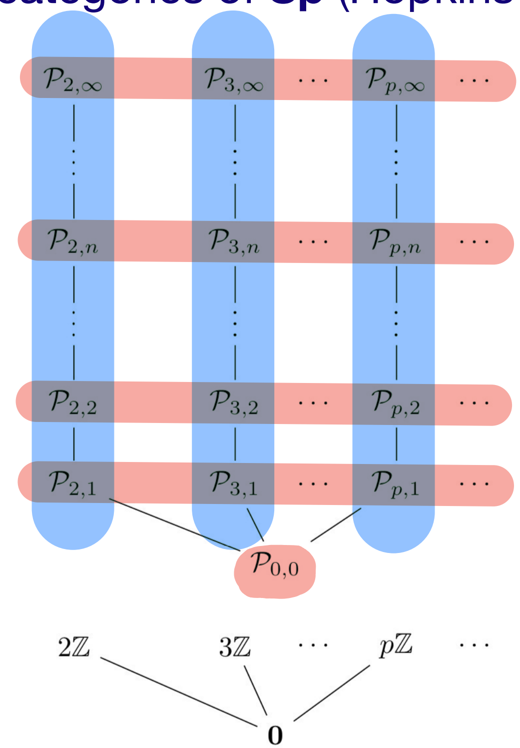

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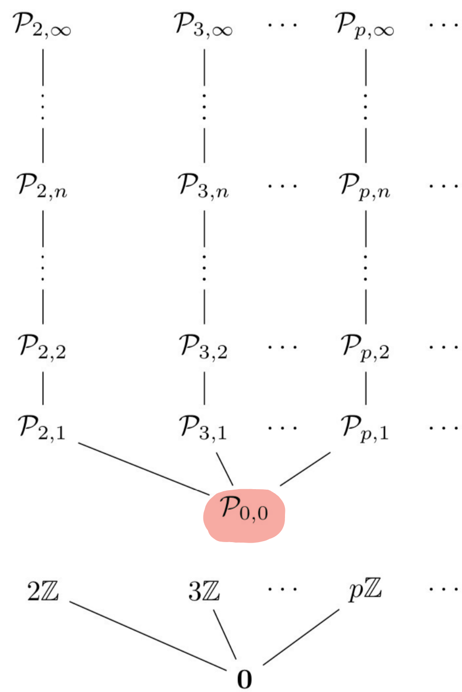

rational cohomology

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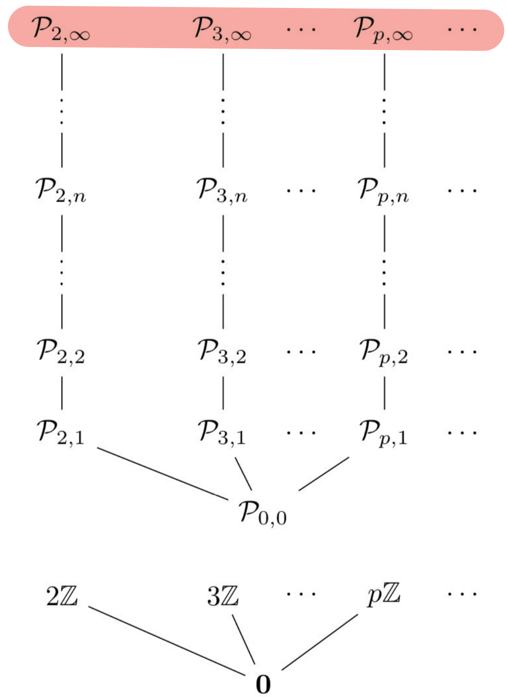

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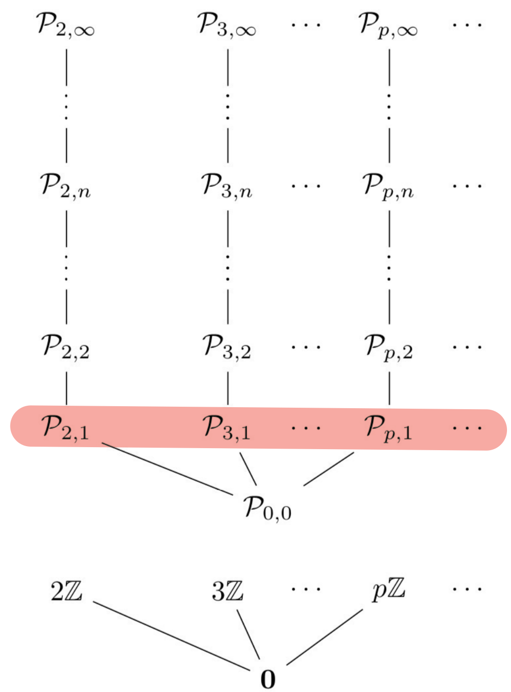

p-adic K-theory

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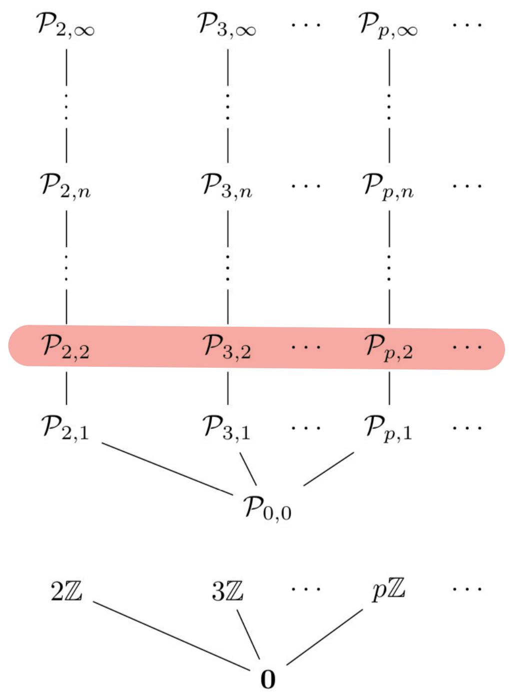

elliptic cohomology
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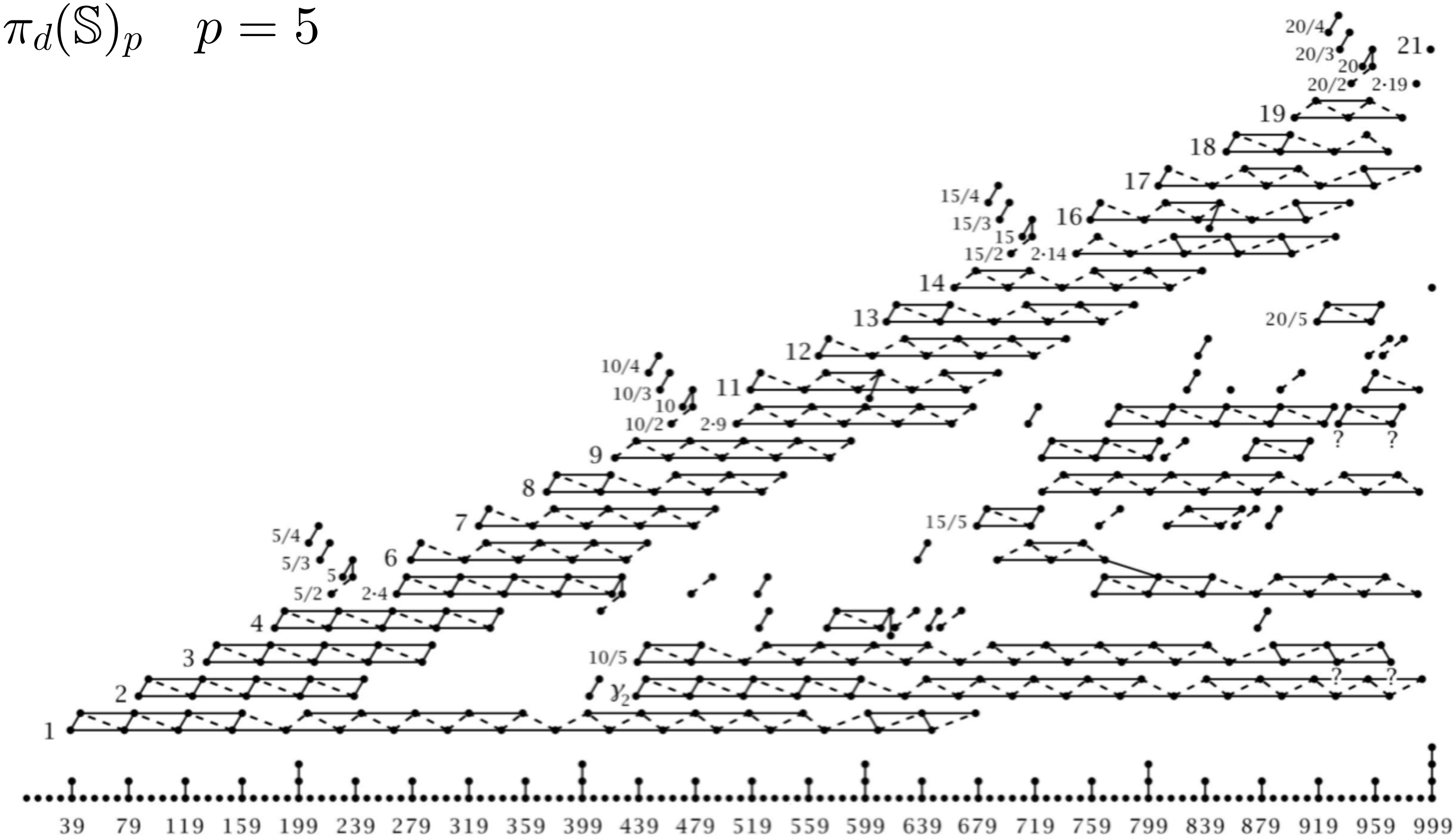

Picture: Hatcher
Computation: Ravenel

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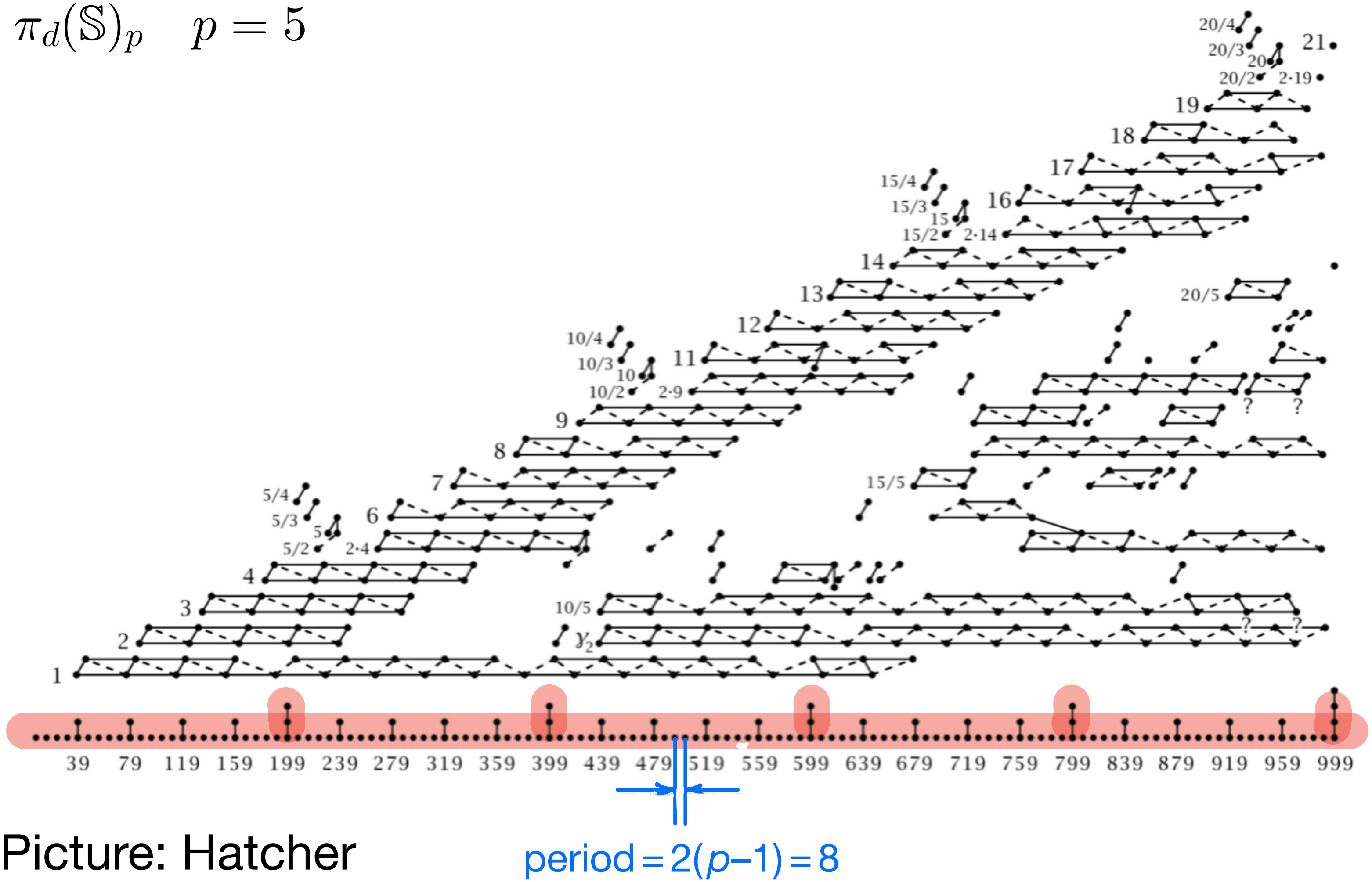

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v_1-periodic elements

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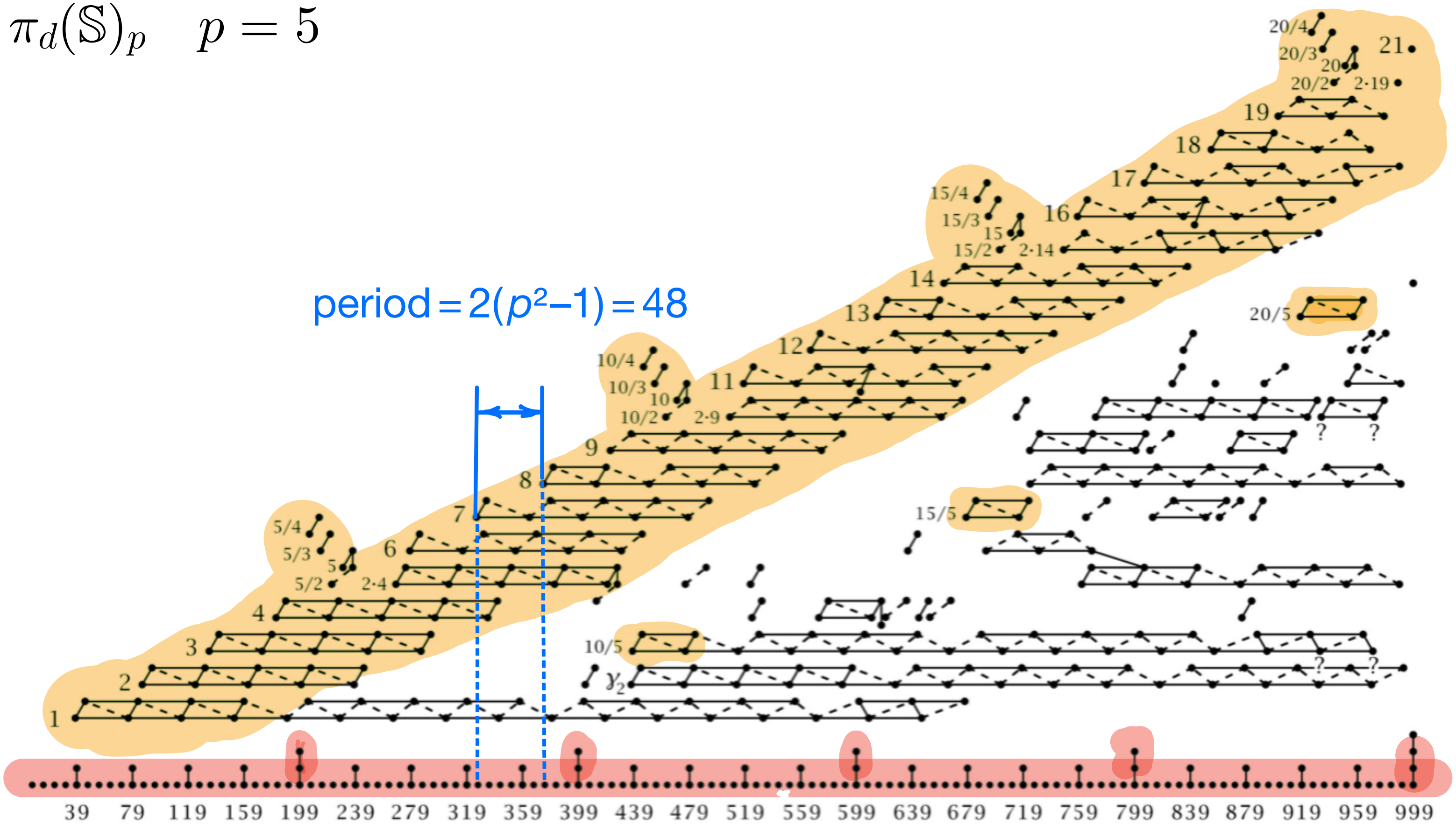

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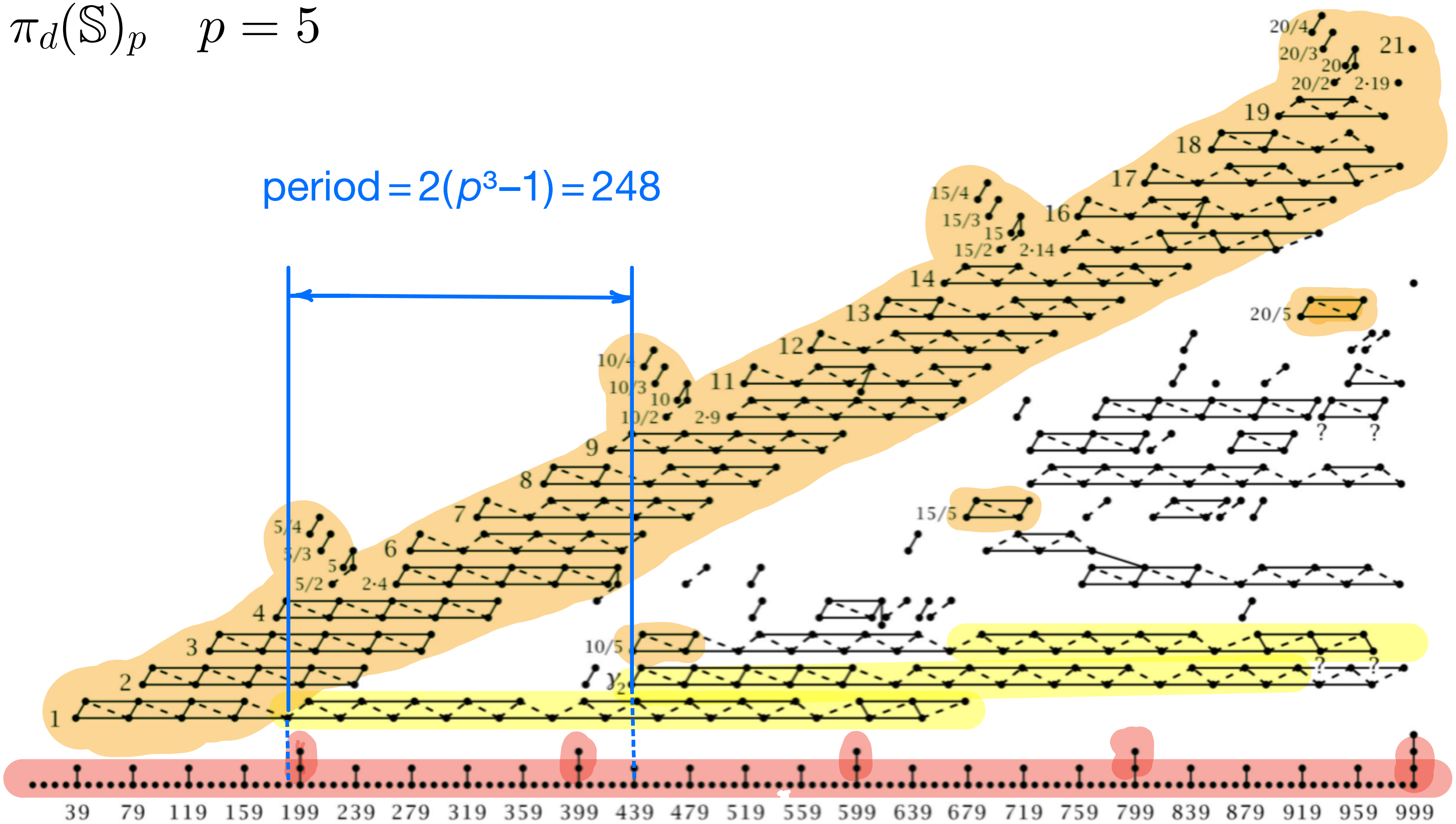

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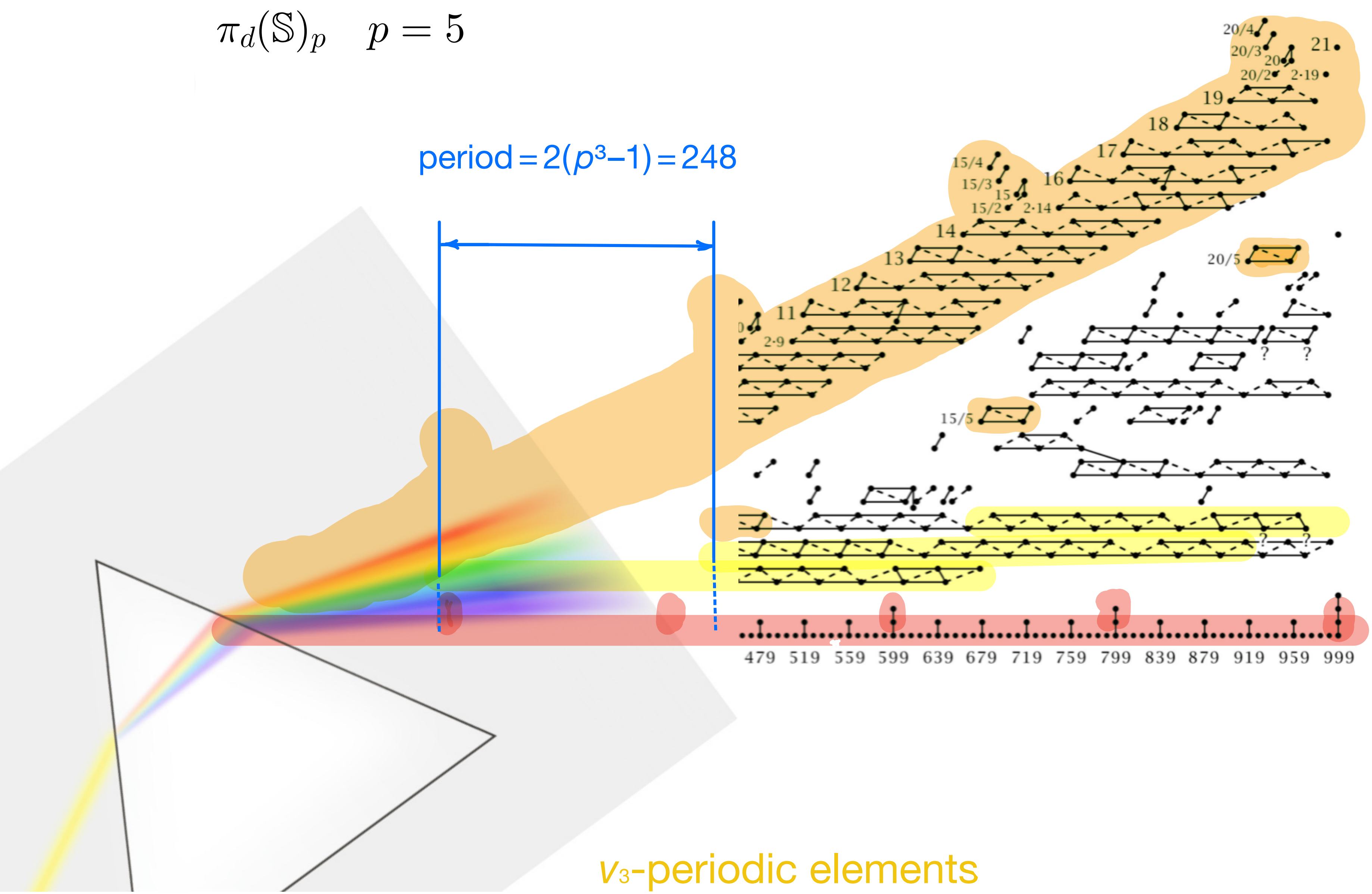

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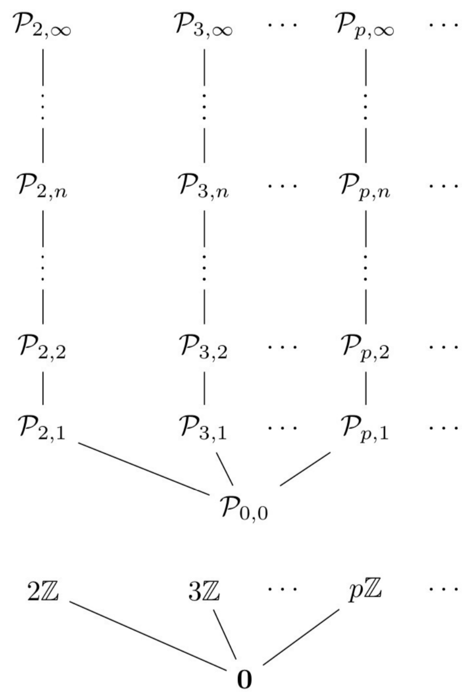

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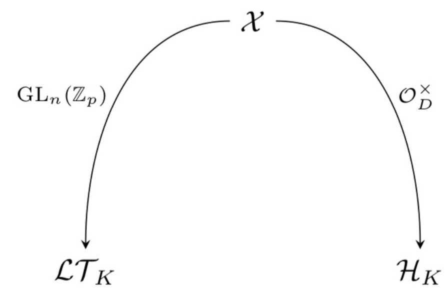

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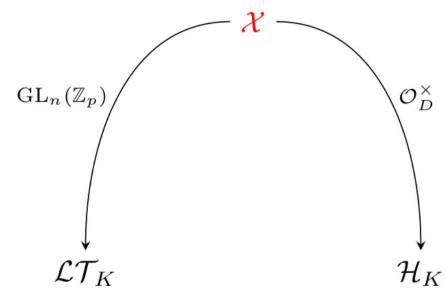

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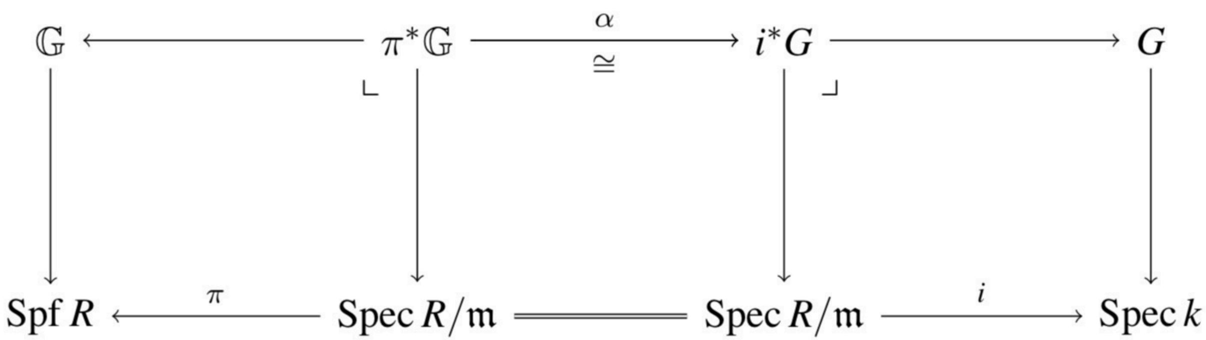

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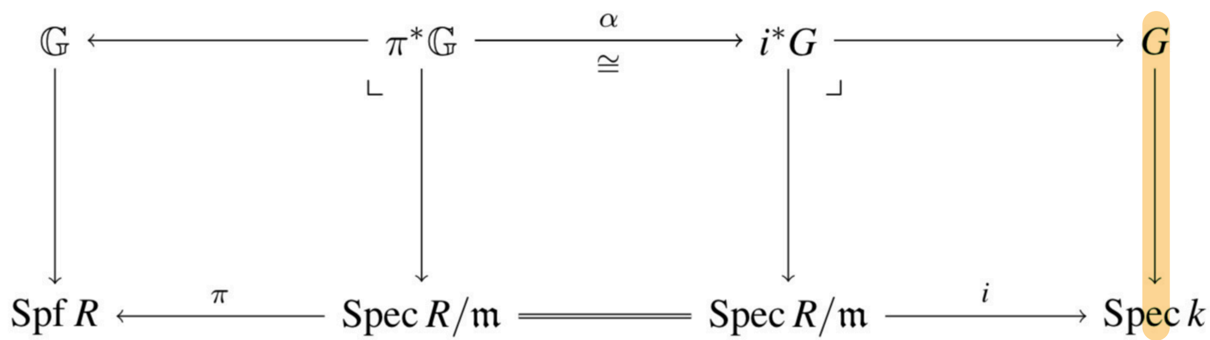

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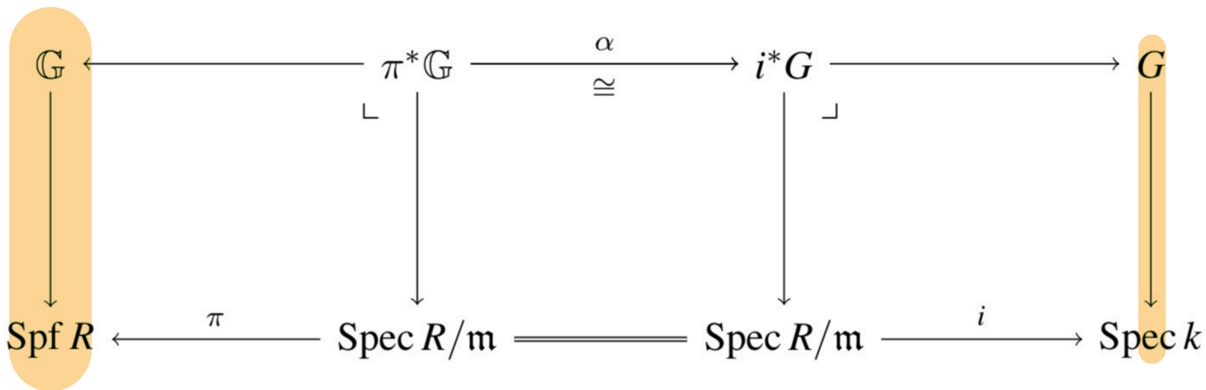

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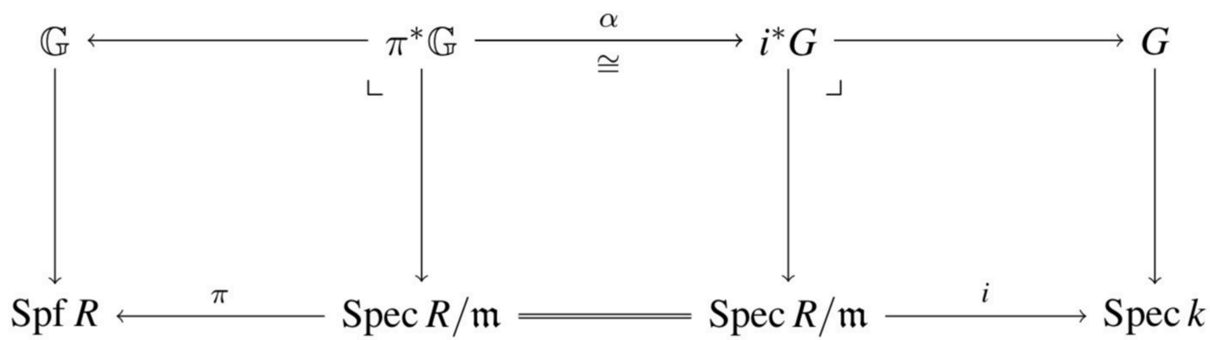

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$$\begin{array}{ccccccc}
\mathbb{G} & \longleftarrow & \pi^*\mathbb{G} & \xrightarrow[\cong]{\alpha} & i^*G & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spf} R & \xleftarrow{\pi} & \operatorname{Spec} R/\mathfrak{m} & = & \operatorname{Spec} R/\mathfrak{m} & \xrightarrow{i} & \operatorname{Spec} k
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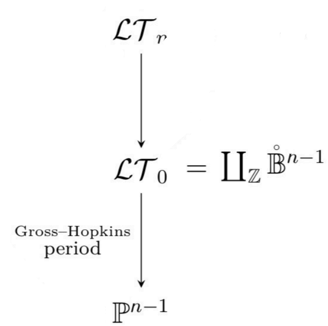

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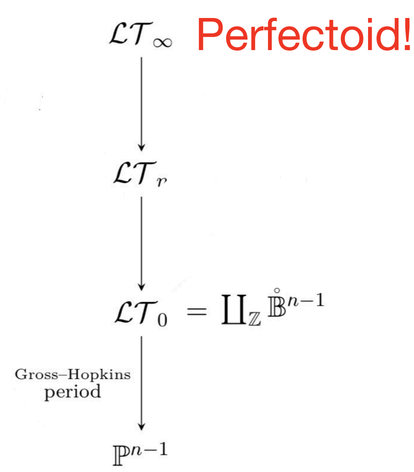

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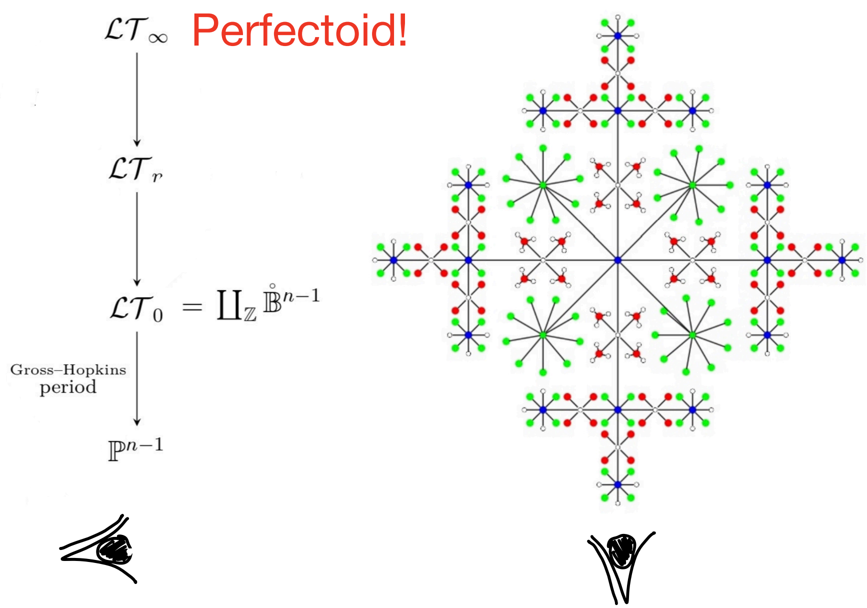
- [Lubin–Tate '66] for $r = 0$, by an affine formal scheme Spf A_0 of dimension $n - 1$ over the Witt ring $\mathcal{O}_F$ of k (with F = max. unrf. ext. of p-adic comp. of k).
- [Strickland '97] for each $r > 0$, by an affine formal scheme Spf A_r of finite rank over A_0.

[Goerss–Hopkins–Miller, Strickland '98] These moduli spaces are (partially) spectrally realized by the Morava E-theory spectrum $E = E_n(G/k)$, i.e.,

$$A_r \cong E^0(B\Sigma_{p^r})/I_{\text{tr}}$$

[Fargues '08, Weinstein '15] As a family of spaces indexed by r, they stack into a tower with all levels literally:



Analytic geometry and homotopy groups of the $K(n)$-local sphere

Symmetries

- The Morava stabilizer group $\mathbb{G}_n \cong \mathcal{O}_D^\times$ (with D / F = cent. div. alg. of inv. $1/n$).

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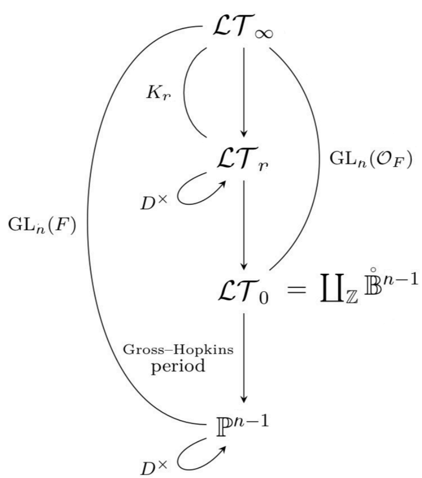

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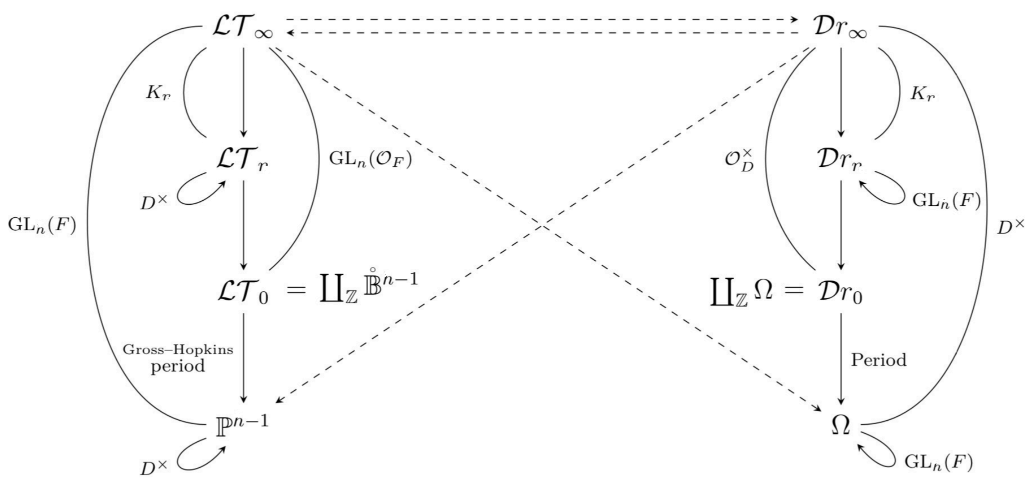
- [Faltings, Fargues '08, Scholze–Weinstein '13] There is an equivariant isomorphism between the Lubin–Tate tower and another Drinfeld tower (parametrizing deformations of shtukas).



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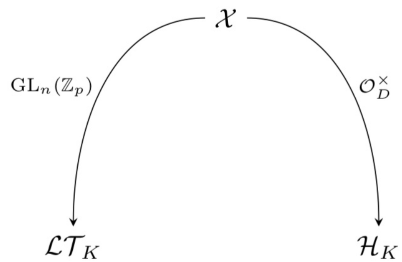

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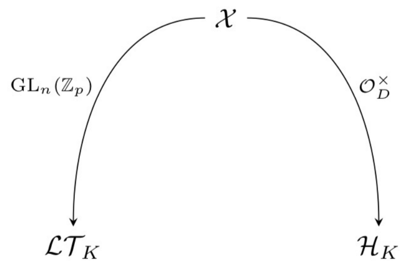

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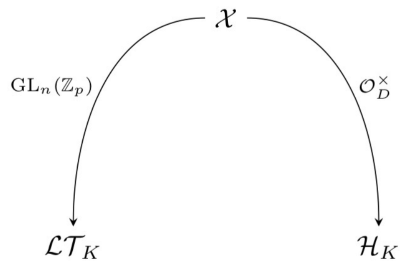

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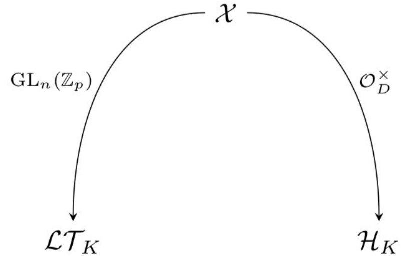

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In general, given a pro-étale G torsor $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$, we have $R\Gamma(Y, \widehat{\mathcal{O}}^+_{\text{cond}})^{hG} \cong R\Gamma(X, \widehat{\mathcal{O}}^+_{\text{cond}})$.

Here taking homotopy fixed points of $R\Gamma(\mathcal{X}_K, \widehat{\mathcal{O}}^+_{\text{cond}})^{h(\mathrm{GL}_n(\mathbb{Z}_p)\times\mathcal{O}_D^\times)}$ in two orders yields the above isomorphism.

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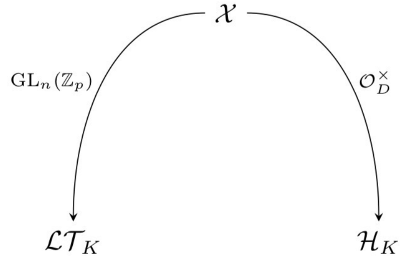

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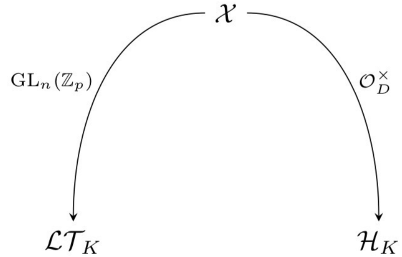

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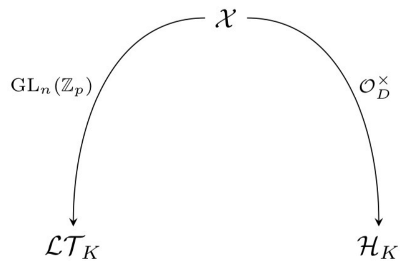

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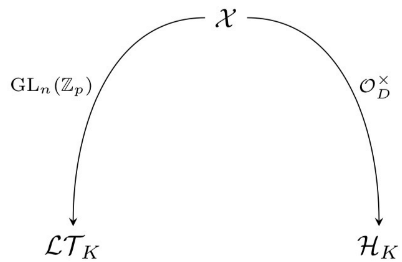

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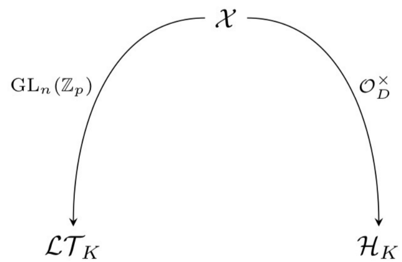

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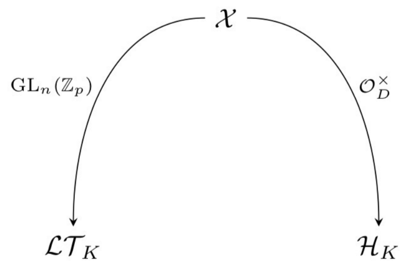

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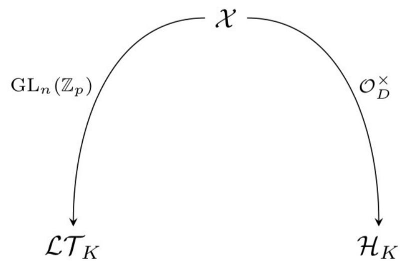

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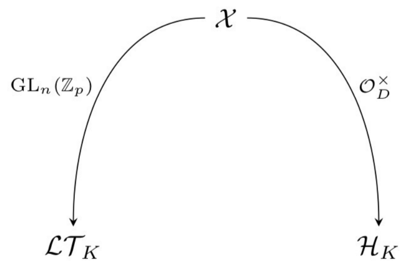

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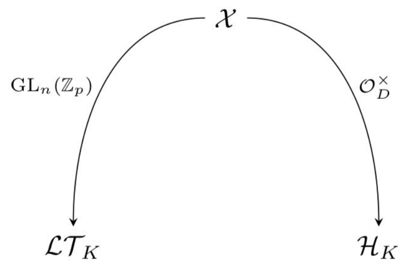

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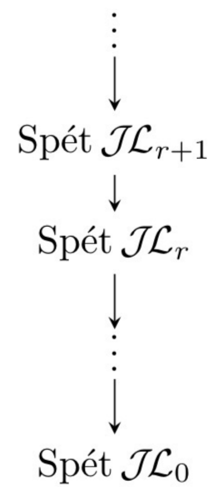

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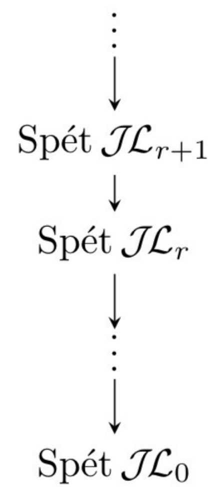

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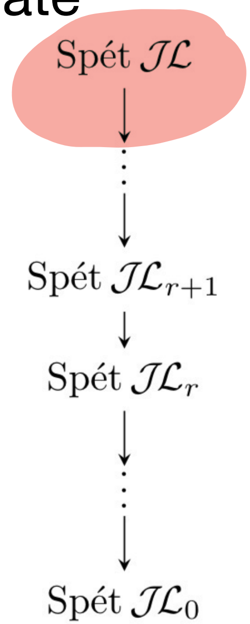

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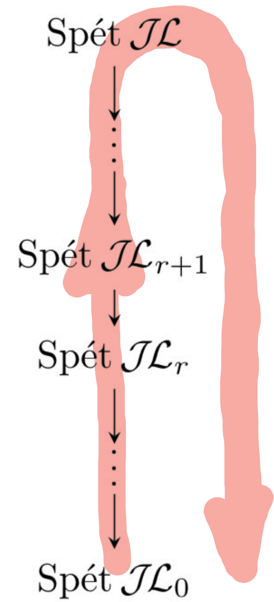

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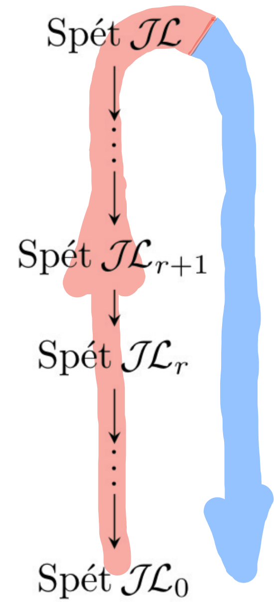

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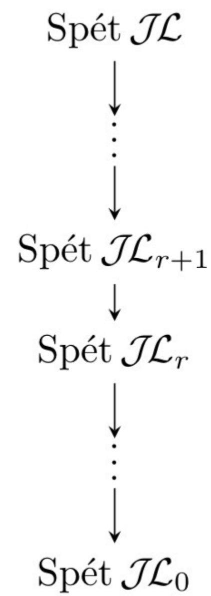

[Hongxiang Zhao ’23] Zhao gave explicit evidence that, along Lubin–Tate towers, certain norm operators from local class field theory have spectral realizations.

Thank you.

Afterthoughts

Mathematics has become so developed and specialized, that the heavy notations of individual subfields make it feel as if each small, distinguished group of people set a rule, a roadmap for themselves, their friends and descendants to play within. That said, we must not give up the effort to increase the understandability of mathematics across subfields, convey the essential ideas, learn and get inspirations from each other, so as to make the better of it for the future.