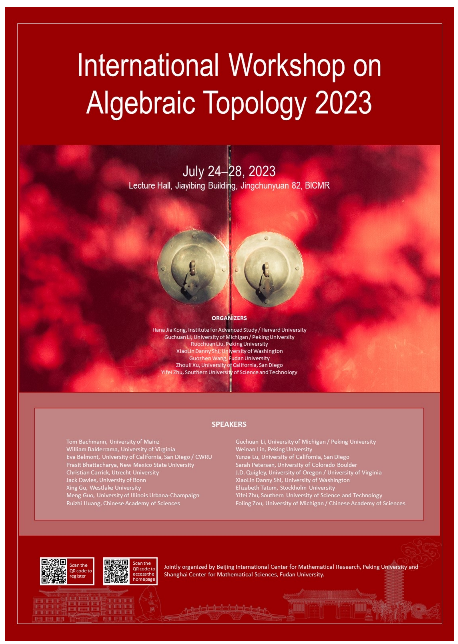
Moduli, moduli, moduli

Yifei Zhu

A moduli space is a space of parameters that label a certain family of structured objects we are interested in. I'll report on using methods of algebraic topology to understand aspects of a diverse set of moduli problems: (1, joint with Guozhen Wang et al.) in connection with p-adic arithmetic geometry, a filtered equivariant quasi-syntomic sheaf of Koszul complexes for computing unstable chromatic homotopy of spheres, over moduli spaces that parametrize deformations of a formal group with level structures; (2, joint with Hongwei Jia et al.) in connection with condensed-matter physics and materials science, monodromy of stratified vector bundles as moduli for gapless quantum mechanical systems, which arise from non-Hermitian symmetries; and (3, joint with Pingyao Feng et al.) in connection with data science, topological distribution spaces for image and speech signals, as revealed from persistent homology, and applied to the design of convolutional layers for deep learning. For each, I will introduce the context of study and describe the mathematical objects in question, with all technical terms above explained.



Moduli, moduli, moduli:

Portraits of moduli spaces

Yifei Zhu

Southern University of Science and Technology

2023.7.25

What is a moduli space?

A moduli space is a space of parameters, that is, a set of parameters with extra structure. These parameters label objects we would like to study, often in a continuous fashion.

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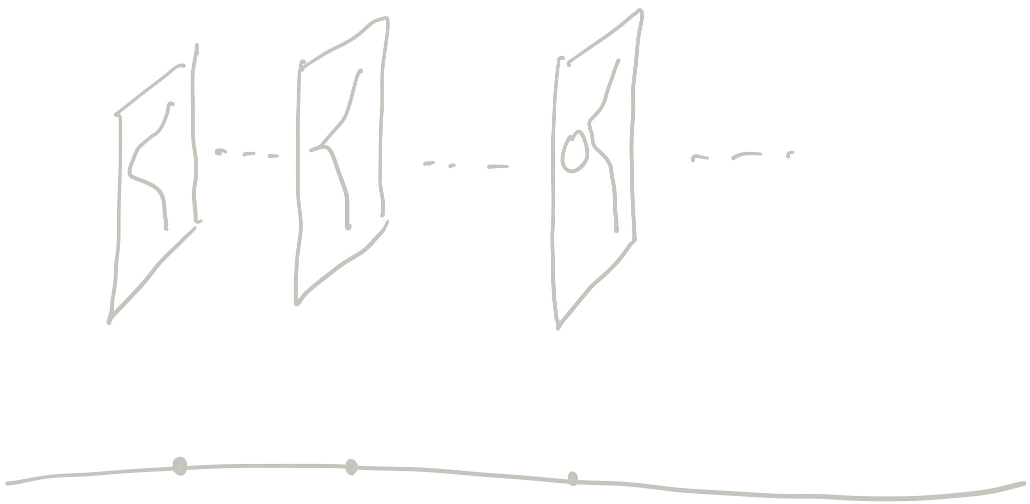

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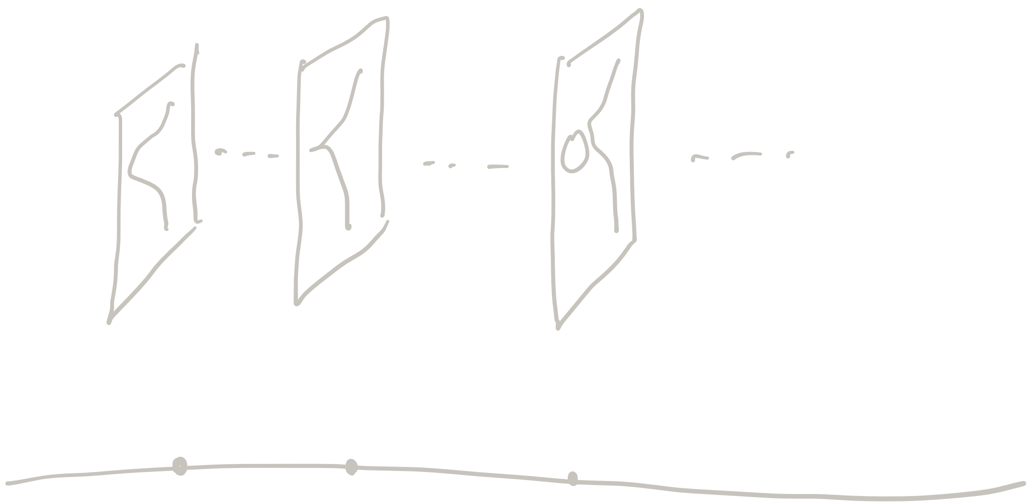

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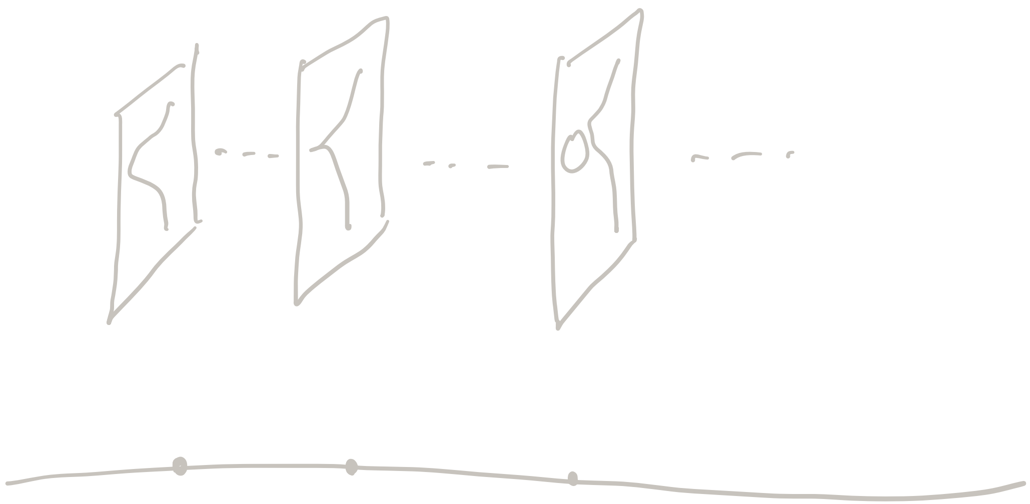

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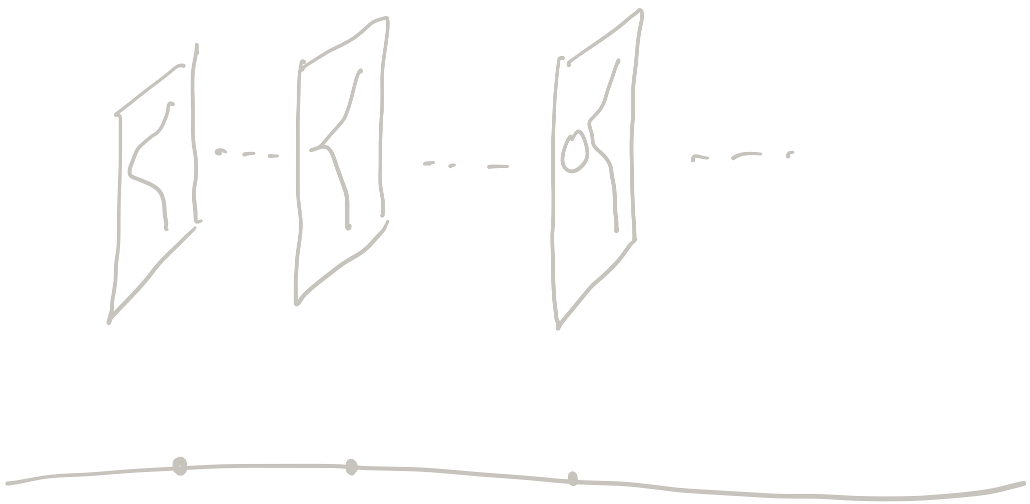

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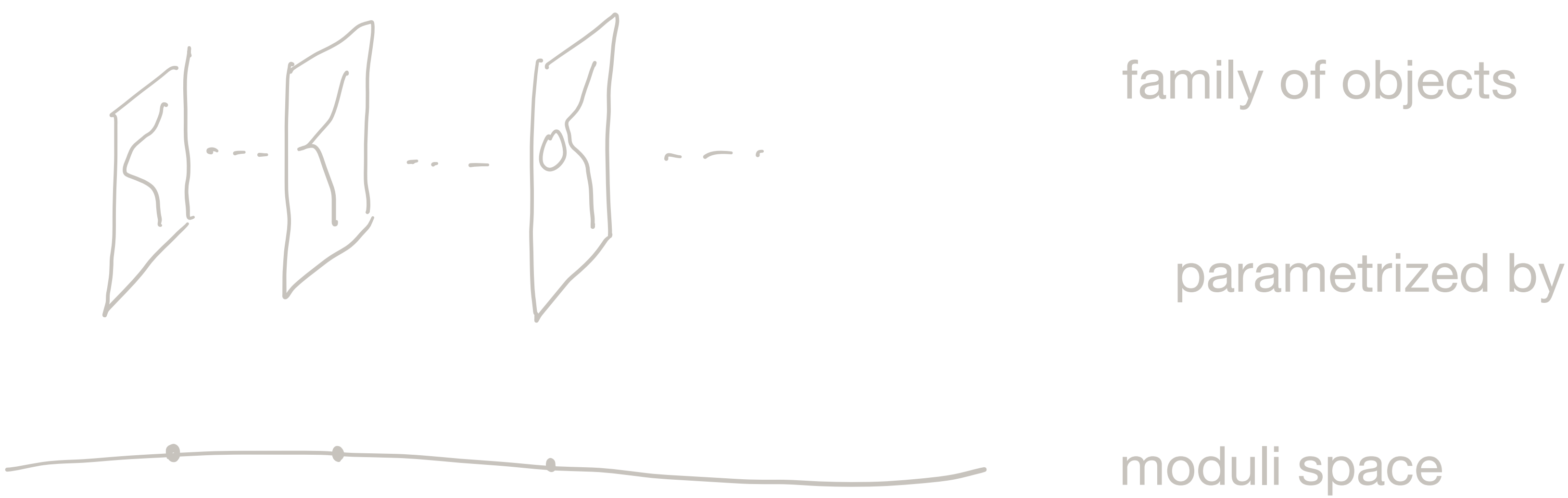

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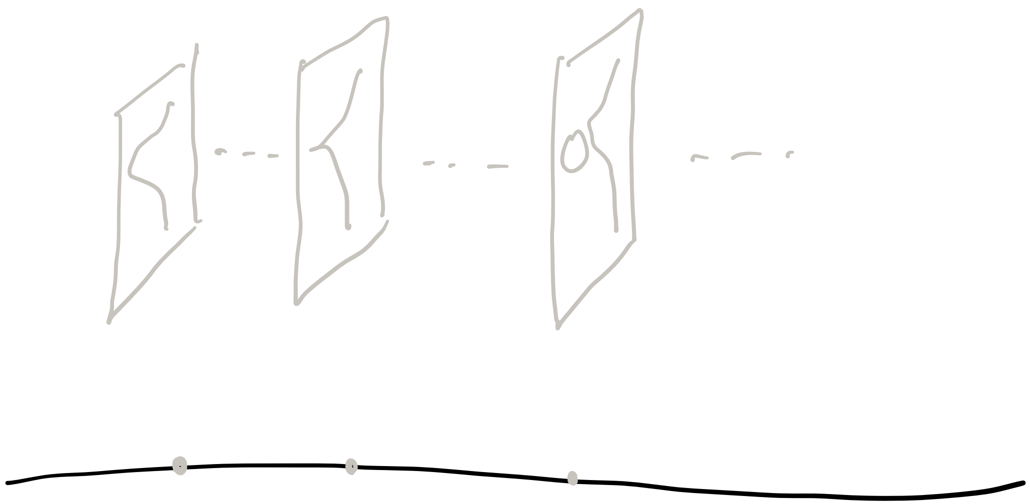

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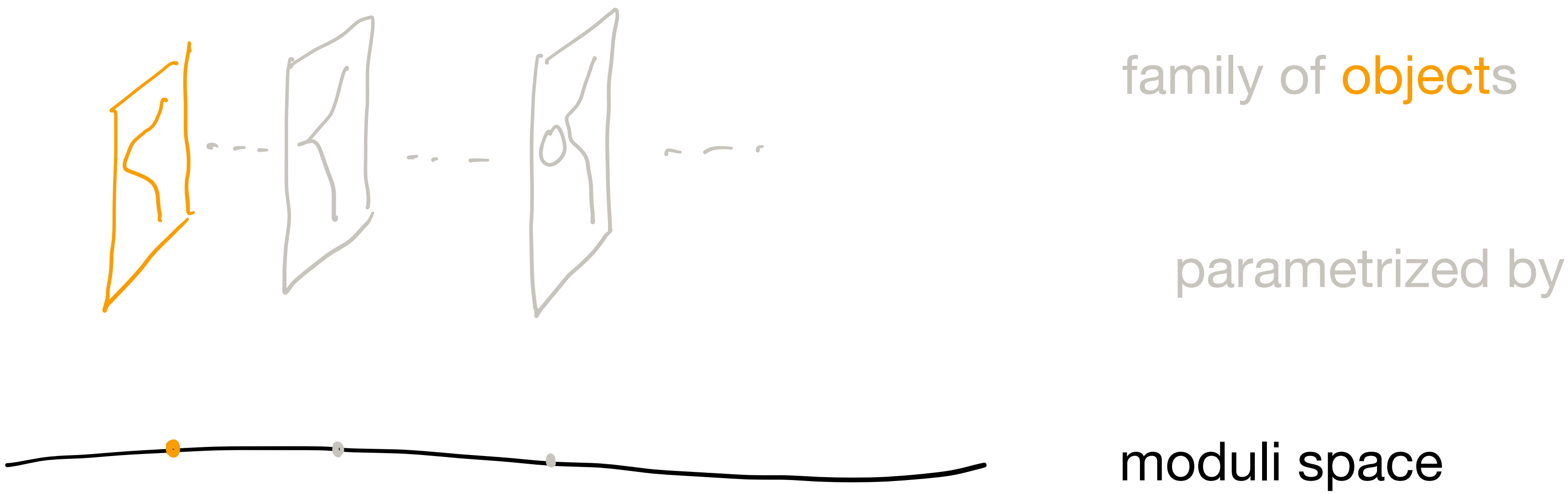

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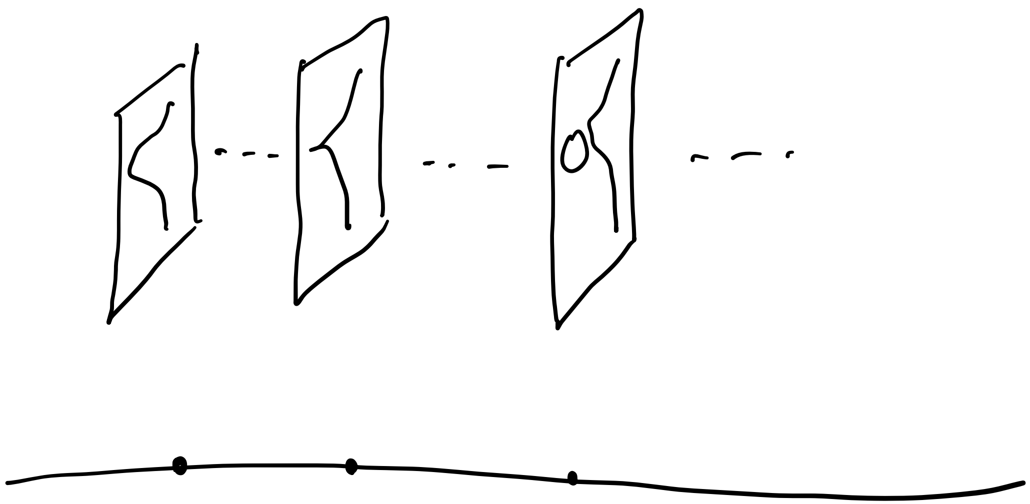

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Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous family of objects, or how an object varies as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure, often rich and distinct from that of the objects.
- Understanding the moduli space offers in turn understanding of the objects individually and as a whole.
- In this sense, studying moduli spaces is of the second-order nature.

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    - In the opposite direction, homotopical methods, especially the theory of infinity categories, have enabled constructions previously ad hoc or impossible in algebra and number theory, leading to recent advances in *p*-adic Hodge theory and algebraic K-theory.
- [Joint with Guozhen Wang et al.] Building upon both computational and conceptual progresses with chromatic homotopy groups of spaces and cohomology operations in connection with explicit arithmetic moduli of algebro-geometric objects, we have constructed a sheaf further binding arithmetic and topological data.

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- Objects to parametrize: Deformations of a formal group with level structure

$G =$ formal group of height $h < \infty$ over a perfect field k of characteristic $p > 0$

$R =$ complete local ring with residue field k, nilpotent maximal ideal $\mathfrak{m}$, and natural projection $\pi\colon R \to R/\mathfrak{m}$

deformation of G/k to $R := (\mathbb{G}, i, \alpha)$ with $\mathbb{G}$ a formal group over R, $i\colon k \hookrightarrow R/\mathfrak{m}$, and $\alpha\colon \pi^*\mathbb{G} \xrightarrow{\cong} i^*G$

deformation of G with a level-$\Gamma_0(p^n)$ structure $:= (\mathbb{G}, \mathbb{H})$ with $\mathbb{H}$ a cyclic degree-p^n subgroup

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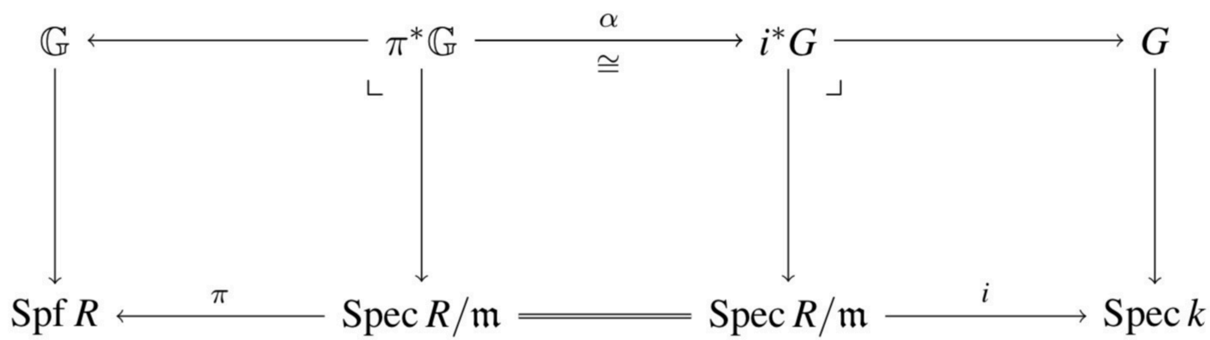

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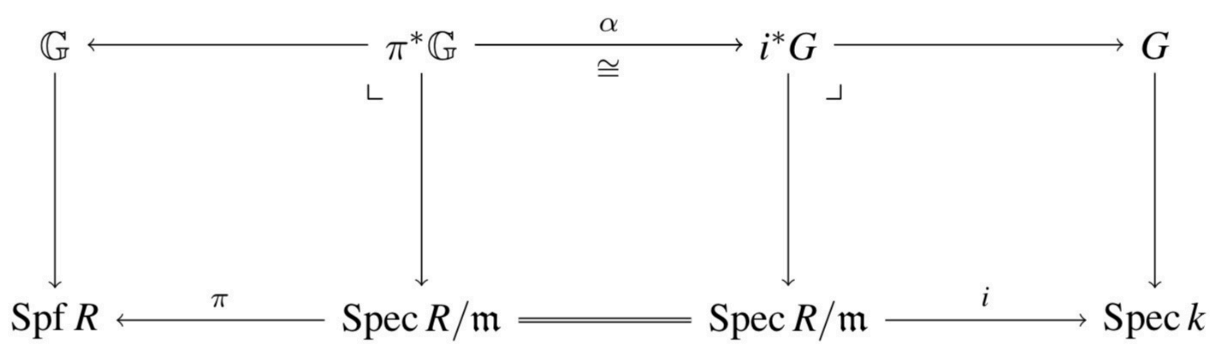

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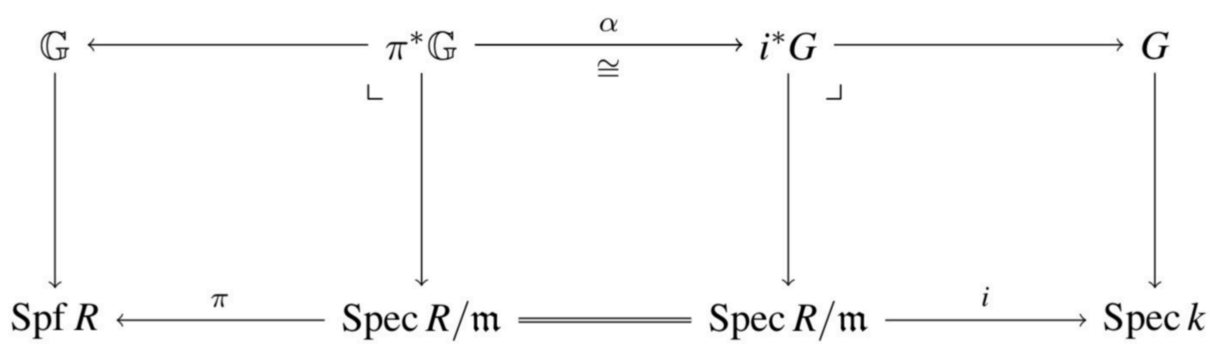

deformation of G with a level-$\Gamma_0(p^n)$ structure $:= (\mathbb{G}, \mathbb{H})$ with $\mathbb{H}$ a cyclic degree-p^n subgroup

$= \psi\colon \mathbb{G} \to \mathbb{G}/\mathbb{H}$ over an extension of R
which lifts the relative Frobenius $\mathrm{Frob}^n\colon G \to G^{(p^n)}$

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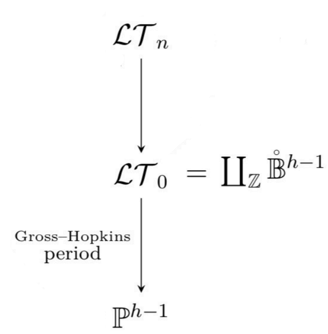

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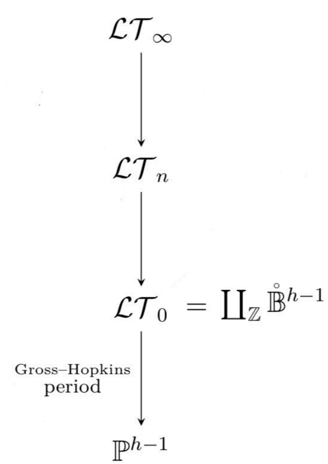

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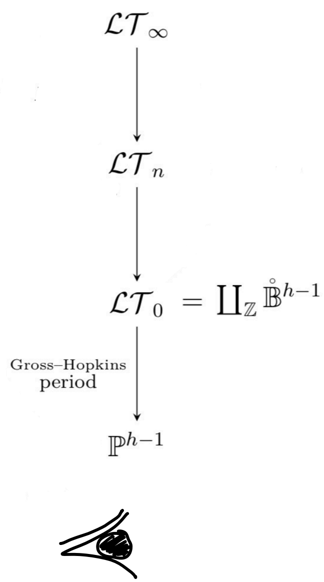

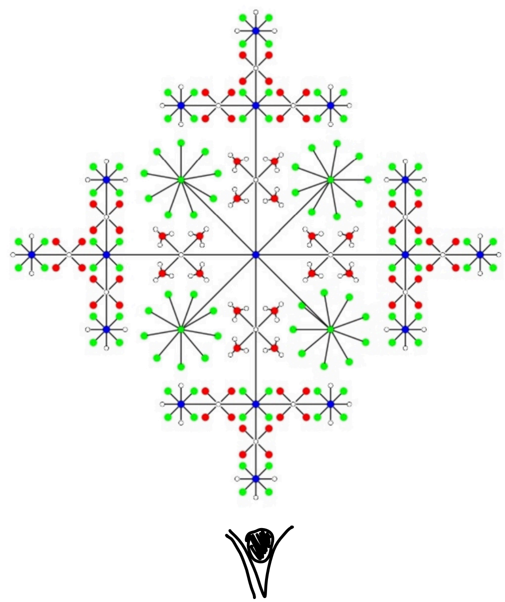

First portraits: A sheaf over the Lubin–Tate tower

- Koszul complexes for computing unstable chromatic homotopy of spheres

[Devinatz–Hopkins '04, Wang '15] There is a homology-to-homotopy SS converging to the v_n-periodic homotopy groups of the q-dimensional sphere

$$H_c^*\big(\mathcal{G}_h; E_*^\wedge \Phi_h(S^q)\big) \implies v_n^{-1}\pi_* S^q$$

whose E_2-page is the continuous group cohomology of the h'th Morava stabilizer group $\mathcal{G}_h$ with coefficients in the completed E-homology of the Bousfield–Kuhn functor Φ_h applied to the q-sphere.

This can be viewed as a homotopy-fixed-point SS with symmetry group $\mathcal{G}_h$.

[Behrens–Rezk '20, Rezk, Z. '17] To compute its E_2-page, the $\mathcal{G}_h$-modules of E-homology groups can be computed from a certain Koszul complex of rings of E-power operations. Moreover, these rings can be derived from A_n in the Lubin–Tate tower, so that with this modular interpretation, the Koszul complex has explicit formulas.

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- Filtration (in the case of $h = 2$ for simplicity)
  - Have a sequence of unstable spheres
  - Applying $E_0^\wedge \Phi_2(-)$, get a sequence of Koszul complexes

$$A_0 = W(\overline{\mathbb{F}}_p)[\![v_1]\!] \cong E^0(\mathrm{pt})$$

$$A_1 = W(\overline{\mathbb{F}}_p)[\alpha_1, \alpha_1']/(\alpha_1\alpha_1' - p) \cong E^0(B\Sigma_p)/I_{\mathrm{tr}}$$

$$\begin{array}{ccccc} E^0 & \xrightarrow{\psi} & E^0(B\Sigma_p)/I & \xrightarrow{\psi} & E^0\big(B(\Sigma_p \wr \Sigma_p)\big)/I' \\ v_1 \mapsto v_1' & & & \alpha_1 \mapsto \alpha_1' & \end{array}$$

  - More concretely, this forms a sheaf of functions over the moduli, filtered by orders m of pole at the irreducible component cut out by α_1, which precisely correspond to dimensions $q = 2m + 1$ of the spheres.

Part of calculations at $p = 3$

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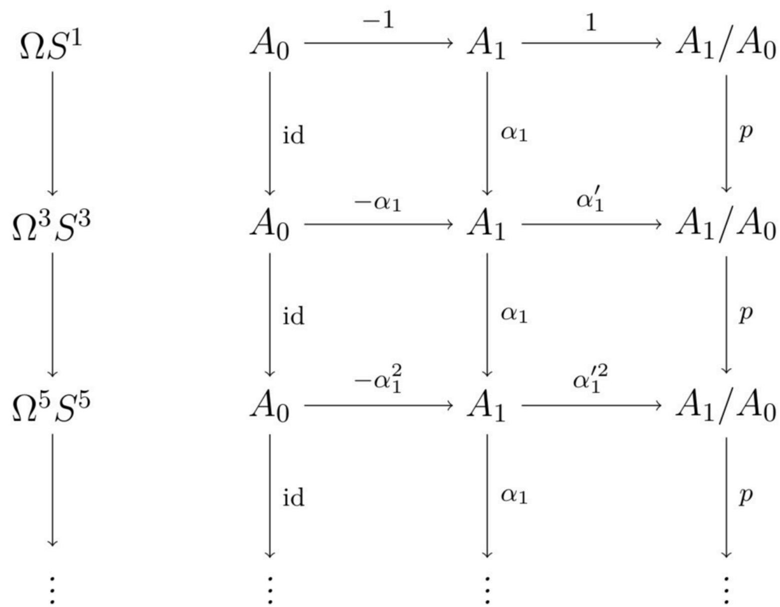

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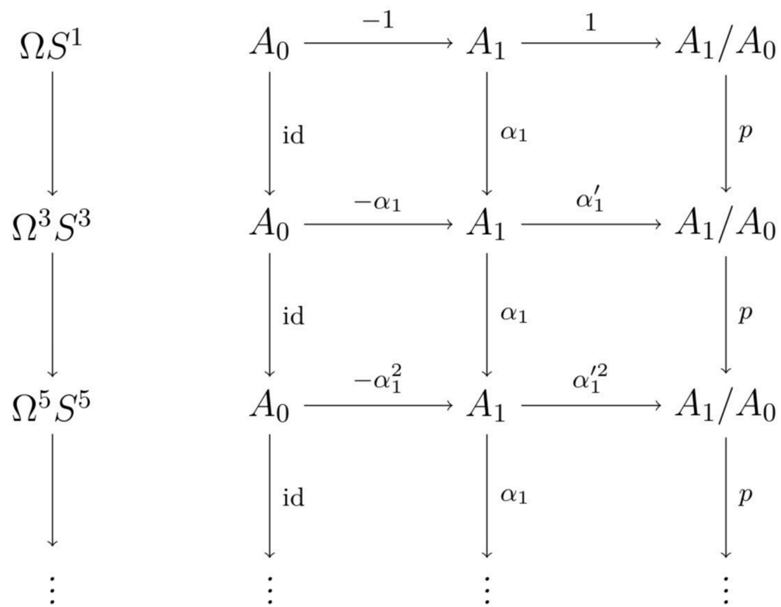

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- Filtration (in the case of $h = 2$ for simplicity)
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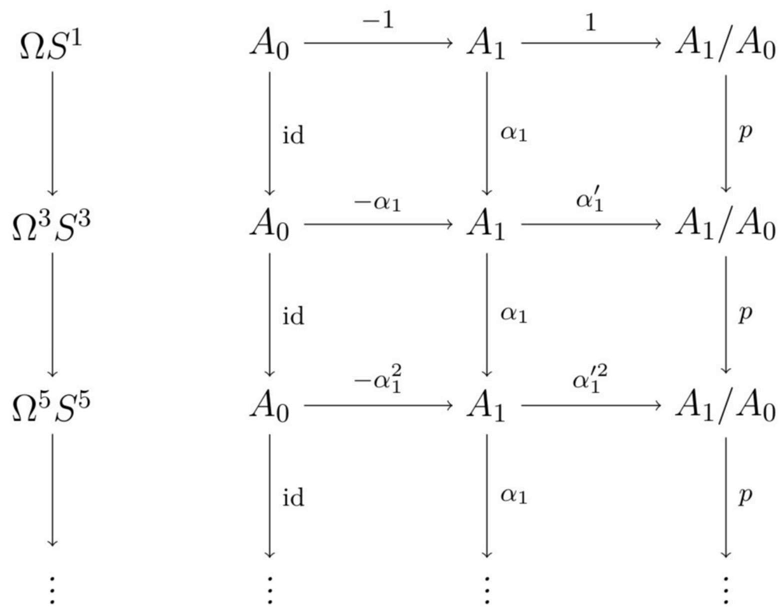

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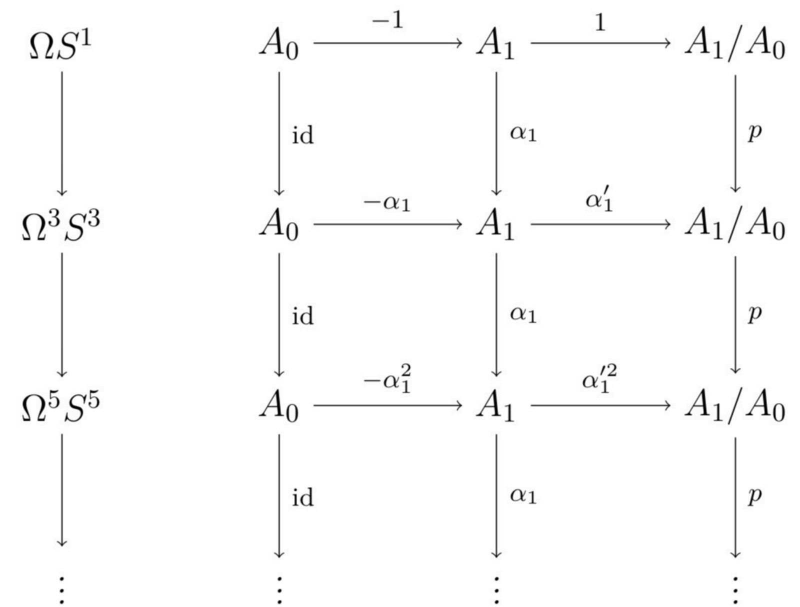

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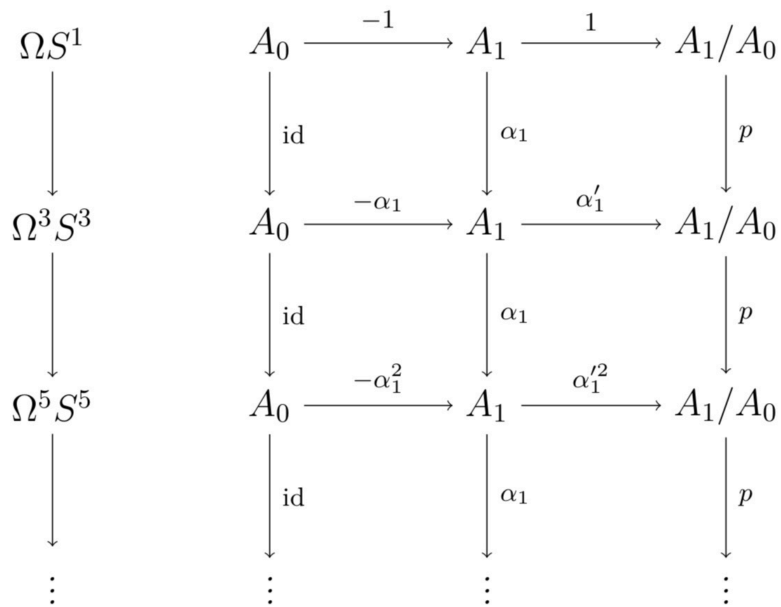

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c_4	c_6	c_4^2	c_4c_6	$\Delta,\ c_4^3$	$c_4^2c_6$	$c_4\Delta,\ c_4^4$	$c_6\Delta,\ c_4^3c_6$
a_1a_3	$9a_3^2$	$a_1a_3c_4$	$9a_3^2c_4$	$a_1a_3c_4^2$	$9a_3^2c_4^2$	$a_1a_3c_4^3$	$9a_3^2c_4^3$
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	a_3^2	$a_2x_0^3 - 2a_4x_0^2$	$a_3^2c_4$	$27a_3^4 \sim a_3^2c_6$	$a_3^2c_4^2$	$a_3^2c_6c_4$	$a_3^2c_4^3$
		x_0^4	$a_1a_3^3(?)$	$9a_3^4$	$a_1a_3^3c_4$	$9a_3^4c_4$	$a_1a_3^3c_4^2$
			x_0^5	$3a_3^4$	$a_1^2a_3^4$	$3a_3^4c_4$	$a_1^2a_3^4c_4$
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First portraits: A filtered, equivariant, quasi-syntomic sheaf

- Symmetries
  - The sequence of Koszul complexes is equivariant with respect to the action of the Morava stabilizer group $\mathcal{G}_h \cong \mathcal{O}_D^{\times}$ (with D / F = cent. div. alg. of inv. $1/h$).
  - Both $GL_h(F)$ (its congruence subgroups K_n) and $D^{\times}$ act on the Lubin–Tate tower, realizing the Jacquet–Langlands correspondence.
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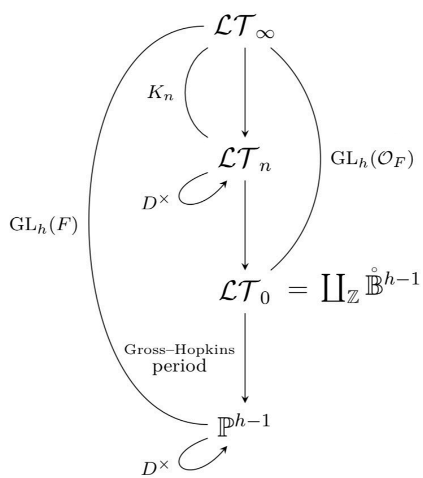

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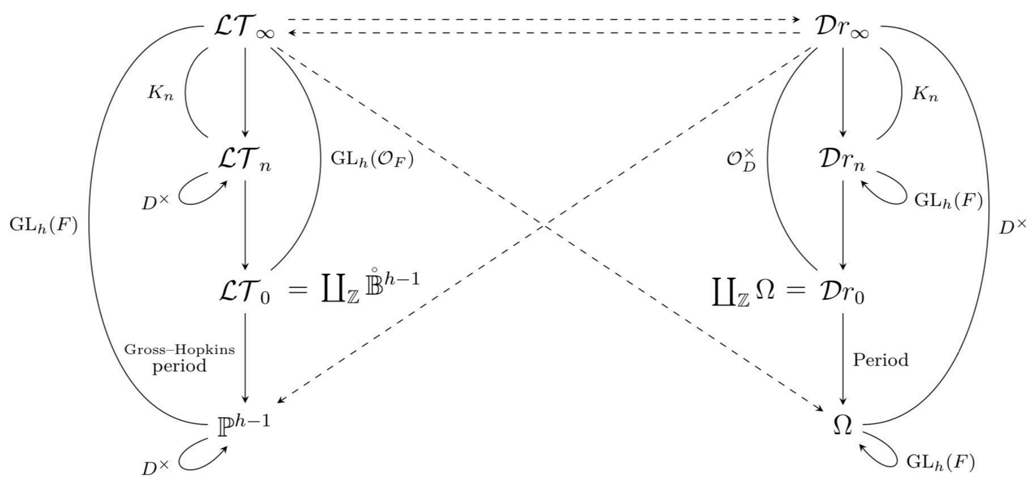

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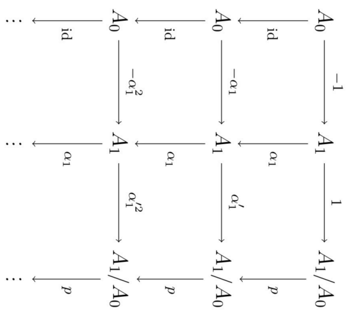

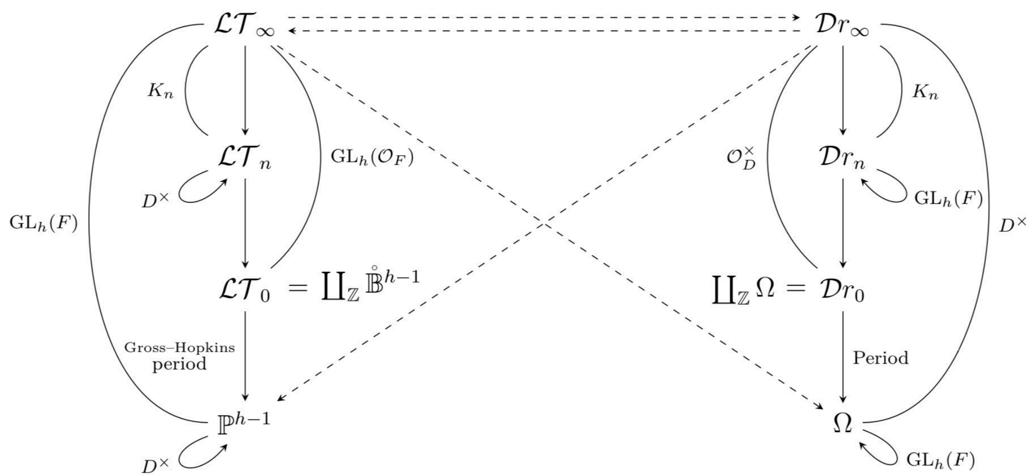

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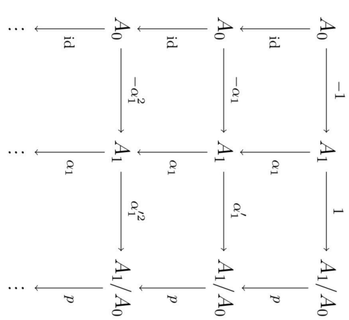

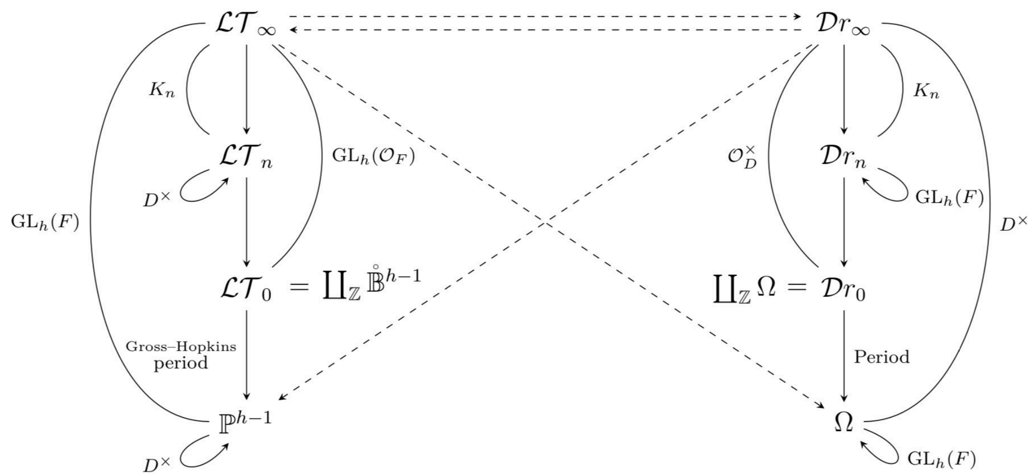

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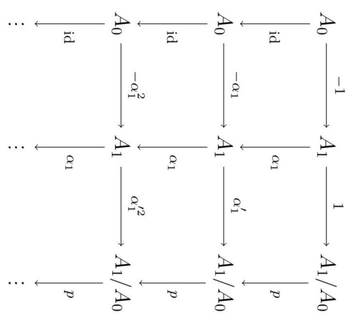

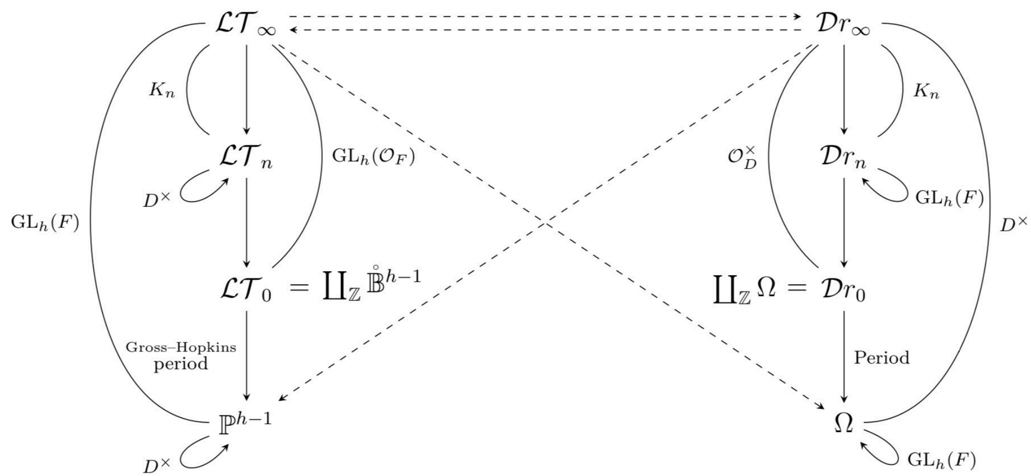

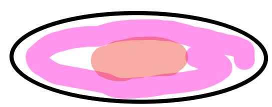

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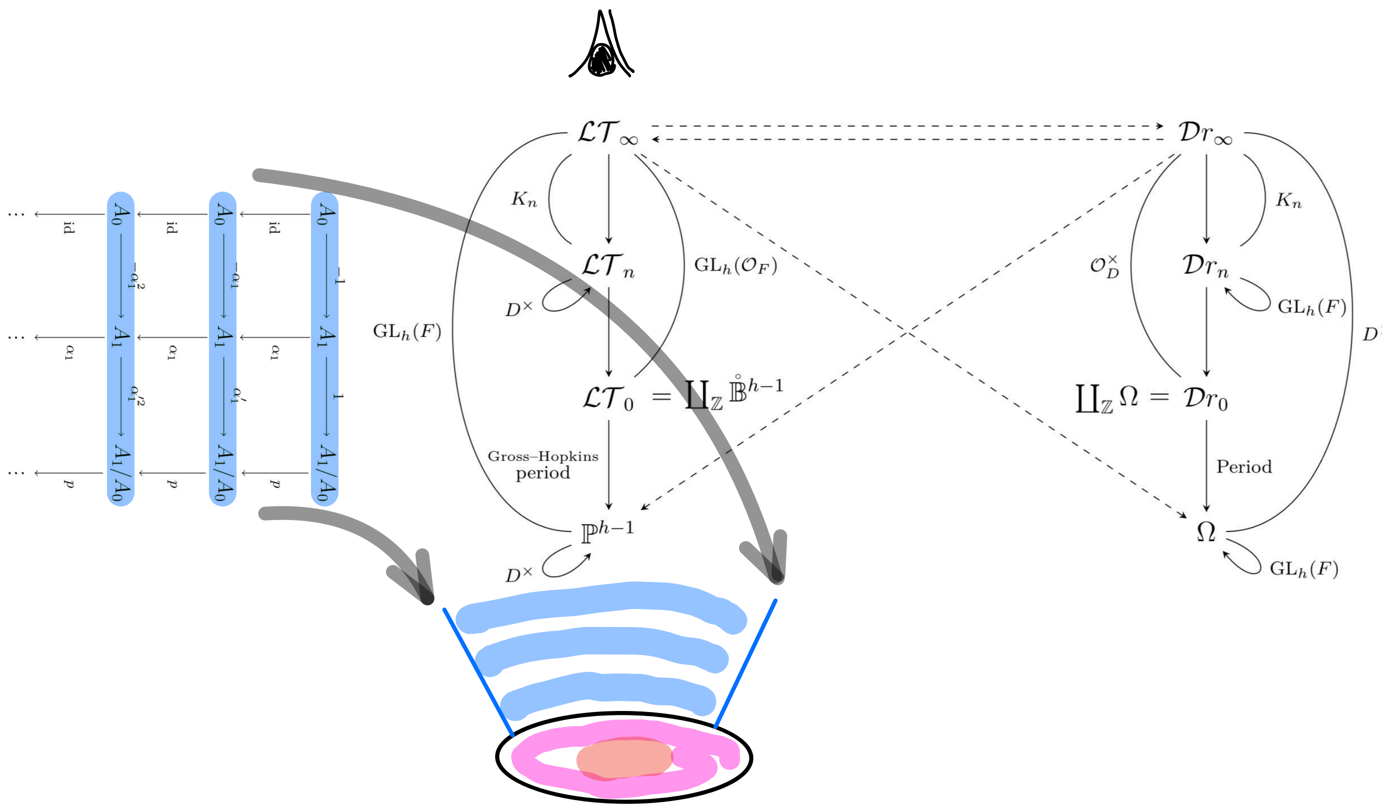

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Second portraits: Context and motivations

- Topological classifications for physical systems

  Understanding continuous evolution of physical systems at the micro or quantum scale has a real impact on the larger-scale properties of materials. For example, holography is made possible via exceptional optical devices.

  Moduli spaces of physical systems, especially their singular loci, play a pivotal role in designing such. Topological classifications enable physicists to fine-tune and create materials that can “do wonders” and cannot be found in nature, e.g., making invisible cloaks and other absorption devices.

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Second portraits: Algebraic setup

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  - Physical systems root in symmetries. For example, quantum mechanical systems can be described by their Hamiltonians, whose mathematical bearings are conventionally Hermitian matrices. Here, Hermiticity guarantees that the eigenvalues are real, corresponding to the fact that energies of the systems are observed to be real.
  - More recently, physicists have begun to model open systems by relaxing the Hermitian symmetry to allow eigenvalues with a nonzero imaginary part. This imaginary part measures energy exchange between the system and its surrounding environment. Still, some sorts of symmetry need to be imposed on the matrices to make them physically meaningful.
  - The size of the matrices corresponds to the number of energy band gaps. It is critical to understand degeneracies of eigenvalues and eigenvectors, across which the gaps close and open.

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- [Joint with Hongwei Jia et al.] By imposing the parity–time symmetry and a pseudo-Hermitian symmetry (with respect to a Lorentz-like transformation), we completely classified a generic family of 2-band systems as well as partially for certain 3-band systems:

$$H_2 = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix} \qquad H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

  where the parameters f_i are functions on the 3D momentum space $\{\boldsymbol{k}_x, \boldsymbol{k}_y, \boldsymbol{k}_z\}$. Moreover, my physics collaborators experimentally realized the above 3-band systems by circuits and verified our mathematical models.

- The moduli spaces keep track of eigen-energies (roots of the characteristic polynomial) as well as their corresponding eigenstates (the eigenvectors). Thus, they are "stratified vector bundles." Interestingly, the stratification of the non-isolated singular loci in the base spaces for these gapless 3-band systems reveals transitions among diverse exceptional physical states, providing a test ground for exotic phenomena and anomalous effects.

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- [Goresky–MacPherson '80, Pflaum '01, Ross '23, ...] A stratified space is a reasonable topological space X together with a reasonable partition Σ into reasonable subspaces such that
  - each stratum $S \in \Sigma$ is a smooth manifold and
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  A stratified morphism is a continuous map between stratified spaces that preserves the stratifications and restricts to be smooth over each stratum.

- [Ross '23] A stratified vector bundle consists of two stratified spaces (E, Σ_E) and (B, Σ_B) together with a stratified morphism $p \colon E \to B$ such that
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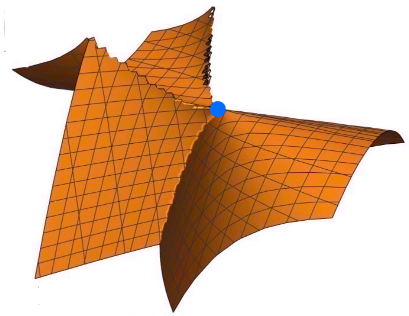

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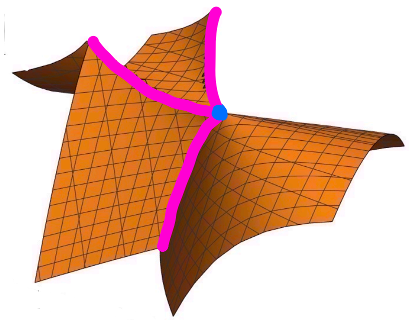

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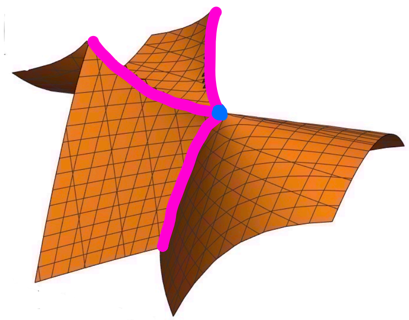

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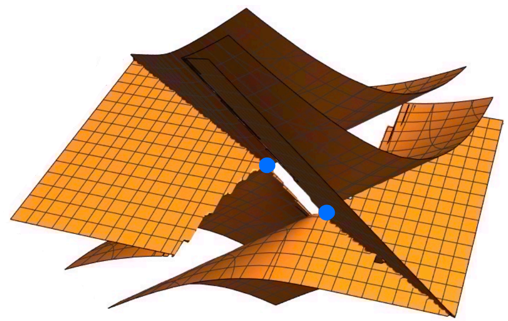

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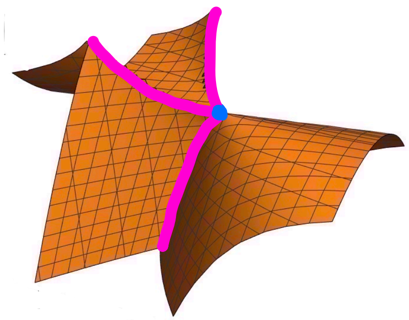

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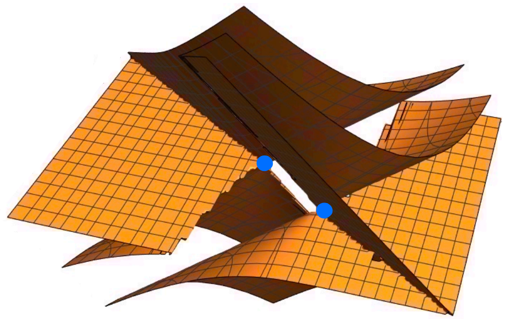

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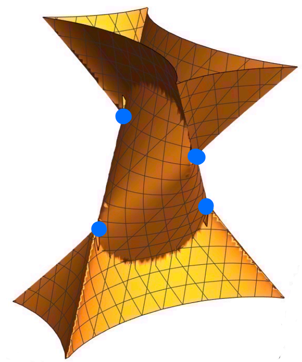

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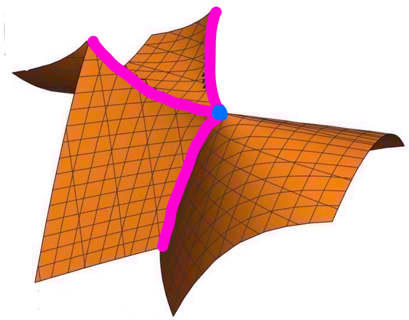

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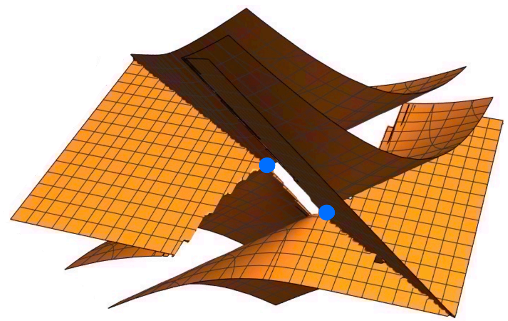

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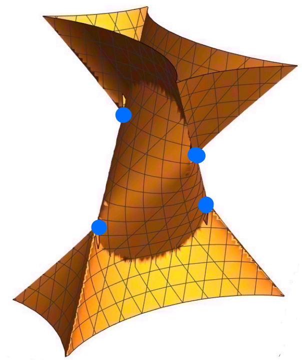

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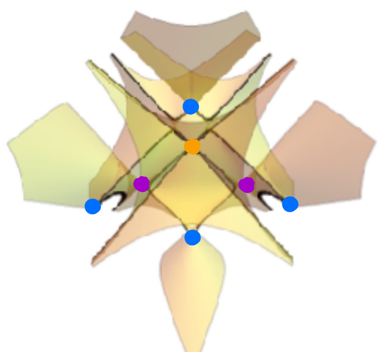

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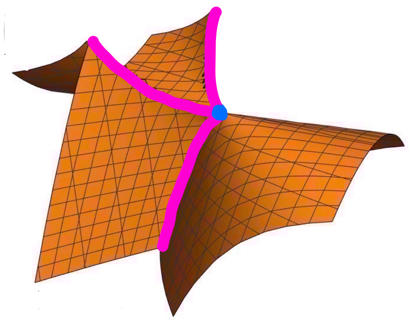

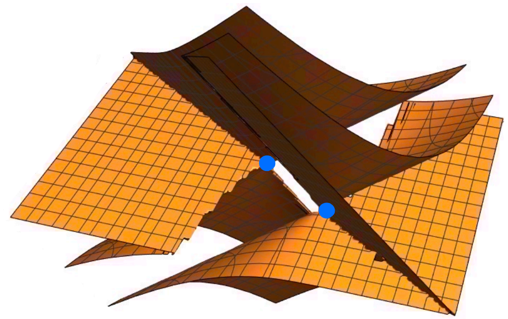

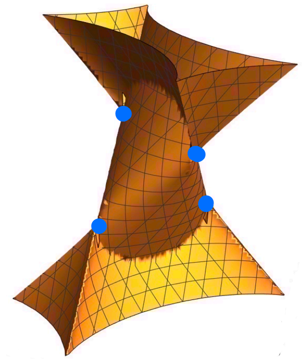

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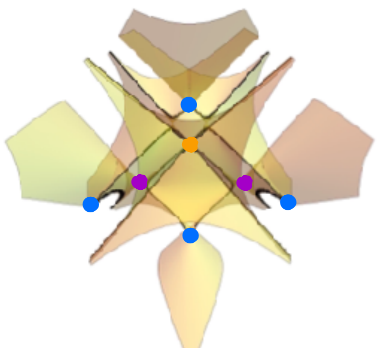

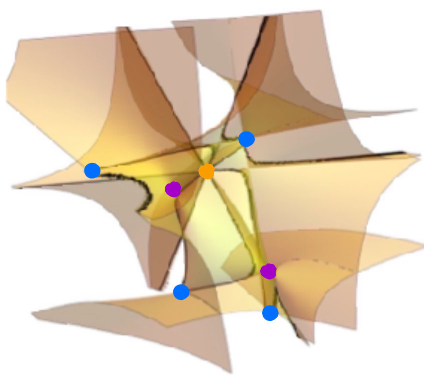

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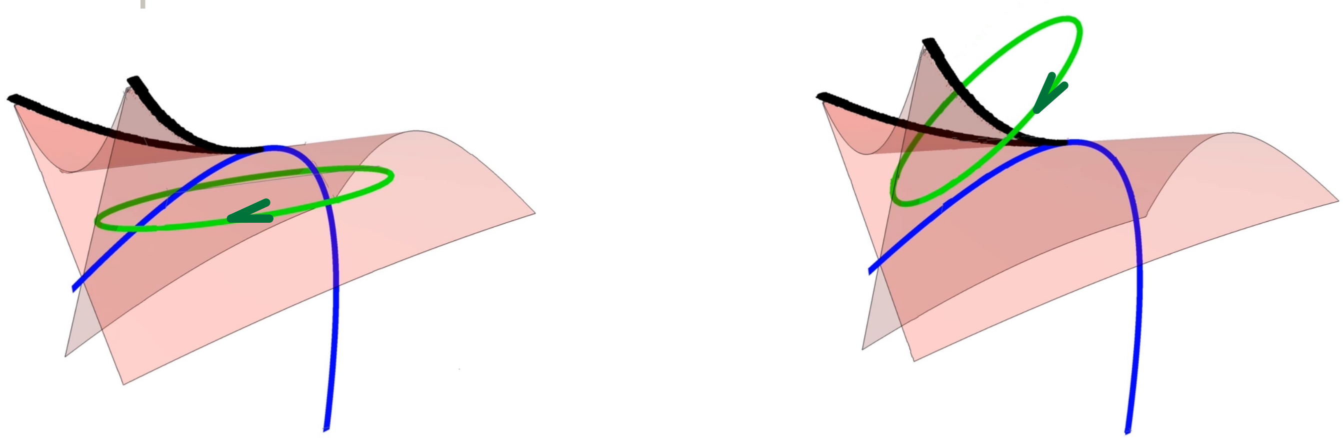

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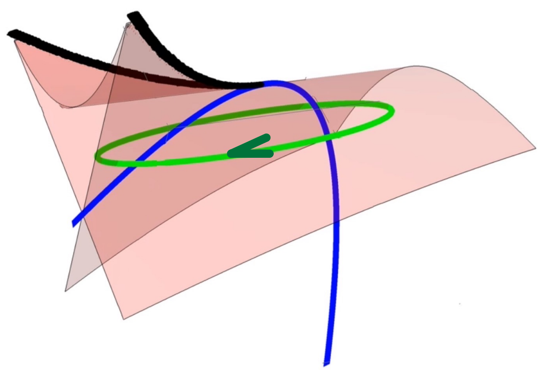

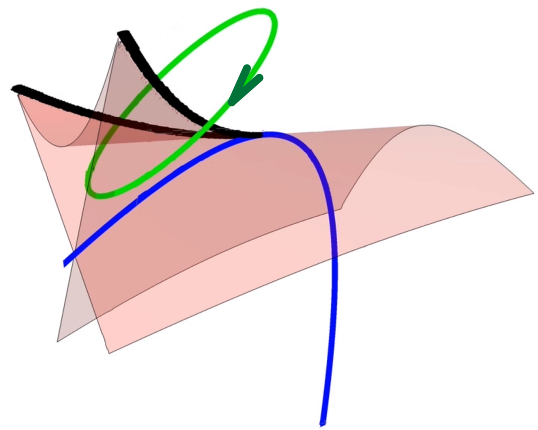

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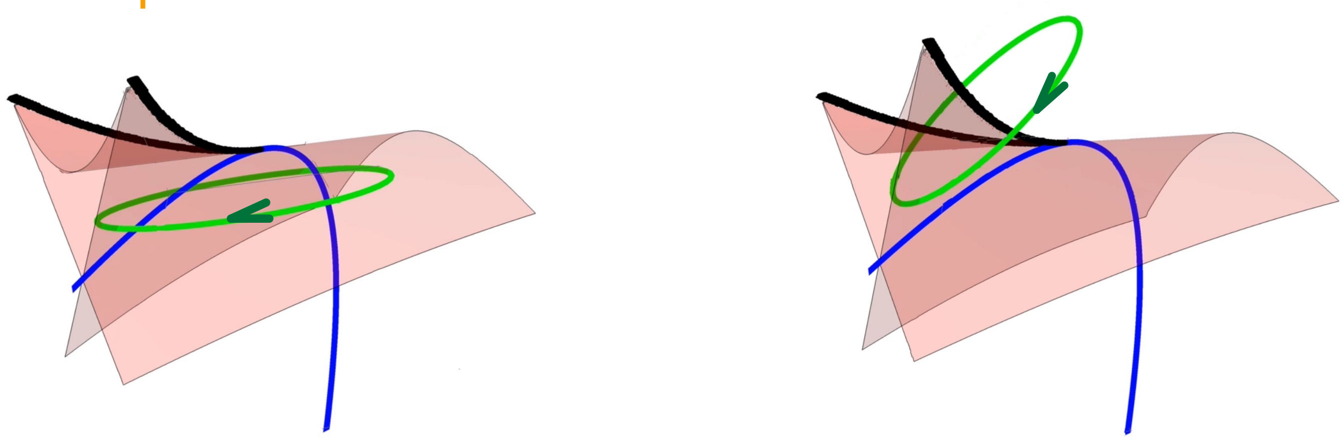

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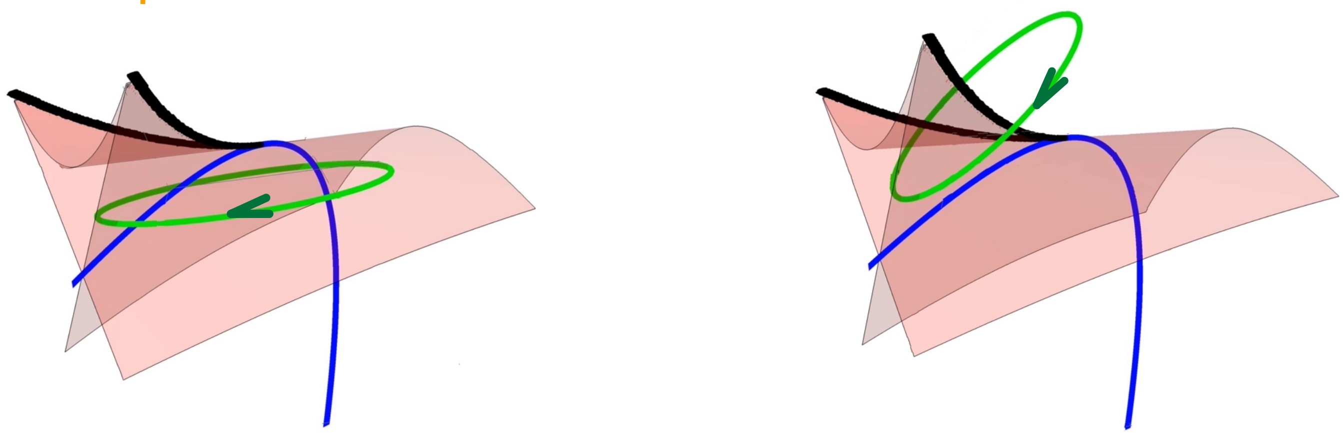

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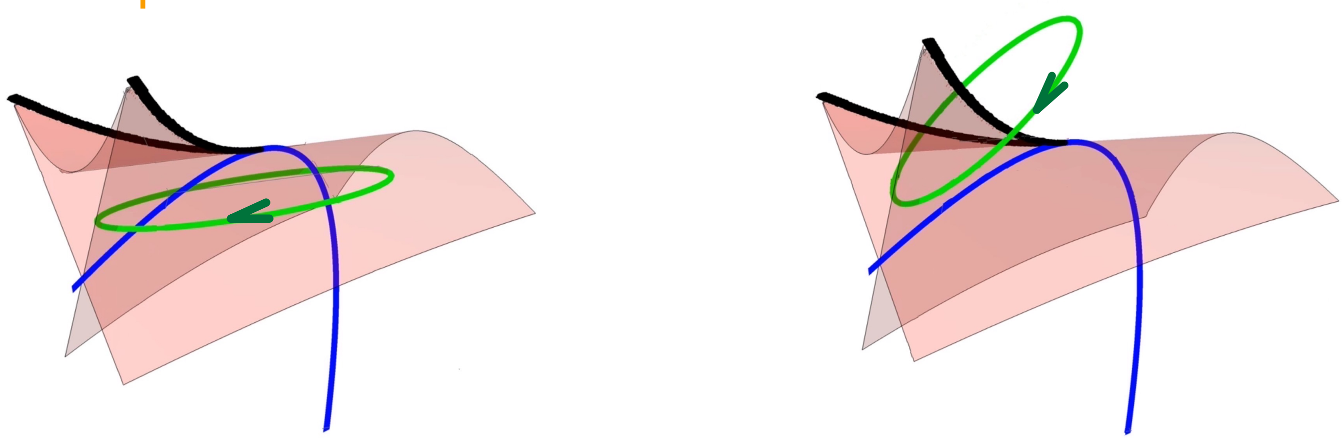

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  - Using persistent homology, Carlsson, Ishkhanov, de Silva, and Zomorodian qualitatively analyzed approximately 4.5×10^6 high-contrast local patches of natural images obtained by van Hateren and van der Schaaf and previously studied by Lee, Mumford, and Petersen. In their 2008 article, they discovered that as vectors of pixels, the image data were unevenly distributed over a Klein bottle within the 7-dimensional Euclidean sphere! We may view the Klein bottle as a moduli space for local image data.
  - A decade later, Love, Filippenko, Maroulas, and Carlsson have made the Klein bottle as a topological input for designing convolutional layers in neural networks that learn image data. Moreover, they have incorporated the tangent bundle of a Klein bottle into TCNNs for learning video data. Both learnings achieved higher accuracies with smaller training sets.

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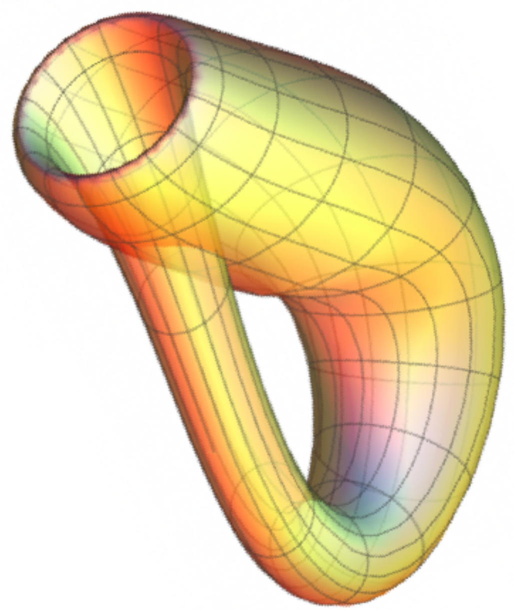

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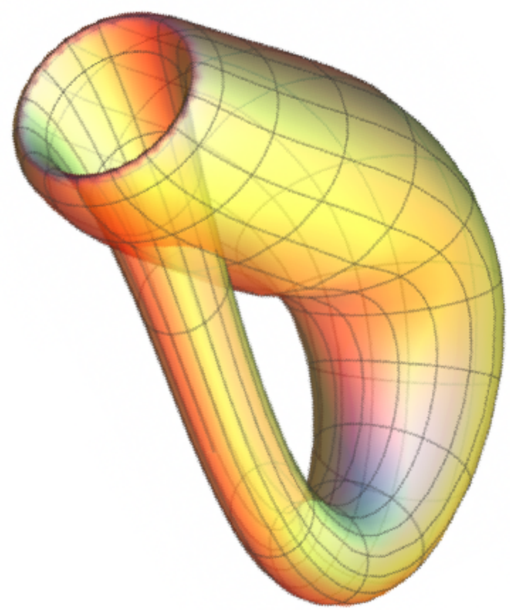

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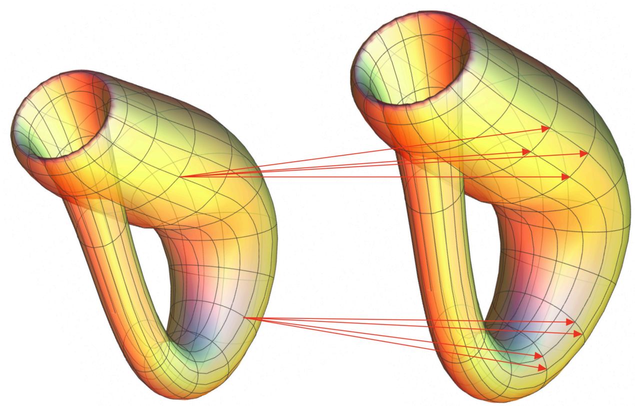

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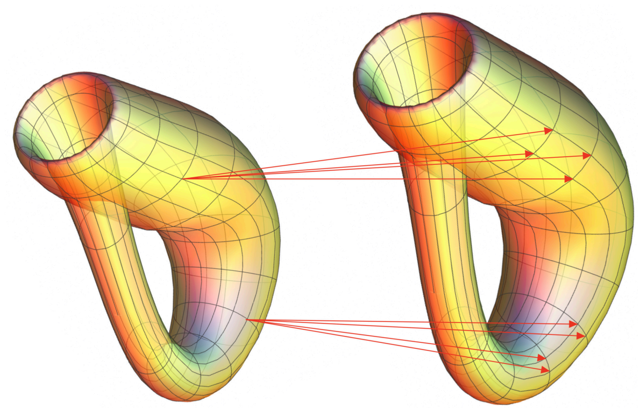

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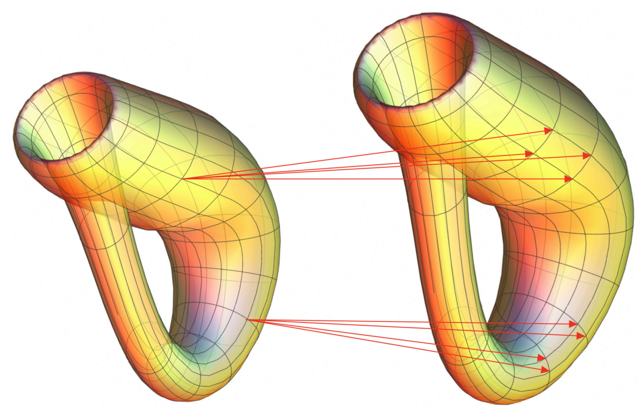

Third portraits: Moduli spaces for speech data

- [Joint with Pingyao Feng et al.] Motivated by the works of Carlsson and his collaborators', in consultation with Meng Yu of Tencent AI Lab, we have been investigating analogous questions for speech signals, with the additional tool of time-delay embedding for turning time series data to point clouds in Euclidean spaces.

  - For phonetic data, linguists created a charted "moduli space" of vowels:

  - Using speech files from SpeechBox, our topological approach achieved an average accuracy exceeding 95% in classifying voiced and voiceless consonants via machine learning.

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The vertical axis of the chart denotes vowel height. Vowels pronounced with the tongue lowered are at the bottom and raised are at the top. The horizontal axis of the chart denotes vowel backness. Vowels with the tongue moved towards the front of the mouth are in the left of the chart, while those with the tongue moved to the back are placed in right. The last parameter is whether the lips are rounded. At each given spot, vowels on the right and left are rounded and unrounded, respectively.

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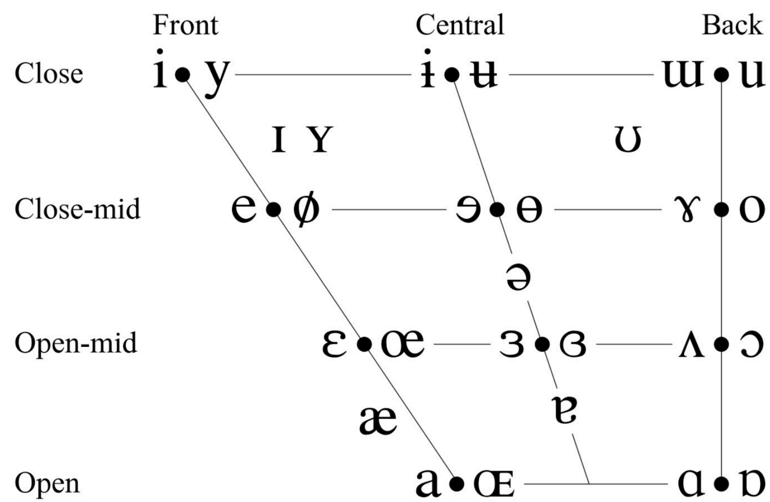

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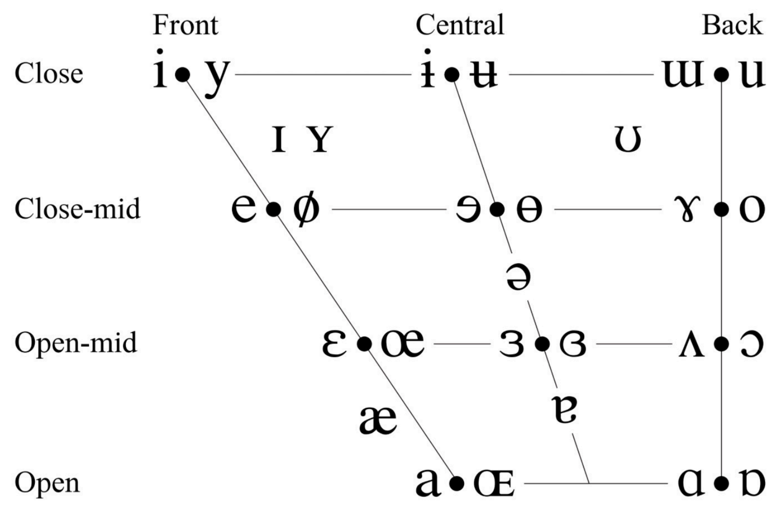

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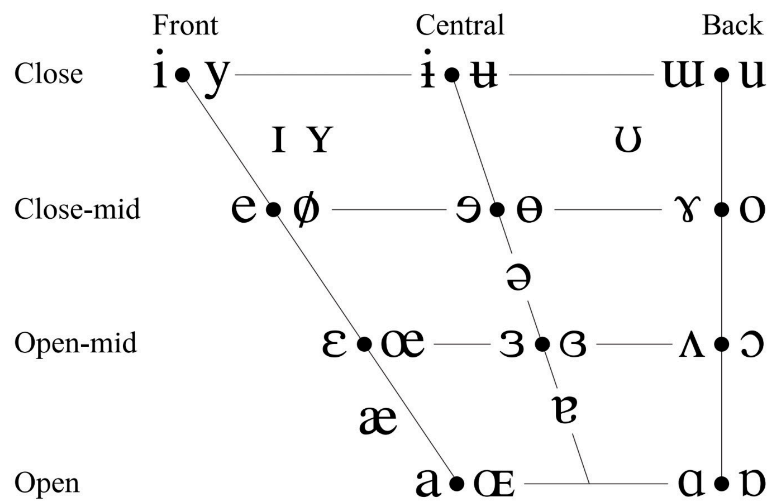

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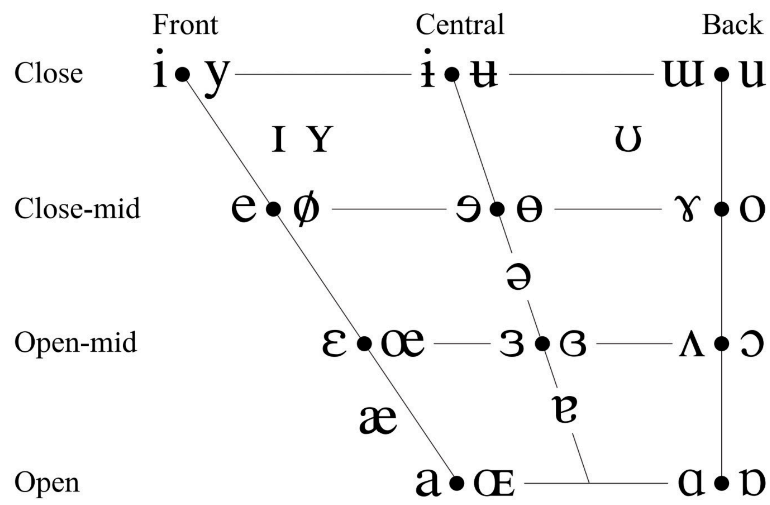

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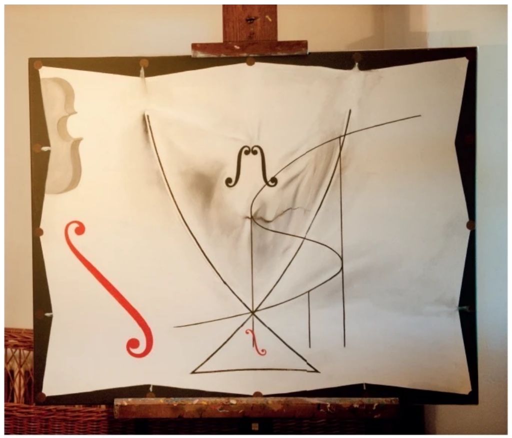

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