Structured ring spectra via spectral moduli problems

joint w/ X. Ma
G. Wang

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Context & Motivations Morava E-theory, power operations

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Results Lifts of rings of power operations as E-infinity ring spectra

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Deformations of Frobenius isogenies
(of formal groups)

Algebraic geometry

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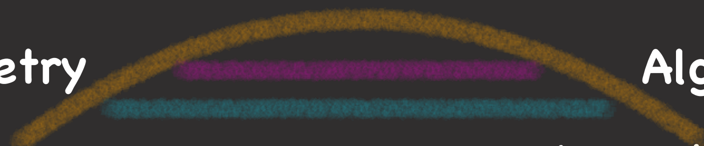

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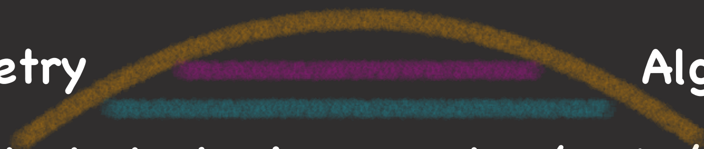

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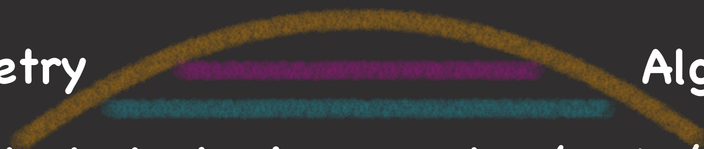

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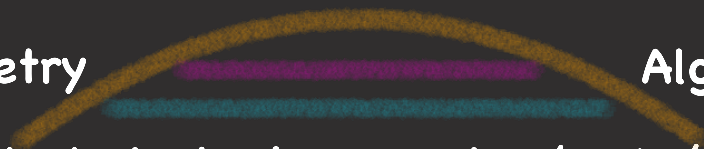
Algebraic geometry  **Algebraic topology**

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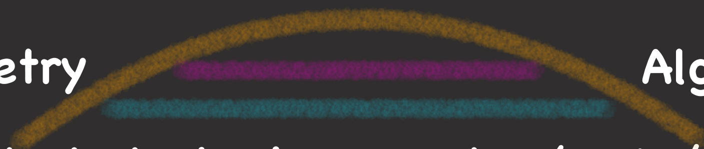

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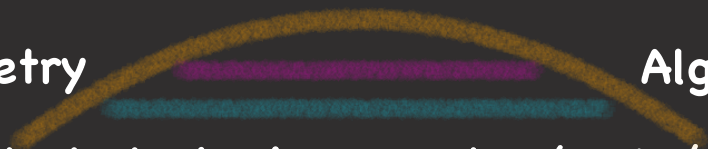

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the n'th total power operation $\psi^n(x)$

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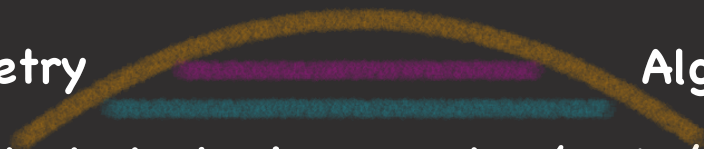

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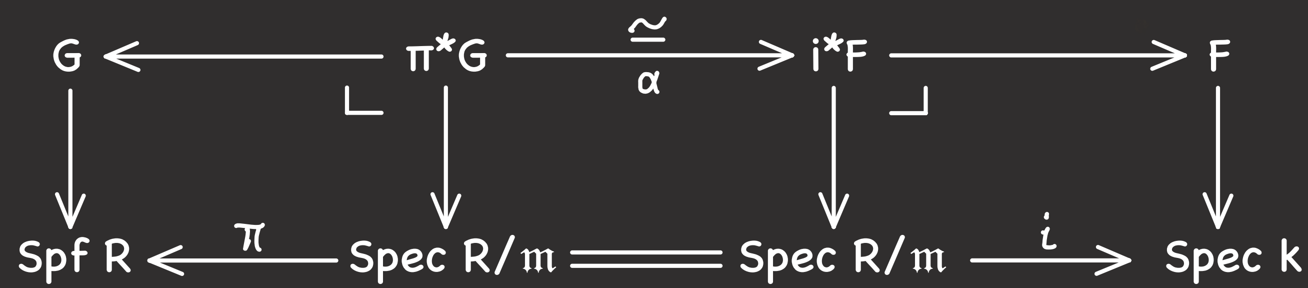

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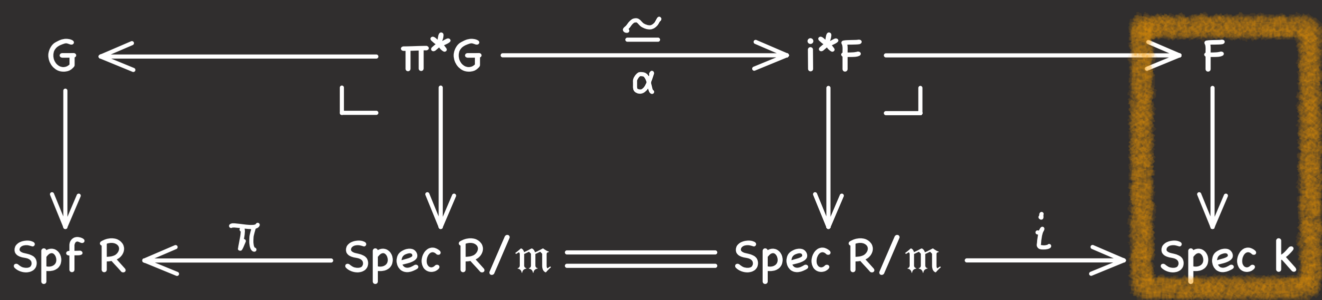

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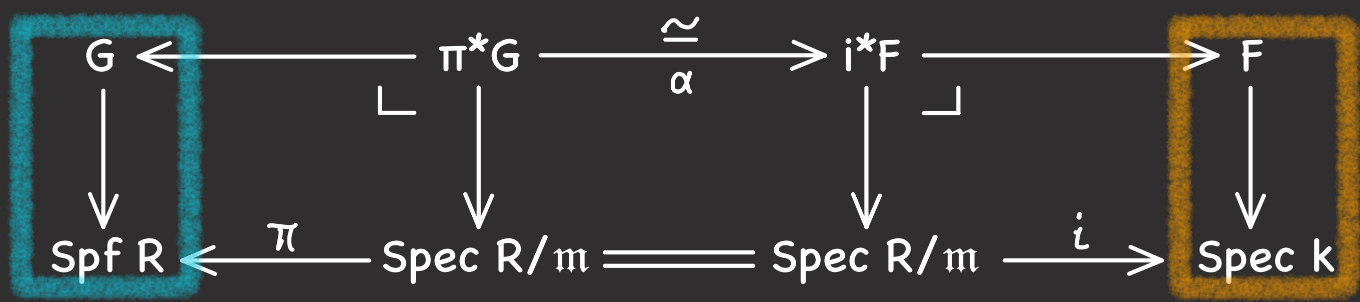

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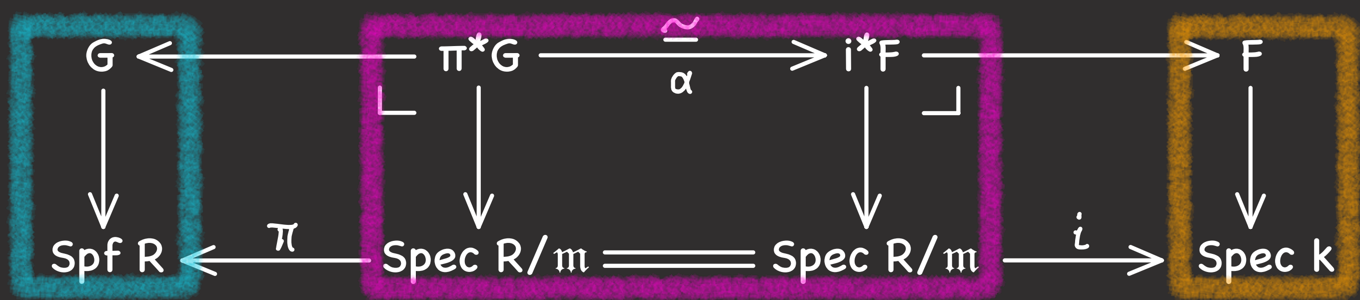

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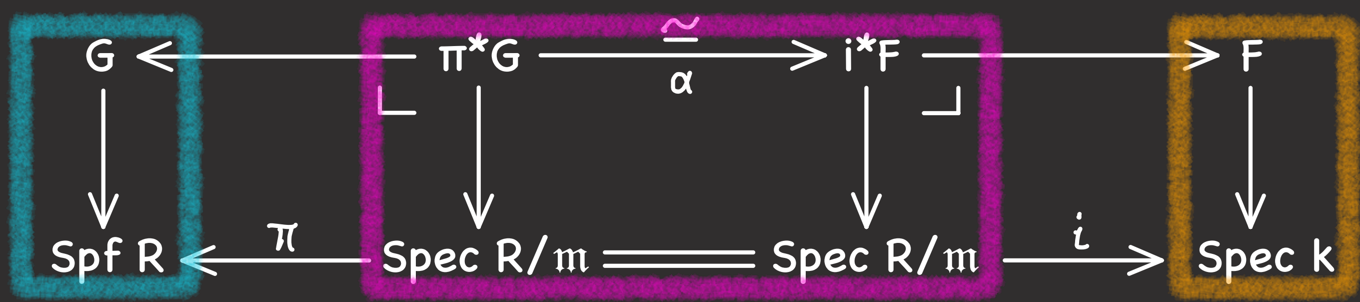

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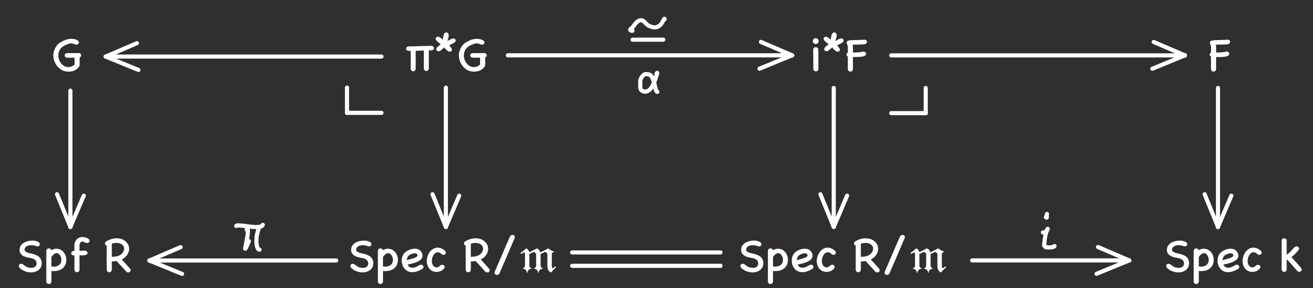

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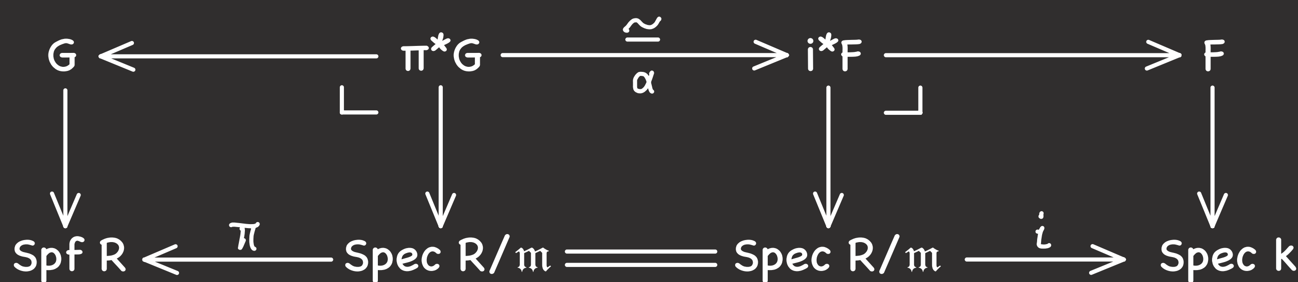

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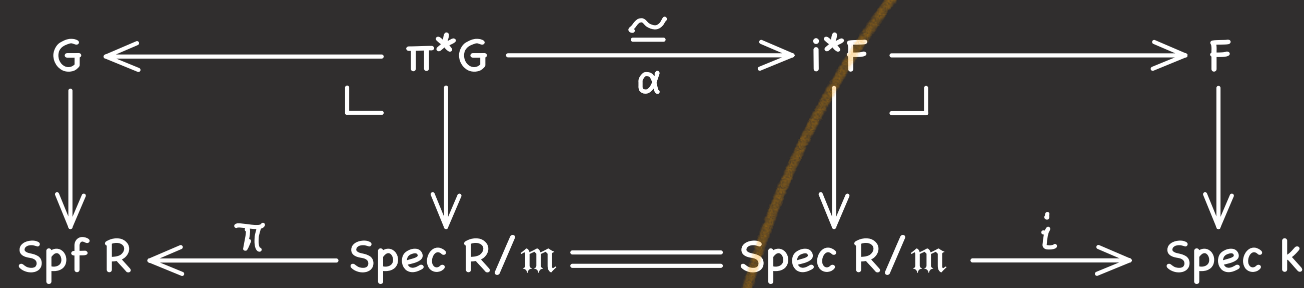

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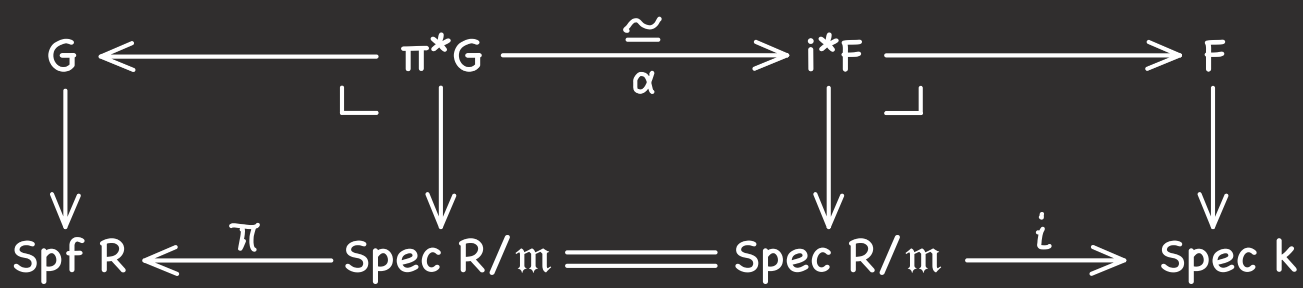

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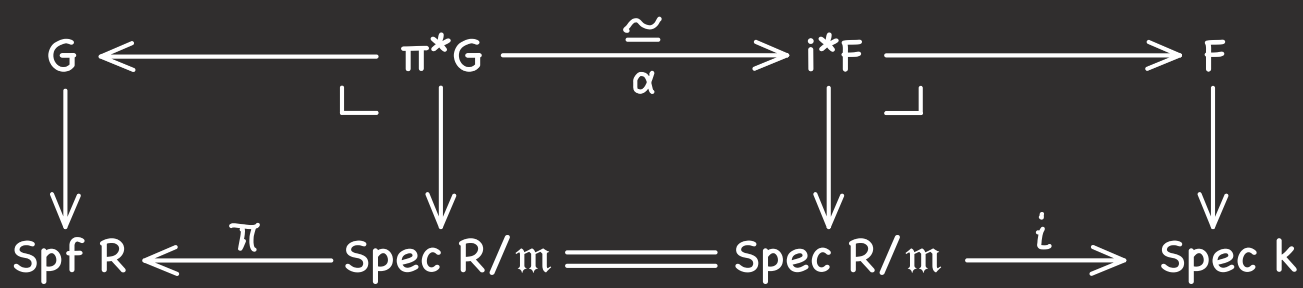

and H a cyclic degree-p^r subgroup of $G\}$

$= \{\psi: G \to G/H$ over an extension of R which lifts the relative Frobenius $\mathrm{Frob}^r: F \to F^{(p^r)}\}$

Example A_r, $r = 1$ (crux for computing E-power operations and the Dyer-Lashof algebra)

$n = 1$, $A_1 \simeq A_0 \simeq W(k)$

$n = 2$ (Rezk '08, Z. '14-19), $A_1 \simeq W(k)[\![u_1]\!][x]/(x-p)(x+(-1)^p)^p - (u_1 - p^2 + (-1)^p)x$

Theorem (Ando '95, Z. '20) Given any F/k, $\exists!$ complex orientation $MUP \to E(F/k)$ that is an H_∞-ring map.

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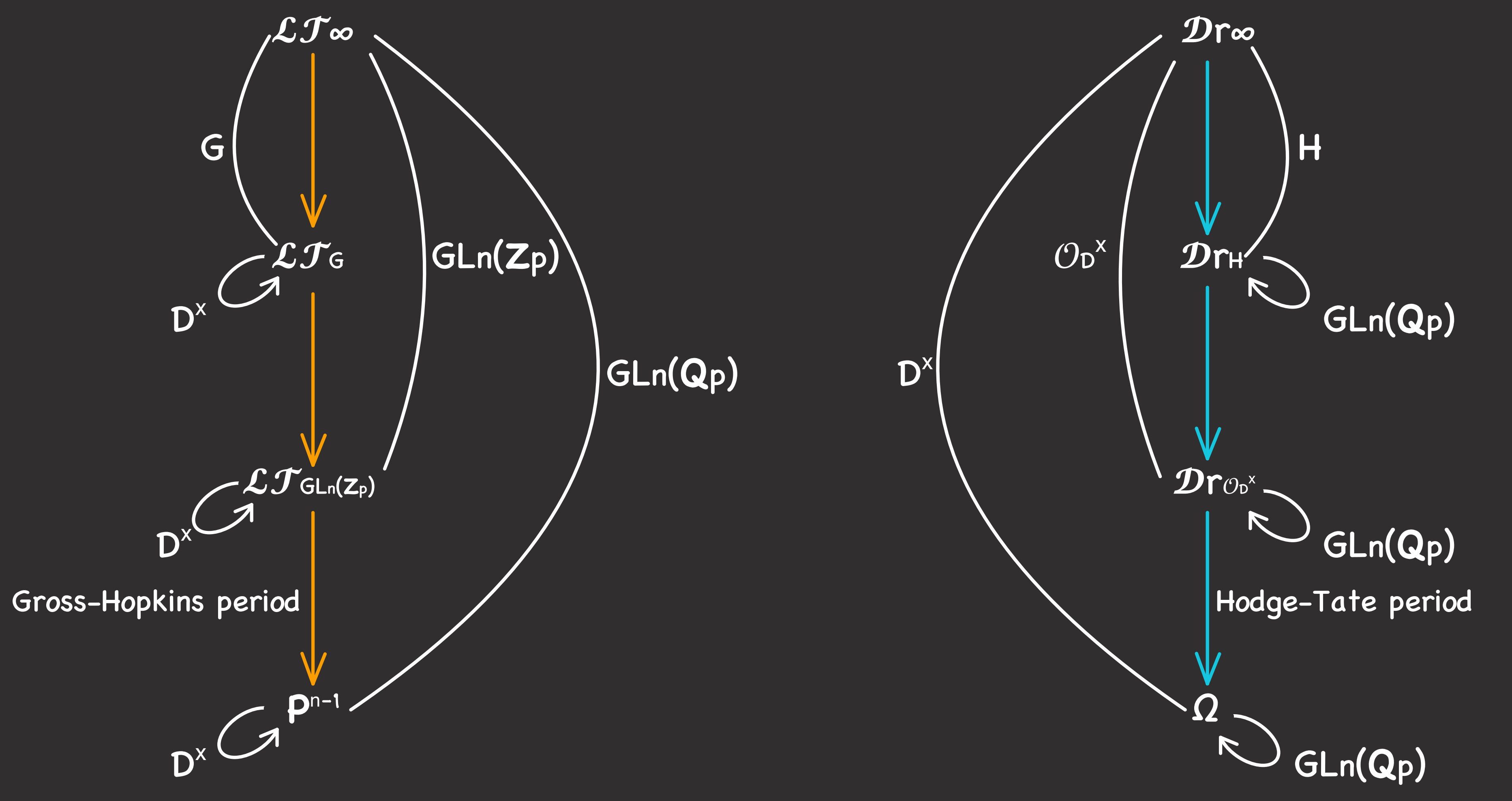
Theorem (Faltings, Fargues '08, Scholze-Weinstein '13) ∃ an isomorphism between the Lubin-Tate tower of moduli of deformations of formal groups and the Drinfeld tower of moduli of deformations of shtukas, which is equivariant with respect to the action of the triple product group $GL_n(\mathbf{Z}_p) \times D^\times \times W_{\mathbf{Z}_p}$.



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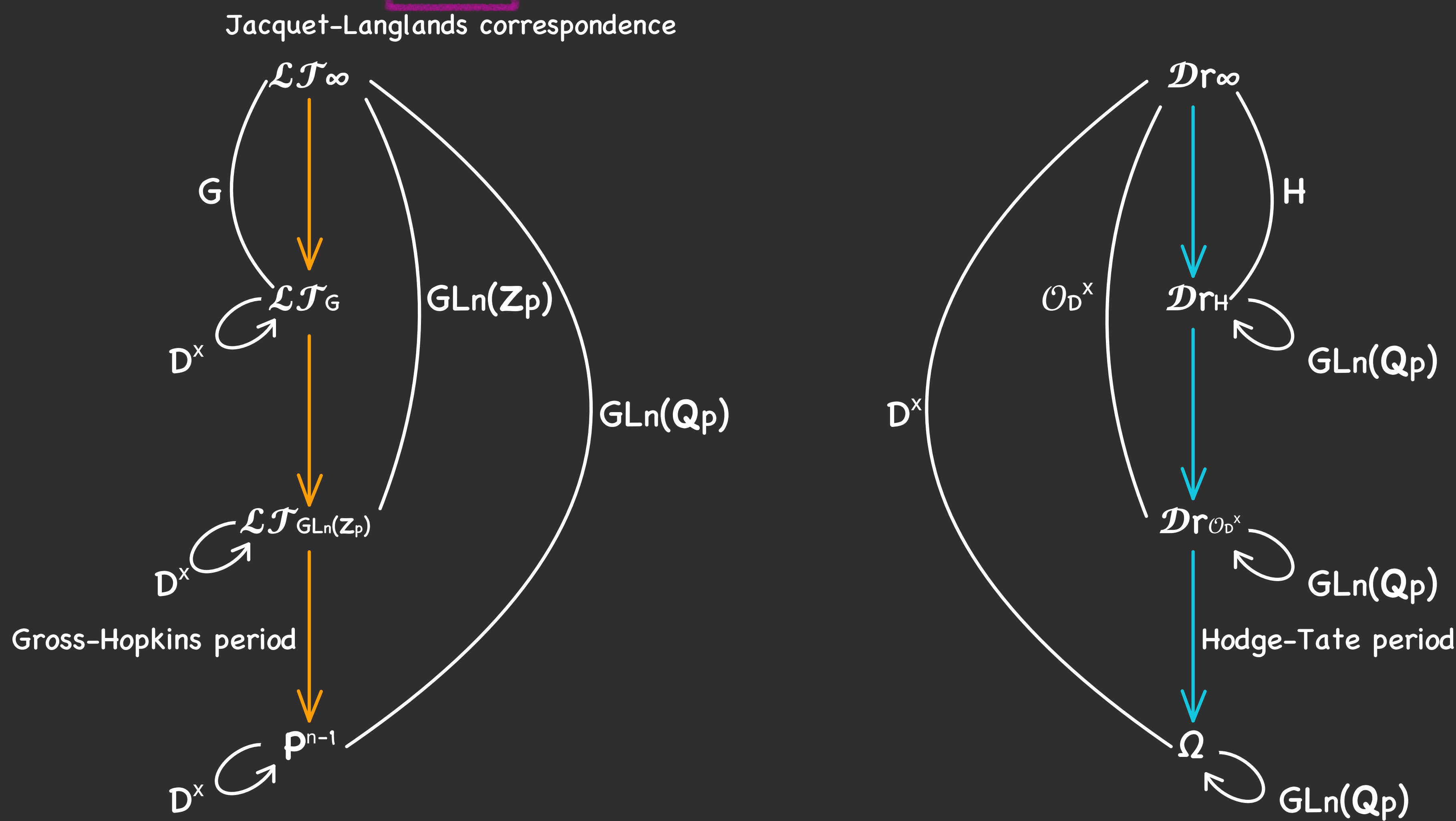

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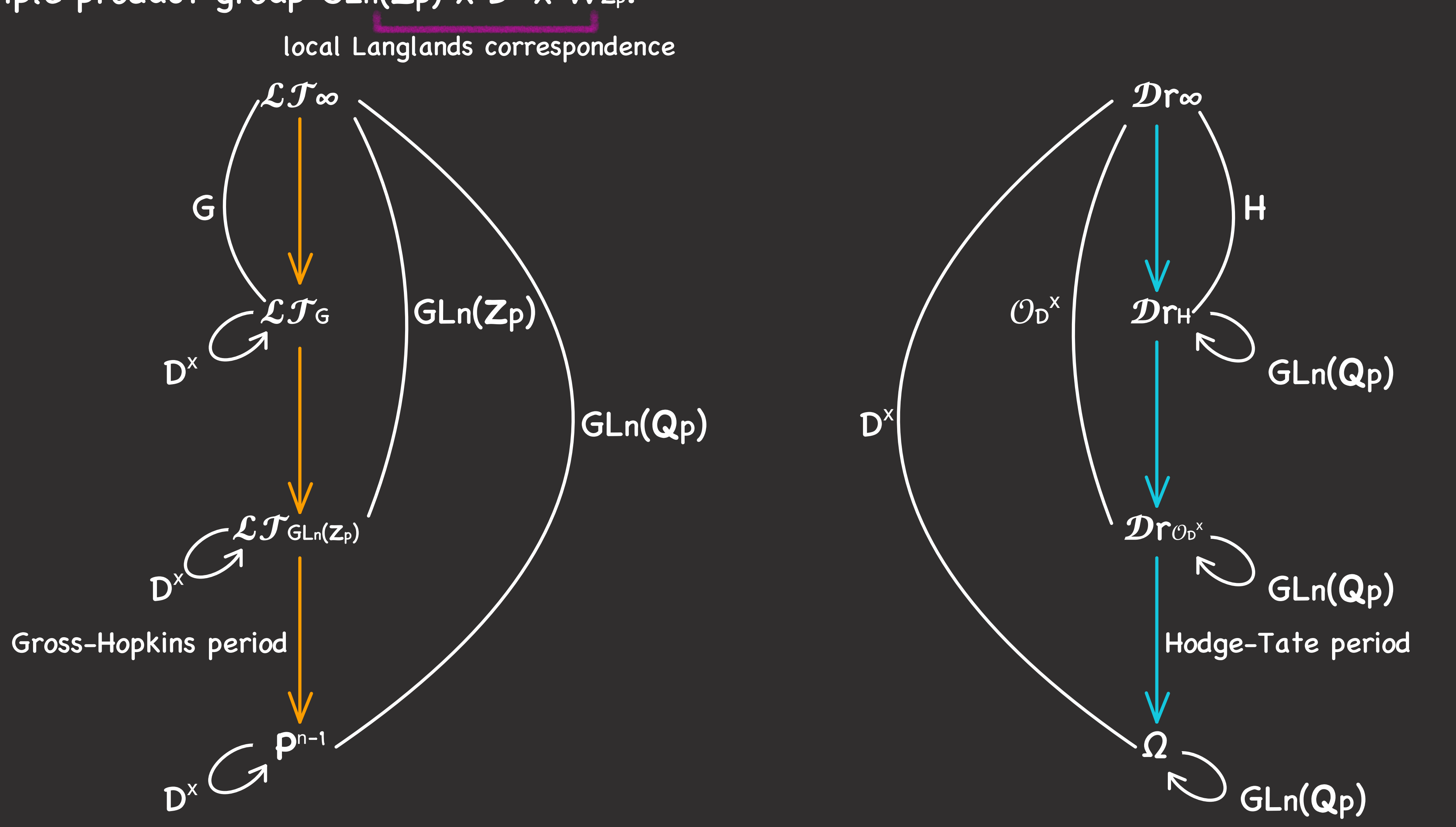

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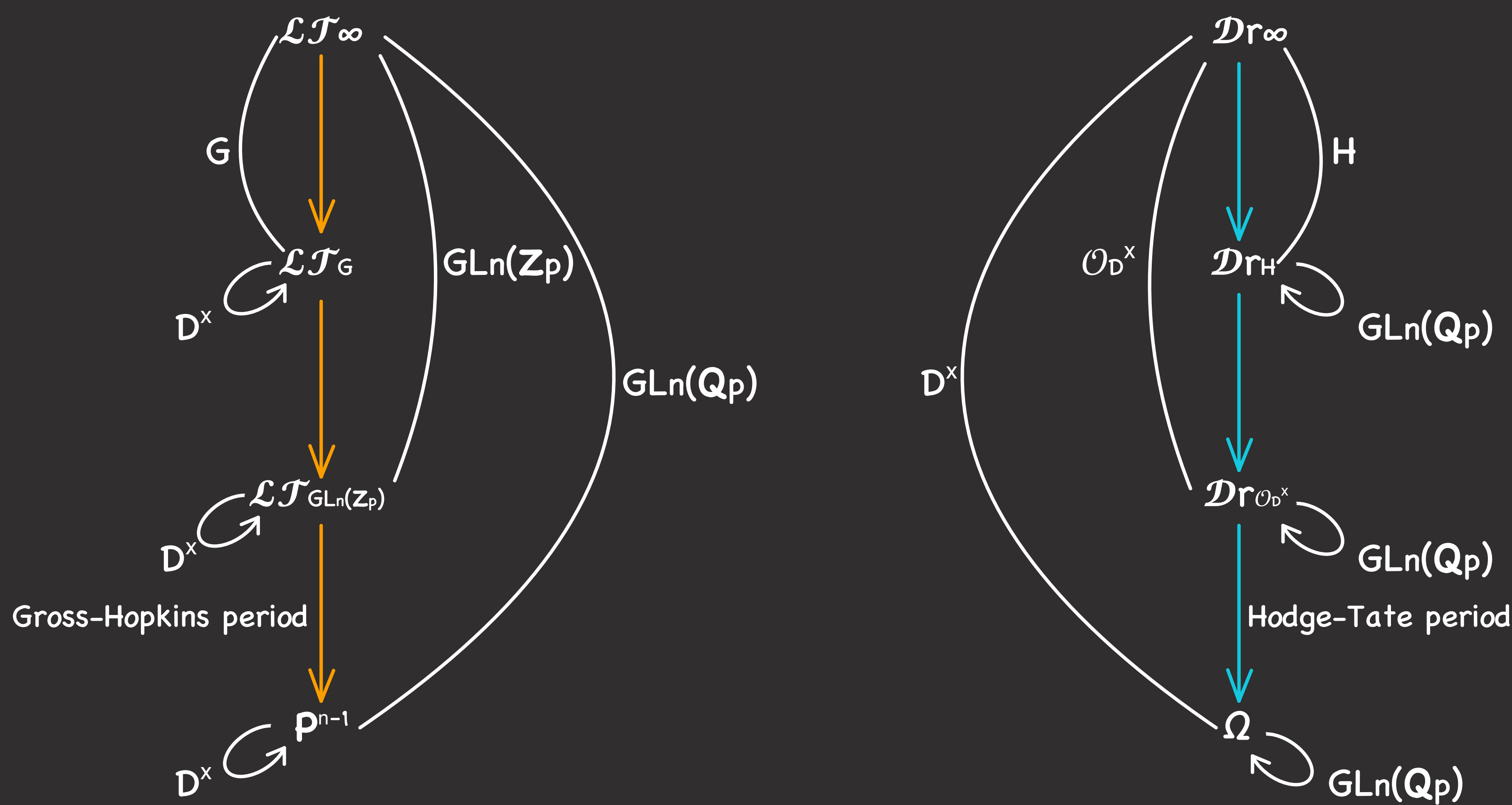

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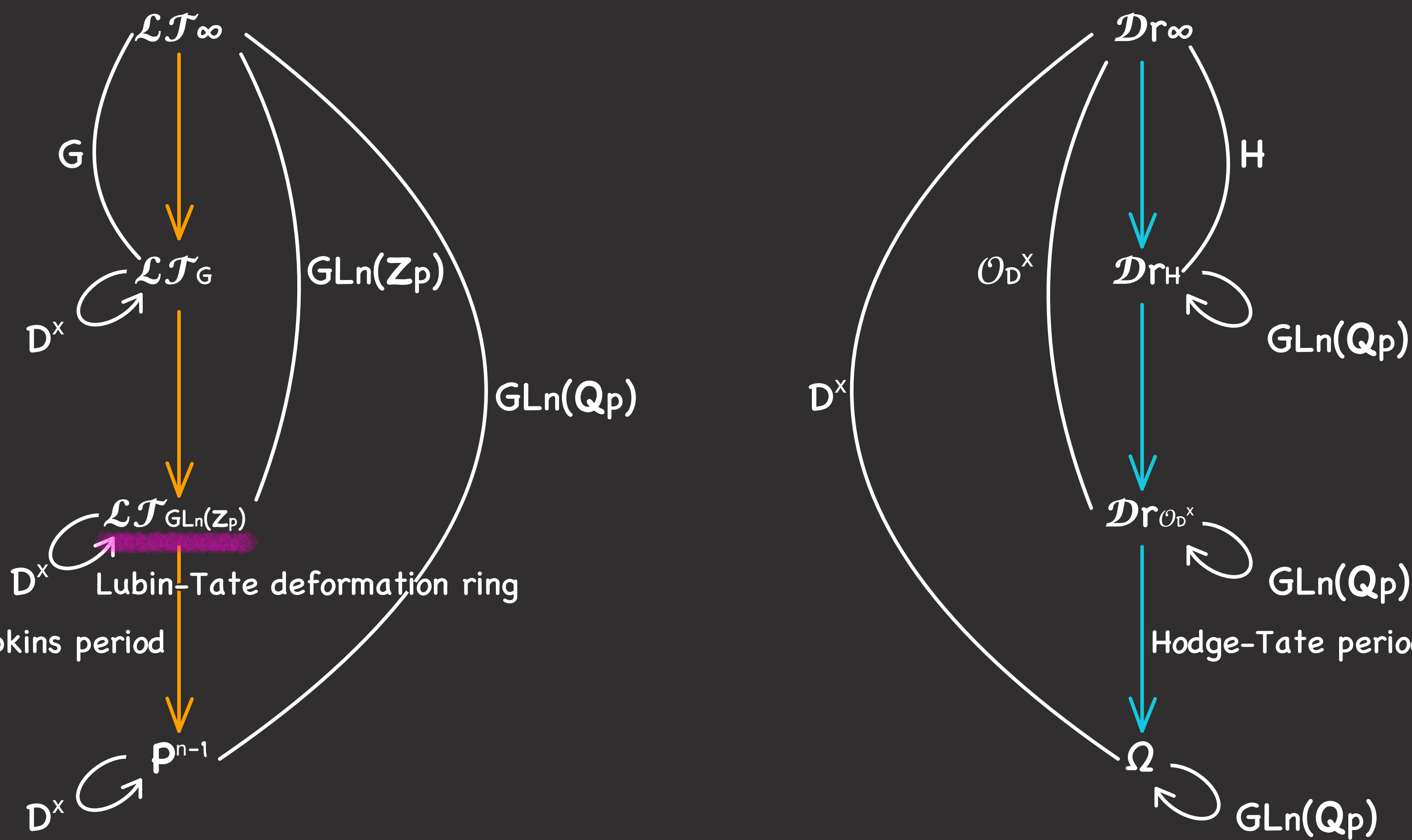

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Corollary ∃ a spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{cts}}\big(\mathrm{GL}_n(\mathbf{Z}_p), \pi_t({}^{\mathrm{L}}E)\big) \implies \pi_{t-s} L_{K(n)}\mathbf{S}$$

where ${}^{\mathrm{L}}E := \mathrm{JL}_\infty^{\mathrm{h}\mathbf{G}_n}$ is a Jaquet-Langlands dual to E.

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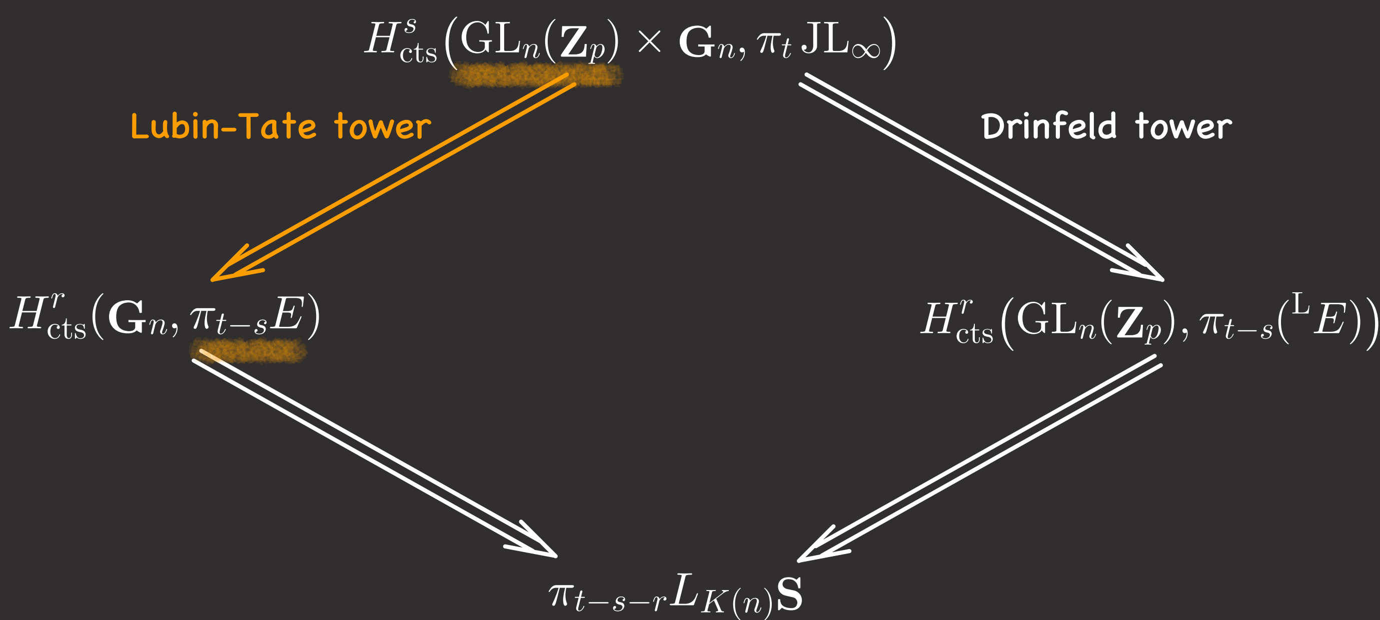

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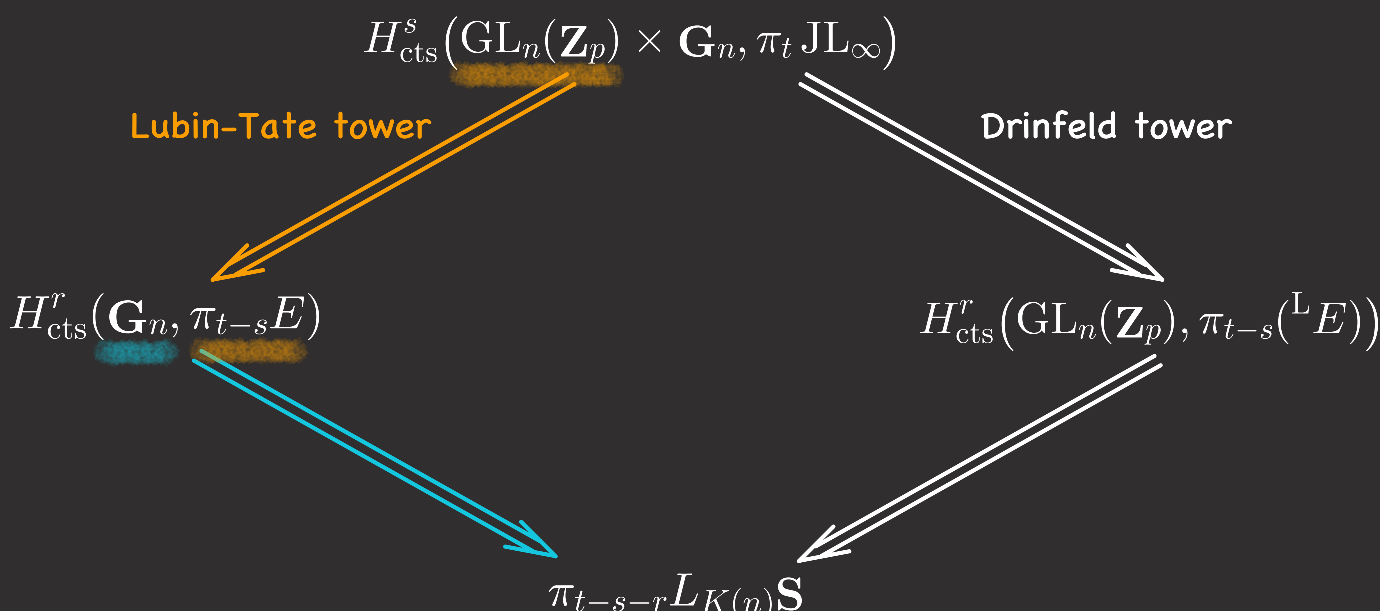

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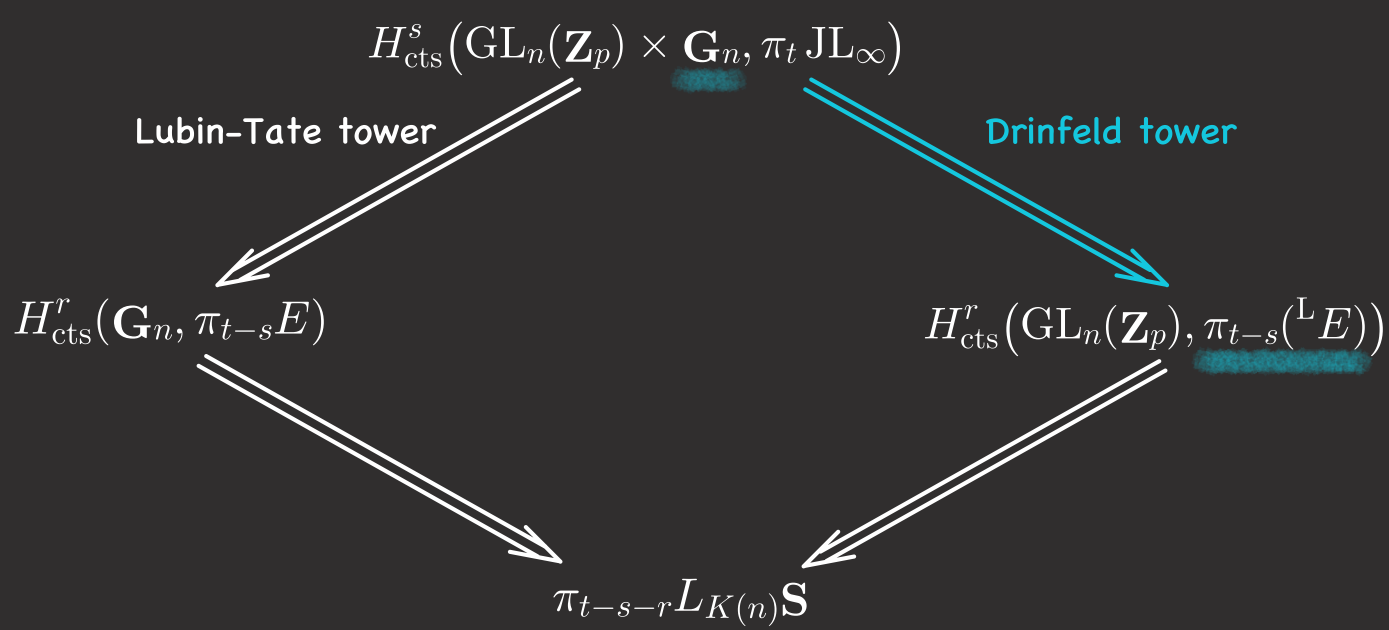

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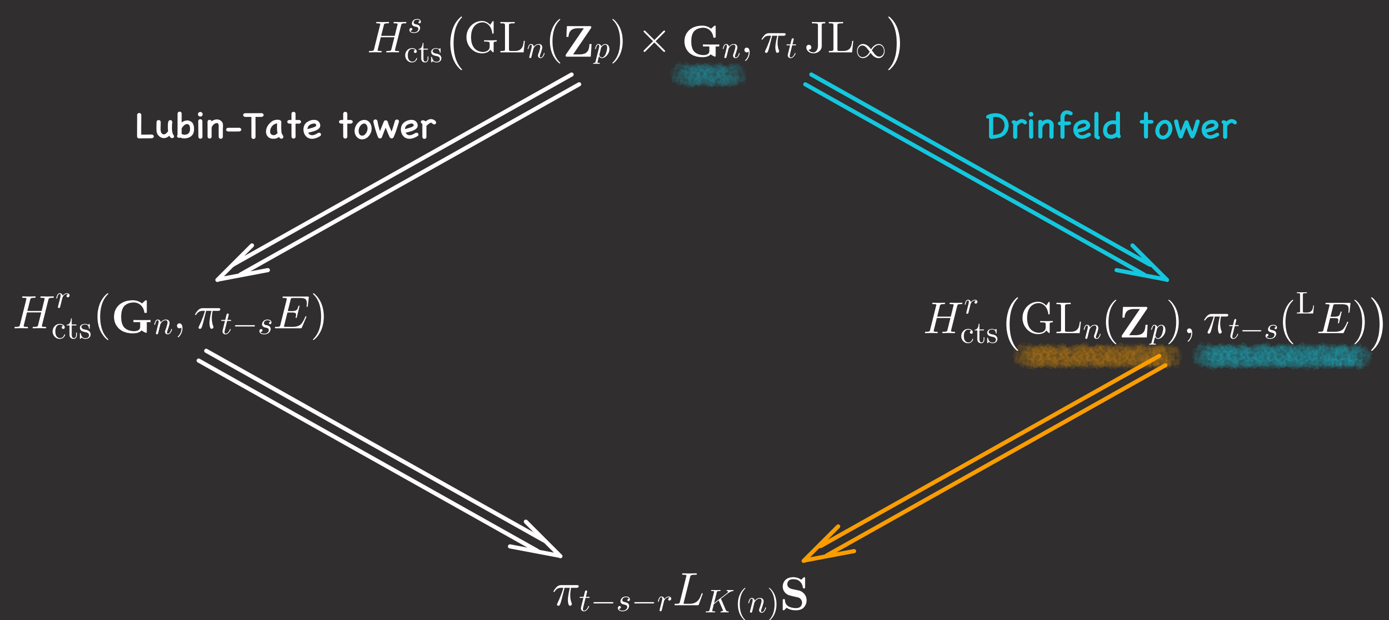
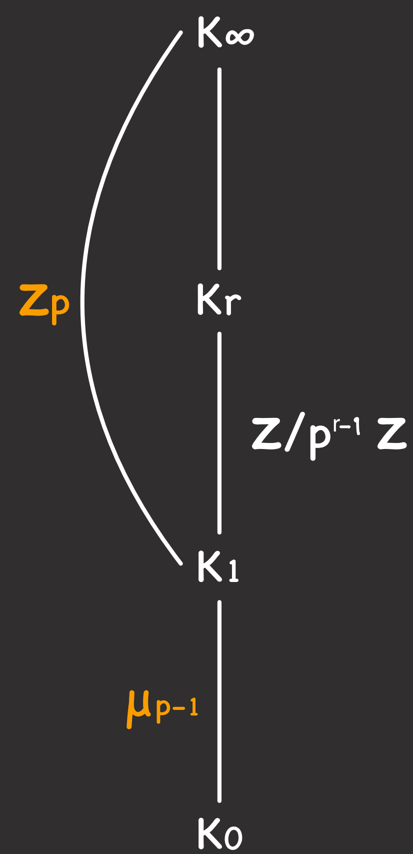


Example ($n = 1$, p odd) The Lubin-Tate tower corresponds to the Iwasawa tower of field extensions



where $K_0 = \mathbb{Q}_p$, $K_r = \mathbb{Q}_p(\zeta_{p^r})$, $K_\infty = \bigcup_{r\geq 0} K_r$, with $\mathrm{Gal}(K_\infty/K_0) \simeq GL_1(\mathbb{Z}_p) \simeq \mathbb{Z}_p^\times$.

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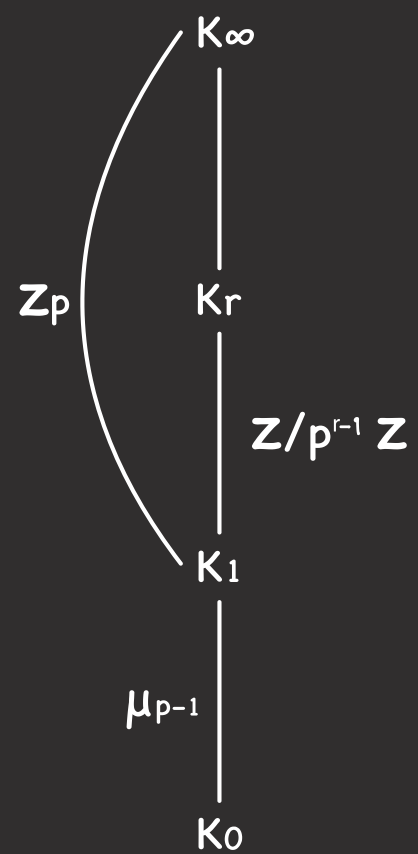

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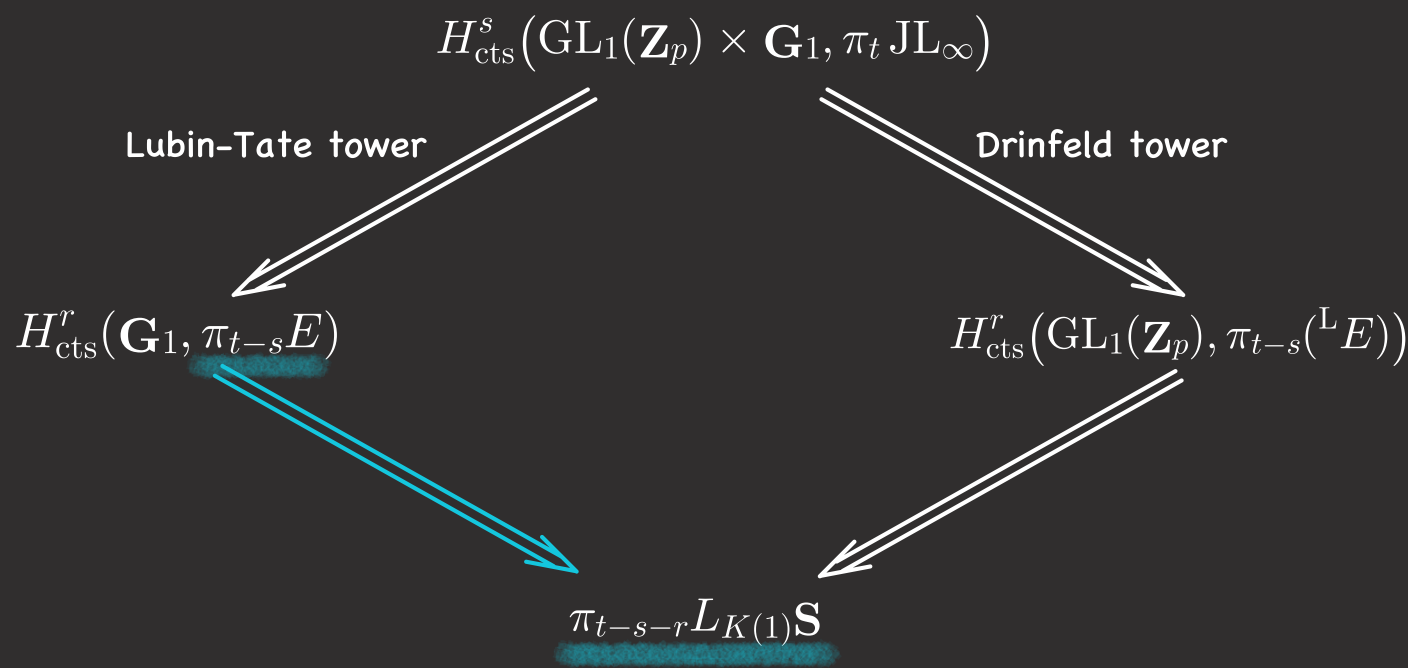
On the other hand, the Morava stabilizer group $\mathbb{G}_1 \simeq \mathbb{Z}_p^\times$, too, but its action on the homotopy groups differs from that by $GL_1(\mathbb{Z}_p)$.



The Devinatz-Hopkins SS is well-understood in this case.

$$H^s_{\mathrm{cts}}\big(\mathrm{GL}_1(\mathbf{Z}_p) \times \mathbf{G}_1, \pi_t\, \mathrm{JL}_\infty\big)$$

Lubin-Tate tower $\Rightarrow$ $H^r_{\mathrm{cts}}(\mathbf{G}_1, \pi_{t-s}E)$

Drinfeld tower $\Rightarrow$ $H^r_{\mathrm{cts}}\big(\mathrm{GL}_1(\mathbf{Z}_p), \pi_{t-s}({}^{\mathrm{L}}E)\big)$

$$\Rightarrow \pi_{t-s-r} L_{K(1)}\mathbf{S}$$

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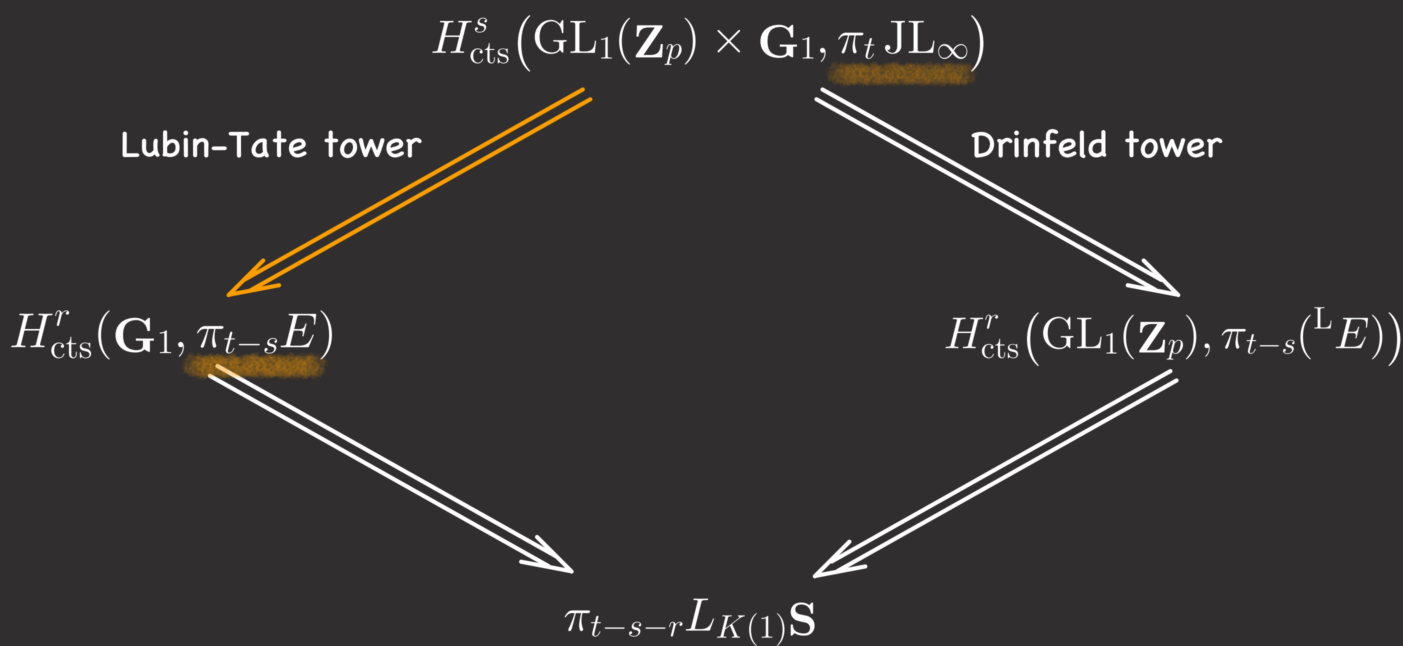

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$$\pi_*\mathrm{JL}_\infty \simeq (\mathcal{O}_{K_\infty} \oplus I_\infty[\epsilon])[u^{\pm 1}] \qquad \text{with } |\epsilon| = 1,\ \epsilon^2 = 0,\ \text{and } |u| = -2$$

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topological invariant encoding
Galois cohomology

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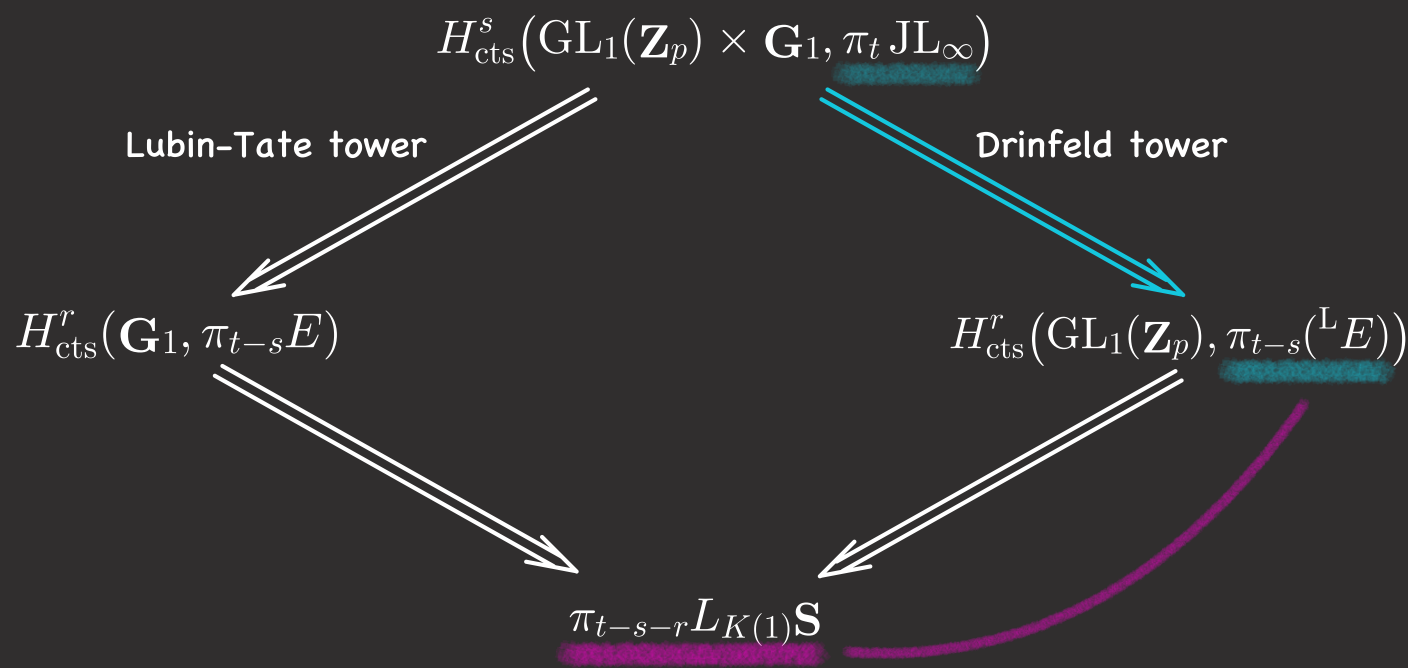

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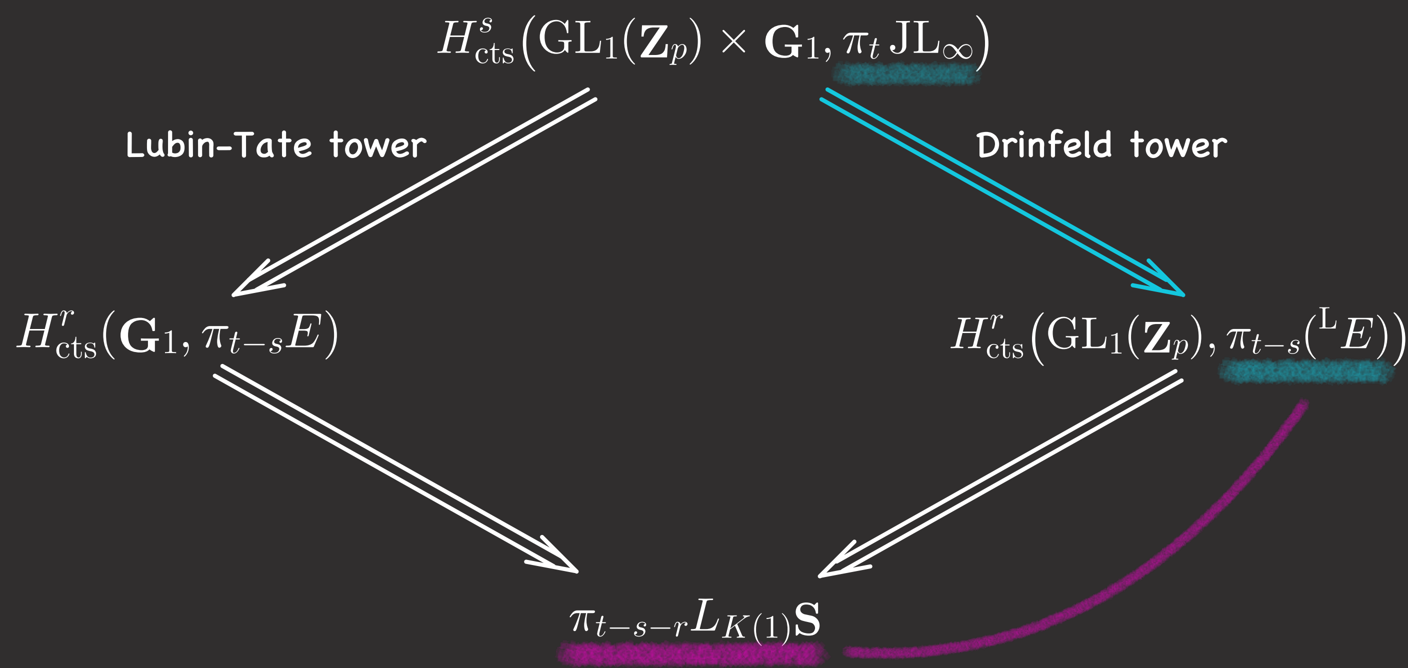

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torsion classes from Galois cohomology

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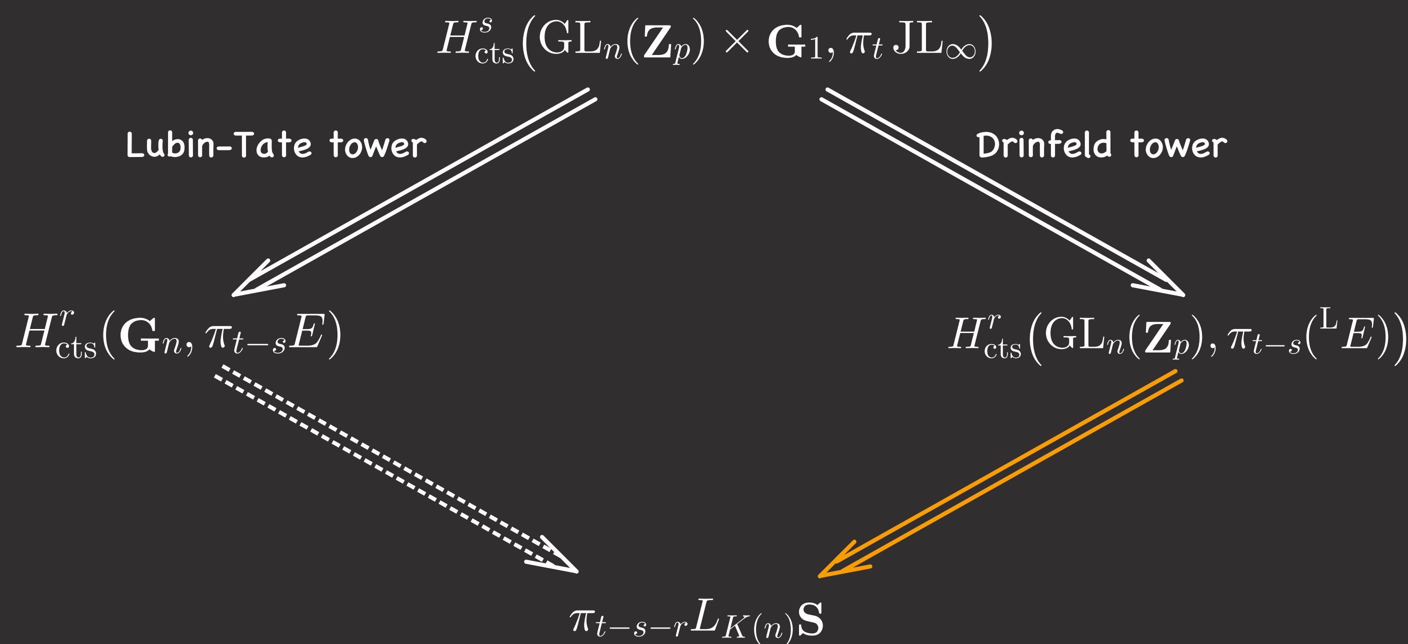

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Pre-Definition Let G be a profinite group. A genuine G-spectrum is a collection of genuine G/U-spectra $X_{G/U}$ for U running through open subgroups of G, s.t. for $U_1 \subset U_2$, $\exists$ an equivariant equivalence

$$X_{G/U_2} \xrightarrow{\simeq} \left(X_{G/U_1}\right)^{\Phi(U_2/U_1)}$$

with compatibility conditions.

Thank you.