Topology, geometry, and physics of eigenbundles for gapless non-Hermitian quantum mechanical systems

Non-Hermitian physics has become an emerging area of research in condensed matter physics, with far-reaching applications to materials science. Resulting from symmetry beyond classical Hermiticity, complexity of the topological structure in such quantum mechanical systems poses challenges to mathematical modeling while affording unconventional physical phenomena. In this talk, I'll discuss stratified singularity in the parameter spaces of such systems using intersection homology for homotopical classifications. Moreover, I'll explain evidence for hyperbolic geometry therein, through Higgs bundles modeling eigenbundles, and Minkowski light-cones modeling exceptional surfaces. This is ongoing joint work with Hongwei Jia et al.

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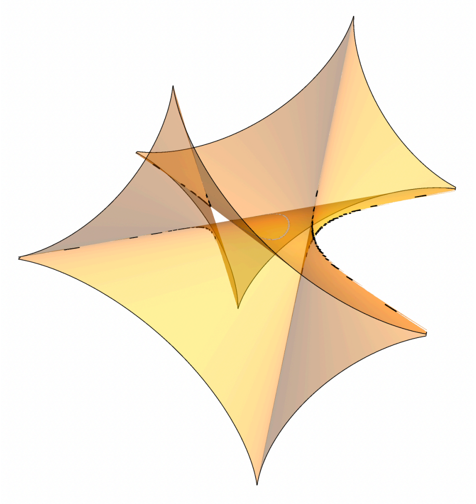

Yifei Zhu

Southern University of Science and Technology

2026.4.30

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This stems from ongoing joint work with
H. Jia, J. Hu, C. T. Chan (physically),
W. Yang, Z. Fang, C. Huang, Q. Qu, Z. Yu (mathematically), et al.

Motivations: Quantum materials and their math modeling

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Holography, optical devices, absorption devices, …

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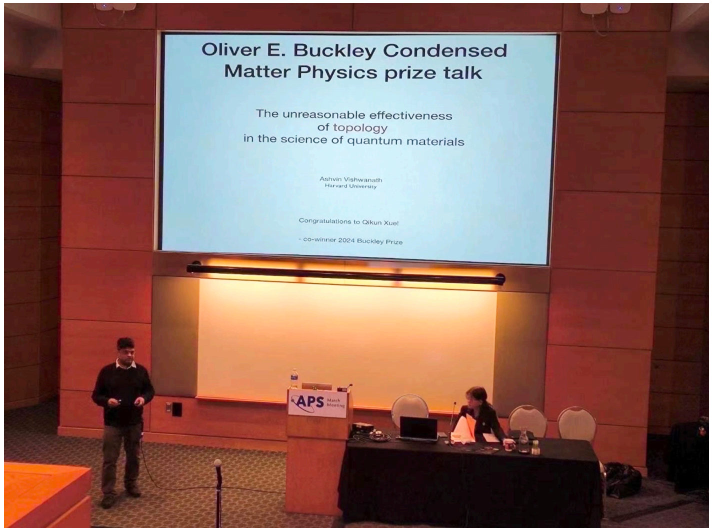

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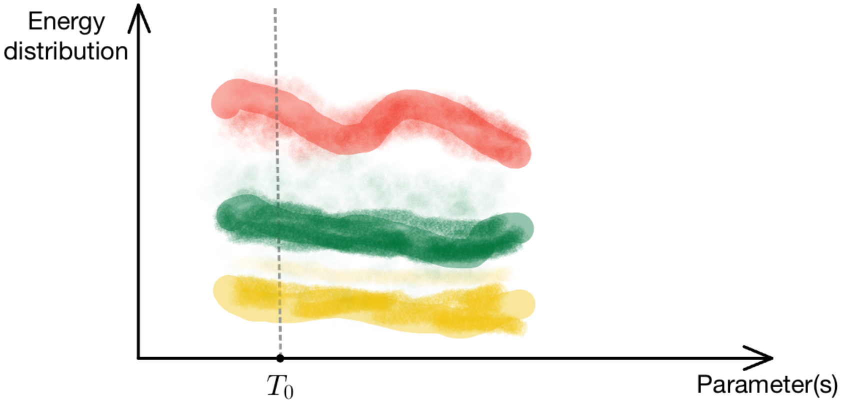

energy $\longleftrightarrow$ *eigenvalue*

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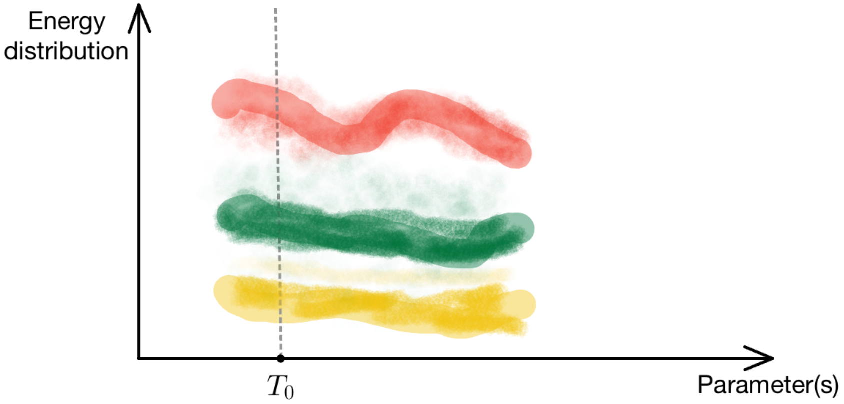

Hermitian

real eigenvalues (observable energies)

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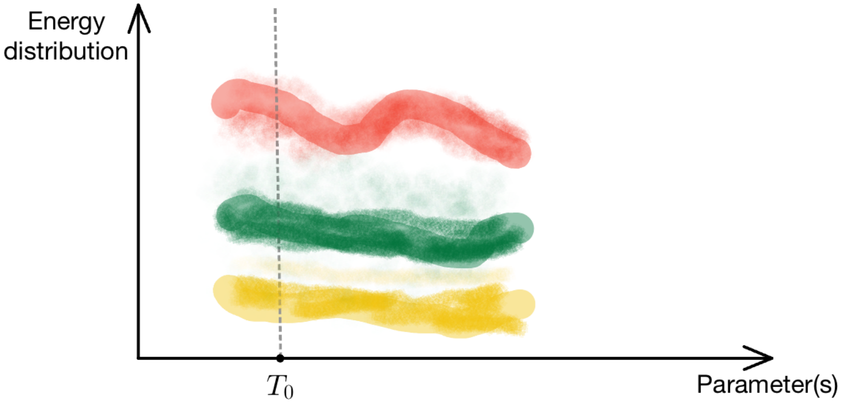

Hermitian vs. non-Hermitian

real eigenvalues (observable energies) vs. eigenvalues with imaginary part (counts for energy exchange with surrounding environment or other systems)

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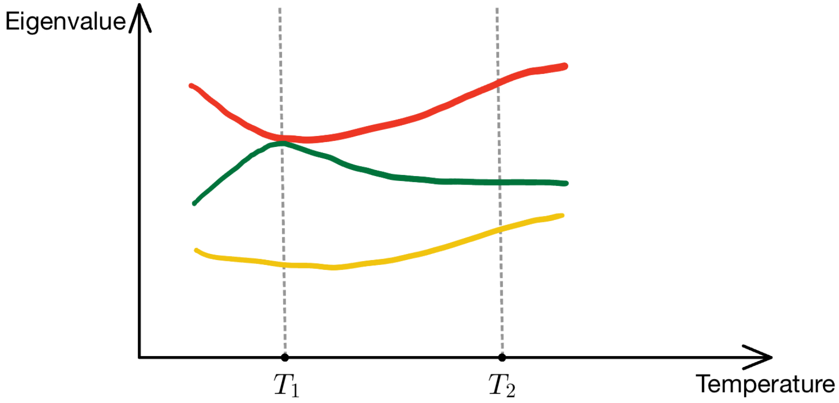

phases of matter
H_2O: *solid, liquid, gas*

T_1 : singular points (points where eigenvalues degenerate)
$H(T_1)$: gapless Hamiltonian
$H(T_2)$: gapped Hamiltonian

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Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, ***PNAS*** *2024.*

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Parity–time symmetry throughout

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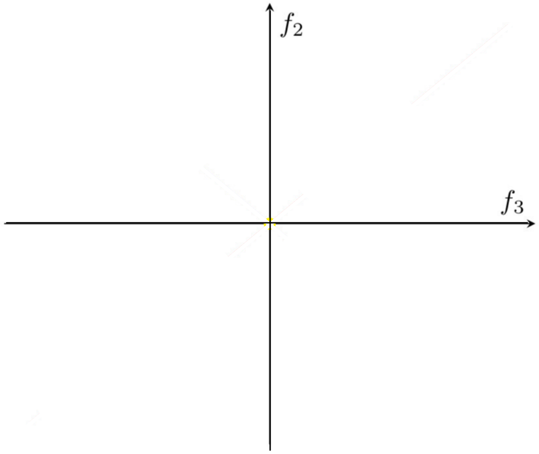

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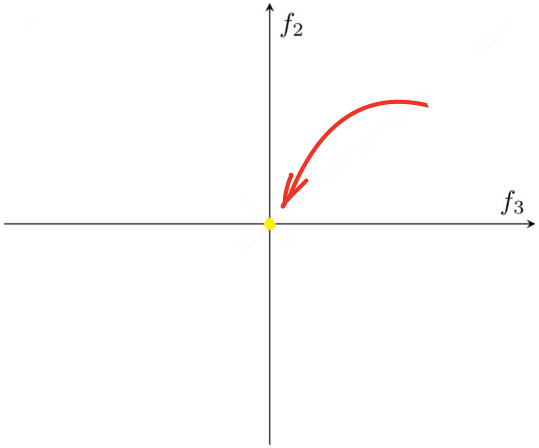

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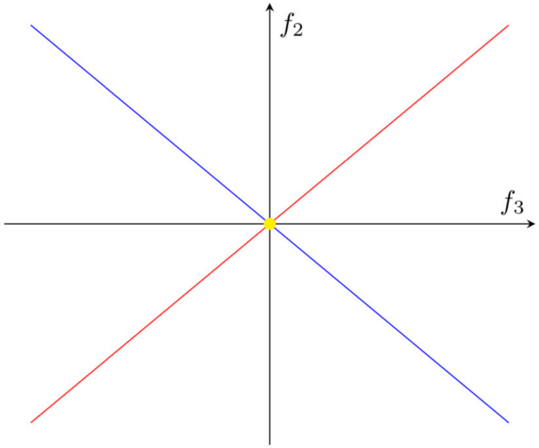

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1. Over $\{f_2 = \pm f_3\} - \{(0, 0)\}$, again *H* has a double eigenvalue, but its eigenspace is of dimension 1.

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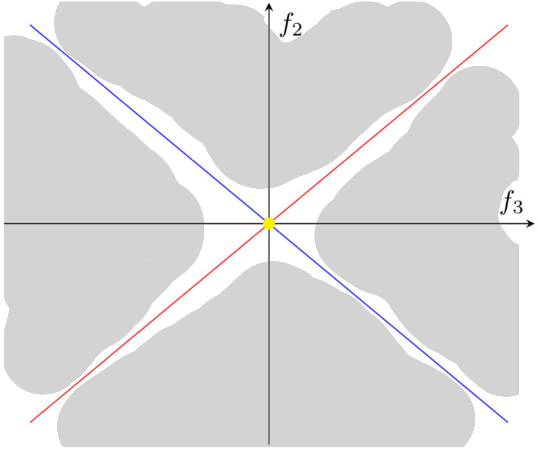

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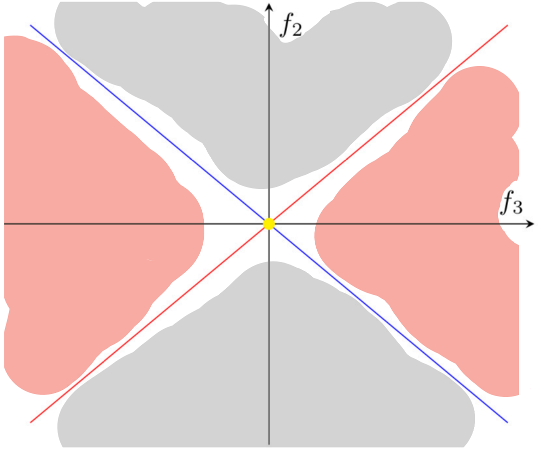

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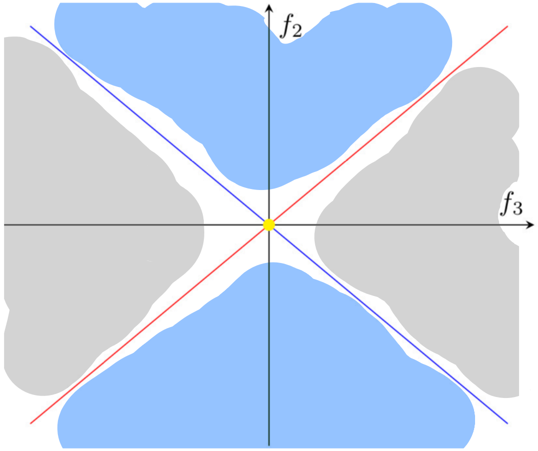

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2. Over $\{f_2 \neq \pm f_3\}$, *H* has 2 distinct eigenvalues. When $|f_2| < |f_3|$, the eigenstates are real. When $|f_2| > |f_3|$, the eigenstates are not real.

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Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

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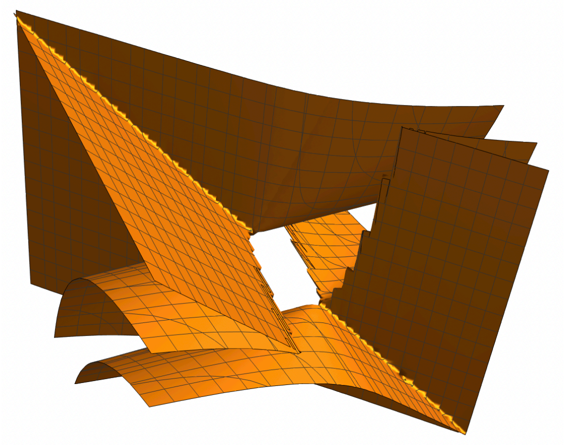
The equation for this surface is a non-homogeneous real polynomial in f_1, f_2, f_3 of degree 6.



Swallowtail couple sw2

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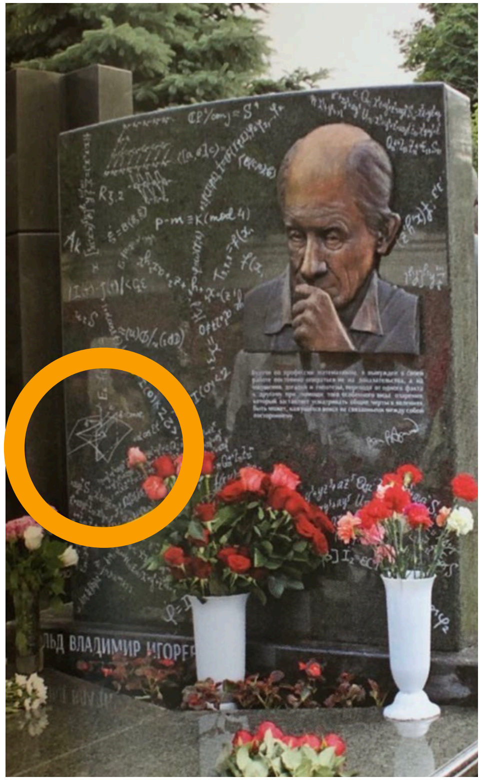

V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

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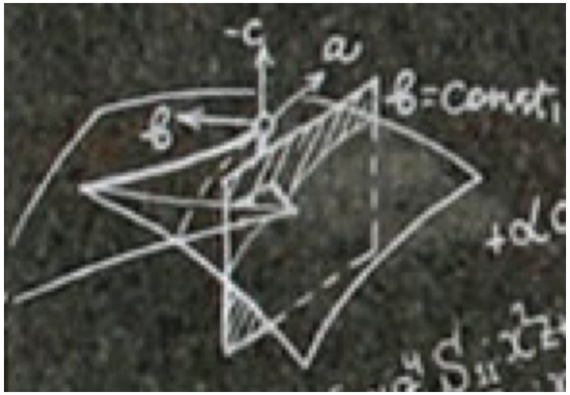

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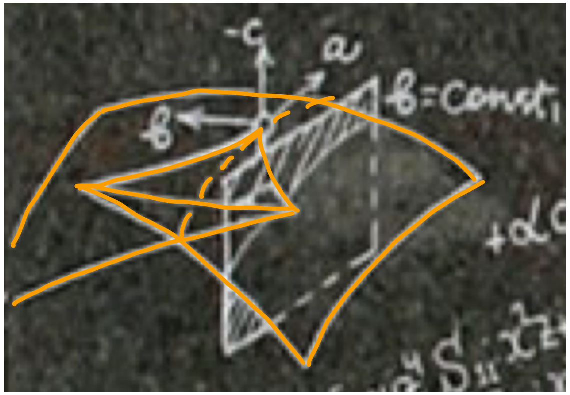

A local model for parameter spaces of 3-band Hamiltonians

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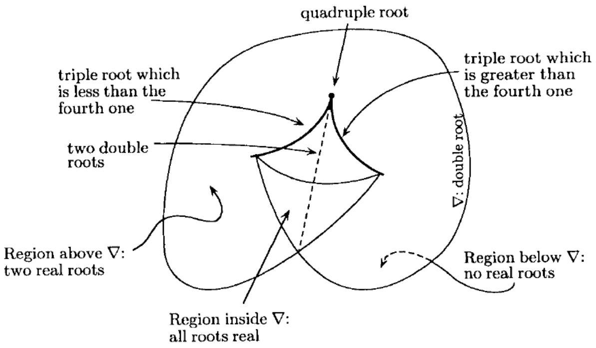

The space of polynomials $x^4 + ax^2 + bx + c$

Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.

Portrait from Gelfand, Kapranov, Zelevinsky,
Discriminants, resultants, and multidimensional determinants.

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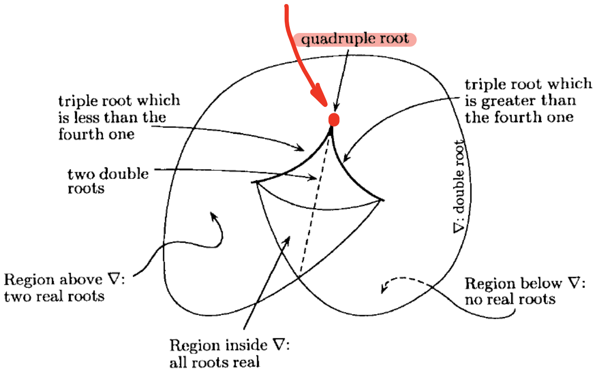

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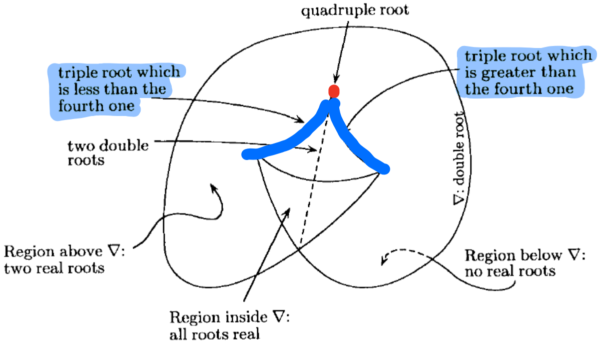

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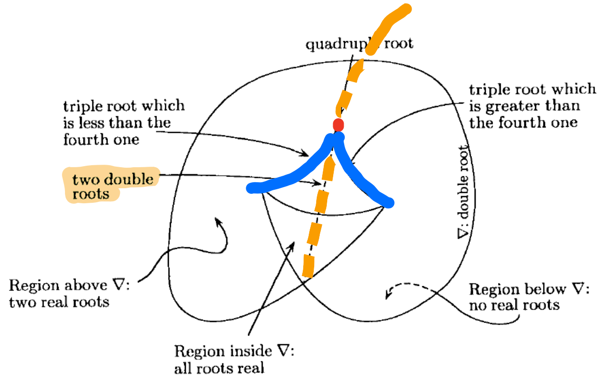

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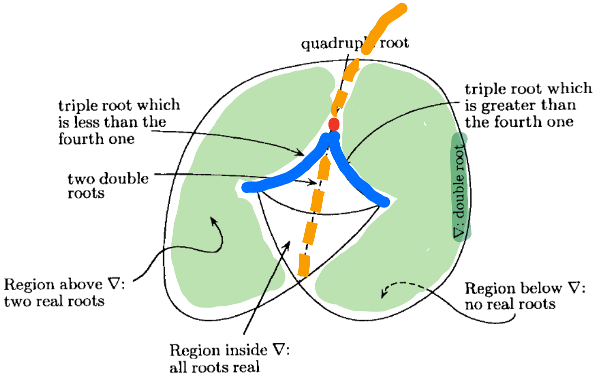

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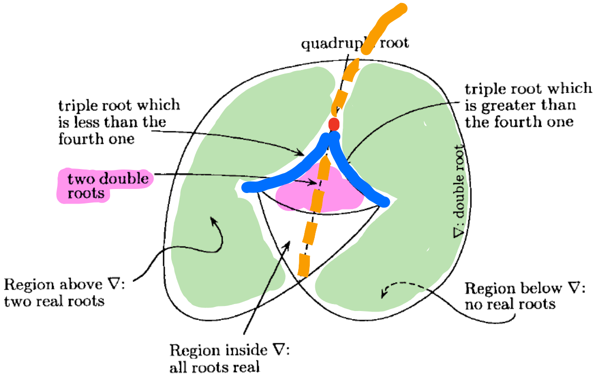

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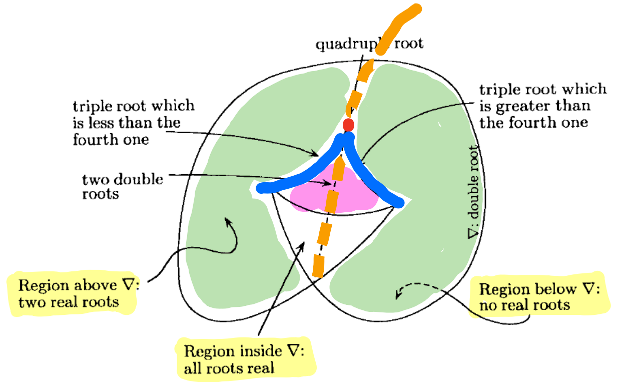

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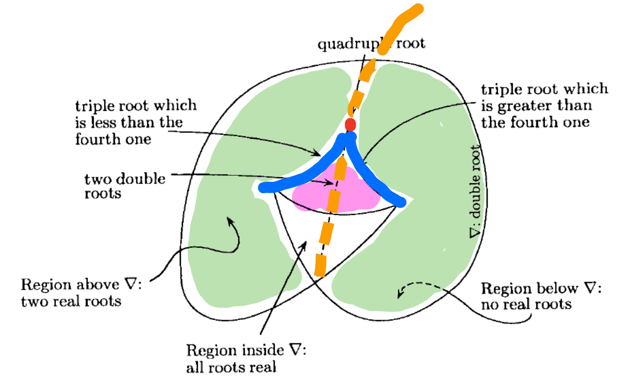

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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenstates along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect, 1995.

Also: Polymathematics, 2000.

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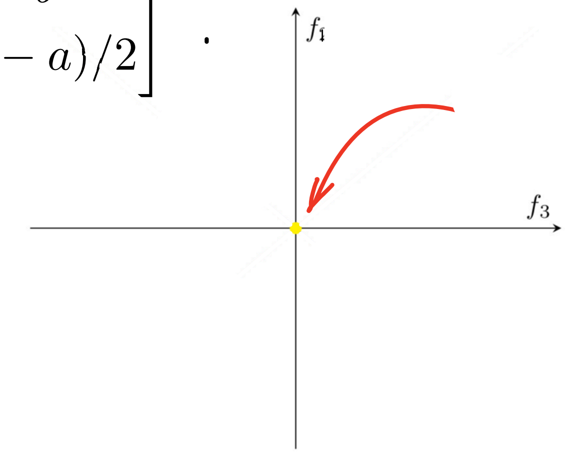

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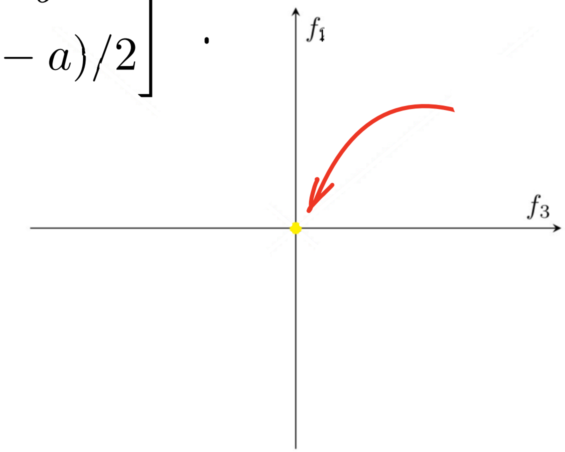

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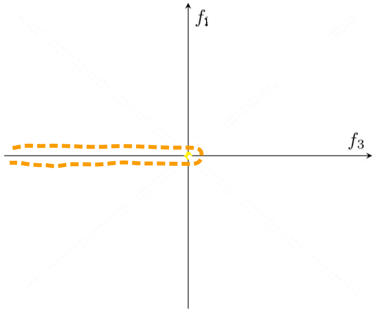

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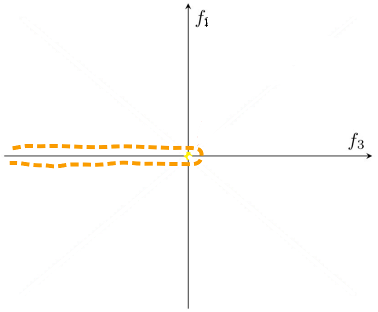

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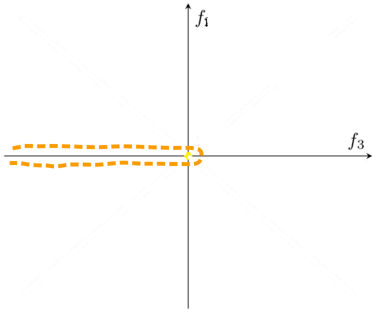

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2+f_3^2} \neq 0 \iff \xcancel{f_1 = 0,\, f_3 \leq 0}}$$

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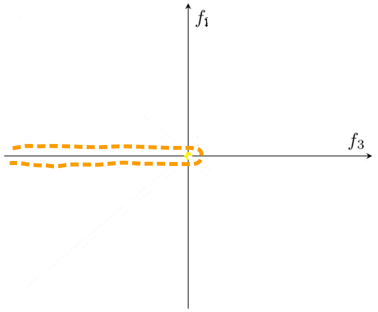

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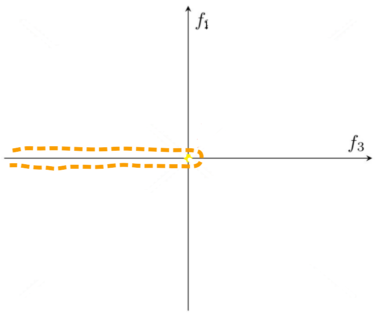

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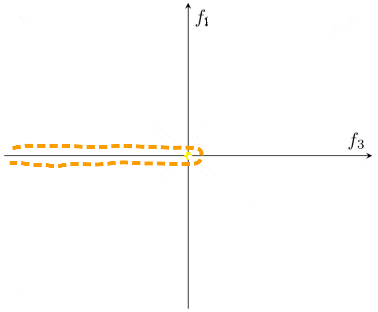

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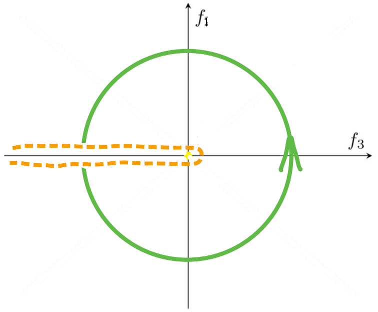

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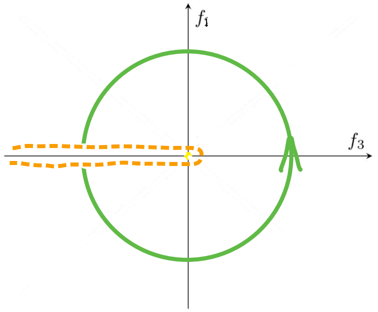

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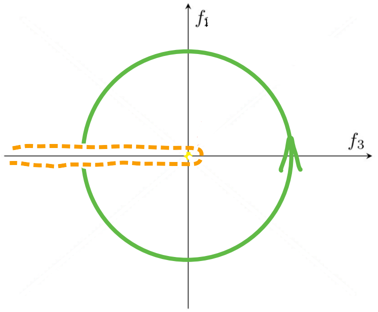

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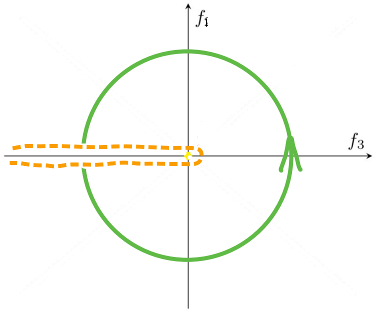

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We compute that

$$\lim_{\theta\to(-\pi)_+} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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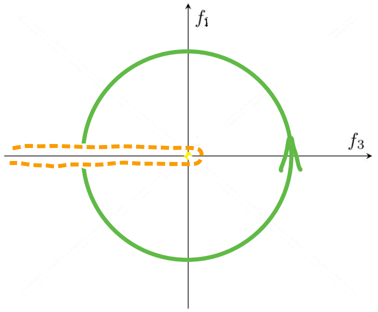

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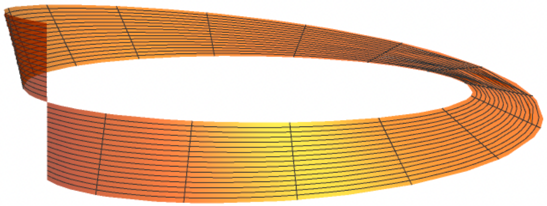

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

Lemma. The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{array}{lcl} S^0 \hookrightarrow S^1 \rightarrow S^1 & & \mathbb{R} \\ S^1 \hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 \hookrightarrow S^7 \rightarrow S^4 & & \mathbb{H} \end{array}$$

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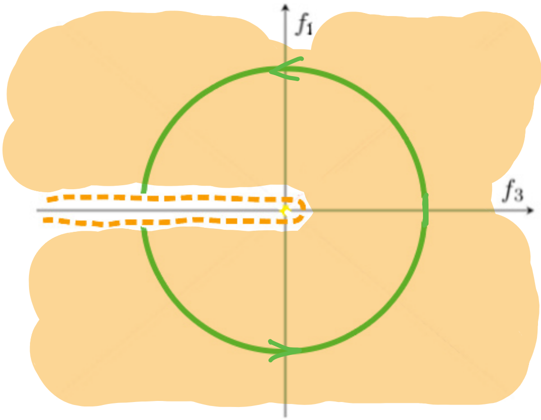

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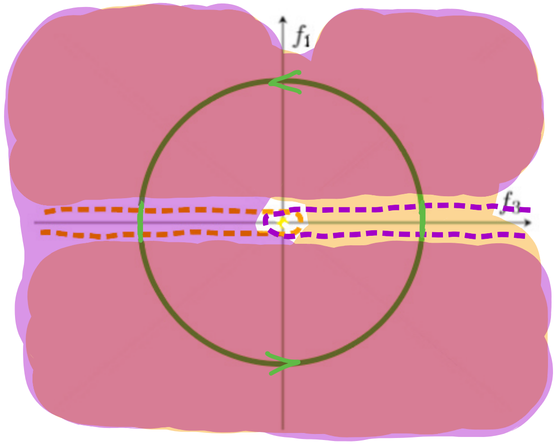

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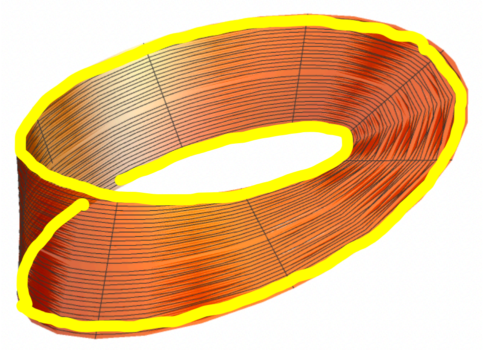

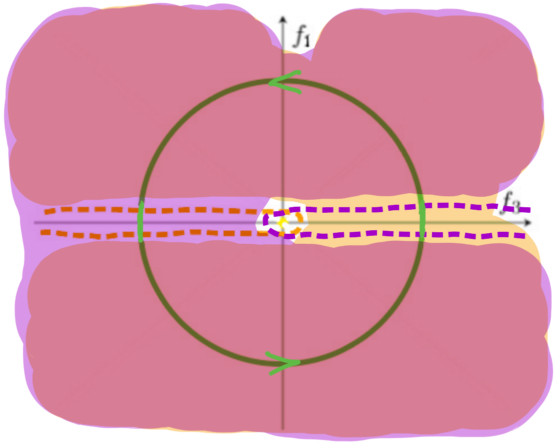

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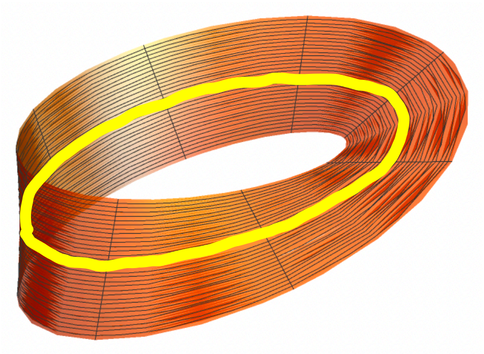

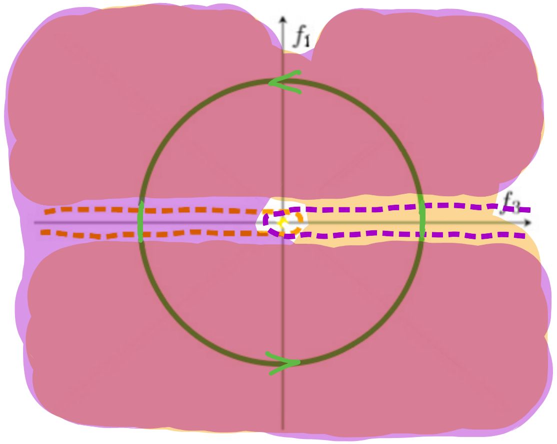

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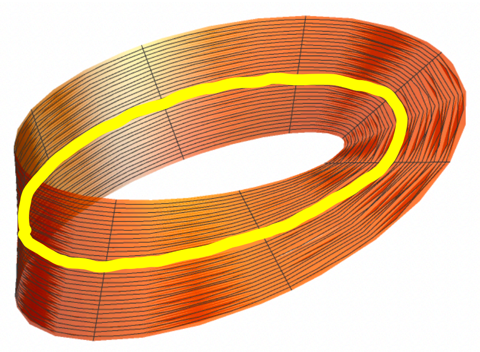

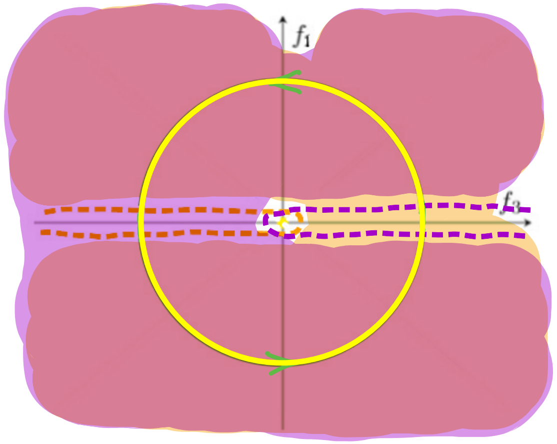

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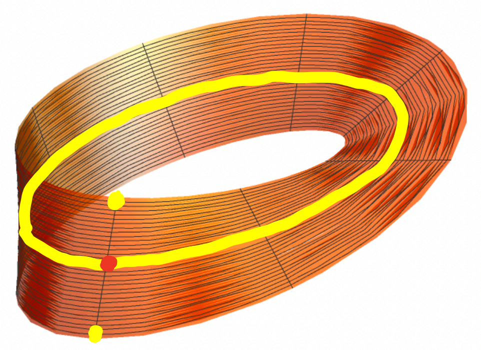

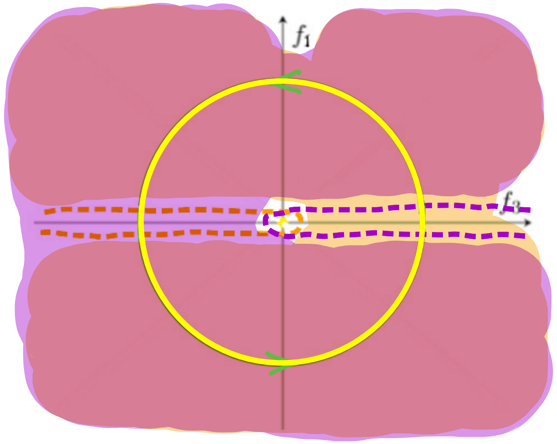

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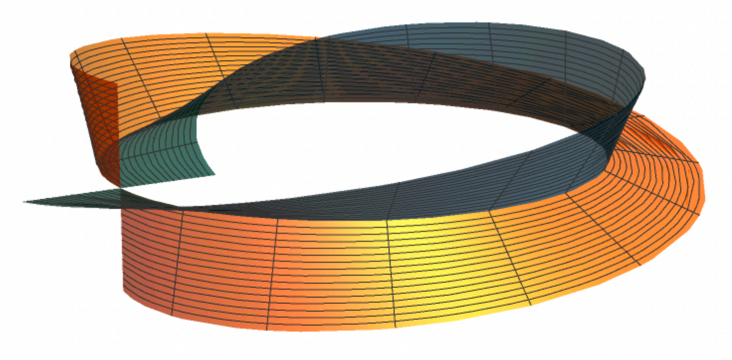
Corollary. The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands



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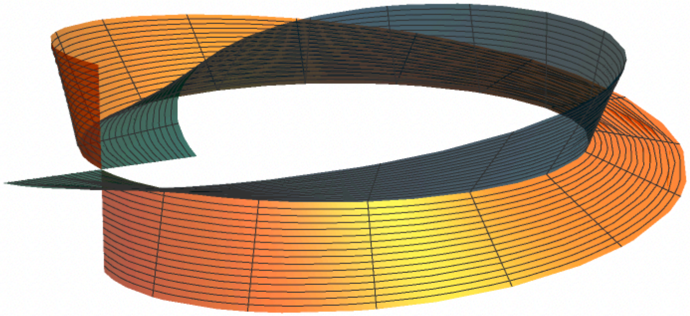
Corollary. The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands over the unit circle in the punctured parameter plane.



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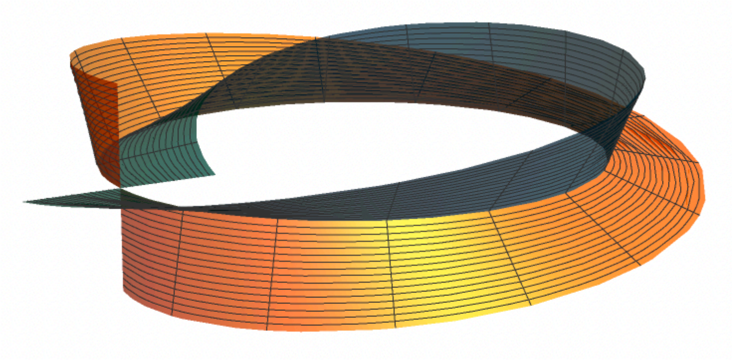

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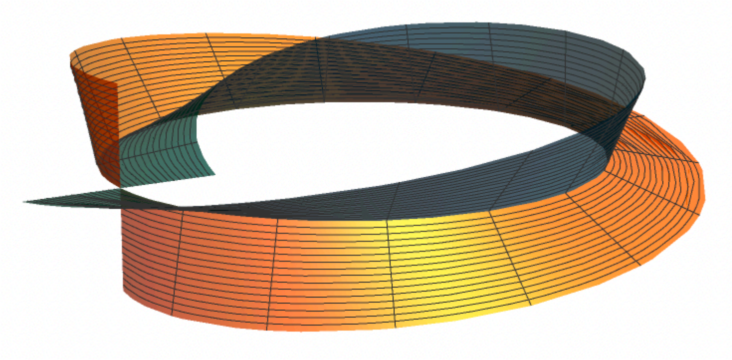

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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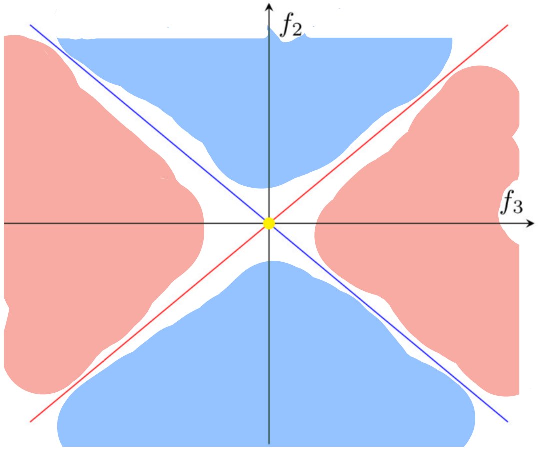
Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$



0. Over {(0, 0)}, *H* has a double eigenvalue, whose eigenspace is 2-dimensional.
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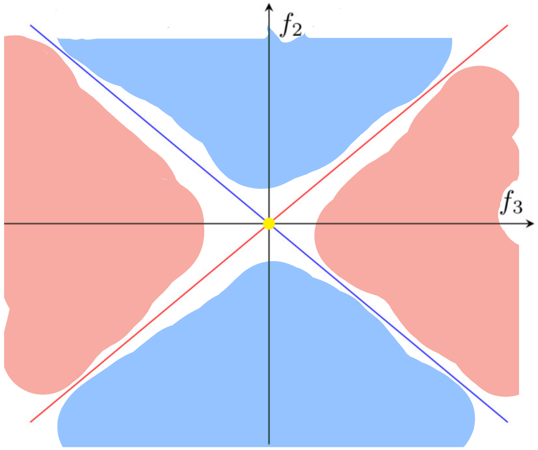
Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:



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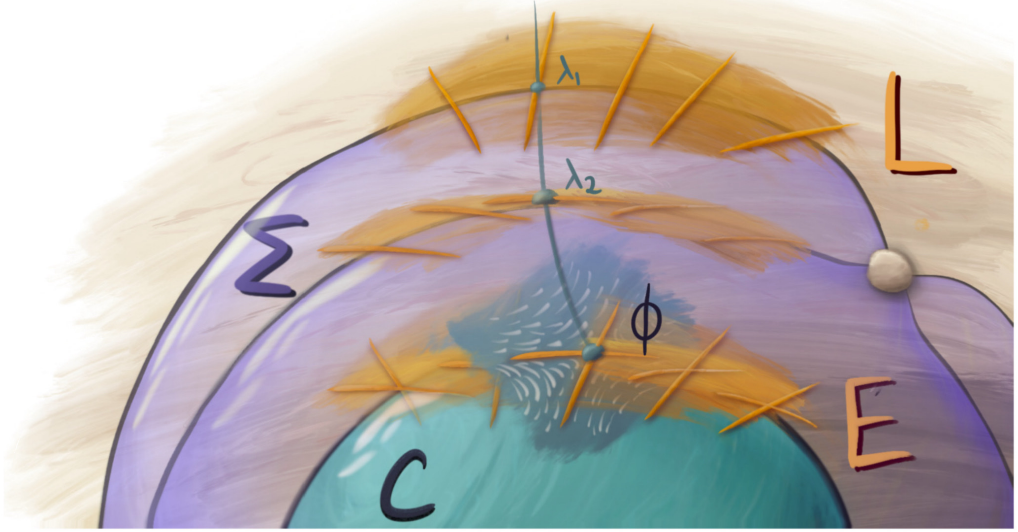
A *Higgs bundle* $(E, \phi) \to C$ is essentially a family of matrices



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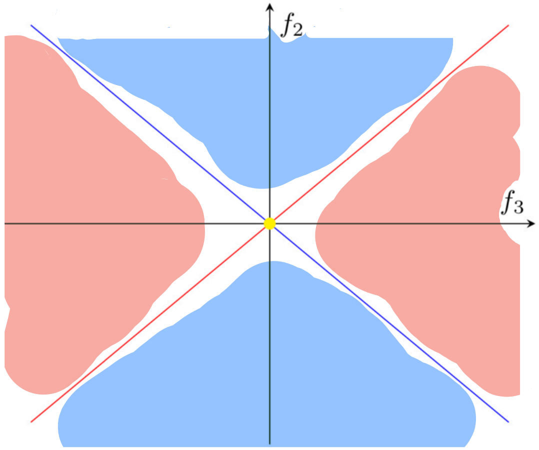

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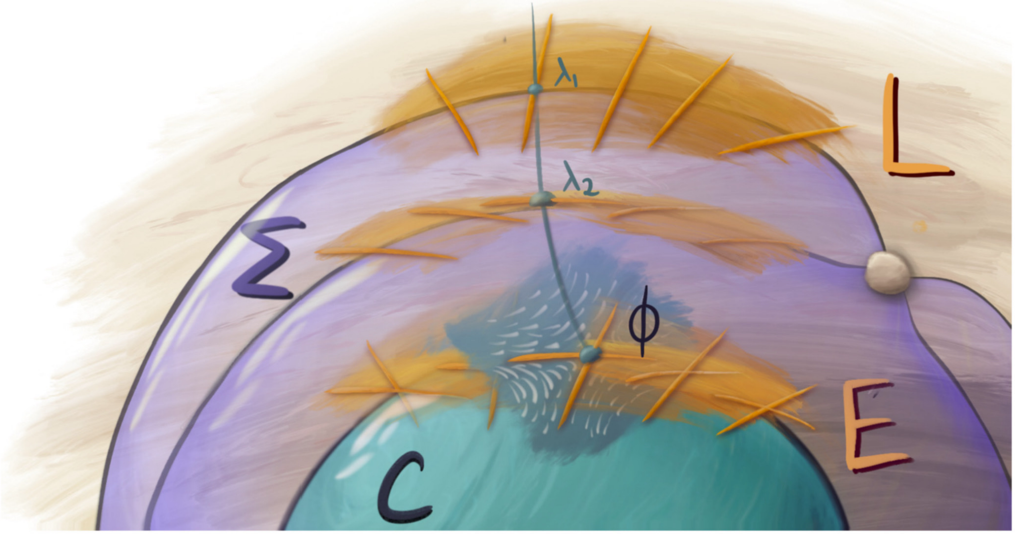

Peter Higgs (bosons)

1929–2024

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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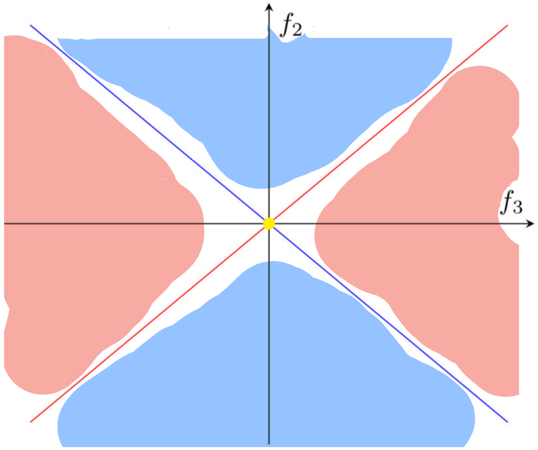

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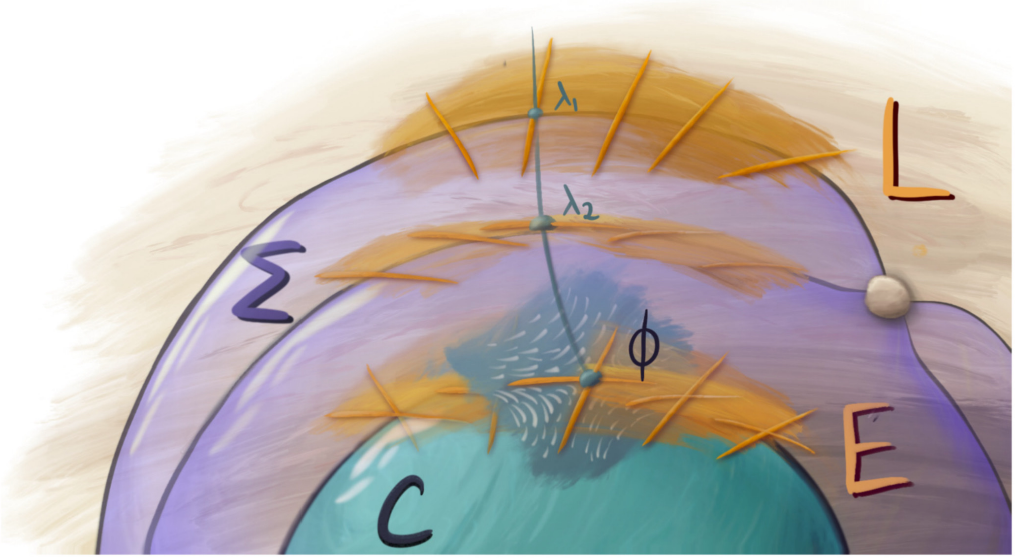
Peter Higgs (bosons)

Nigel Hitchin 1987



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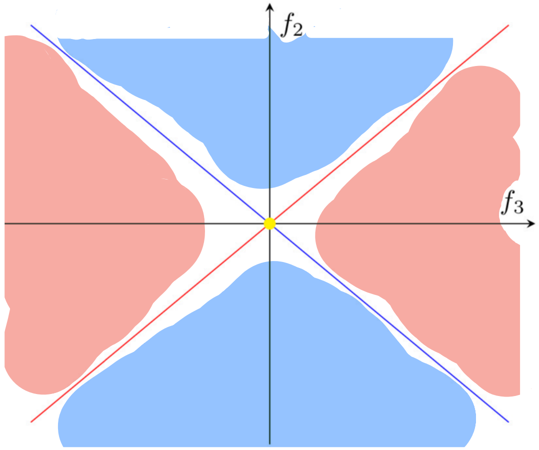

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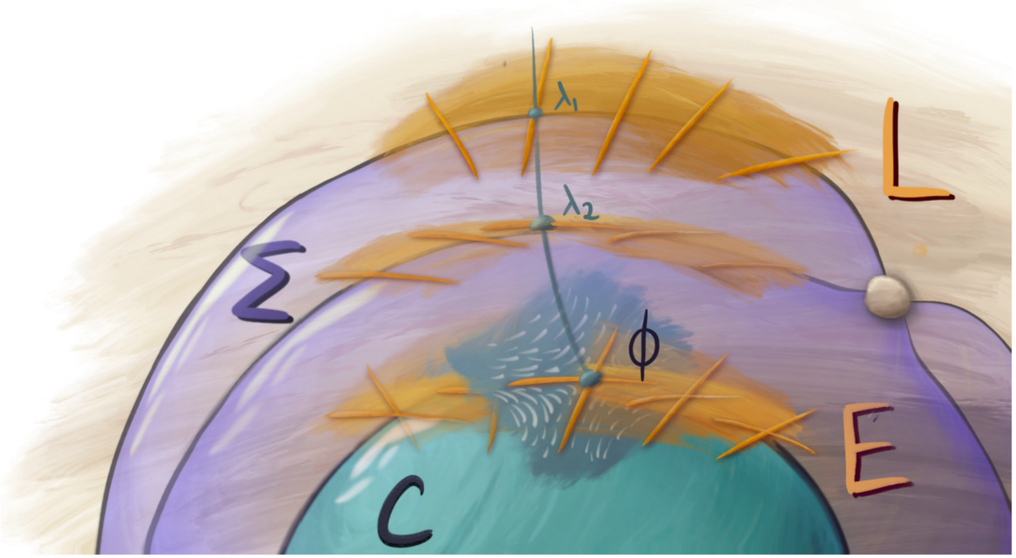
Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface



Eigenframe evolution as Higgs bundles: The non-Hermitian case

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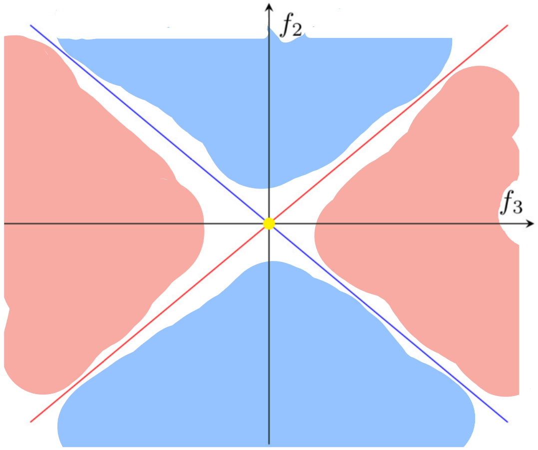

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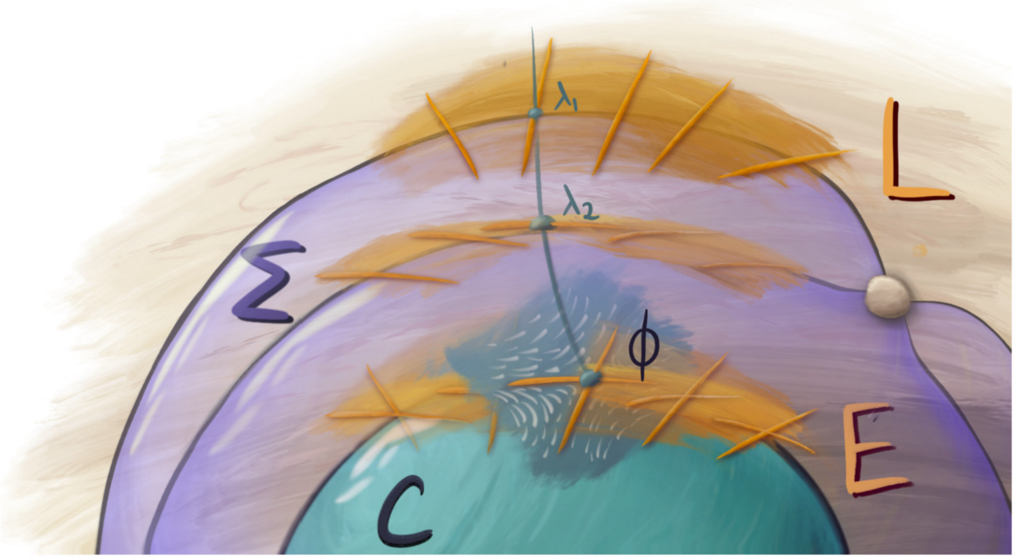

Peter Higgs (bosons)

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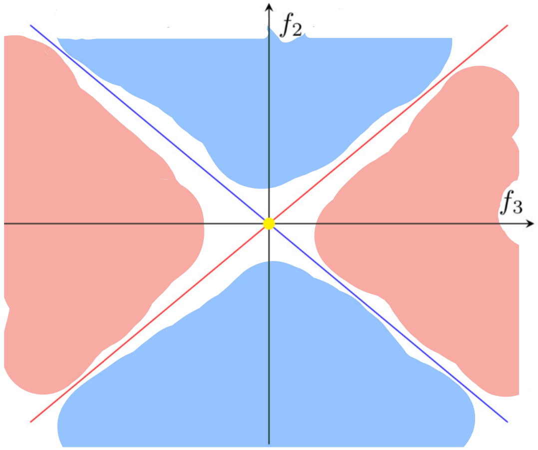

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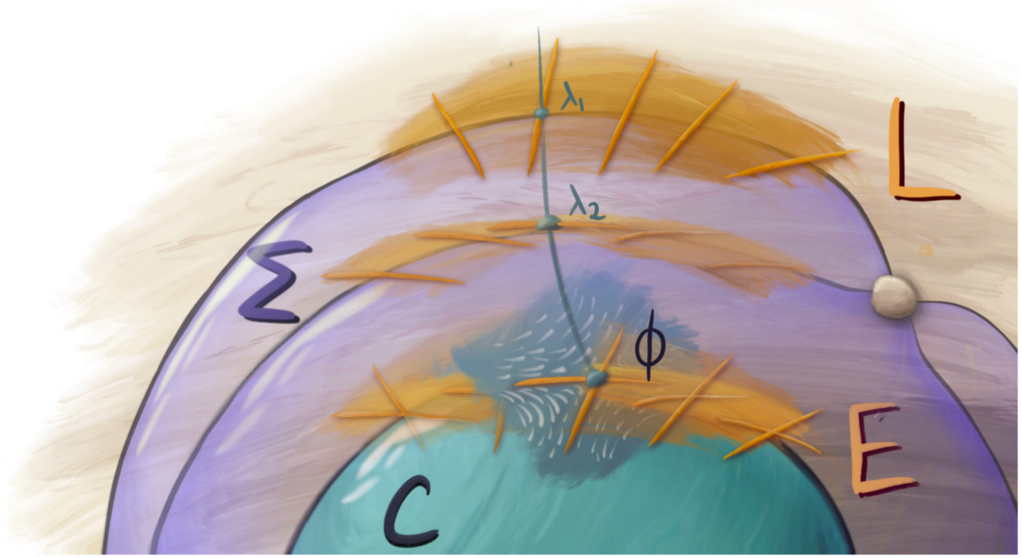

Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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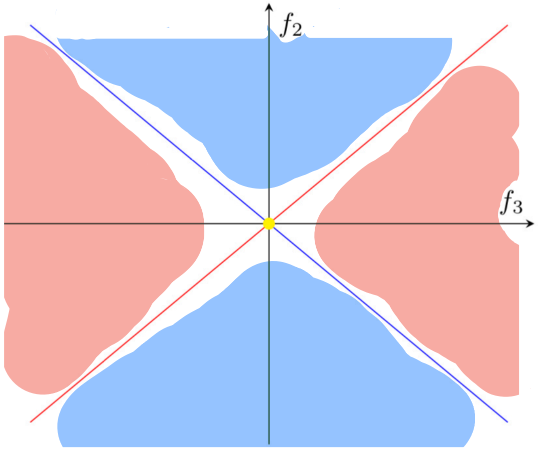

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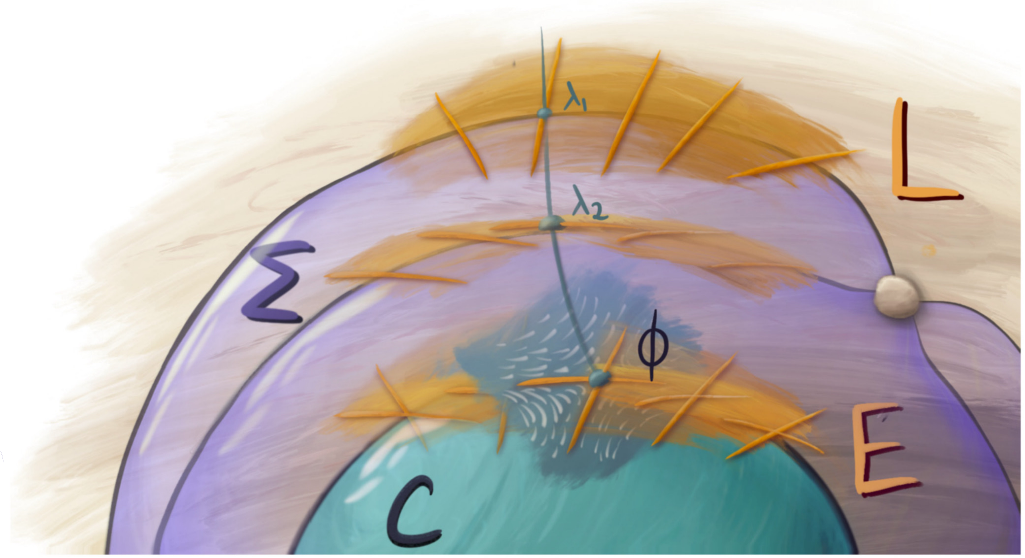

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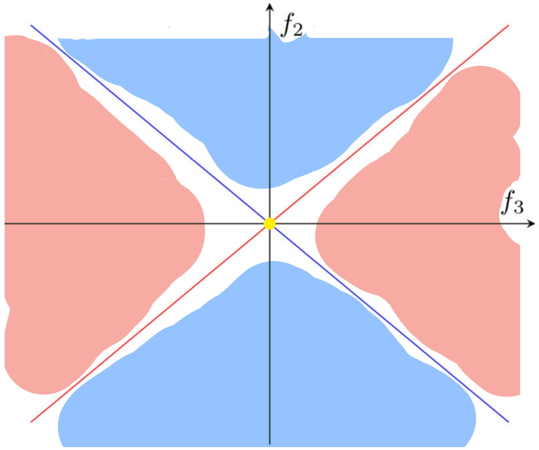

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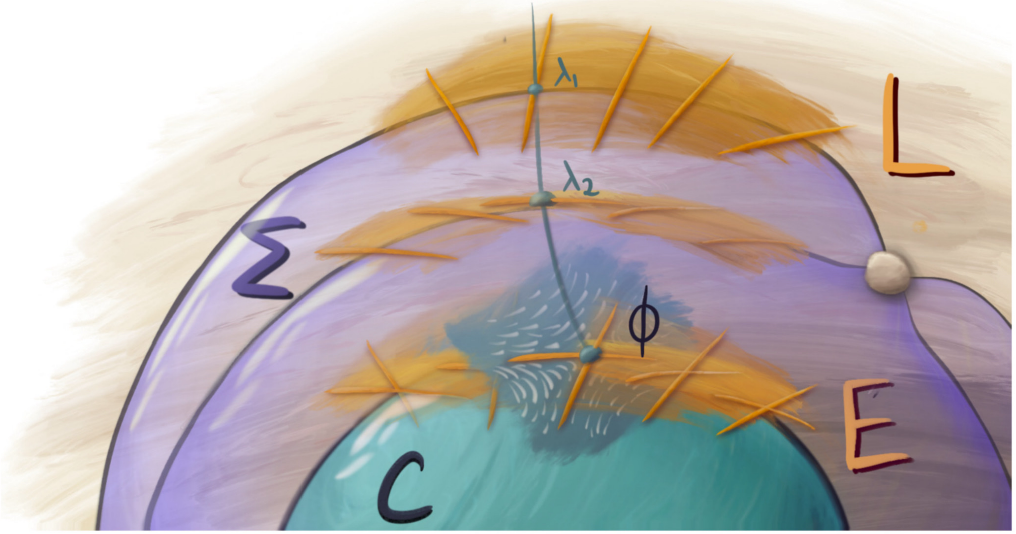

Peter Higgs (bosons)

Nigel Hitchin 1987
Carlos Simpson

C compact Riemann surface (or more generally Kähler manifold)
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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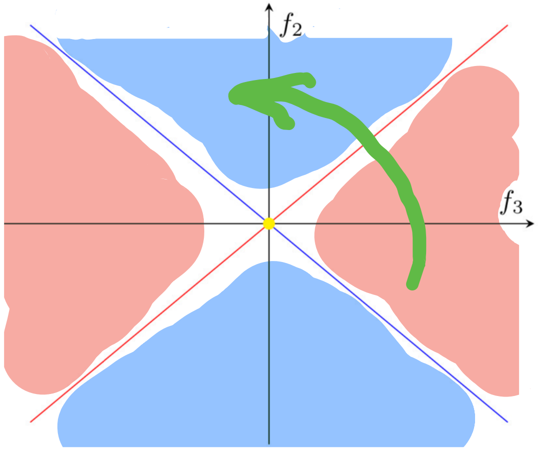

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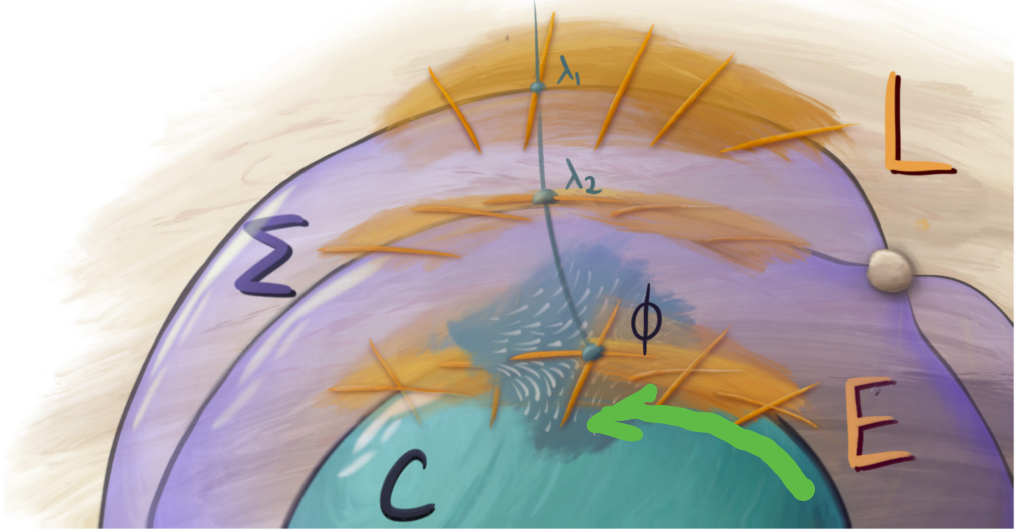
$$\phi_x \in \mathrm{End}(E_x), x \in C$$



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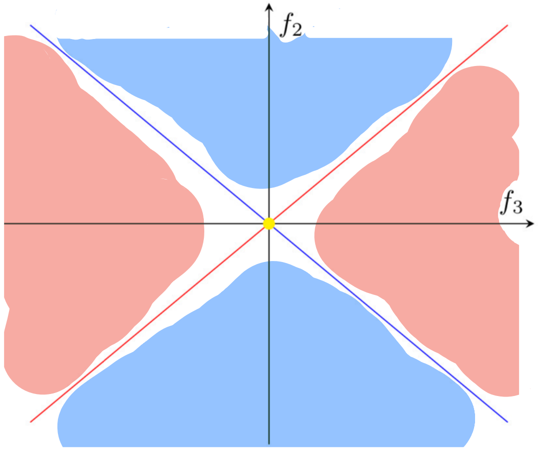

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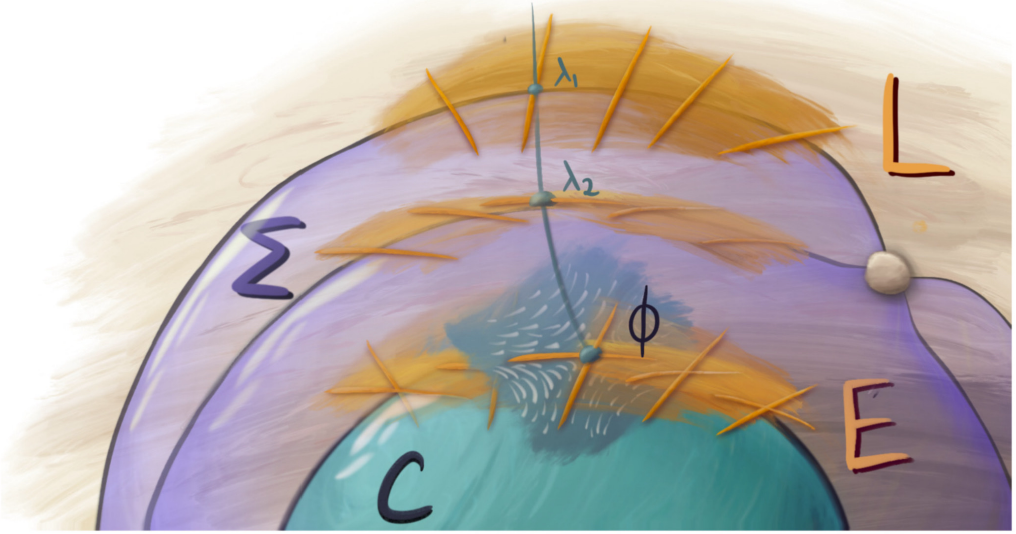

Portrait from Kienzle and Rayan, *Hyperbolic band theory through Higgs bundles*, ***Adv. Math.***, 2022.

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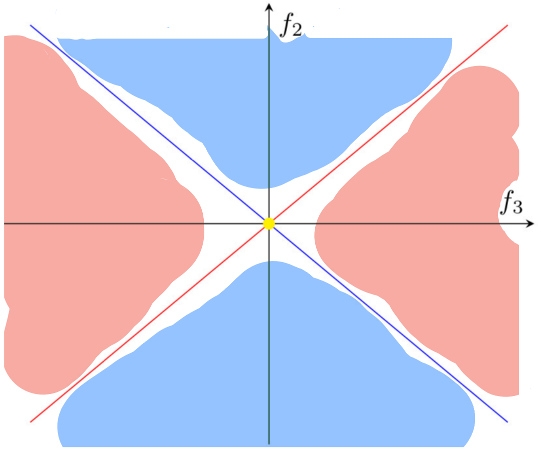

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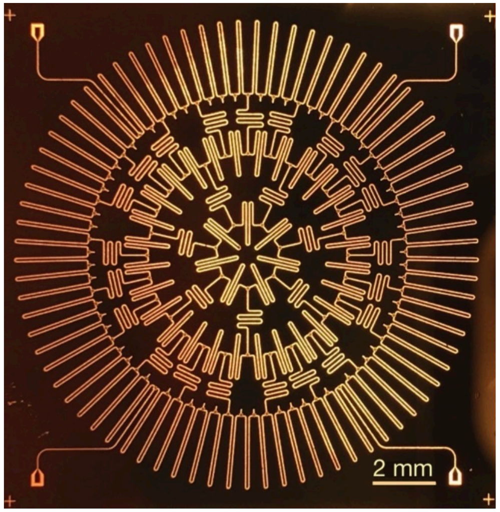
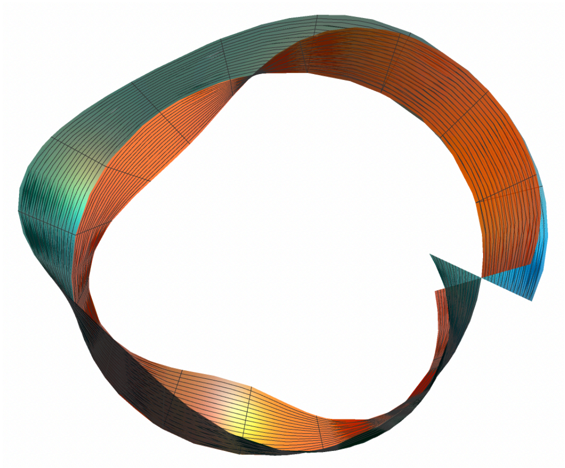
*Hyperbolic metric on the base C. Kollár et al., **Nature**, 2019.*



Eigenframe evolution as Higgs bundles: The non-Hermitian case

Proposition. The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands



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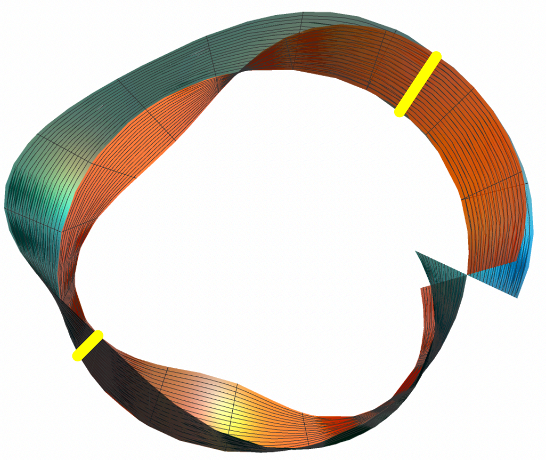

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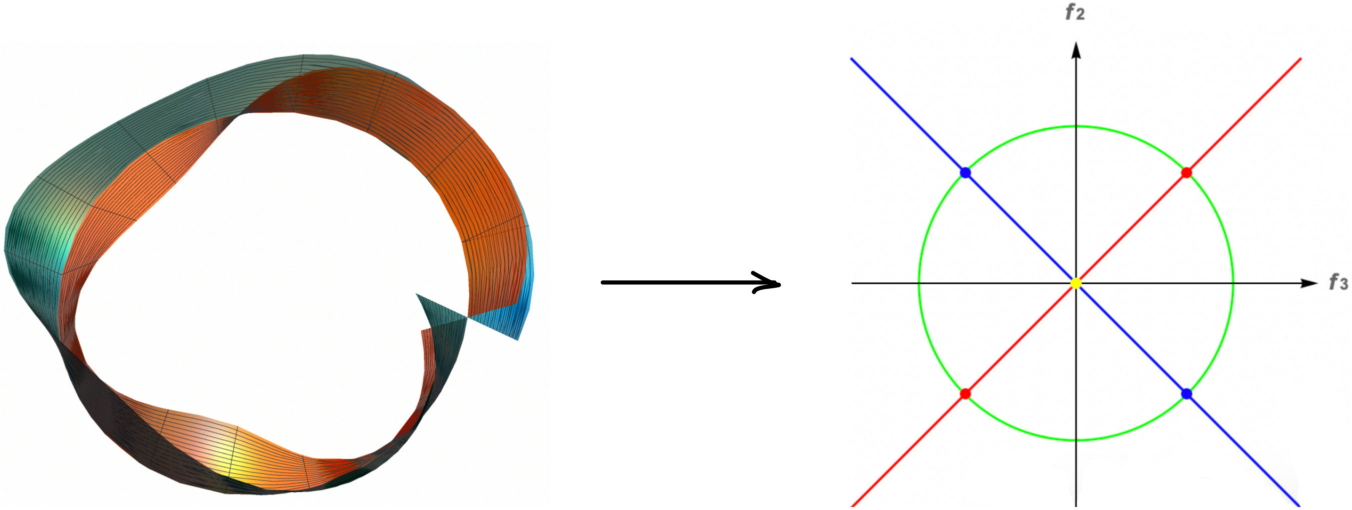
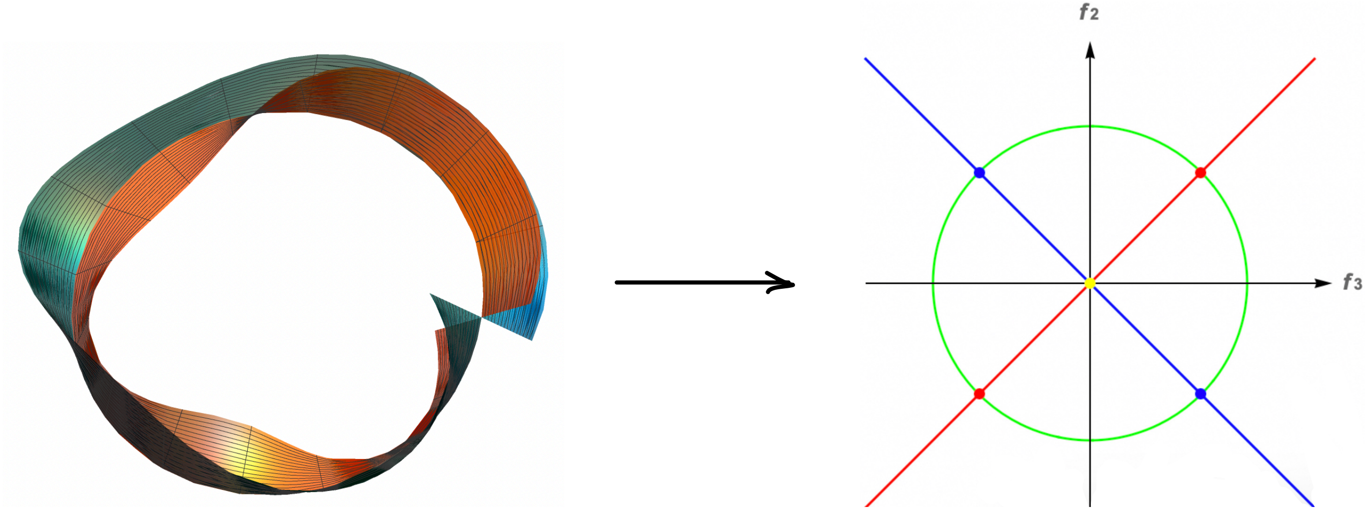


Eigenframe evolution as Higgs bundles: The non-Hermitian case

Proposition. The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



Here is a video showing the eigenframe evolution: https://yifeizhu.github.io/swallowtail/rotate.mp4

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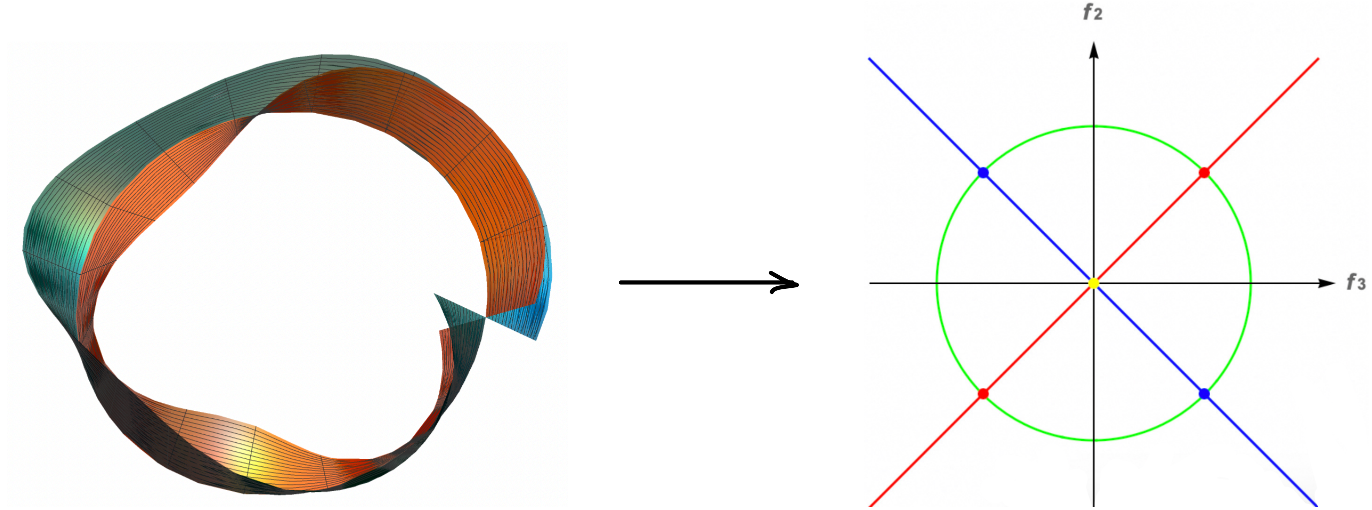

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Note. In the non-Hermitian case, since the eigenstates are in $\mathbb{C}^2$, we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+|\,|v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\mathrm{Herm}} := \rho$$

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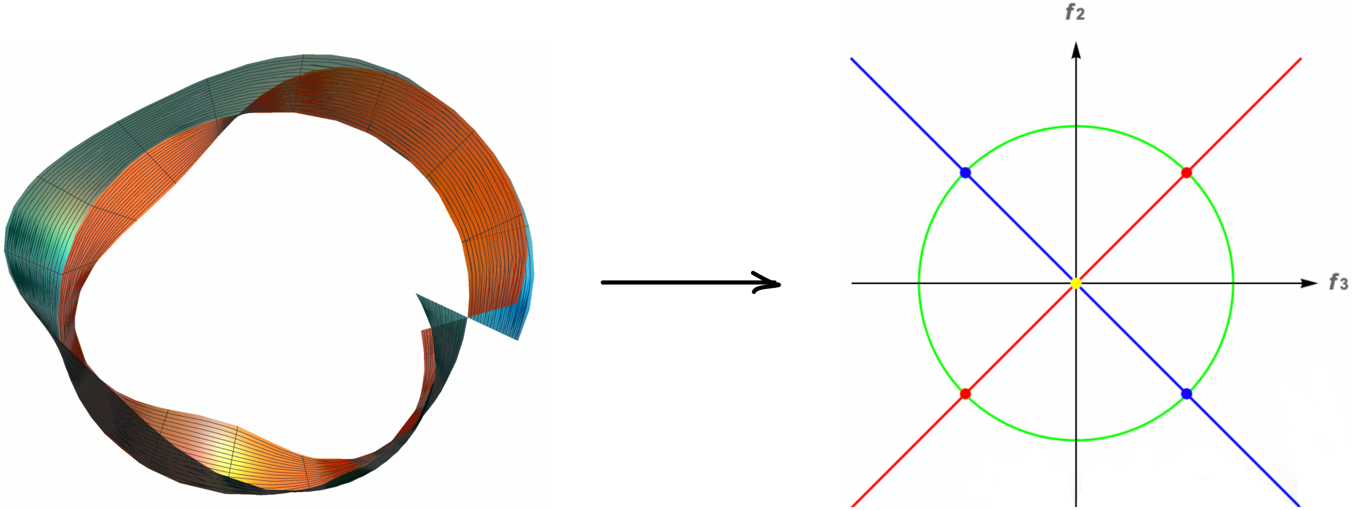

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The intrinsic geometry should be independent of real/complex coordination, though.

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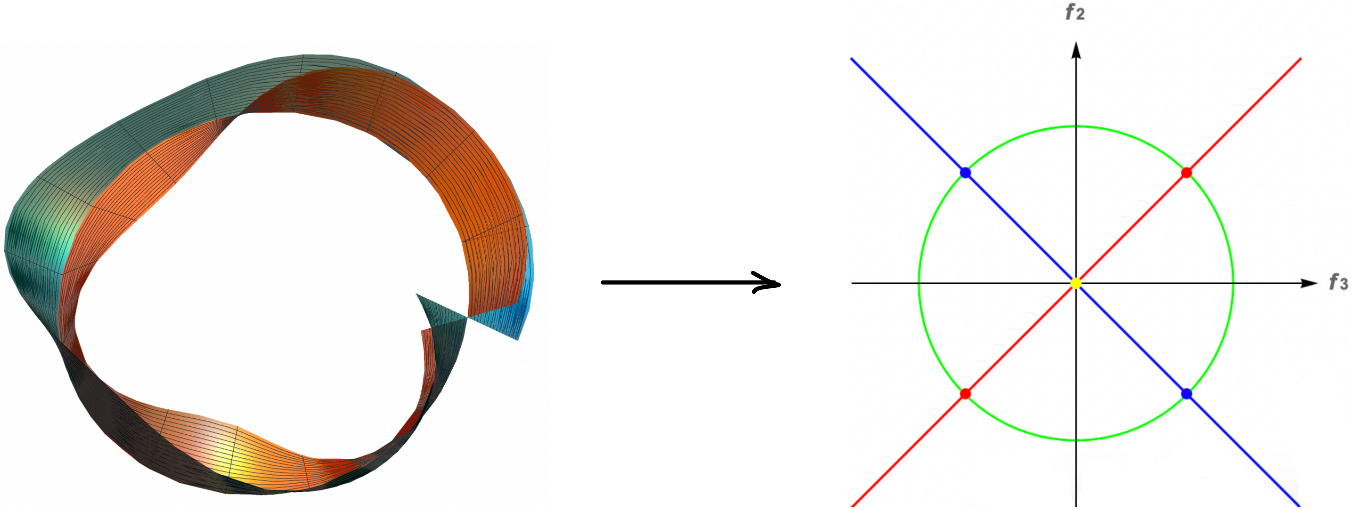

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Note. Recently, we observed that, with respect to the Minkowski-like inner product from the pseudo-Hermiticity, the eigenframe behaves the same way as that in the Hermitian system with the usual Euclidean metric, regardless of the degeneracy lines. This indicates that a change of geometry in the parameter space may lead to new understanding both mathematically and physically.

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Question. How to compute the topological charge?

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd}$$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

$$\begin{tikzcd}
V \arrow[d] & E \arrow[d] & \textit{universal bundle} \\
X & B & \textit{classifying space}
\end{tikzcd}$$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Question. How to compute the topological charge?

Mathematical interlude: Classification of bundles

$$\begin{tikzcd}
V \arrow[d] & E \arrow[d] & \textit{universal bundle} \\
X \arrow[r, "f"] & B & \textit{classifying space} \\
\textit{classifying map} & &
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Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
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universal bundle

classifying space

classifying map

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{isomorphism classes of bundles $V \to X$} ≅ {homotopy classes of maps $X \to B$}

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universal bundle

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For eigenframe evolution, we take $X = S^1$, and the right side becomes $\pi_1(B)$.

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

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universal bundle

classifying space

classifying map

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- Find computable and effective algebraic invariants (topological charge) for the classifying/parameter space

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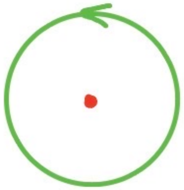
In progress: Need to compute the *intersection fundamental group* of the stratified parameter space.

Gajer, The intersection Dold–Thom theorem,
***Topology**, 1996. (Ph.D. student of Blaine Lawson, 1993)*

Goresky and MacPherson, 1974.

Eigenframe evolution as Higgs bundles: The non-Hermitian case



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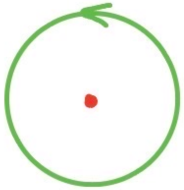

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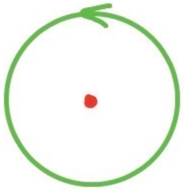
$$\begin{array}{c|c|c|c} \color{red}{\mathbb{Z}} & \color{blue}{\mathbb{Z}} & \color{blue}{\mathbb{Z}} & \color{blue}{\mathbb{Z}} \\ \color{red}{\mathbb{Z}} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} \\ \color{red}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} \\ & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} \quad \longrightarrow \bar{p} \end{array}$$

Intersection homology of $\mathbb{R}^2$ with one singular point: from top to bottom are $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2,$ where $\bar{p}$ is the perversity function.

From blue to red regions, they detect the singular point.

Eigenframe evolution as Higgs bundles: The non-Hermitian case



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$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
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0	0	0	0
	0	1	2 $\longrightarrow \bar{p}$

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Tolerance of ill-behaved cycles

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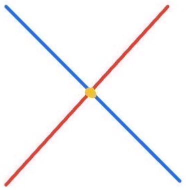

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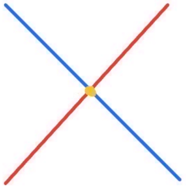

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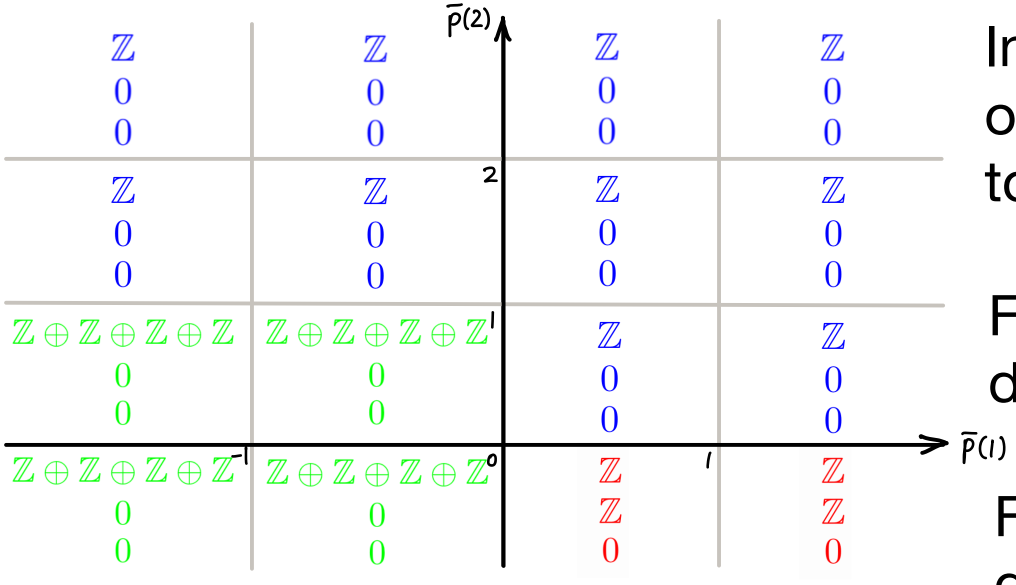


$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0
$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0
$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ 0 0	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0	$\mathbb{Z}$ 0 0
$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ 0 0	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ 0 0	$\mathbb{Z}$ $\mathbb{Z}$ 0	$\mathbb{Z}$ $\mathbb{Z}$ 0

Intersection homology of $\mathbb{R}^2$ with a pair of intersecting singular lines: from top to bottom are $I^{\bar{p}}H_*$ with $* = 0, 1, 2.$

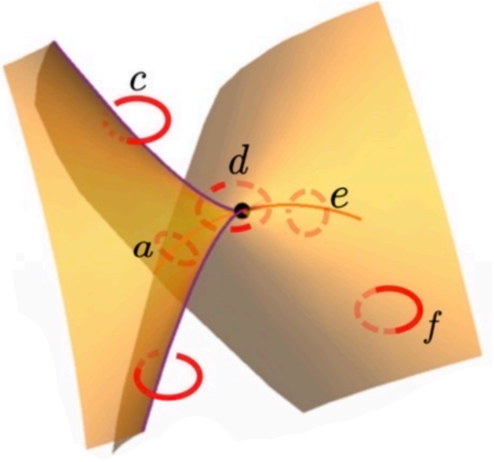
From green to blue regions, they detect the singular lines.

From blue to red regions, they detect the intersection point.

Mathematical set-up: Eigenframe evolution of non-Hermitian systems

We have explicitly calculated the intersection homology of the swallowtail catastrophe, which appears locally in many 3-band non-Hermitian systems.

$$I^{\bar{p}}H_1(\text{swallowtail}) = \begin{cases} \mathbb{Z}e & \text{if } \bar{p}(1) < 0, \bar{p}(2) < 0 \\ \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c & \text{if } \bar{p}(1) \geqslant 0, \bar{p}(2) < 0, \bar{p}(3) < 1 \\ \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c/(a+b+c=0) & \text{if } \bar{p}(1) \geqslant 0, \bar{p}(2) < 0, \bar{p}(3) \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$



$$a + b + c = d, \quad d = e$$

since the 2D chains that witness these equations are *allowable*.

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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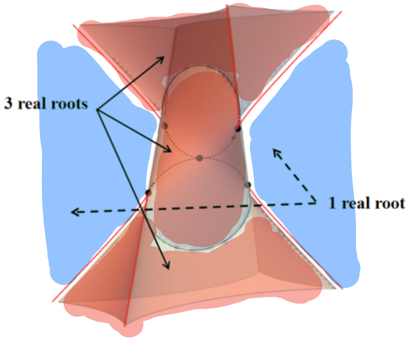

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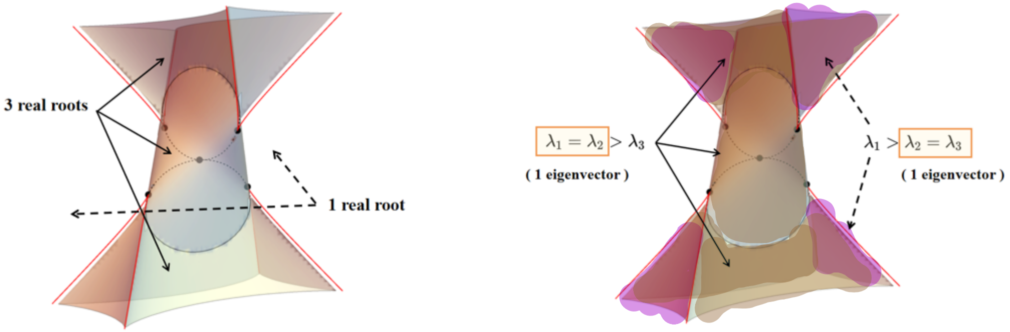

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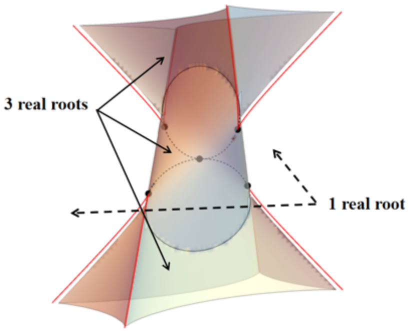

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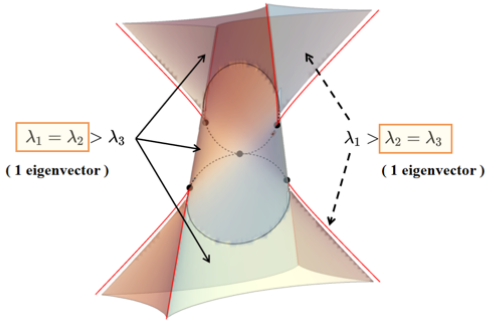

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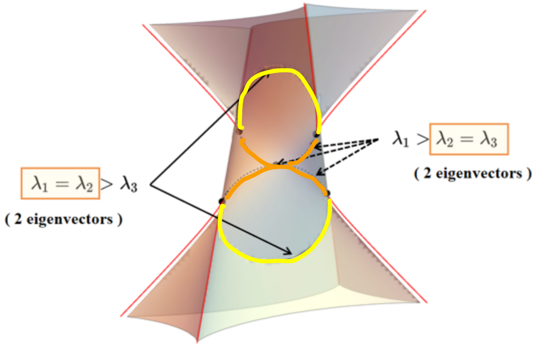

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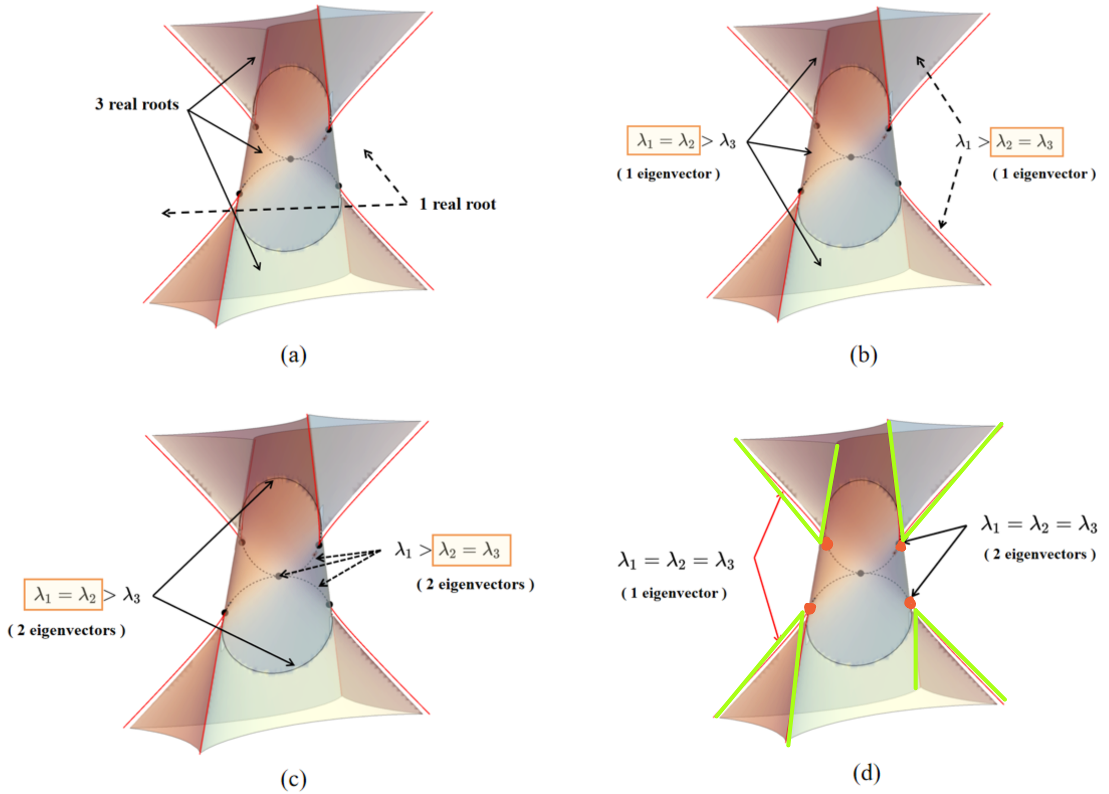

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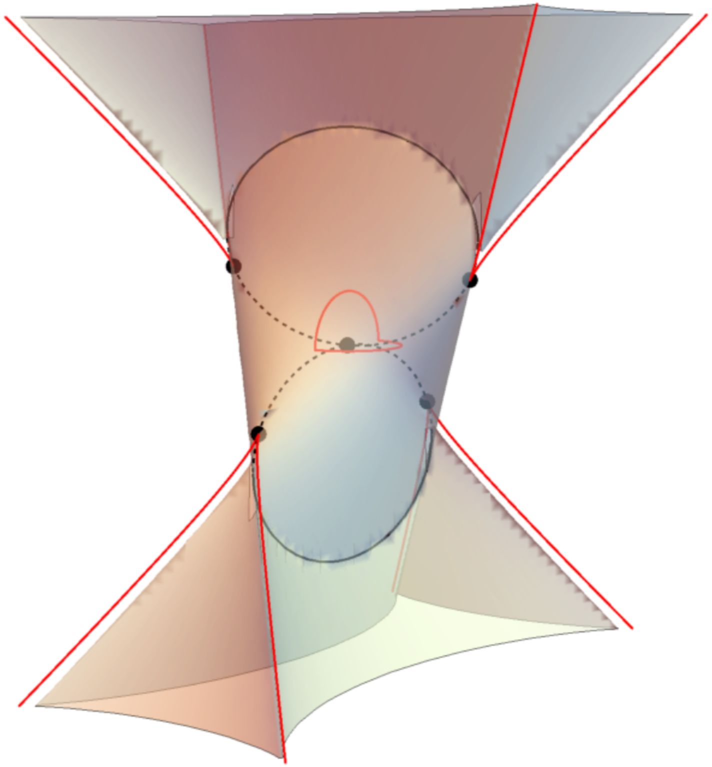

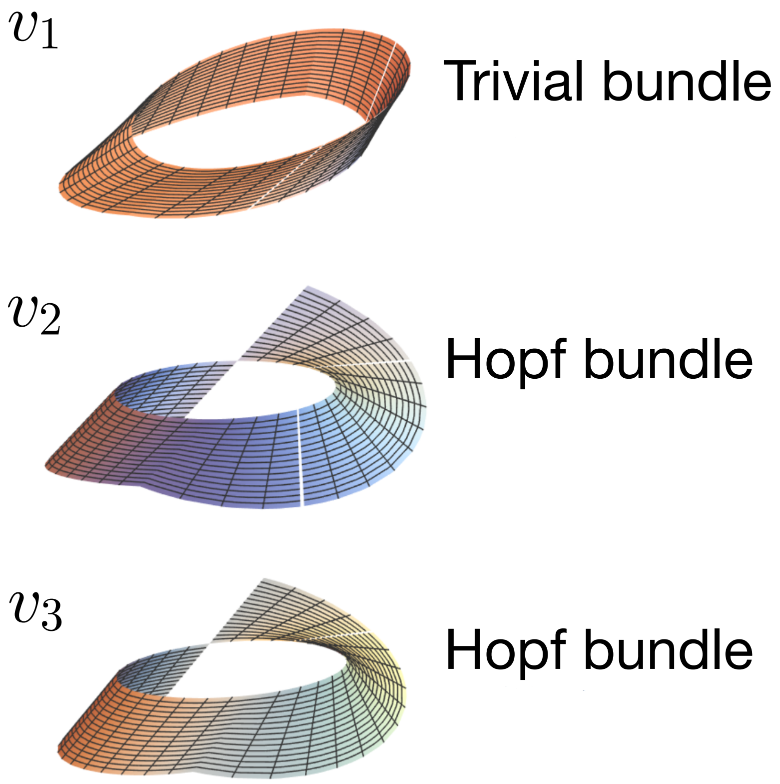

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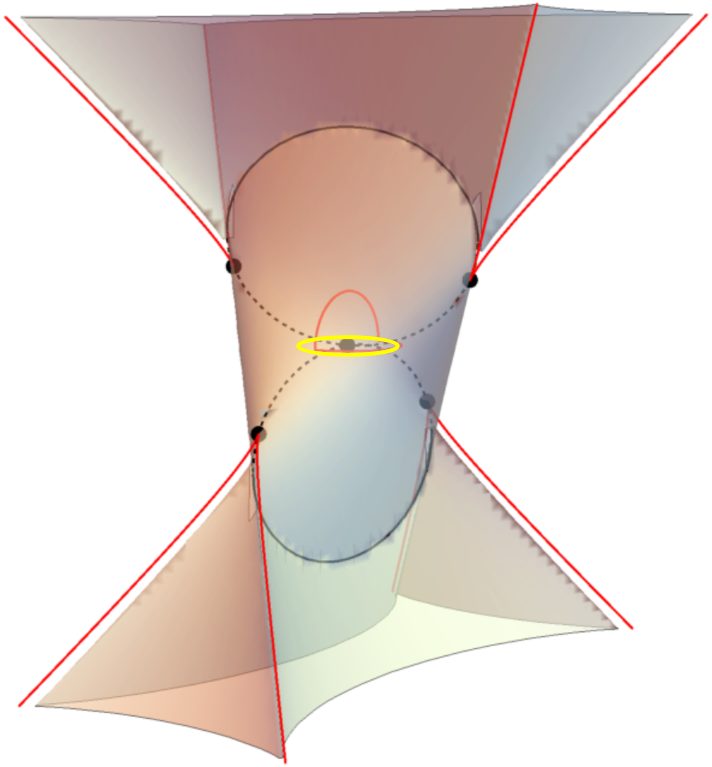
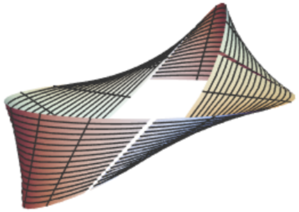
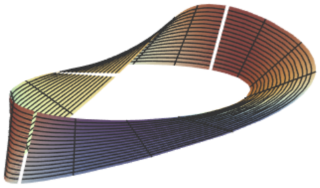


v_1



Trivial bundle

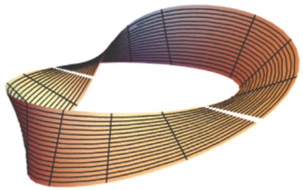
v_2



~~Hopf bundle~~

Trivial bundle

v_3



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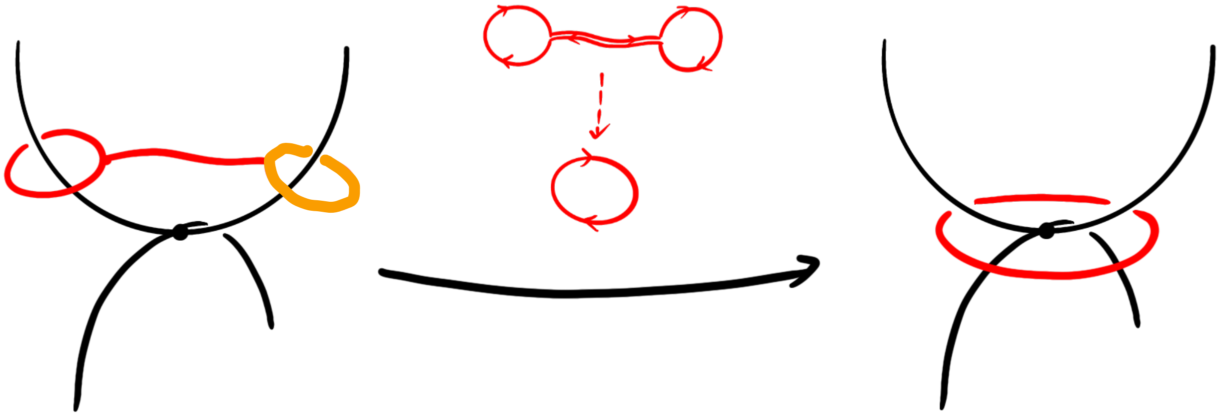

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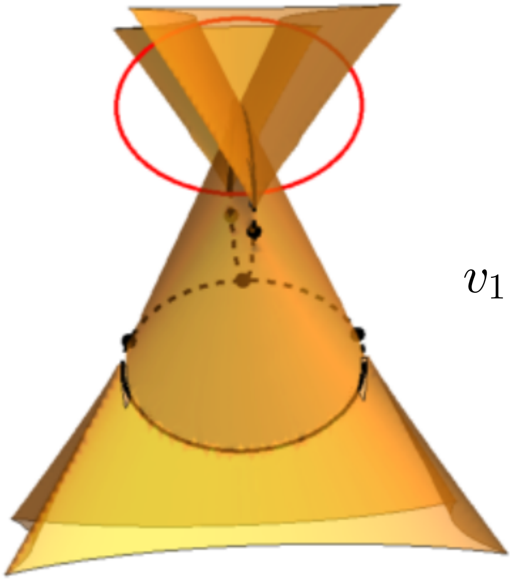

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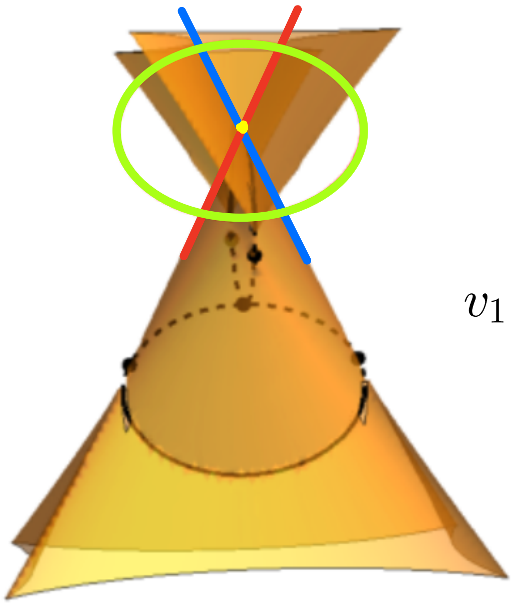

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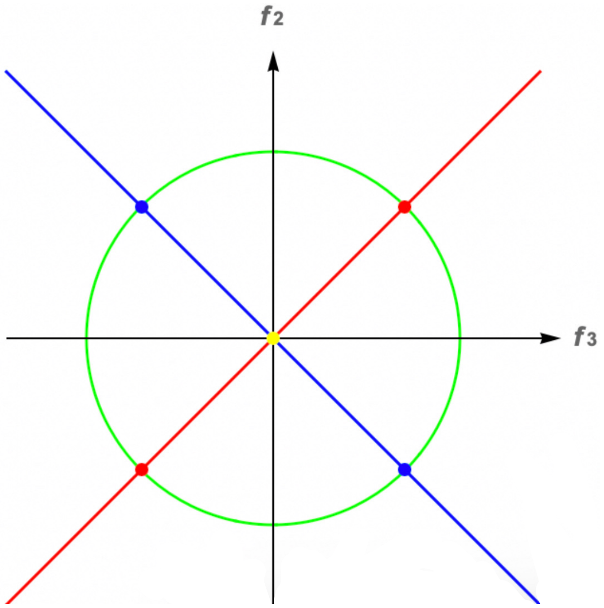

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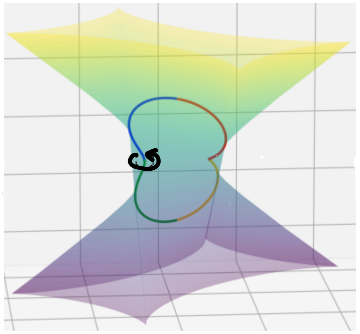

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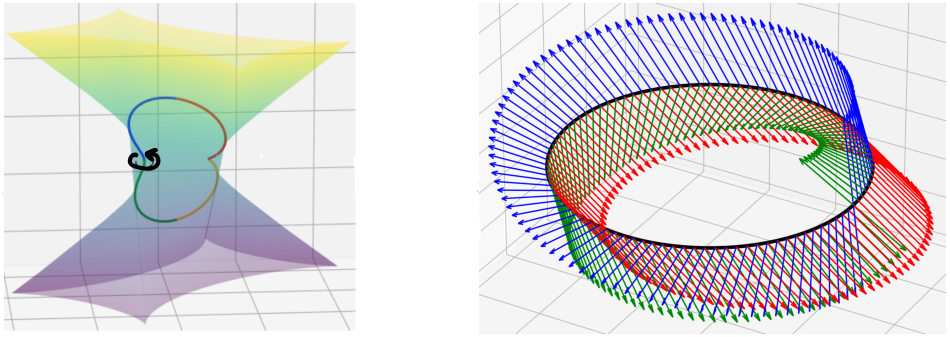

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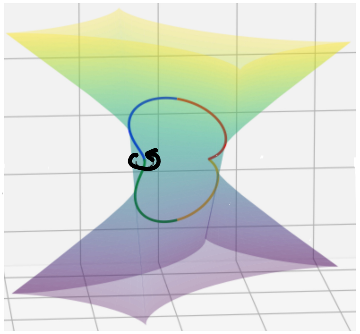

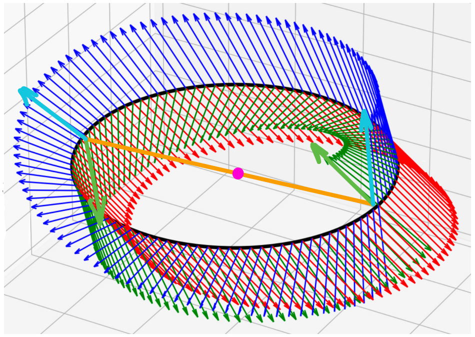

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Across the center (nodal line), the blue and green eigenstates swap — "band inversion."

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Visualizing at https://www.wolframcloud.com/env/zhuyf0/Presentation.nb

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
- Ruledness (specific parametrization)

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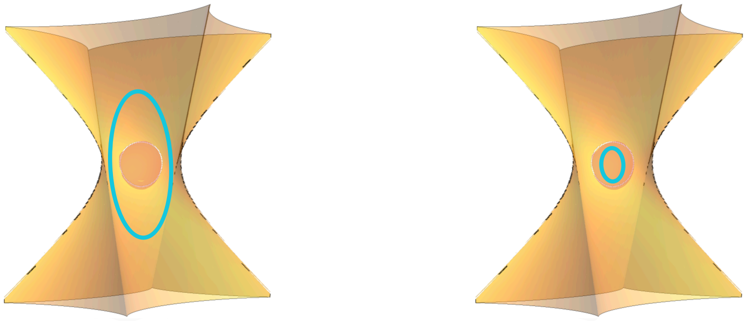

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Shrinking this loop into the enclosed region, we find the eigenbundle along it remains trivial.

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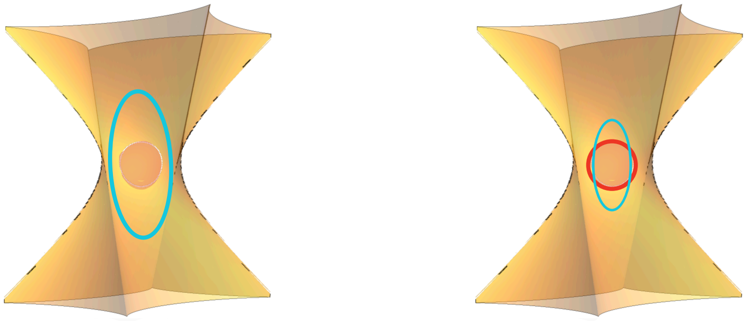

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What about loops transversing the nodal intersection lines? Band inversion again?

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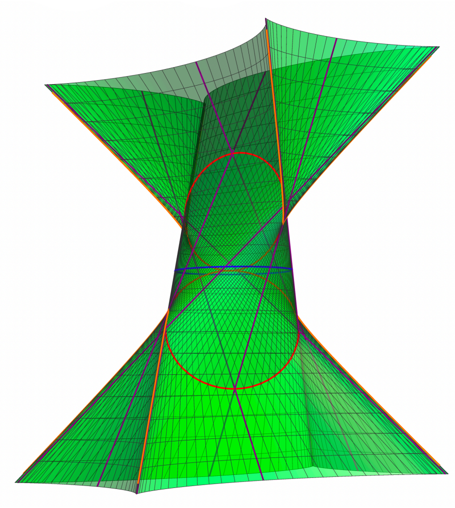
- Improved precision with graphing and engineering



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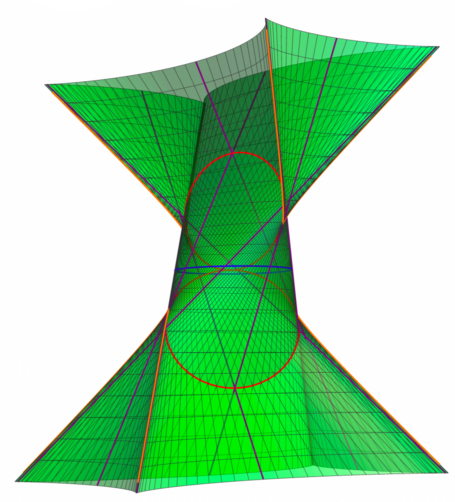

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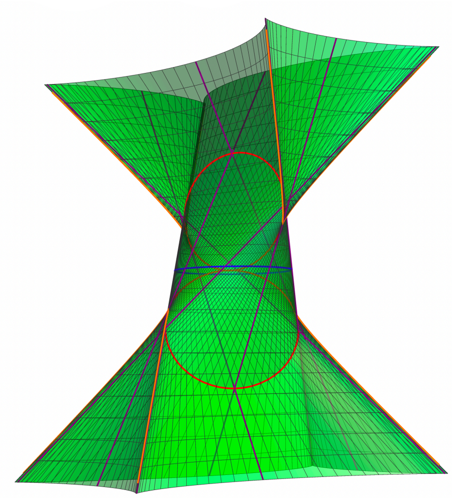

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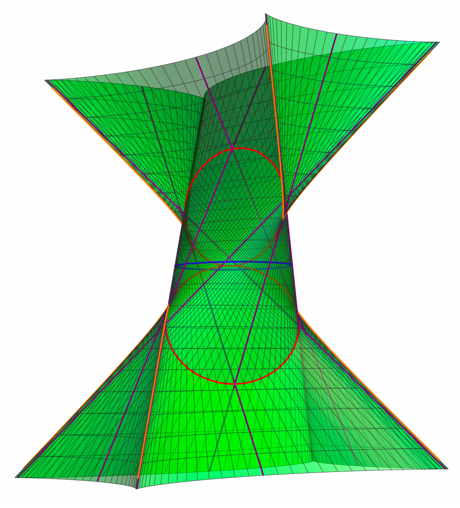

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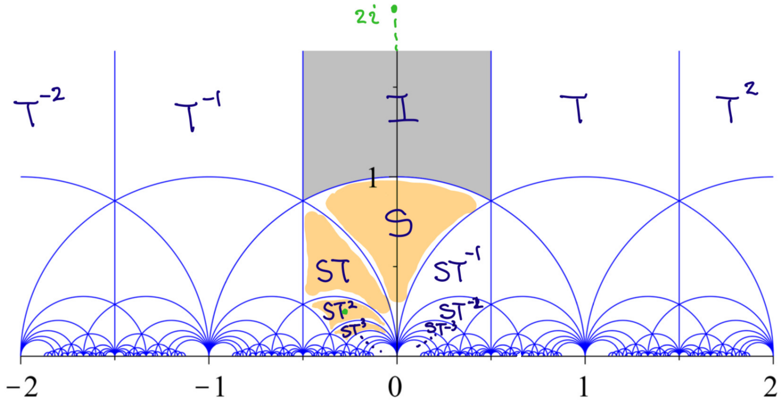

A prototypical 2D hyperbolic lattice with a straight-line boundary

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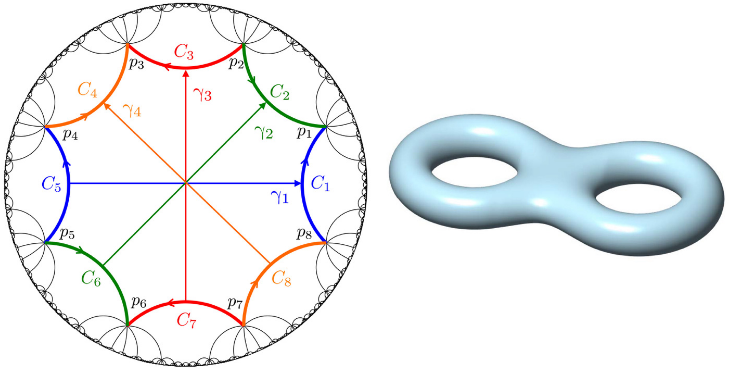

Another basic example of a hyperbolic lattice associated to a genus-2 surface
(from Maciejko and Rayan, *Hyperbolic band theory*, ***Sci. Adv.***, 2021)

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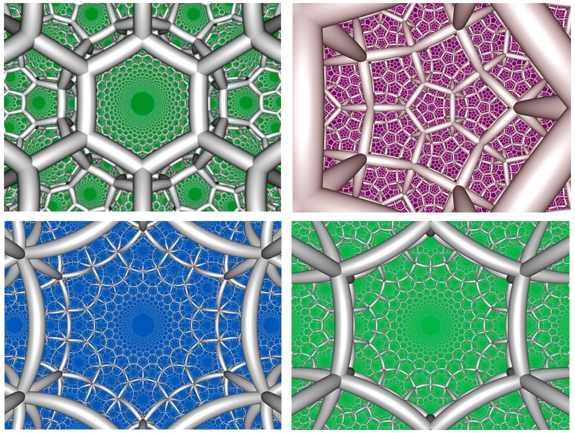

Four 3D hyperbolic lattices tiling up the hyperbolic 3-space $\mathbb{H}^3$ (from John Baez's blog)

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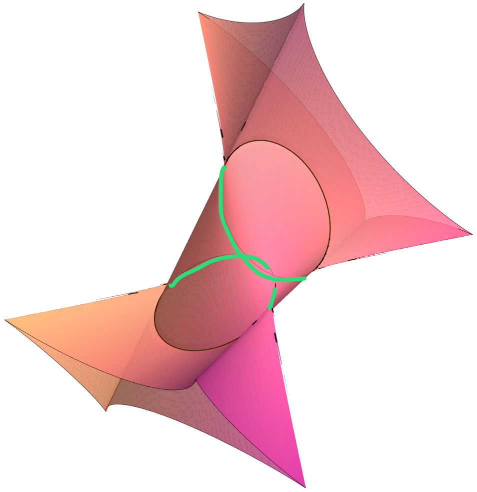

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Existence of nodal curves inside also gives evidence, supporting nontrivial loops around (generating a free group on 3 letters) acting on a 3D hyperbolic lattice.

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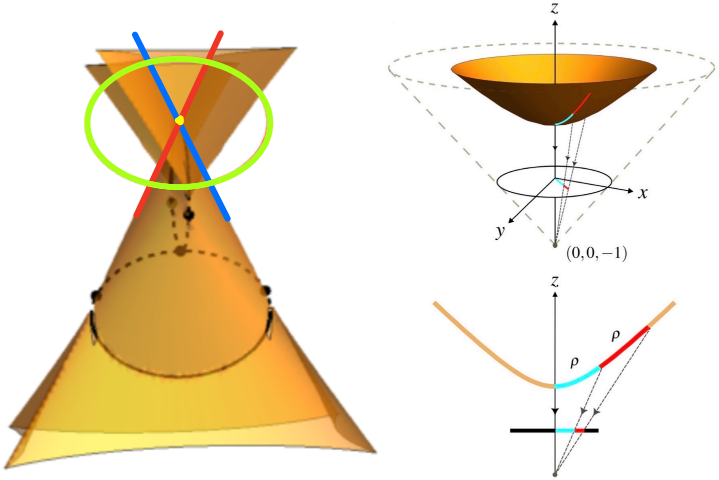

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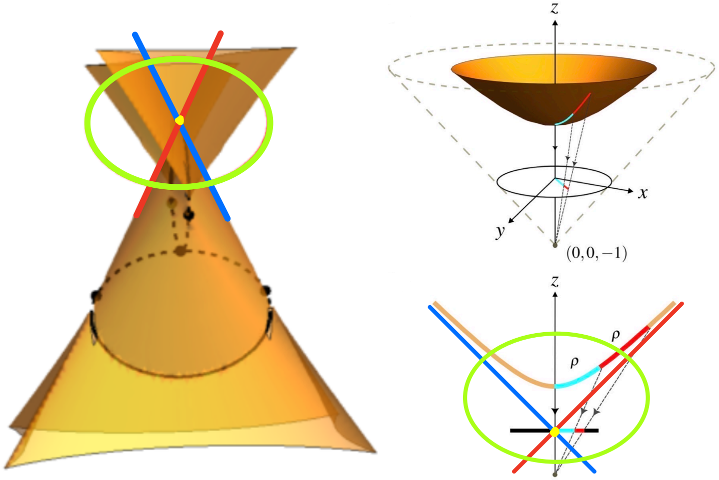

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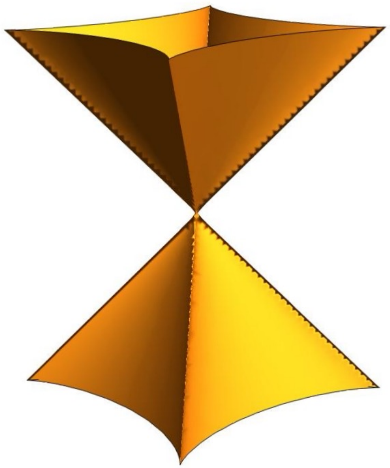

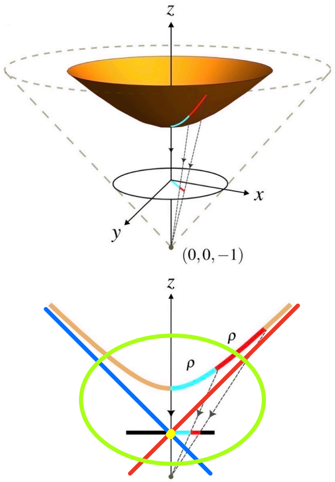

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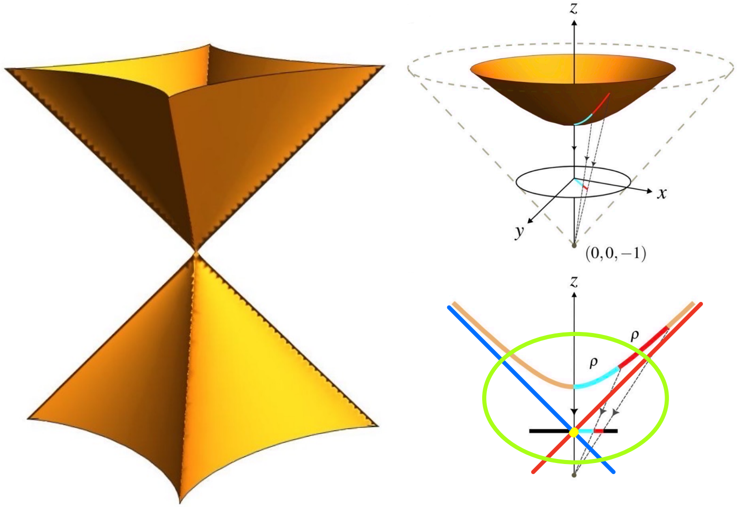

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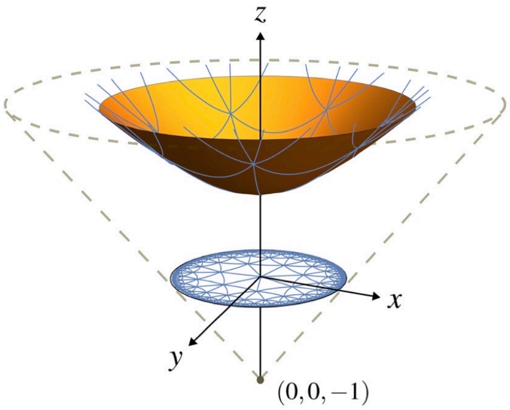

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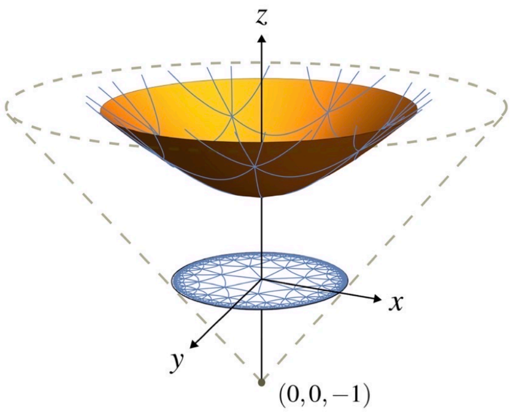

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Implications of ruledness:

- Exceptional surface as a light-cone in Minkowski space? It would be interesting to explicitly construct hyperbolic lattices as initiated by Kollár et al. We have started to explore the possibilities with the assistance of AI tools.

Classifying Higgs bundles

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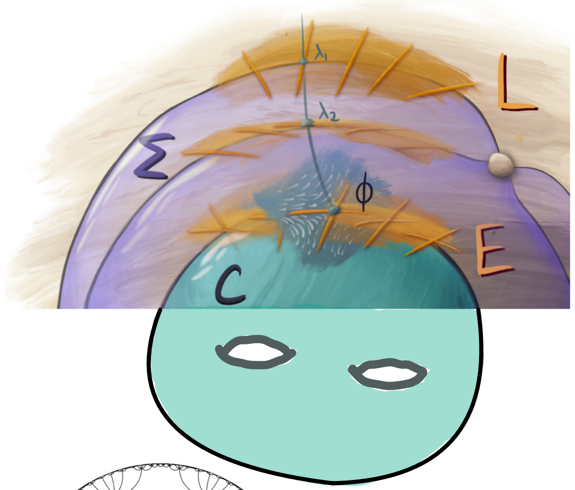
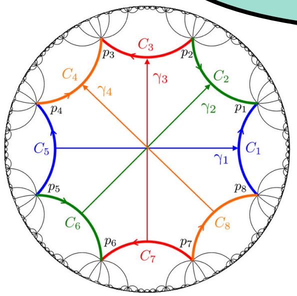
In fact, the non-Abelian Hodge correspondence gives analytic isomorphisms

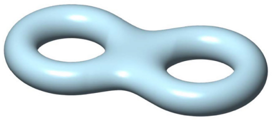
$$\mathscr{M}^{\mathrm{S}}_{\mathrm{Higgs}}\big(\mathrm{SL}_n(\mathbb{C})\big) \cong \mathrm{Rep}\big(\pi_1(C), \mathrm{SL}_n(\mathbb{C})\big) \cong \mathcal{H}$$



where $\mathscr{H}$ is the space of equivariant harmonic maps from the universal cover $\tilde{C}$ to $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$, modulo isometries. Here, the equivariance is with respect to a representation $\rho\colon \pi_1(C) \to \mathrm{SL}_n(\mathbb{C})$.

Thus, given a Higgs bundle (E, ϕ), we get a harmonic map $f\colon \tilde{C} \to \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$. The negative semi-definite Killing form on the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ gives rise to a metric on $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$, which then pulls back along f and descends to C. This metric has non-positive Gaussian curvature (constant negative in favorable cases).





Maciejko–Rayan ’21

Bulk–edge correspondence

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> *More precisely, e.g., the bulk–edge correspondence relates a topological invariant of the bulk insulator (the first Chern number of the Bloch eigenbundle, also called the Hall conductance) with an invariant of a surface state (the winding number about the Fermi energy in the complex Bloch variety).*

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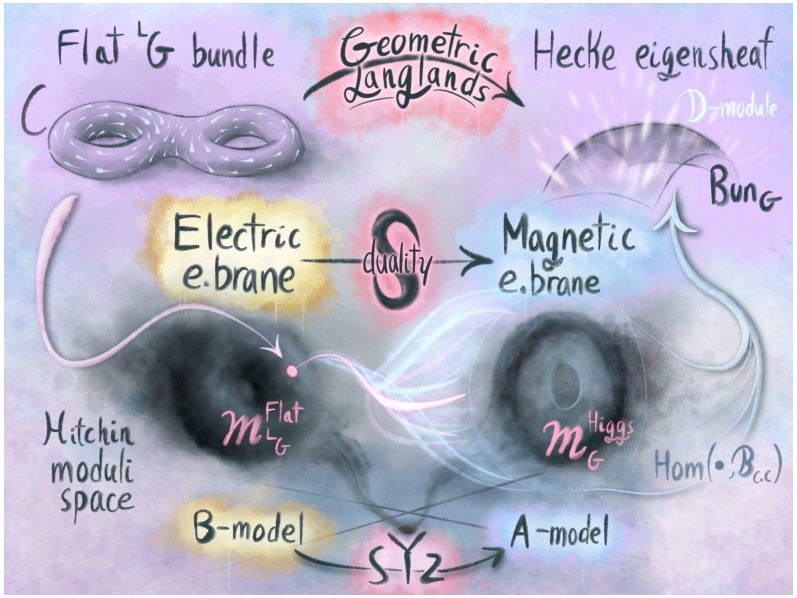
There has not been a rigorous mathematical explanation for such a correspondence in general, but it is reminiscent of the Langlands duality.

Cf. Ikeda, Quantum Hall effect and Langlands program, 2018



Portrait by Elliot Kienzle

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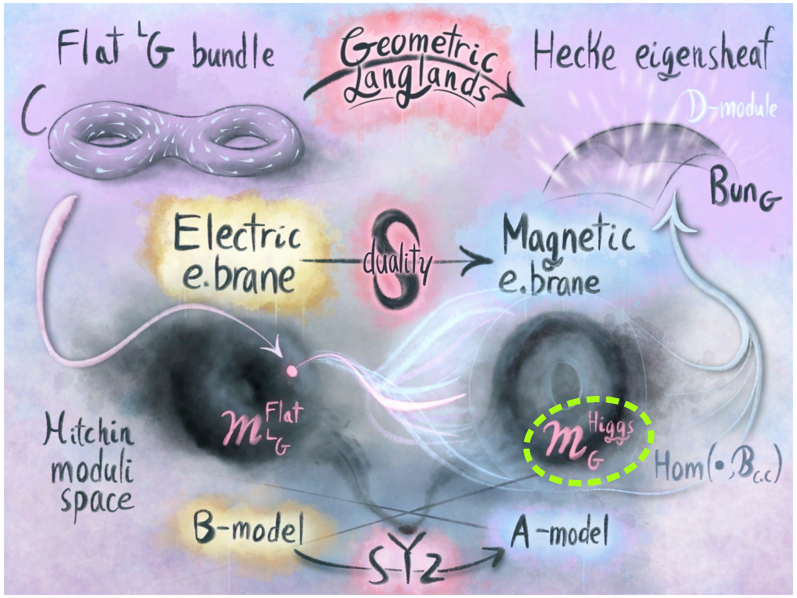

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Recap

Main approach. We modeled adiabatic evolution of quantum mechanical systems using **bundle theory**, topologically classifying **eigenbundles** associated to Hamiltonians with prescribed symmetry (non-Hermitian):

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

universal bundle = Higgs bundle
real-analytic variant

classifying space = parameter space
stratified nonisolated singularity

classifying maps = homotopical invariants of B
intersection homology and homotopy groups

Geometric aspects. Non-Abelian Hodge correspondence (hyperbolic metric), Minkowski light-cone (hyperbolic lattice), and **hyperbolic band theory**

Further perspectives. Bulk–edge correspondence and geometric Langlands through Higgs bundles

Thank you.