Toward calculating unstable higher-periodic homotopy types

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Electronic computational homotopy theory seminar

Rational homotopy theory

Theorem (Quillen '69)

There are equivalences of homotopy categories

$$\mathrm{Ho}(\mathrm{Top}_{\mathbb{Q}}^{\geq 2}) \simeq \mathrm{Ho}(\mathrm{DGCoalg}_{\mathbb{Q}}^{\geq 2}) \simeq \mathrm{Ho}(\mathrm{DGLie}_{\mathbb{Q}}^{\geq 1})$$

between simply-connected rational spaces, simply-connected differential graded cocommutative coalgebras over $\mathbb{Q}$, and connected differential graded Lie algebras over $\mathbb{Q}$.

simply-connected $X \rightsquigarrow C_{\mathbb{Q}}(X) \in \mathrm{DGCoalg}_{\mathbb{Q}}$

$$H_*(C_{\mathbb{Q}}(X)) \cong H_*(X;\mathbb{Q})$$

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simply-connected $X \rightsquigarrow L_{\mathbb{Q}}(X) \in \mathrm{DGLie}_{\mathbb{Q}}$

$$H_*\big(L_{\mathbb{Q}}(X)\big) \cong \pi_{*+1}(X) \otimes \mathbb{Q}$$

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simply-connected $X \rightsquigarrow A_{\mathbb{Q}}(X) \in \mathrm{DGAlg}_{\mathbb{Q}}$, Sullivan '77, minimal models
finite type

$$H^*\big(A_{\mathbb{Q}}(X)\big) \cong H^*(X; \mathbb{Q})$$

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simply-connected $X \rightsquigarrow$ models for the $\mathbb{Q}$-homotopy type of X

v_n-self maps

Example

$$S^d \xrightarrow{p} S^d \to S^d/p \quad \text{for any } d$$

induces an isomorphism in $H_*(-;\mathbb{Q})$.

S^d admits v_0-*self maps*, with $v_0 = p$ a prime.

Theorem (Hopkins-Smith '98)

Let V be a p-local finite complex *of type* n, i.e. $K(n)_*(V) \neq 0$ but $K(i)_*(V) = 0$ for $i < n$. Then V admits a v_n-*self map*

$$v\colon \Sigma^k V \to V$$

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Example (Adams '66)

$$v\colon \Sigma^q S^d/p \to S^d/p \quad \text{for } q = \begin{cases} 2p-2 & \text{if } p \text{ is odd} \\ 8 & \text{if } p = 2 \end{cases}$$

induces an isomorphism in K-theory.
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The stable composite

$$\mathbf{S}^q \to \Sigma^q \mathbf{S}^0/p \xrightarrow{v} \mathbf{S}^0/p \to \mathbf{S}^1$$

is α_1 (the first element of order p in π_*^s) for p odd, and 8σ (where σ is the generator of π_7^s) for $p = 2$.

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v_n-periodic homotopy groups

Recall rational homology $H_*(-;\mathbb{Q})$
rational homotopy $\pi_*(-) \otimes \mathbb{Q}$

Now v_n-periodic homology $K(n)_*(-)$ There is an underlying prime p.
v_n-periodic homotopy ?

Observe (Bousfield '01, Kuhn '08) $v_n^{-1}\pi_*(X;V) \cong \pi_*\Phi_V(X)$

Define *Bousfield-Kuhn functor* $\Phi_n(X) \coloneqq \operatorname{holim}_i \Phi_{\mathbf{V}_i^\vee}(X)$
with $\mathbf{V}_i$ such that $\operatorname{holim}_i v_n^{-1}\mathbf{V}_i \simeq \mathbf{S}_{T(n)}$

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Now v_n-periodic homology $K(n)_*(-)$ There is an underlying prime p.
v_n-periodic homotopy $v_n^{-1}\pi_*(-;V) := [\Sigma^* V, -] \underset{\mathbb{Z}[v]}{\otimes} \mathbb{Z}[v^{\pm 1}]$

Observe (Bousfield '01, Kuhn '08) $v_n^{-1}\pi_*(X;V) \cong \pi_* \Phi_{\Sigma^\infty V}(X)$

Define *Bousfield-Kuhn functor* $\Phi_n(X) := \operatorname{holim}_i \Phi_{\mathbf{V}_i^\vee}(X)$
with $\mathbf{V}_i$ such that $\operatorname{holim}_i v_n^{-1}\mathbf{V}_i \simeq \mathbf{S}_{T(n)}$

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Consider $\mathbf{S}_{T(n)}^{(-)} \colon \mathrm{Ho}(M_n^f \mathrm{Top}_*)^{\mathrm{op}} \to \mathrm{Ho}\big(\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Sp}_{T(n)})\big)$

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Main strategy and ingredients in Behrens and Rezk's proof

(1) Do induction up the Goodwillie towers of both the source and target of the comparison map.

(2) Use the Bousfield-Kan cosimplicial resolution

$$X \to Q^{\bullet+1}X = (QX \rightrightarrows QQX \Rrightarrow \cdots), \quad QX = \Omega^\infty\Sigma^\infty X$$

to reduce to proving the comparison map on QX, for which one needs the Morava E-theory Dyer-Lashof algebra in an essential way.

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Stages $P_k\Phi_{K(n)} \simeq \Phi_{K(n)} P_k \mathrm{Id}$

Layers $D_k\Phi_{K(n)}(X) \simeq (s^{-1}\mathrm{Lie}_k \wedge_{h\Sigma_k} X^{\wedge k})_{K(n)}$

Convergence (Arone-Mahowald '99)

$$\Phi_{K(n)}(S^q) \xrightarrow{\sim} \Phi_{K(n)} P_k \mathrm{Id}(S^q)$$

where $k = p^n$ if q is odd, and $k = 2p^n$ if q is even.

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Convergence (Arone-Mahowald '99)

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An application of the Behrens-Rezk model

Theorem (Z.)

Let E be a Morava E-theory spectrum of height 2, and write $E_*^\wedge(-) := \pi_*(E \wedge -)_{K(2)}$. Then $E_0^\wedge(\Phi_2 S^{2d+1}) \cong 0$ for any $d \geq 0$, and $E_1^\wedge(\Phi_2 S^{2d+1})$ is given by

$$\begin{cases} 0 & \text{if } d = 0 \\ (E_0/p)^{\oplus p-1} & \text{if } d = 1 \end{cases}$$

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$$H_c^s\Big(\mathbb{G}_2; E_t^\wedge(\Phi_2 S^{2d+1})\Big) \Longrightarrow v_2^{-1}\pi_{t-s}(S^{2d+1}) \qquad p \geq 5$$

An application of the Behrens-Rezk model

Idea of proof

(1) Apply Behrens-Rezk, reduced to computing E-theory of TAQ—the algebraic model.

(2) Set up two spectral sequences to compute this, reduced to calculating $\mathrm{Ext}^*_\Gamma(M, N)$.

(3) Compute $\mathrm{Ext}^*_\Gamma(M, N) \cong H^*\mathcal{C}^\bullet(M, N)$.

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Joint work in progress with Wang

- Have a sequence of unstable spheres

$$\Omega S^1 \to \Omega^3 S^3 \to \Omega^5 S^5 \to \cdots$$

- Apply $E_0^\wedge \Phi_2(-)$, get a sequence of Koszul complexes

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Have a $K(n)$-local equivalence

$$\operatorname*{hocolim}_k \Omega^k \Phi_n S^k \simeq \mathbf{S}_{K(n)}$$

- $\underline{n=1}$ (Heuts, Bousfield) The $K(1)$-local sphere $\mathbf{S}_{K(1)}$ admits a "Kuhn filtration" with associated graded

$$L_{K(1)} \Sigma^{-2d-1} \bigoplus_{m \geq 1} (\mathbf{S}^{2d}/p^d)^{\wedge m}_{h\Sigma_m} \qquad (p \text{ odd})$$

- $\underline{n=2}$ In the "EHP filtration" define the fiber $K_d \to \Omega^{2d-1} S^{2d-1} \to \Omega^{2d+1} S^{2d+1}$. Using the *double Koszul complex* we calculated $E_0^{\wedge} \Phi_2(K_d)$ explicitly modulo p.

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  $bb' \equiv (a' - a^p)(a'^p - a) \mod p$

Thank you.