

Structured ring spectra from moduli problems

Motivated by multiplicative complex orientations valued in Morava E-theory and computations in unstable chromatic homotopy theory, we construct lifts of rings of power operations as E-infinity ring spectra, via moduli problems for derived level structures on spectral deformations of formal groups. As applications, we obtain homotopy fixed point spectral sequences and genuine equivariant spectra with respect to the action of profinite groups that appear in the Jacquet–Langlands correspondence. This is joint work with Xuecai Ma and Guozhen Wang.



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joint w/ Xuecai Ma
Guozhen Wang

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Context & Motivations Morava E-theory, power operations

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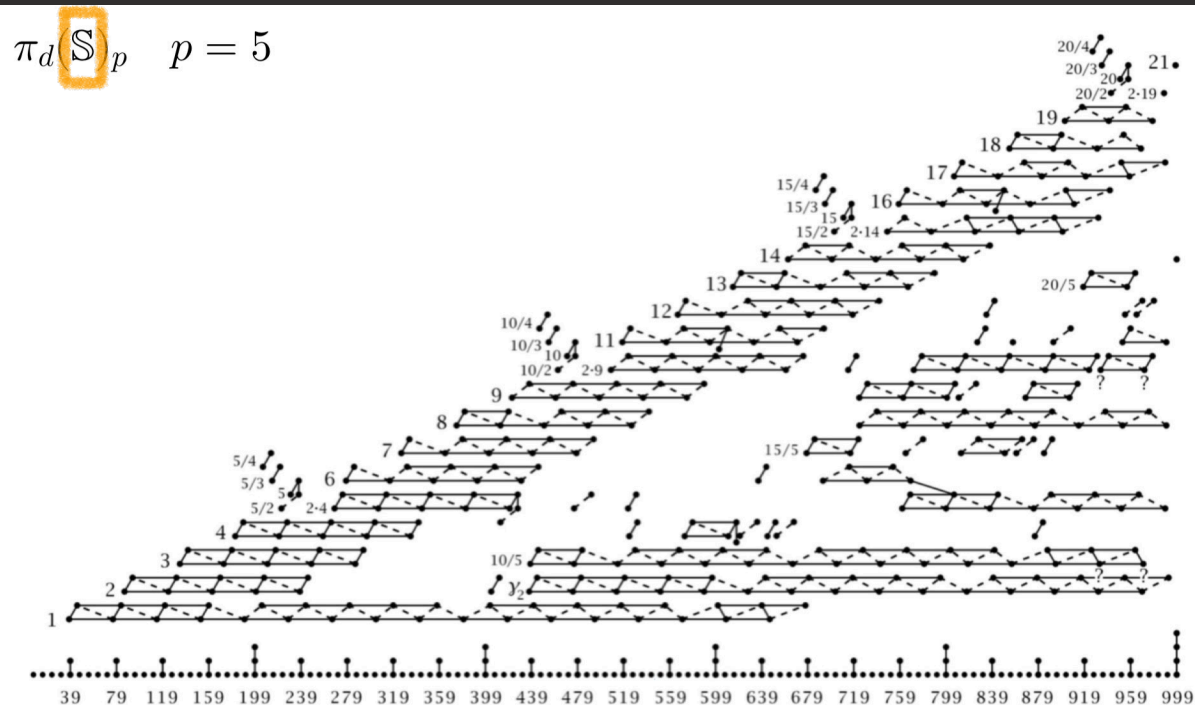
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Structured ring **spectra** from moduli problems

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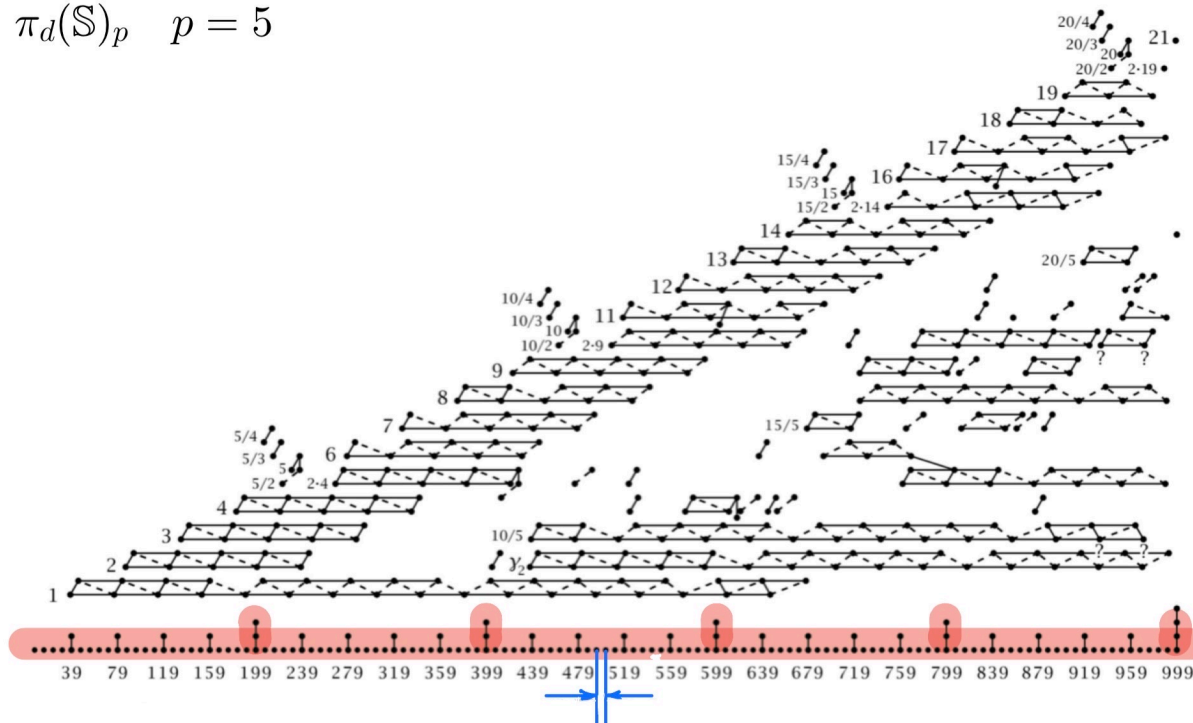
Picture: Hatcher
Computation: Ravenel

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$$\pi_d(\mathbb{S})_p \quad p = 5$$



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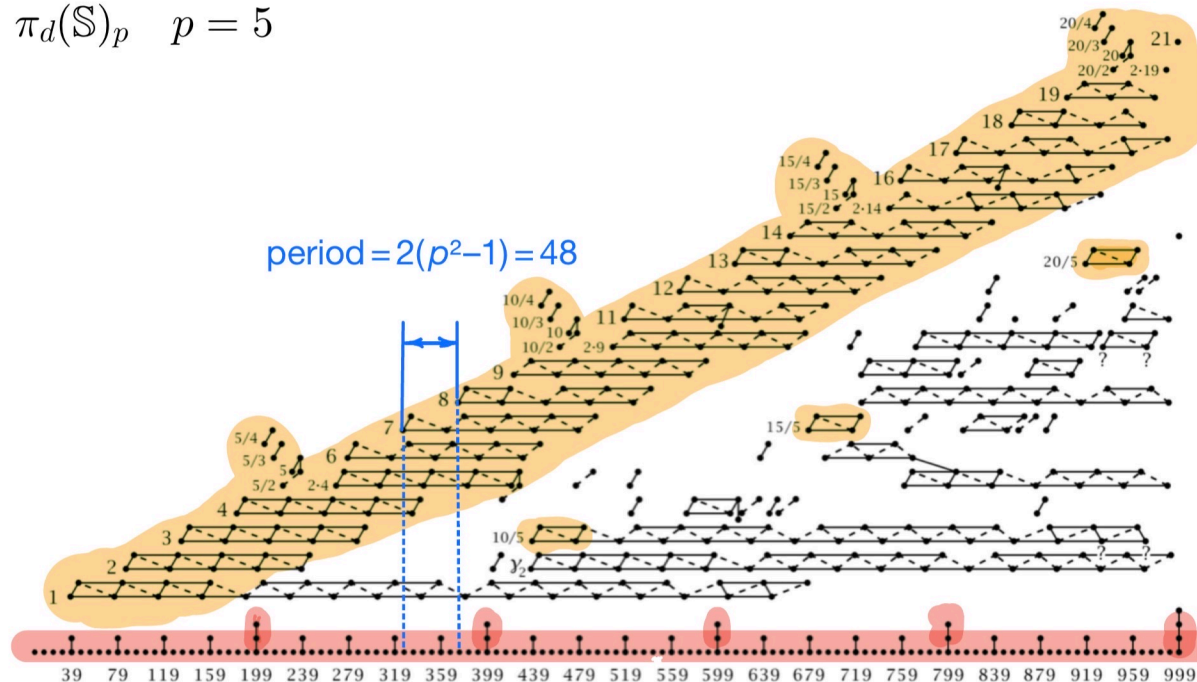
v_1 -periodic elements

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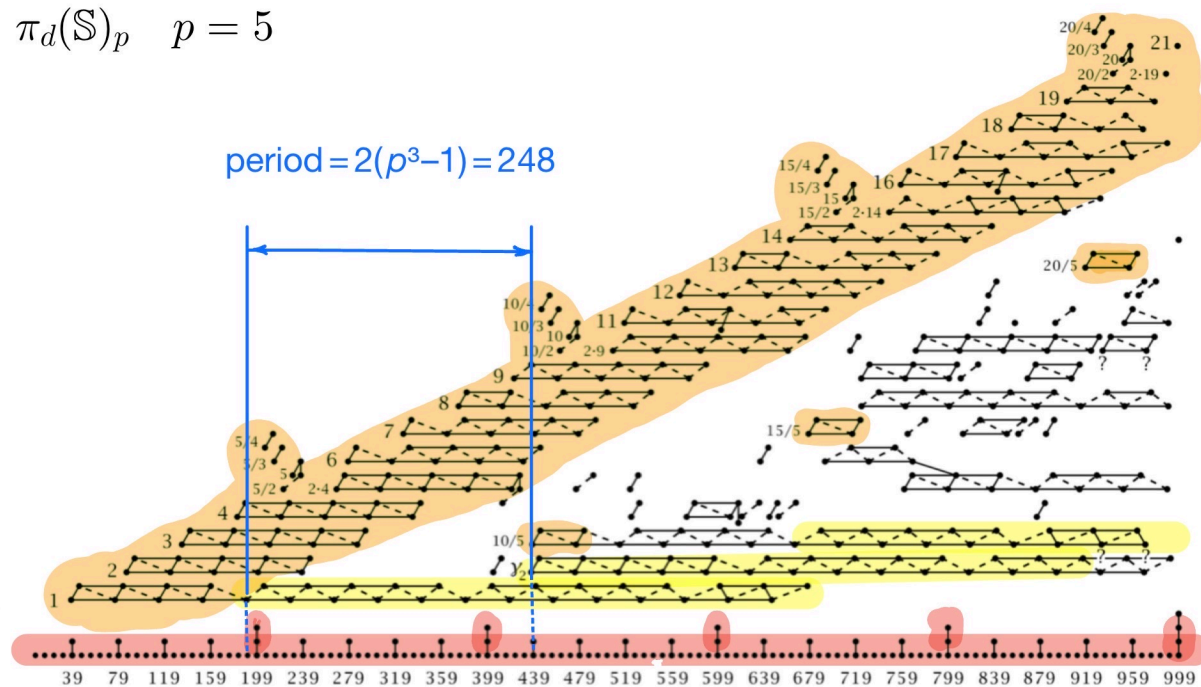
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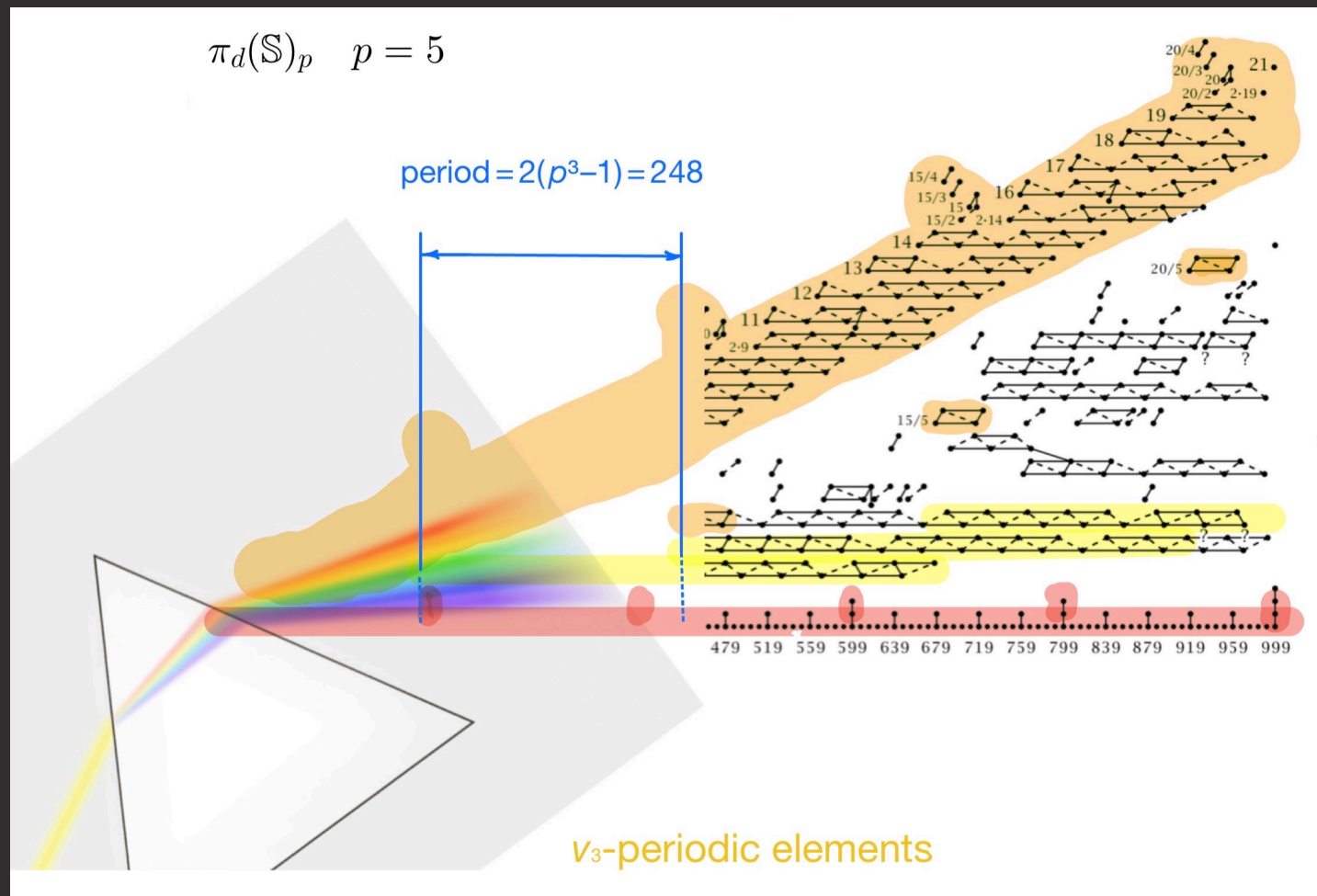
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v_3 -periodic elements

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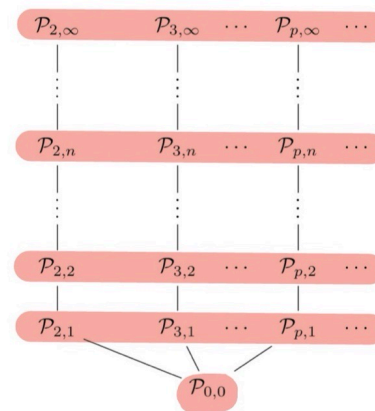


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Thick subcategories of $\mathbf{Sp}$ (Hopkins–Smith '98)

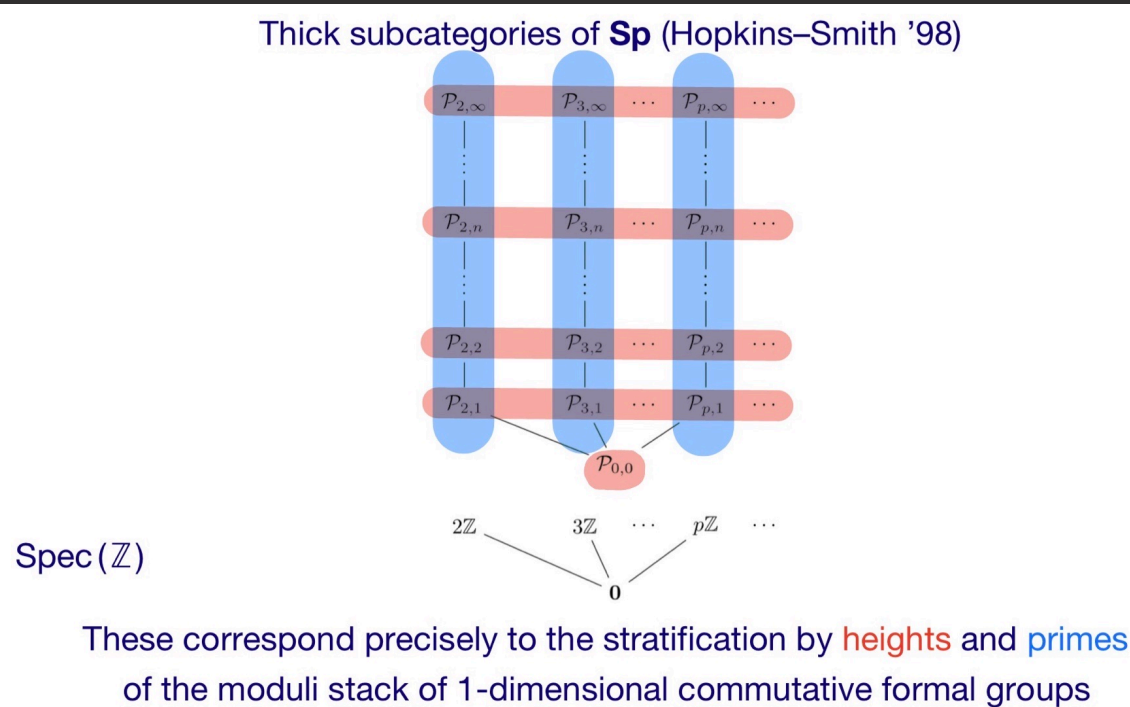


These correspond precisely to the stratification by heights and primes
of the moduli stack of 1-dimensional commutative formal groups

Structured ring spectra from moduli problems

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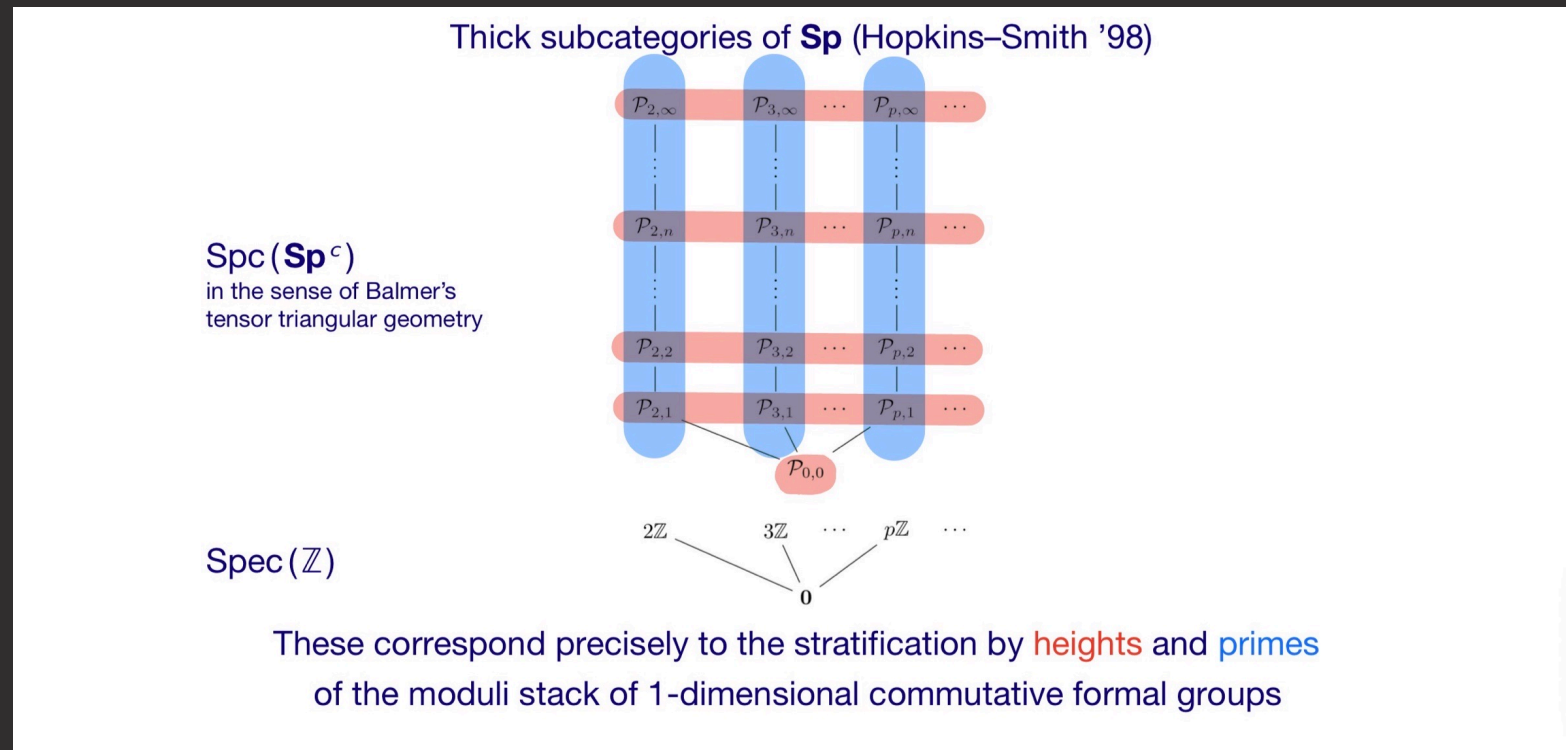
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Structured ring **spectra** from moduli problems

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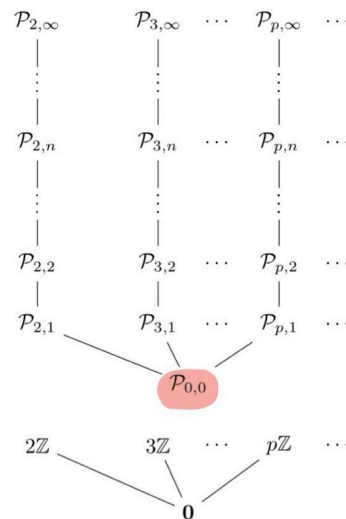
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Thick subcategories of $\mathbf{Sp}$ (Hopkins–Smith '98)

$\mathbf{Spc}(\mathbf{Sp}^c)$

in the sense of Balmer's
tensor triangular geometry

$\mathrm{Spec}(\mathbb{Z})$



rational cohomology

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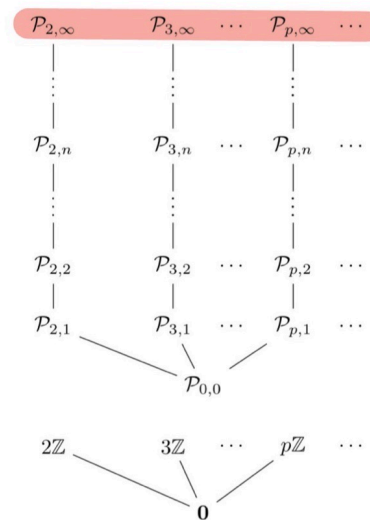
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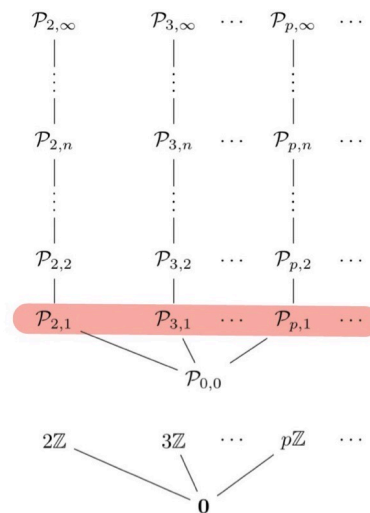
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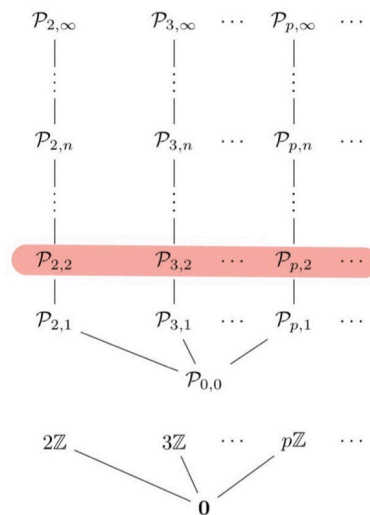
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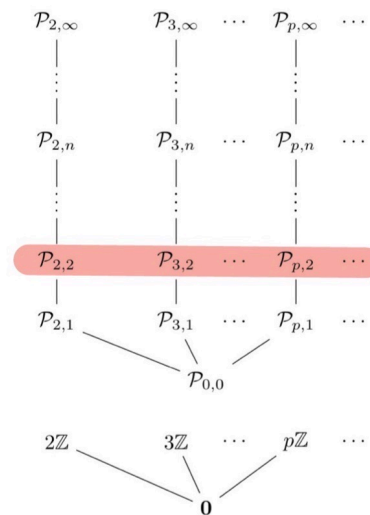
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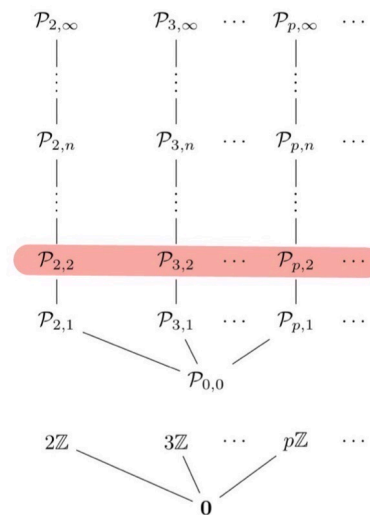
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Takeshi Torii '99

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- The structure of genuine equivariant spectra with respect to the action of profinite groups (e.g. $GL_n \mathbb{Z}_p$)

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topological realization / lift of the **Lubin-Tate deformation ring**

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converging to the homotopy groups of the sphere spectrum localized with respect to the Morava K -theory spectrum of height n at the prime p

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converging to the homotopy groups of the **sphere spectrum localized with respect to the Morava K -theory spectrum** of height n at the prime p . Indeed, this homotopy fixed point SS agrees with the $K(n)$ -local E -Adams SS.

Theorem (Ando-Hopkins-Strickland '04, Rezk '09) $\exists$ an equivalence of categories

$$\left\{ \begin{array}{l} \text{quasicoherent sheaves of} \\ \text{graded commutative algebras over} \\ \text{the moduli problem of deformations} \\ \text{of } F/k \text{ and Frobenius isogenies} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{graded commutative algebras over} \\ \text{the Dyer-Lashof algebra for } E(F/k) \end{array} \right\}$$

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$$M = E\text{-module}, \quad \pi_0 M = [S, M]_S \simeq [E, M]_E$$

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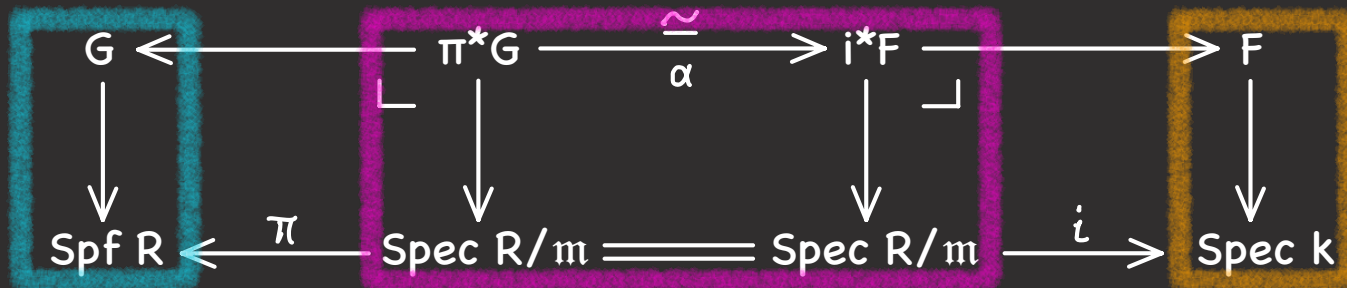
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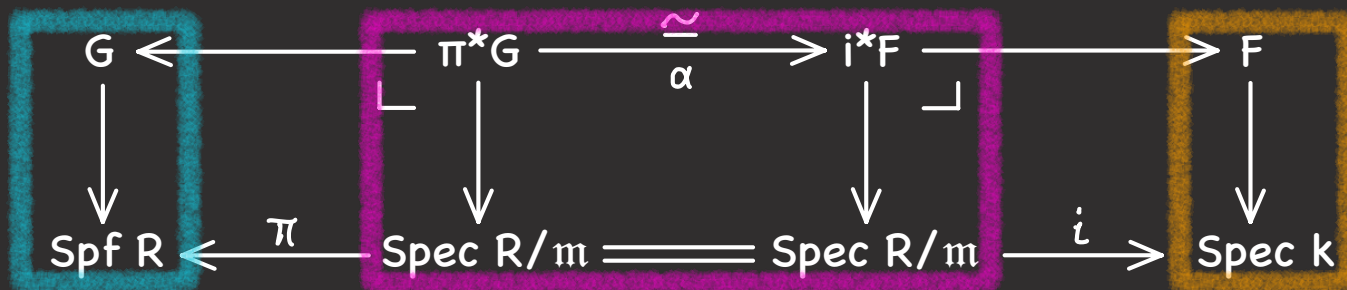


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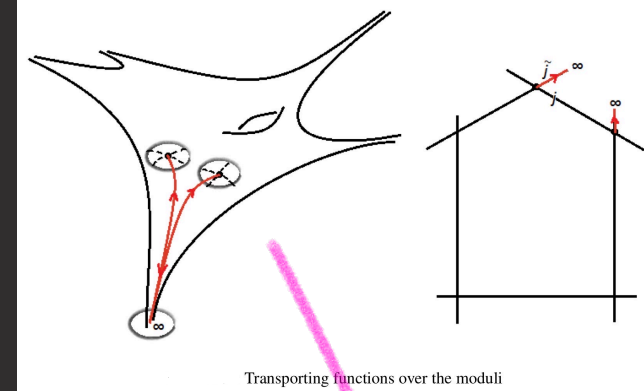
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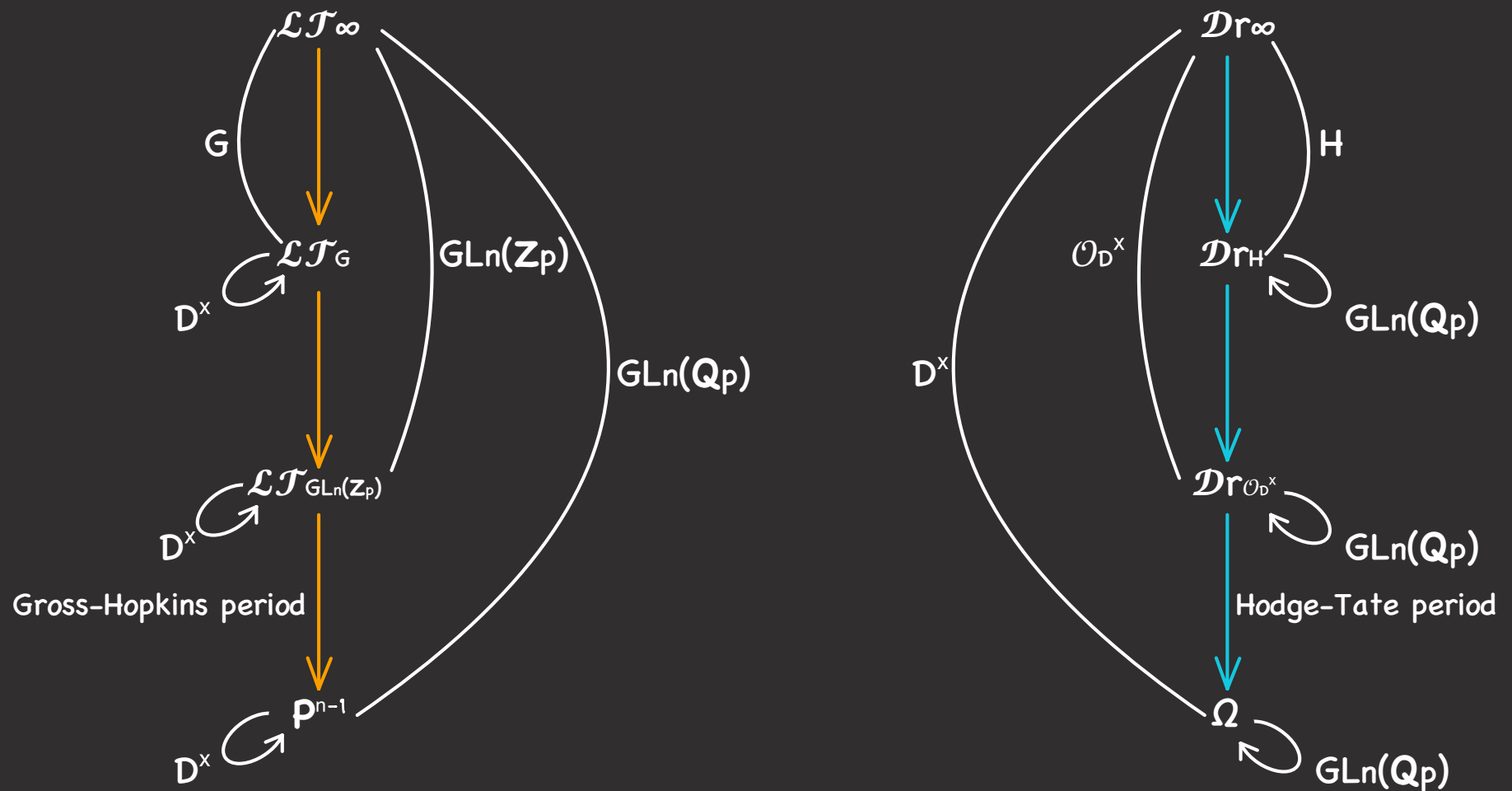
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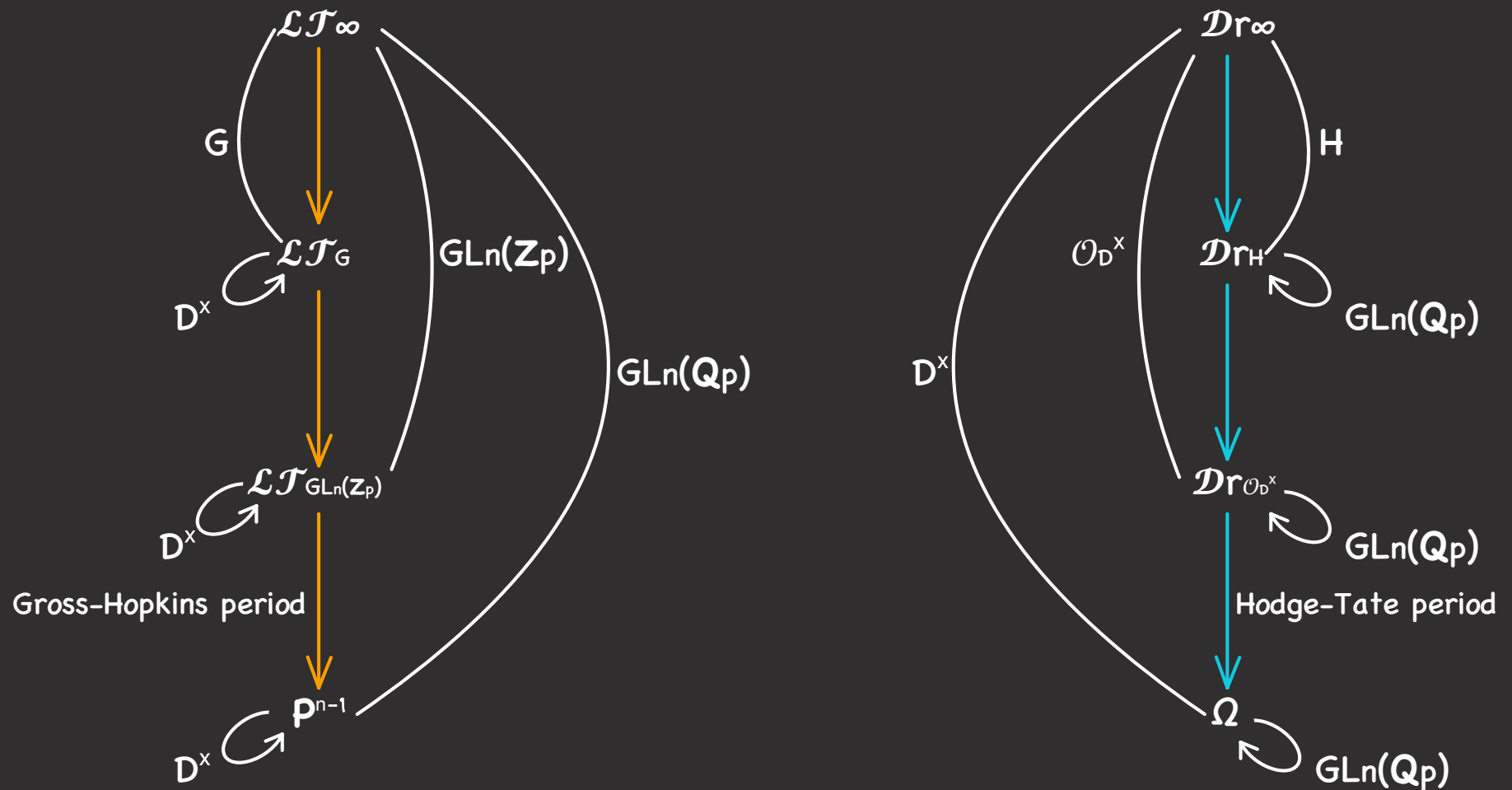
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Theorem (Faltings, Fargues '08, Scholze-Weinstein '13) $\exists$ an isomorphism between the Lubin-Tate tower of moduli of deformations of formal groups and the Drinfeld tower of moduli of deformations of shtukas, which is equivariant with respect to the action of the triple product group $GL_n(\mathbb{Z}_p) \times D^\times \times W_{\mathbb{Z}_p}$.



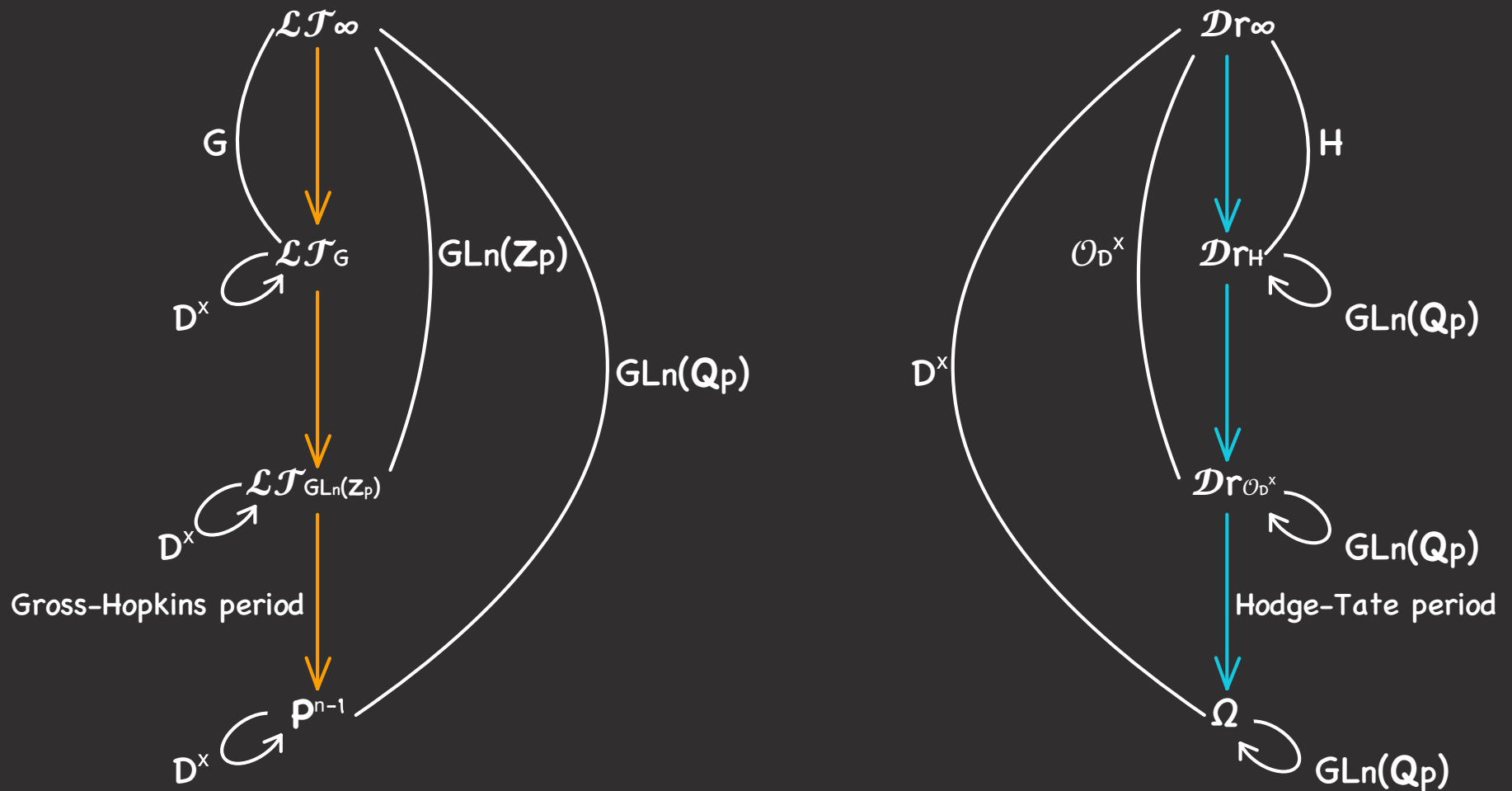
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Jacquet-Langlands correspondence



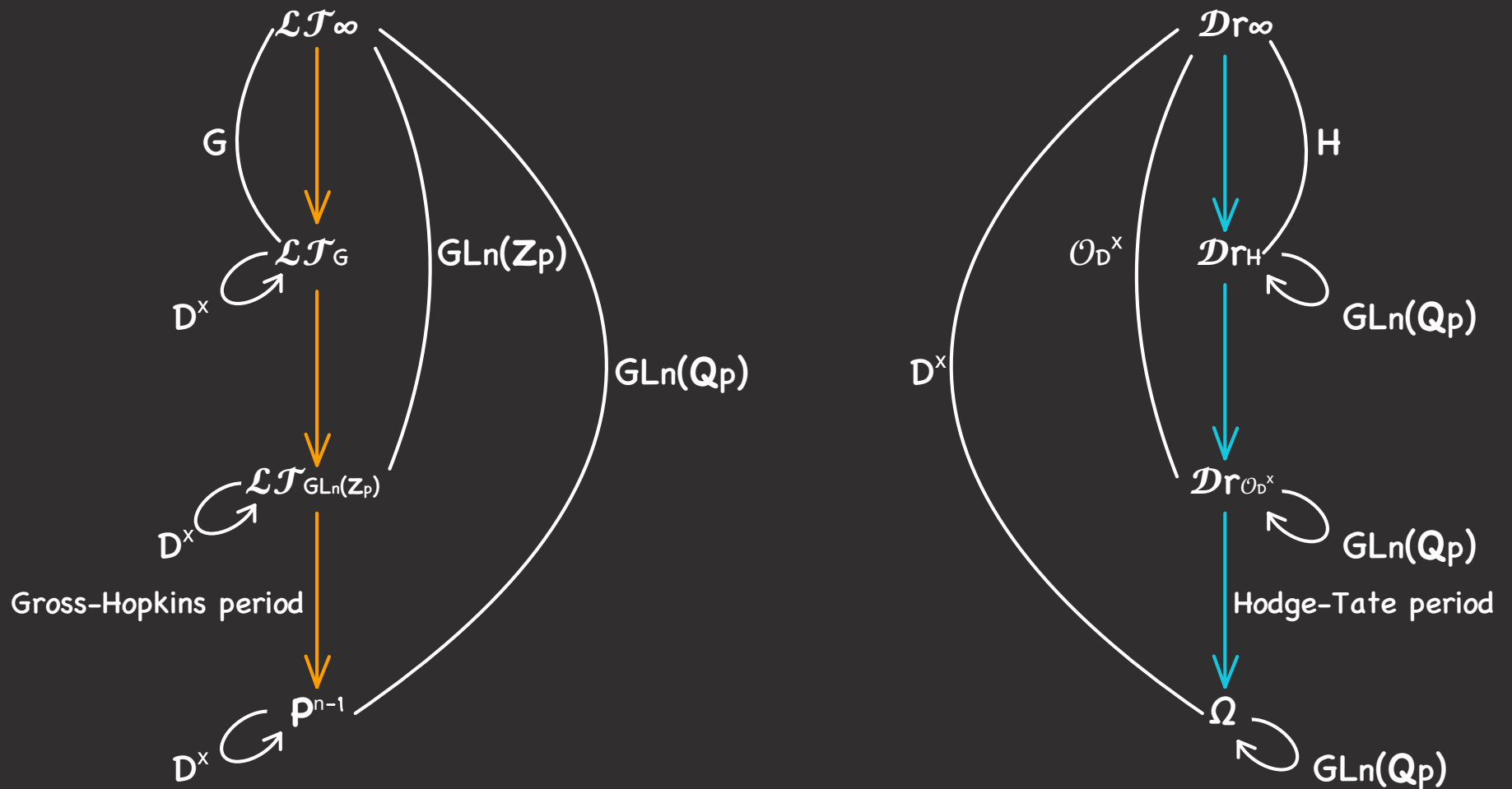
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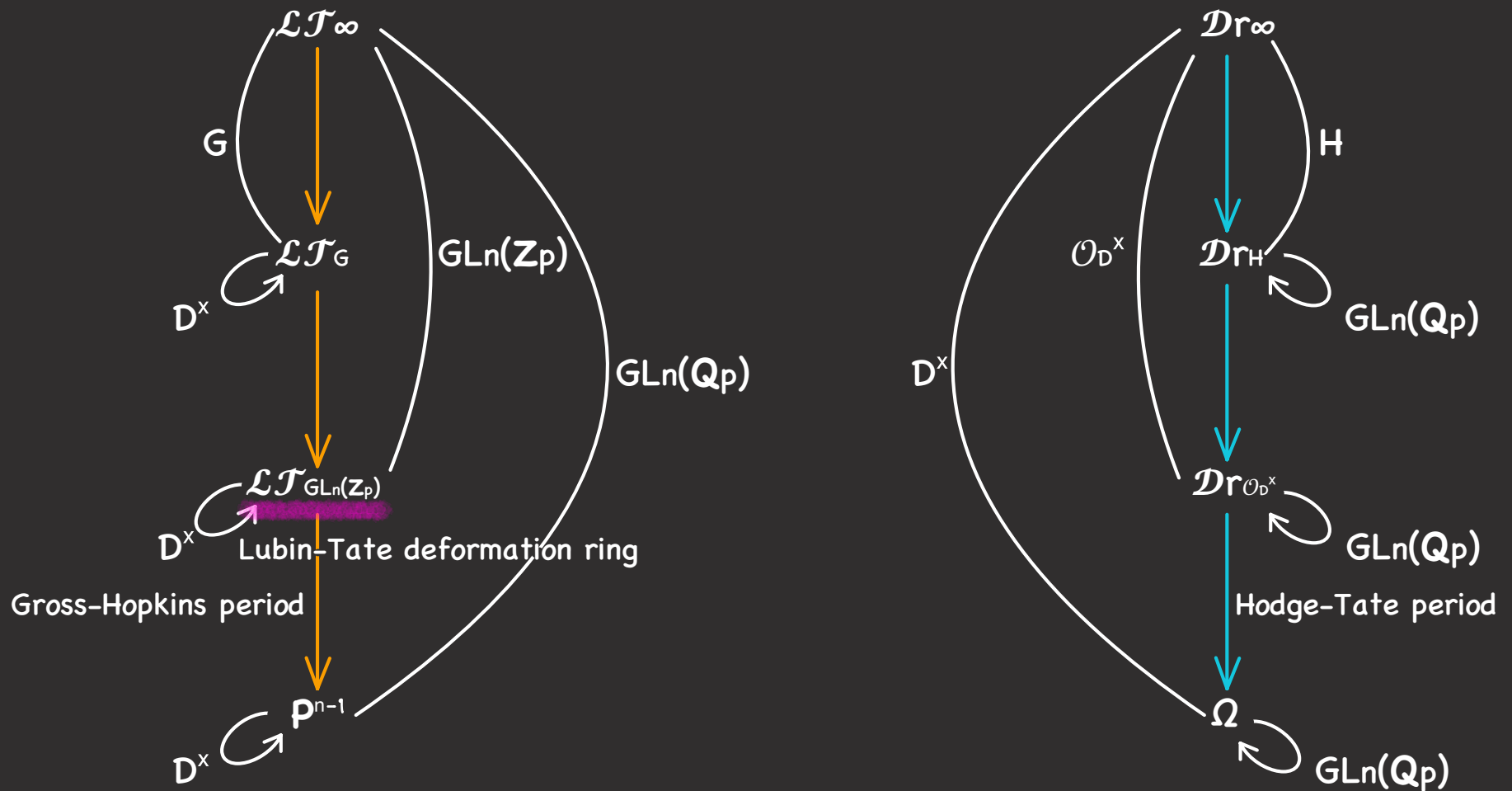
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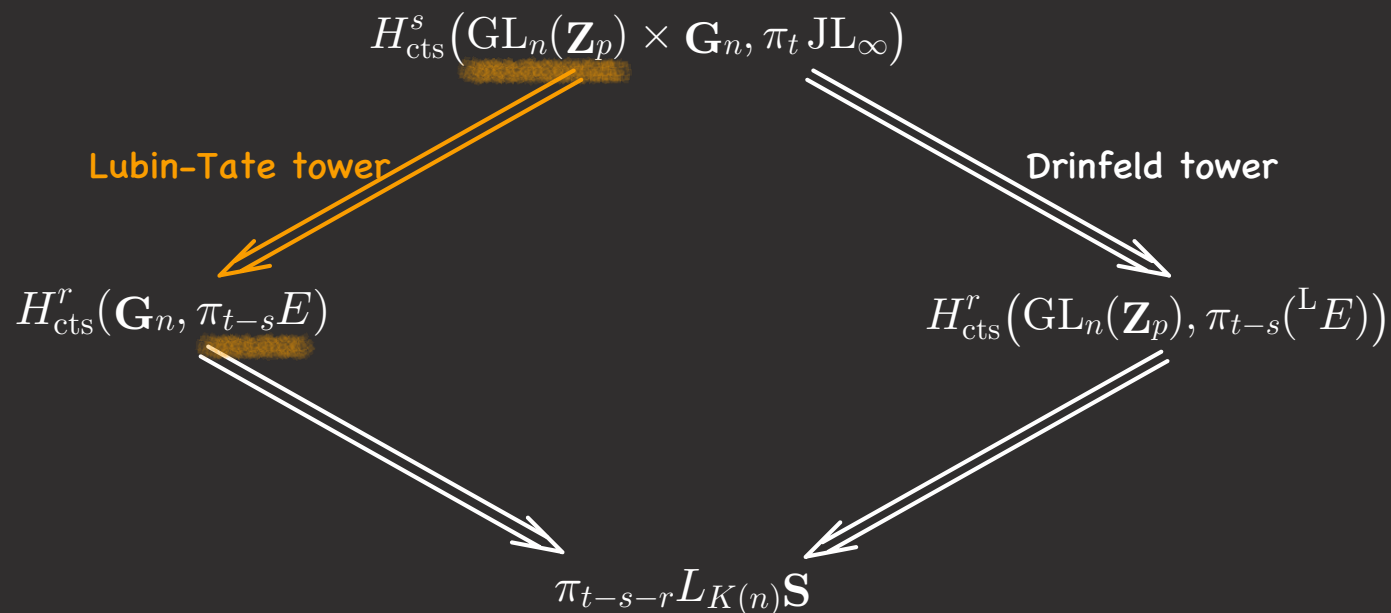
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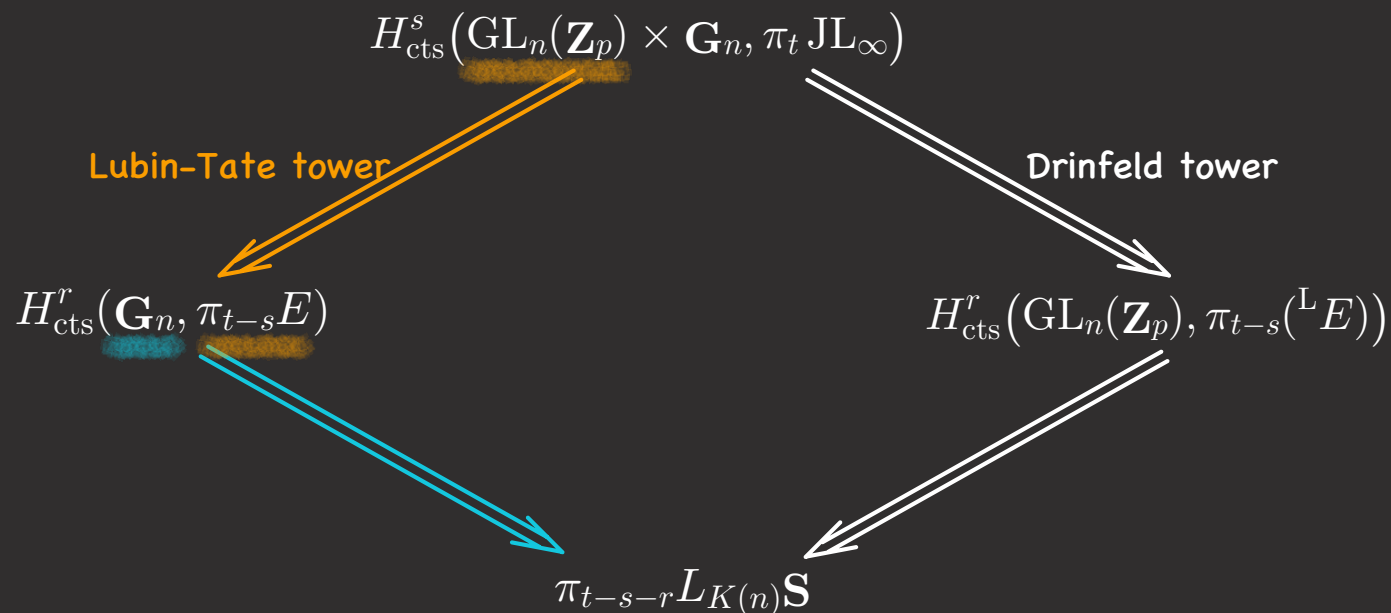
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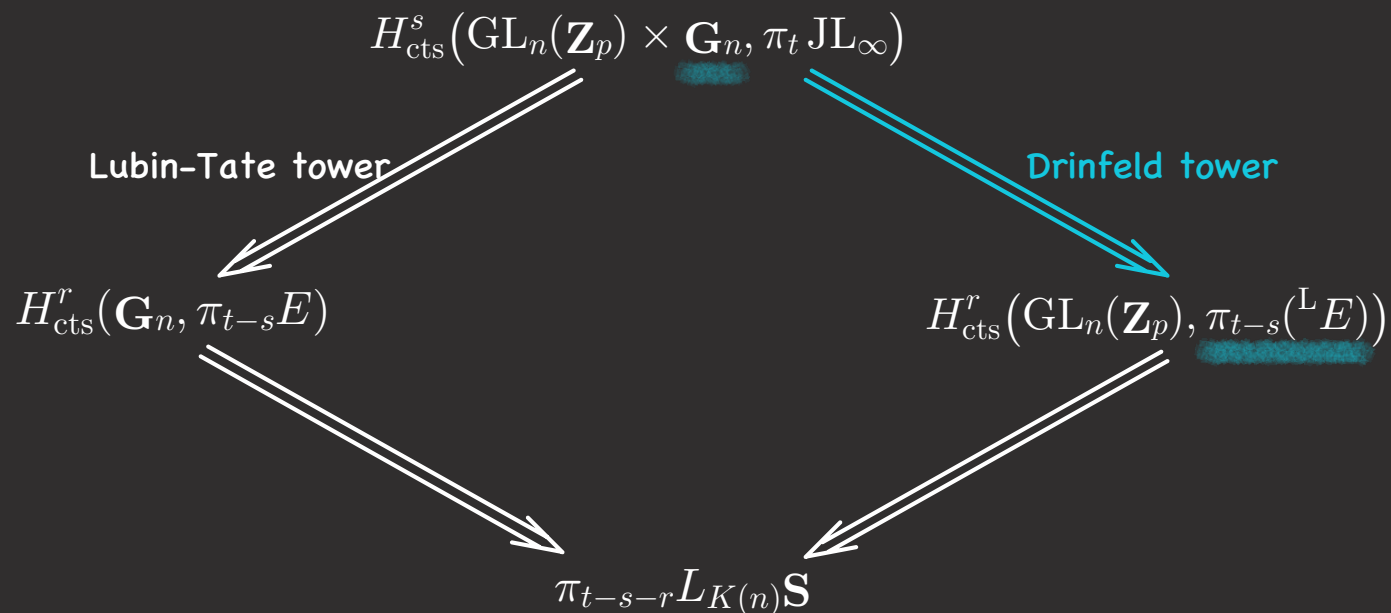
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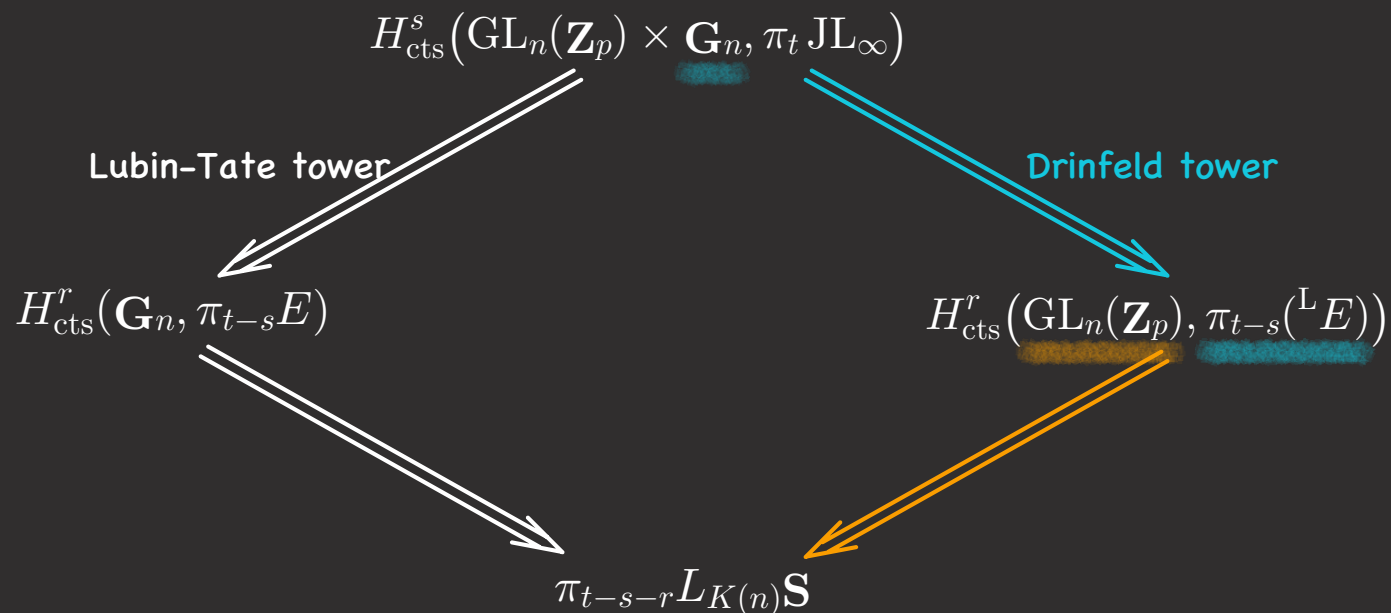
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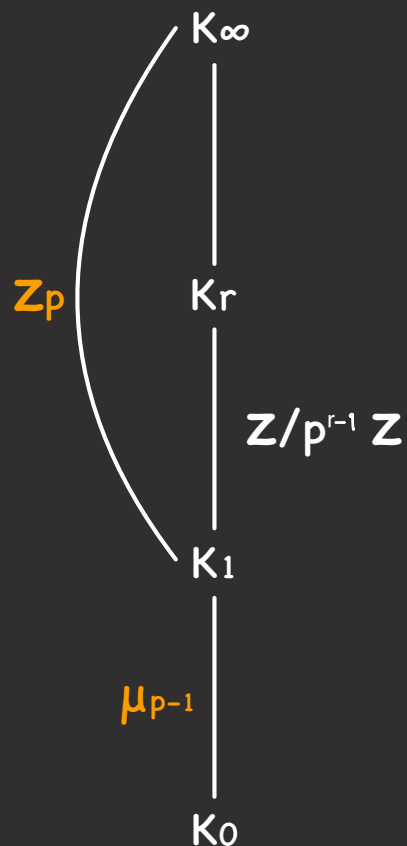
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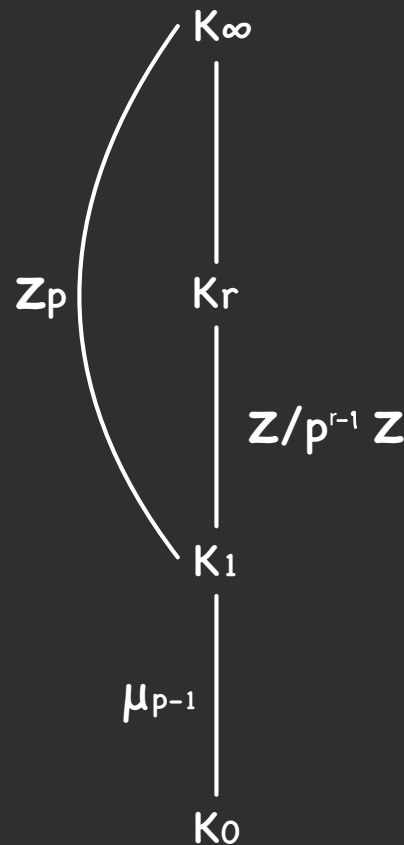


Heuristic Example ($n = 1$, p odd) The Lubin-Tate tower corresponds to the **Iwasawa tower** of cyclotomic field extensions



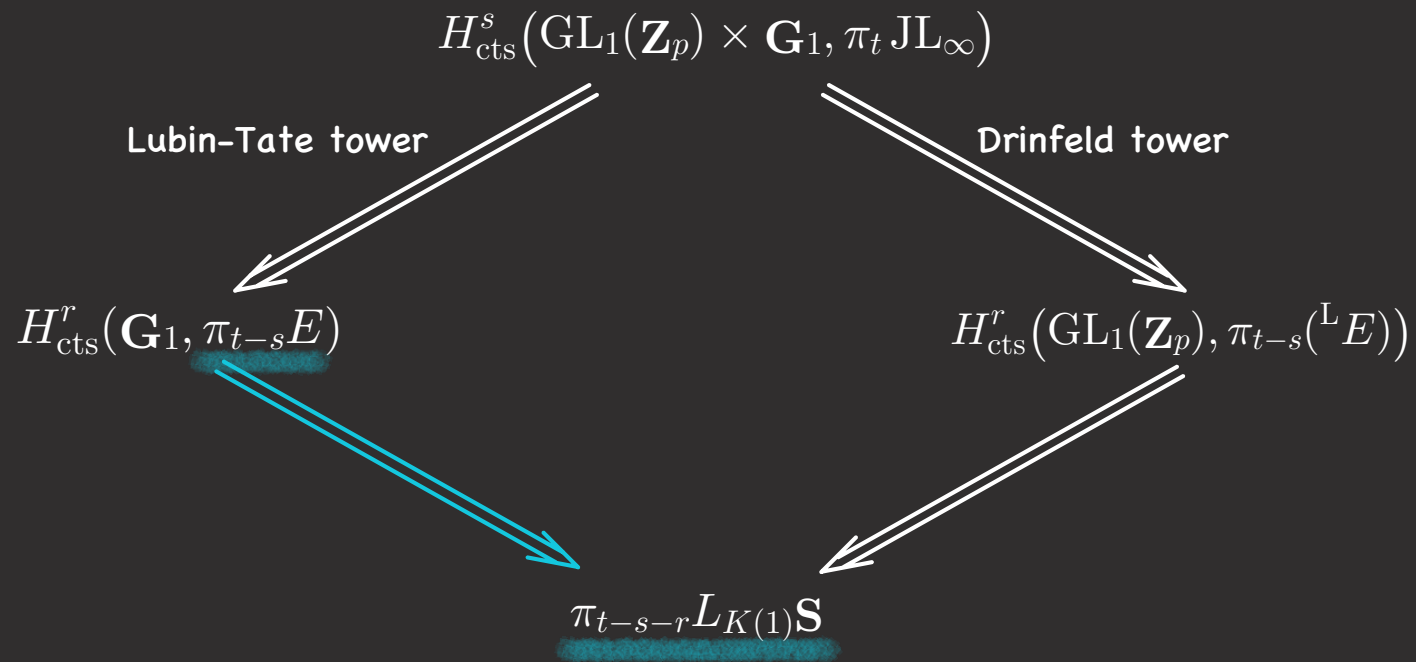
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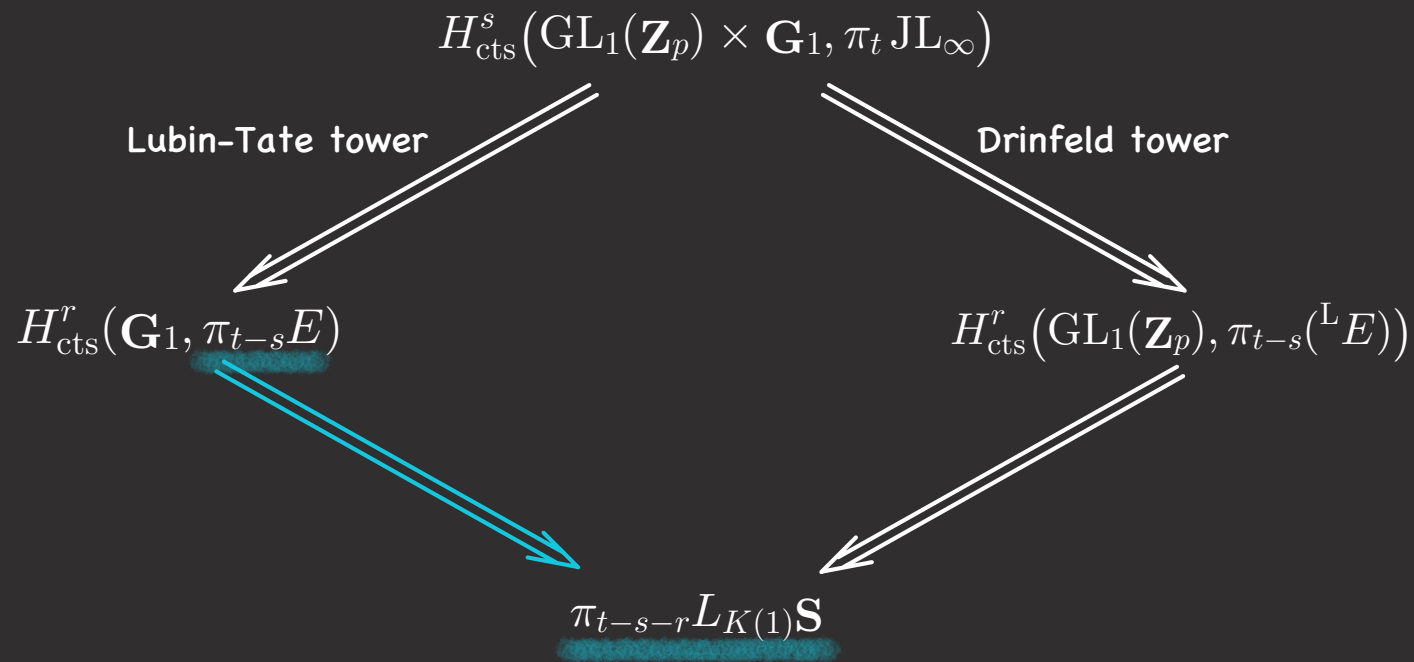


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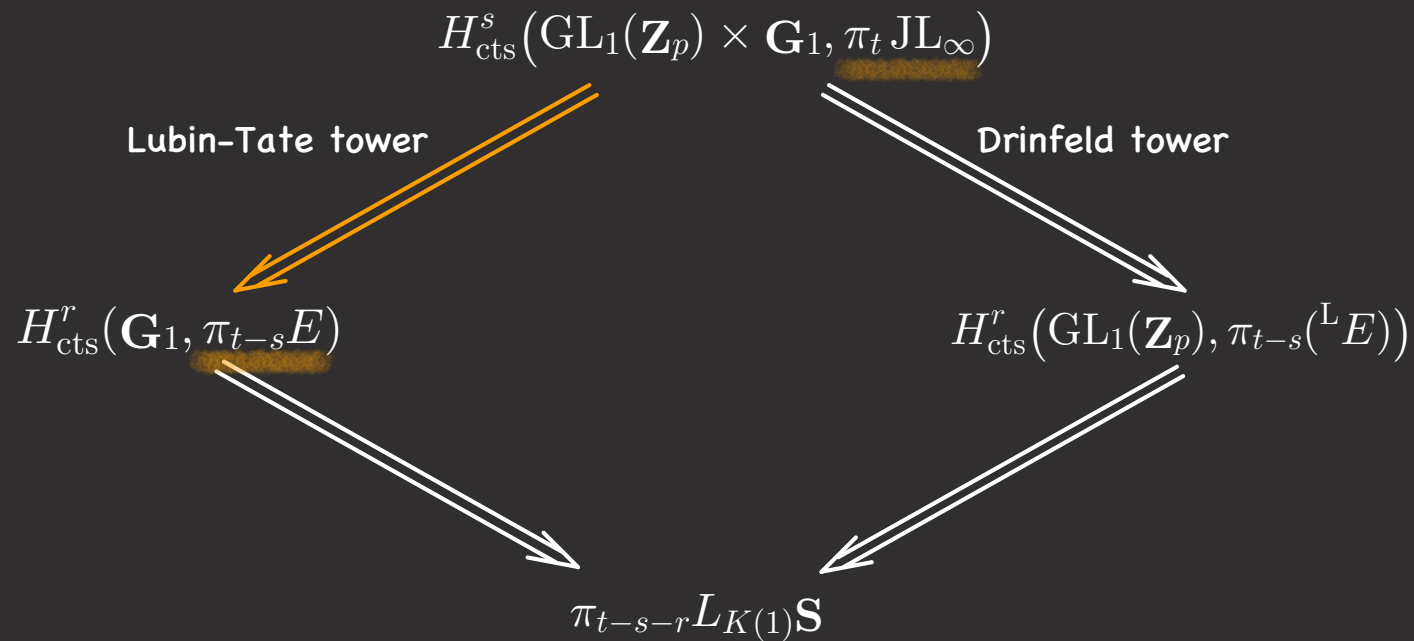
On the other hand, the Morava stabilizer group $G_1 \cong \mathbb{Z}_p^\times$, too, but its action on the homotopy groups differs from that by $\text{GL}_1(\mathbb{Z}_p)$.



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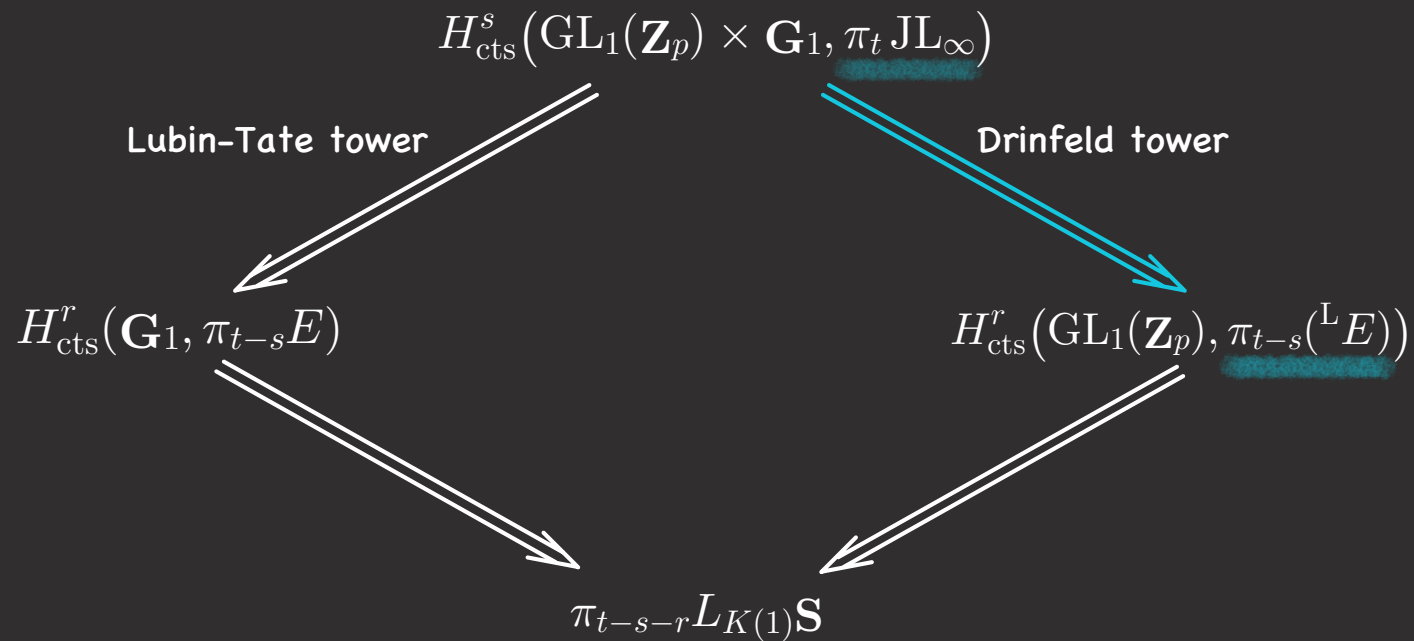
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topological invariant encoding
and repackaging
Galois cohomology

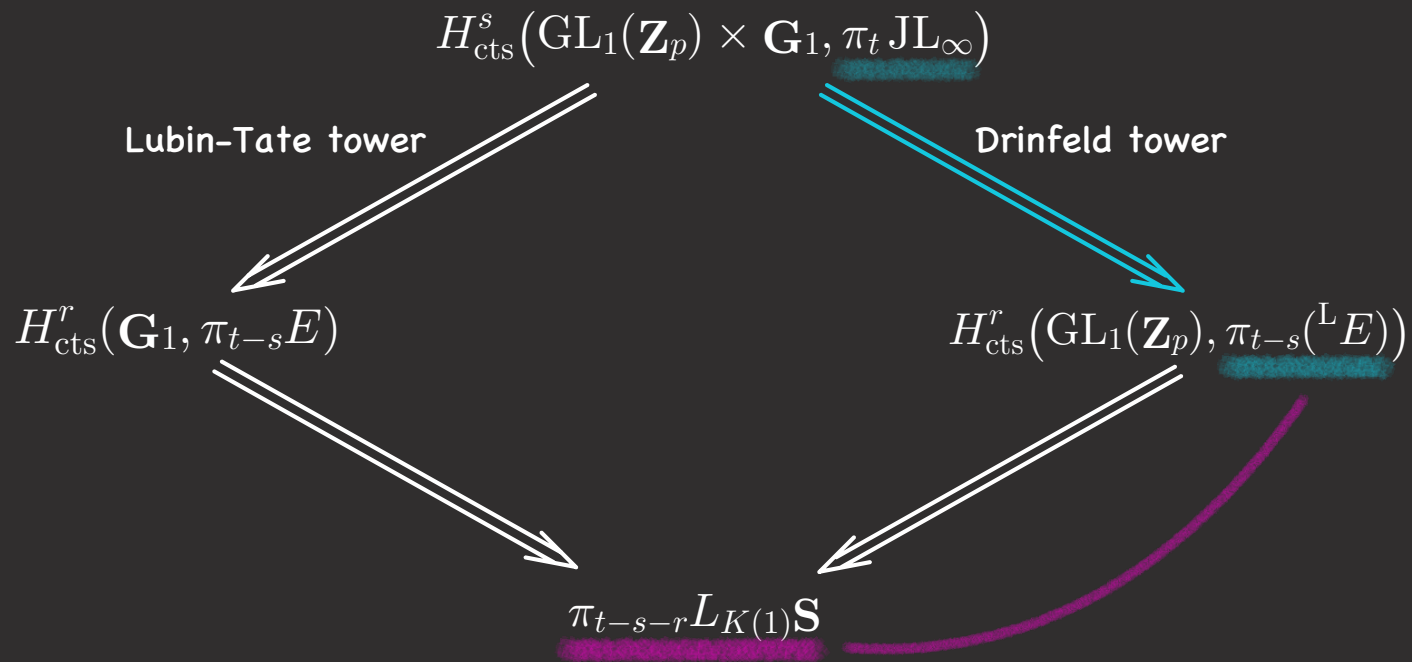


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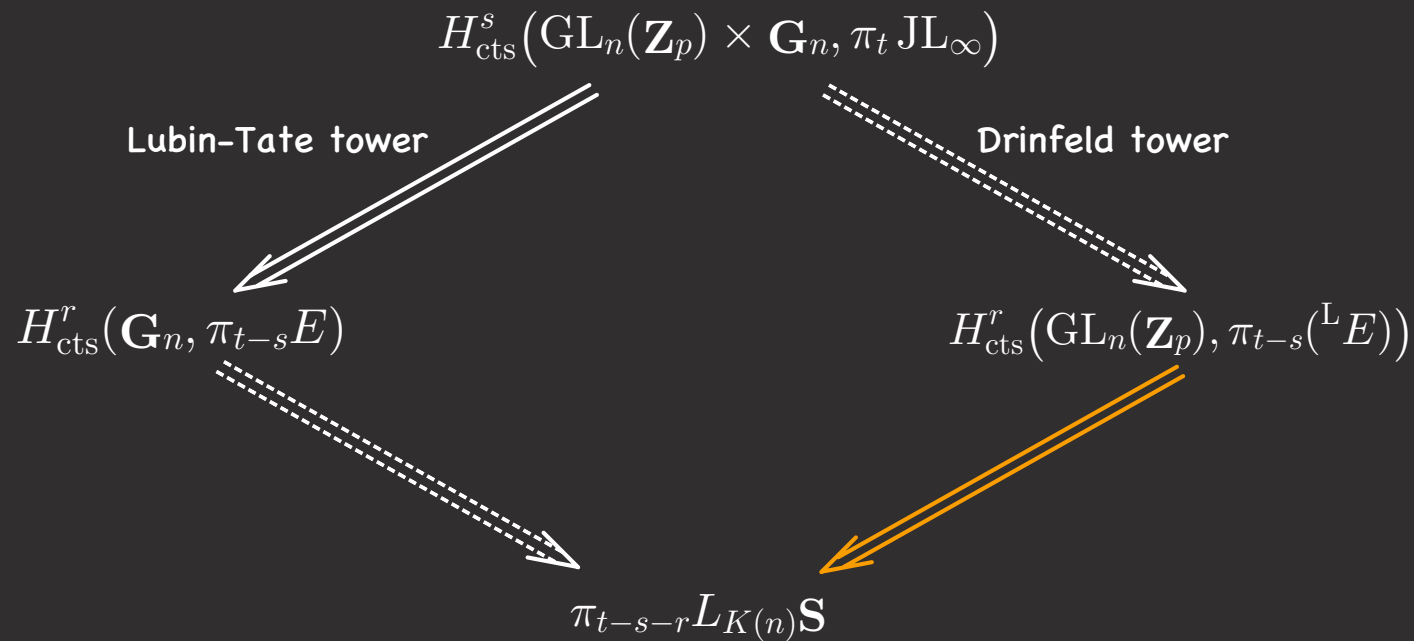
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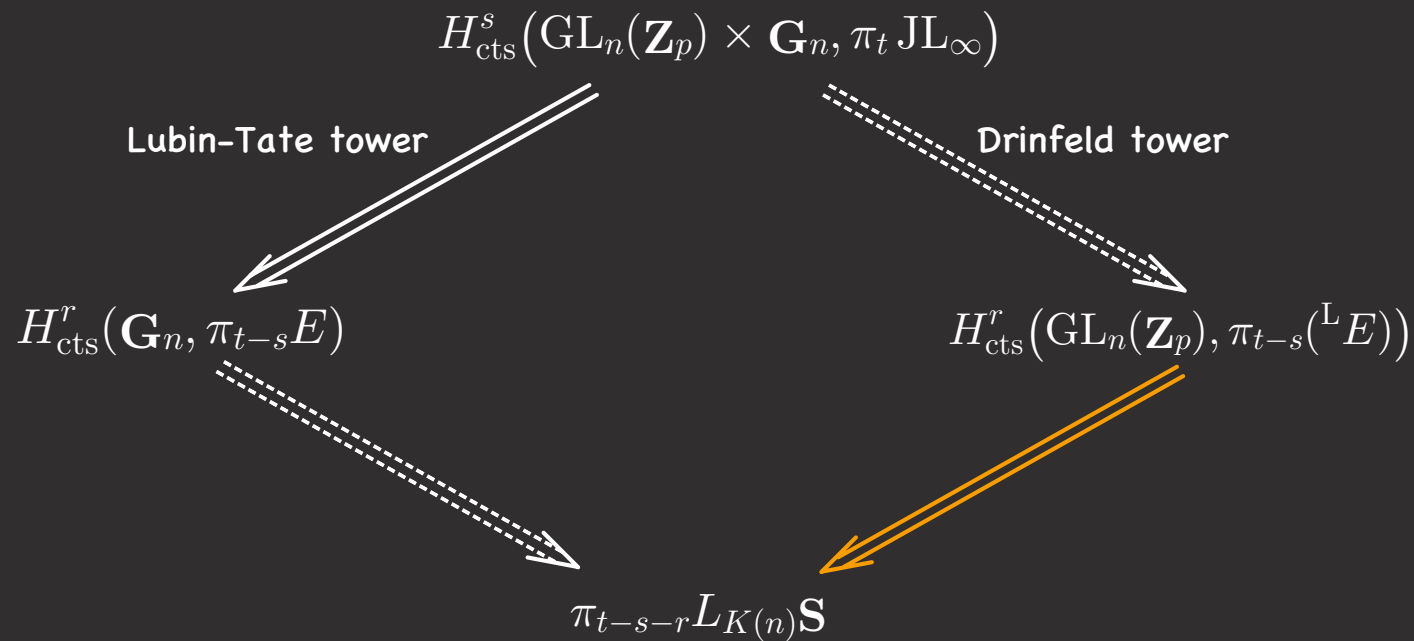
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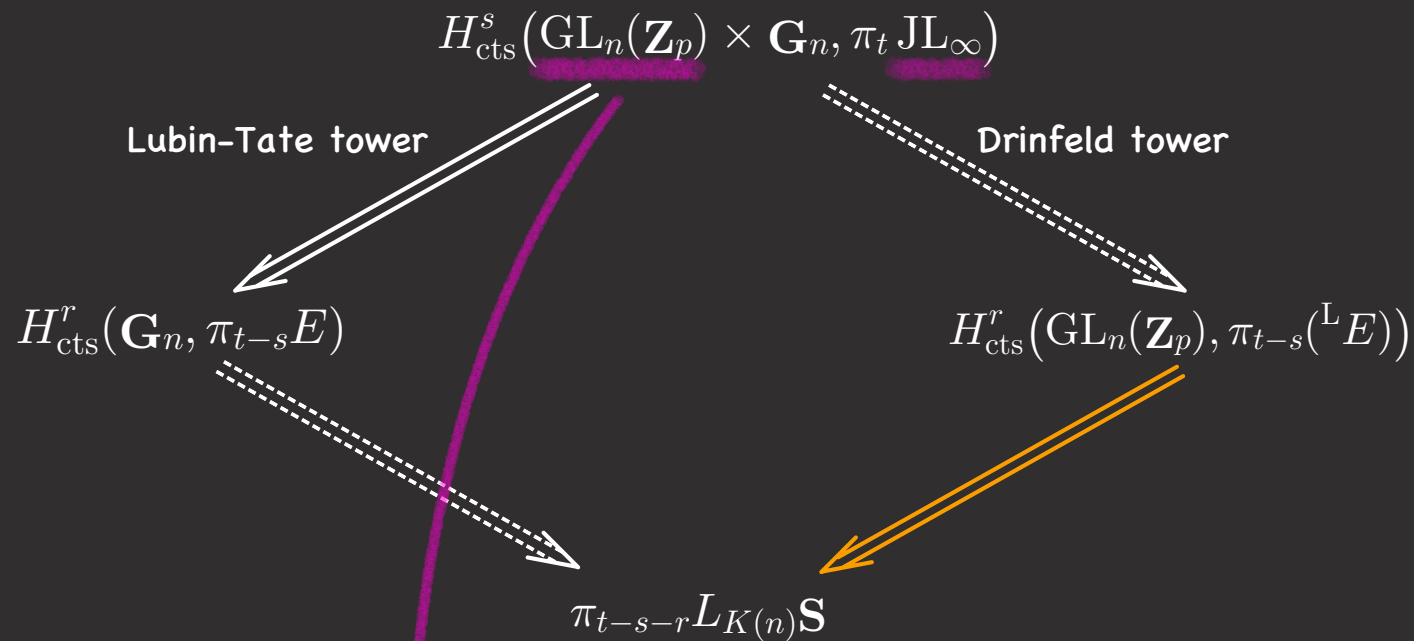


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