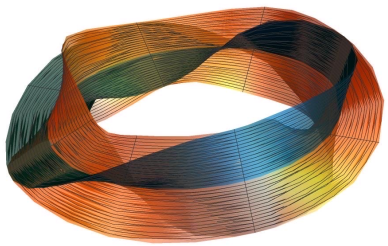
Explicit examples of Higgs bundles from physics and bulk–edge correspondence



Yifei Zhu

Southern University of Science and Technology

2024.7.2

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Holography, optical devices, absorption devices, …

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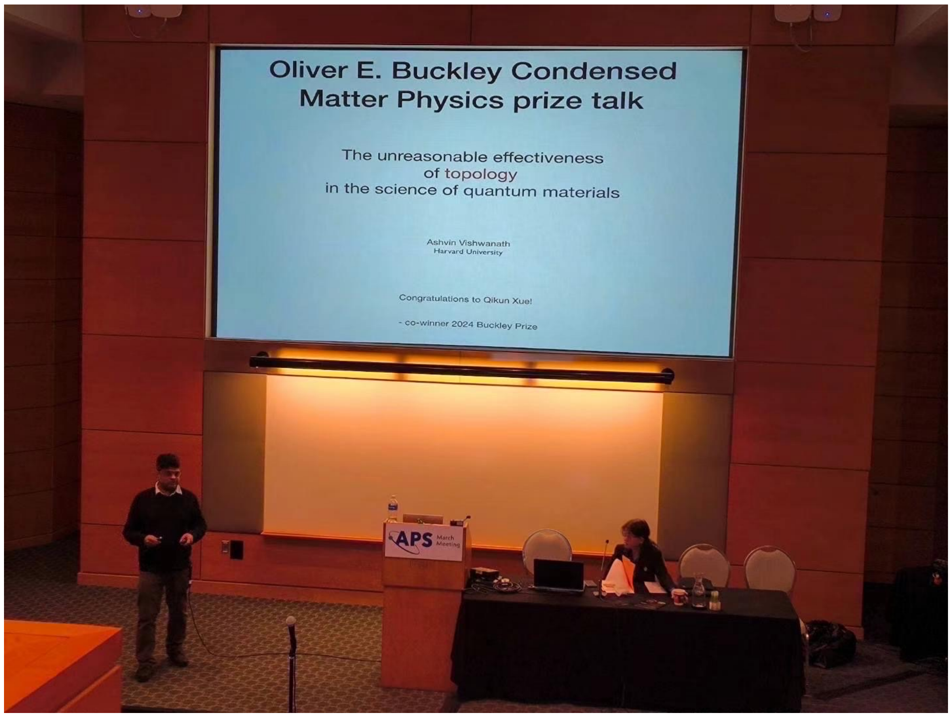

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- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at this year's APS March Meeting in Minneapolis (薛其坤 of Tsinghua and SUSTech was the co-winner of the prize)
- U.S. Department of Energy, Office of Science. *Basic research needs for quantum materials: Research to discover, harness, and exploit exotic electronic properties* (brochure), 2016.
- 方忠 等，“拓扑电子材料计算预测”，2023年度国家自然科学奖一等奖
- 第一届魅丽数学与交叉应用会议“数学与生物医药、数学与先进材料”，2024年5月，苏州

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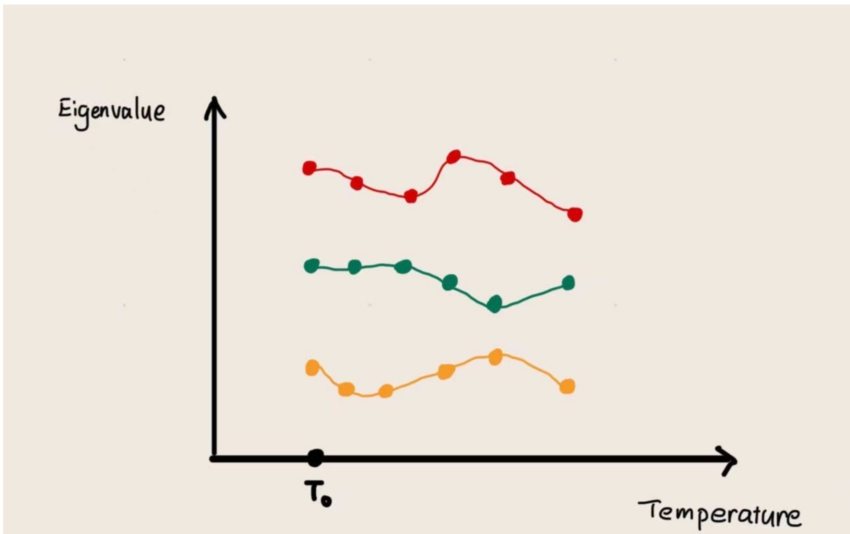
Mathematical modeling of electronic energy *band structures* therein



energy $\longleftrightarrow$ *eigenvalue*

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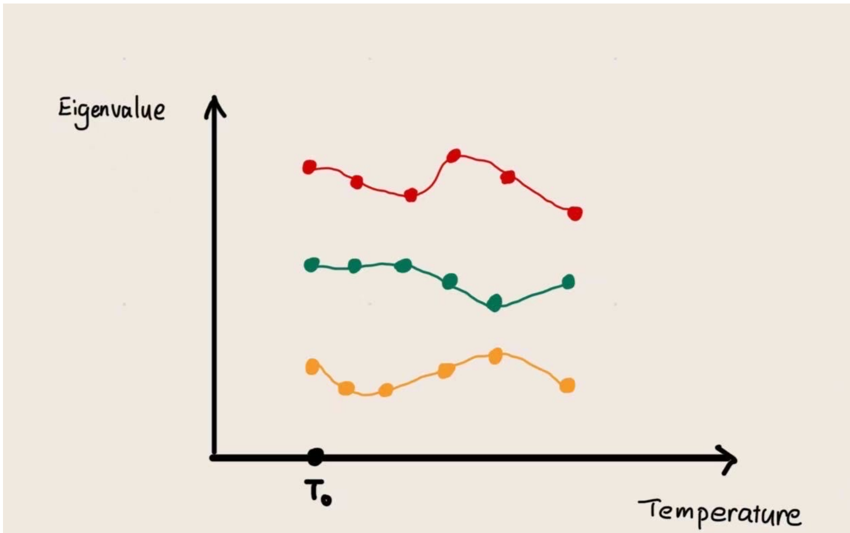

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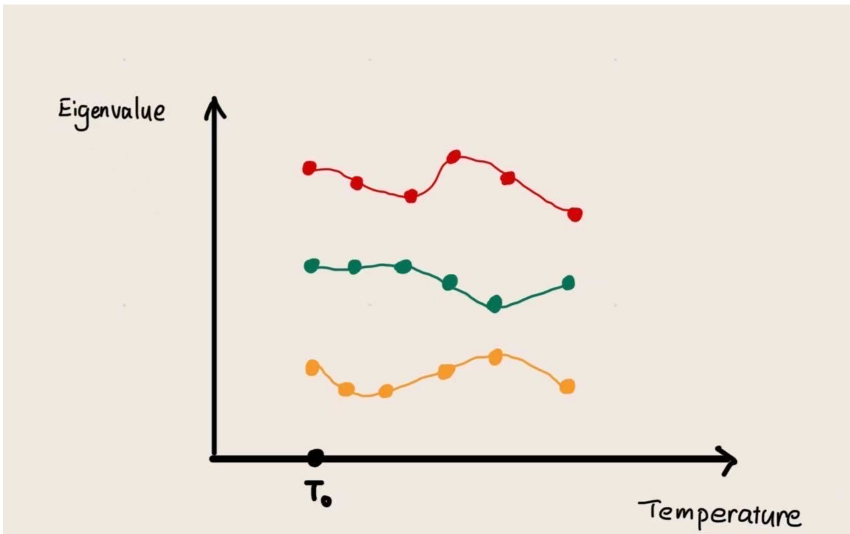

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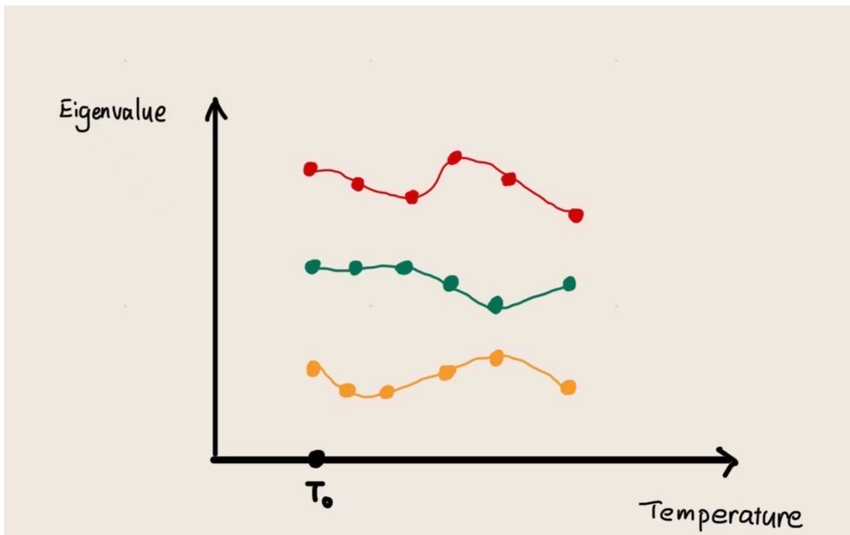

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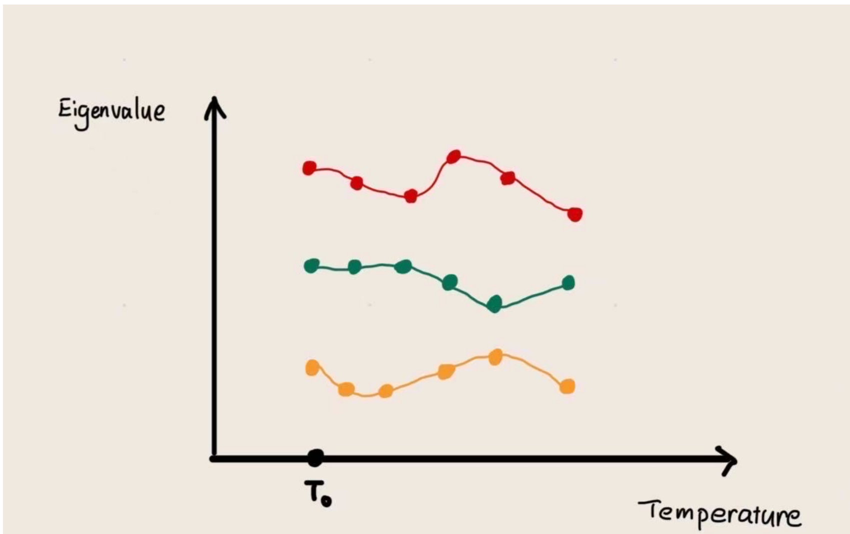

Hermitian vs. non-Hermitian

real eigenvalues (observable energies) vs. eigenvalues with imaginary part (counts for energy exchange with surrounding environment or other systems)

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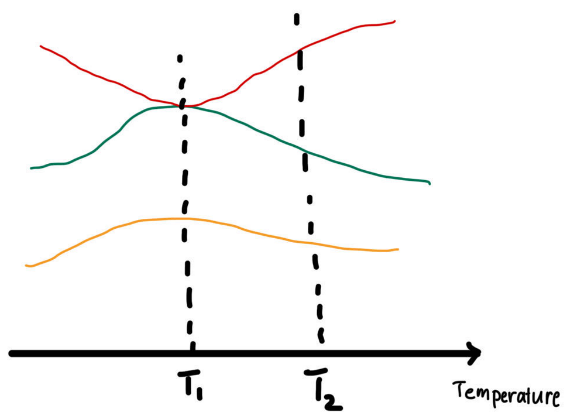

phases of matter
H_2O*: solid, liquid, gas*

T_1 : singular points (points where eigenvalues degenerate)
$H(T_1)$: gapless Hamiltonian
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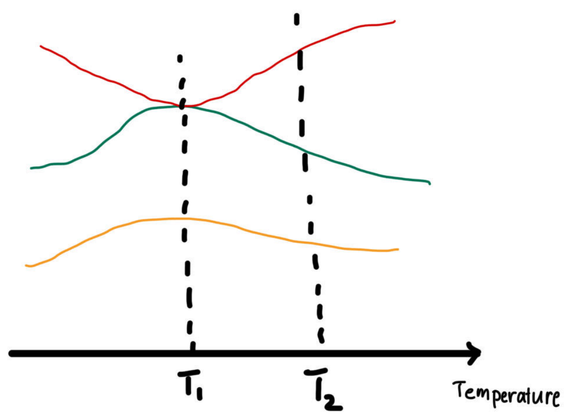

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Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, PNAS 2024.

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Thanks to Hopf bundles and Higgs bundles as *eigenbundles*, we now have a conceptually more systematic, visibly more intuitive understanding of the topic.

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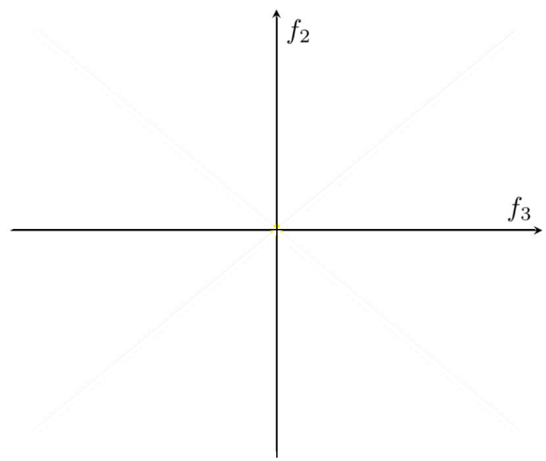

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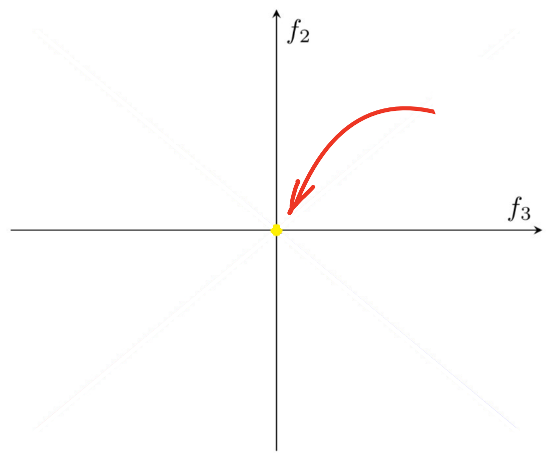

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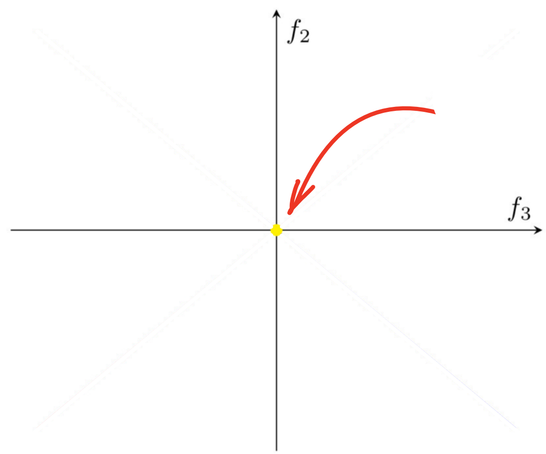

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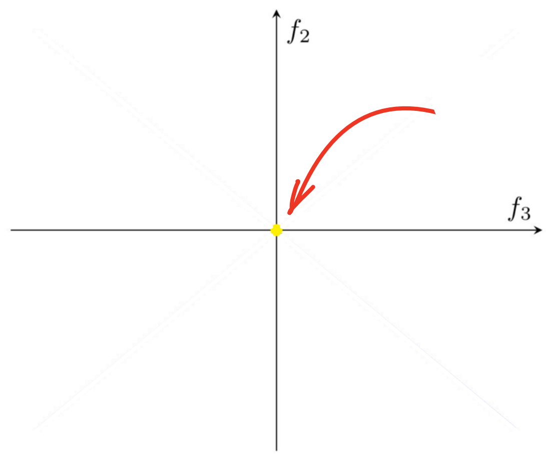

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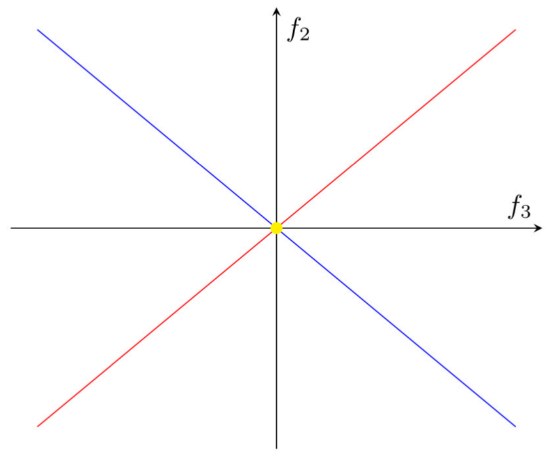

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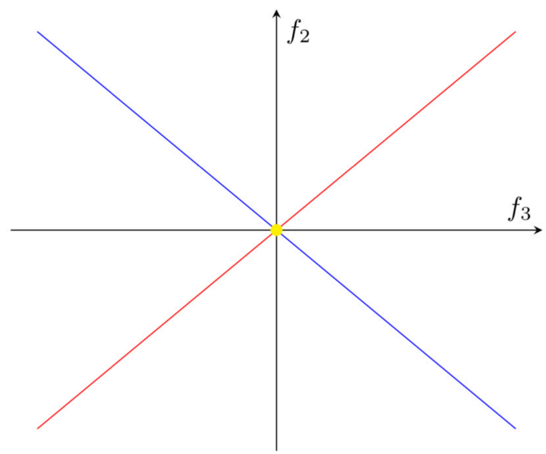

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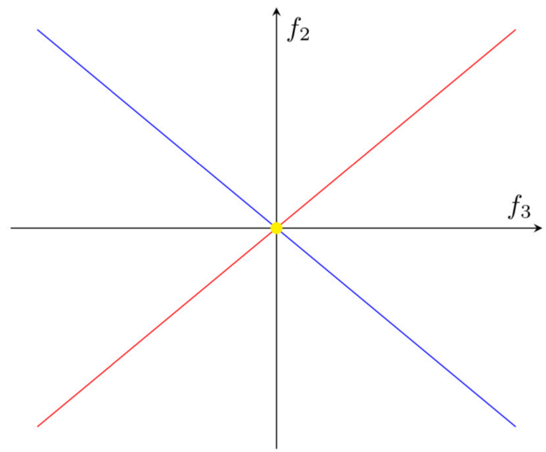

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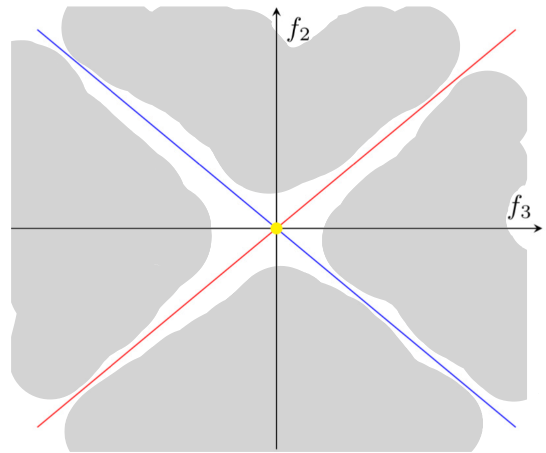

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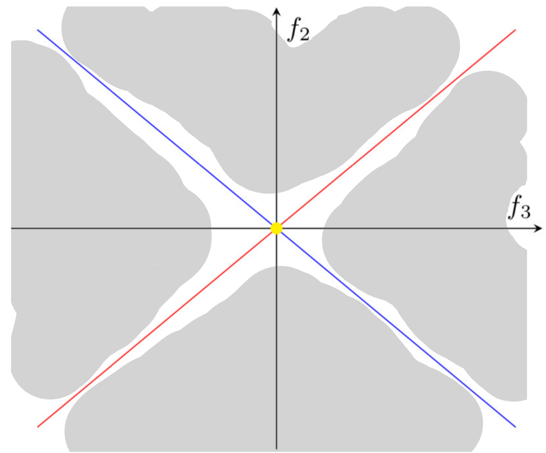

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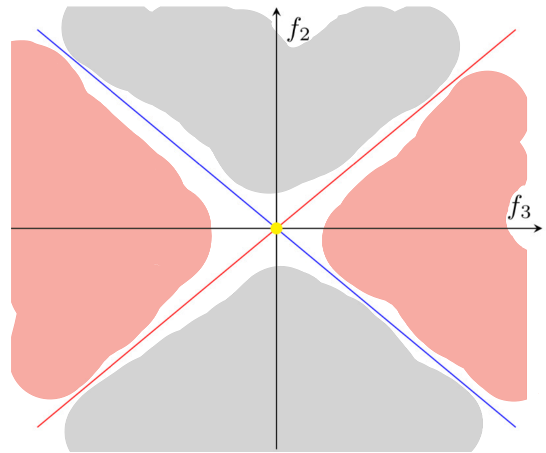

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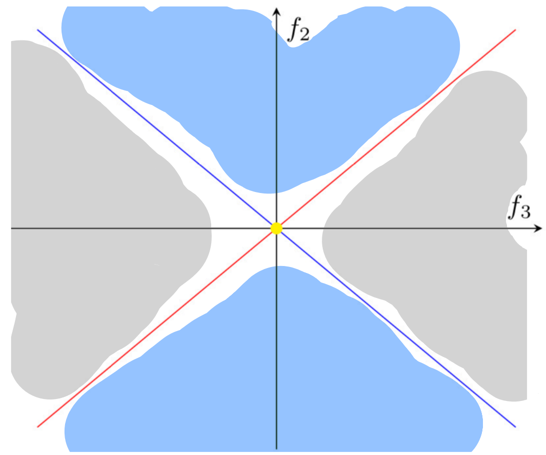

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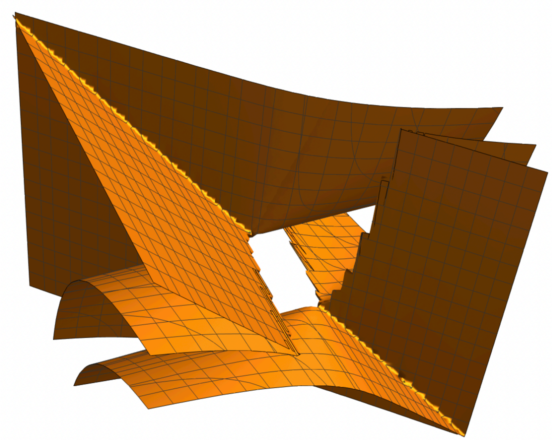
Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the $f_1 f_2 f_3$-space:

The equation for this surface is a non-homogeneous real polynomial in f_1, f_2, f_3 of degree 6.



Swallowtail couple $\mathrm{sw2}$

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V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

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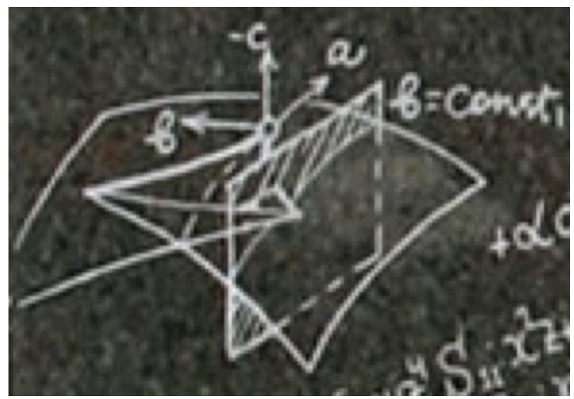

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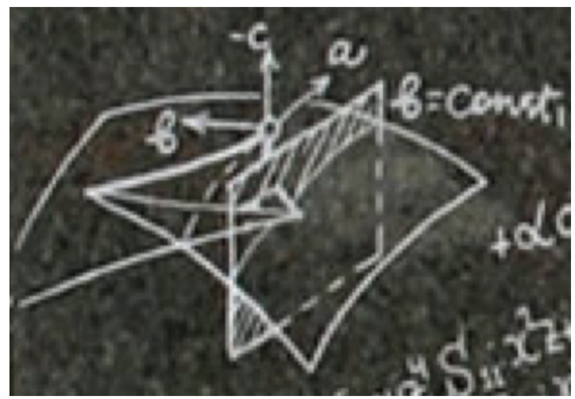

A local model for moduli spaces of 3-band Hamiltonians

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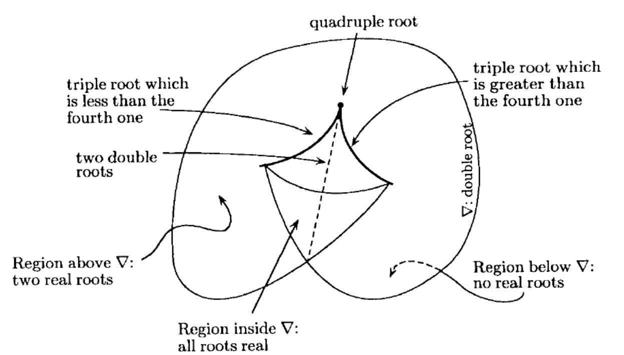
Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.

Homological stability of braid groups

Portrait from Gelfand, Kapranov, Zelevinsky,
Discriminants, resultants, and multidimensional determinants.



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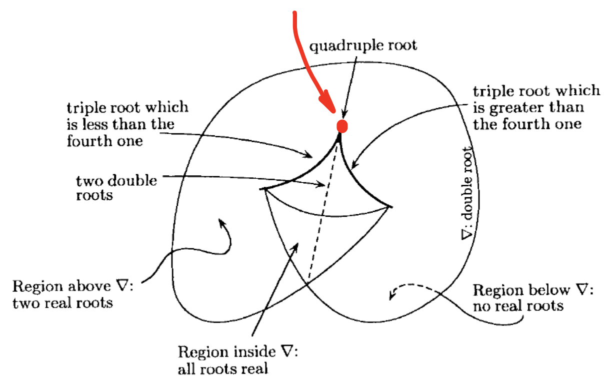

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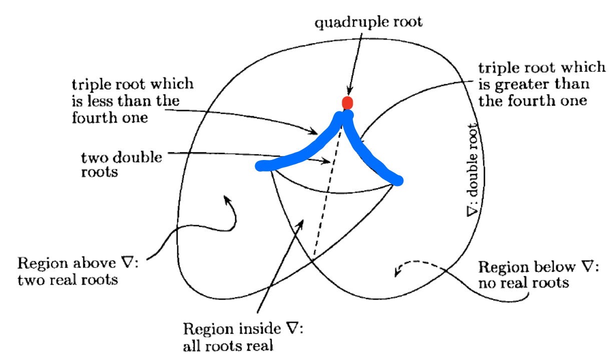

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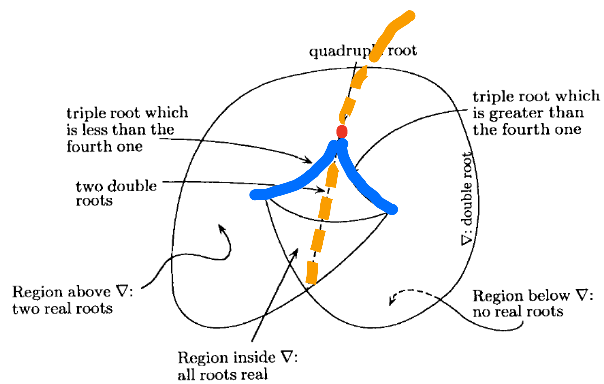

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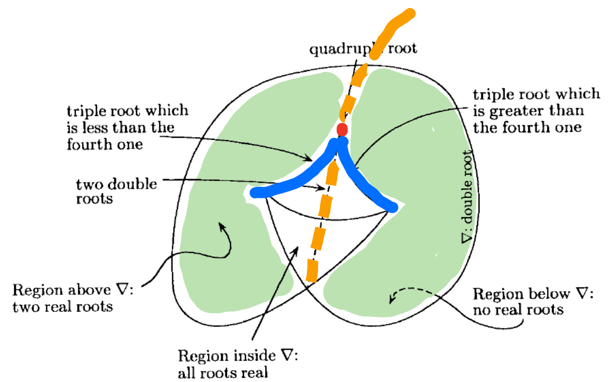

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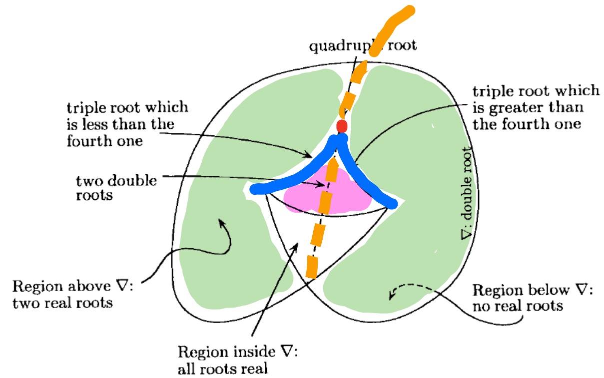

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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold.

Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect, 1995.

Also: Polymathematics, 2000.

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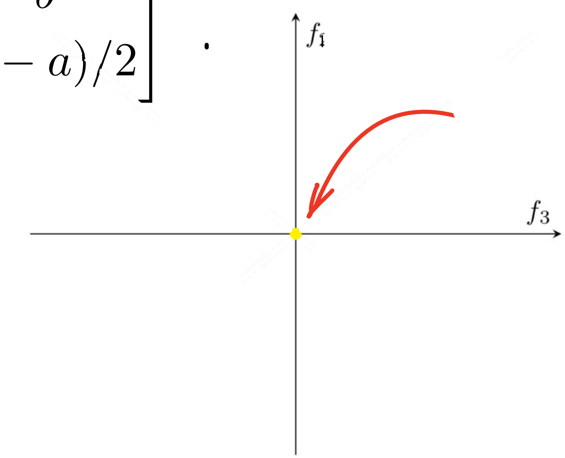

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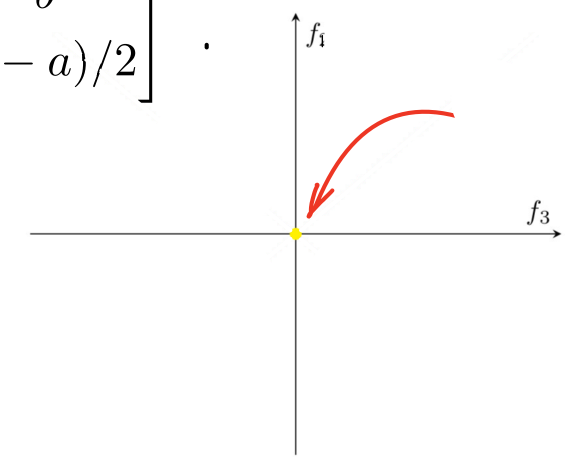

The parameter $f_1 f_3$-plane thus has an isolated singular point $(0,0)$ and is a particularly simple stratified space.

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One of our key steps is a more conceptual understanding of the above moduli spaces in the case of $n = 2$ through bundle theory.

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Moreover, they obtained non-Abelian "topological charge" for *n*-band Hermitian systems when $n > 2$, such as

$$\pi_1\Big(SO(3)/\big(O(1) \times O(1)\big)\Big) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the $O(1)^n$-action in constructing the moduli spaces was rather ad hoc.

One of our key steps is a more conceptual understanding of the above moduli spaces in the case of $n = 2$ through bundle theory. To see how they rotate, let us compute the unit eigenvectors explicitly.

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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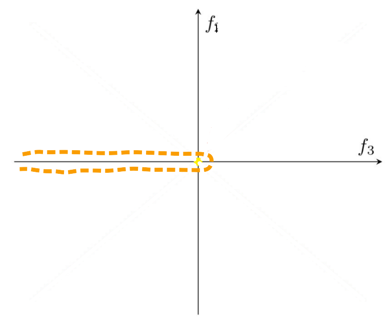

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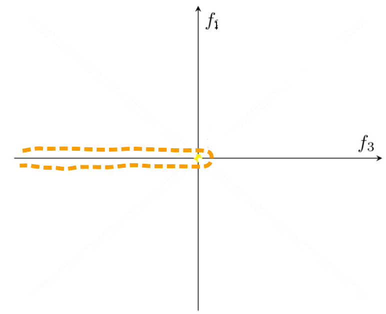

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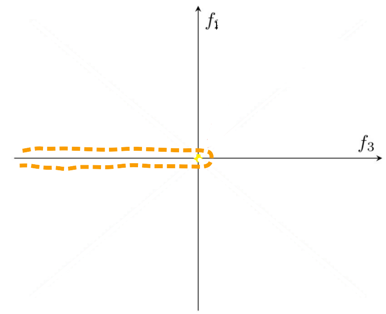

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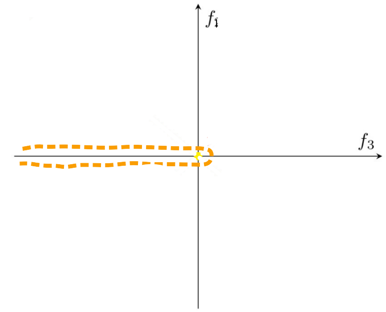

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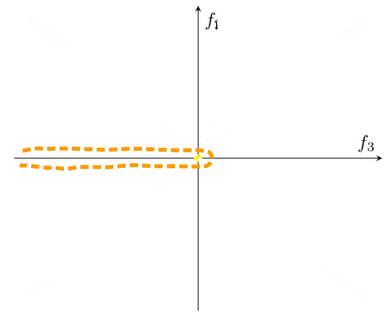

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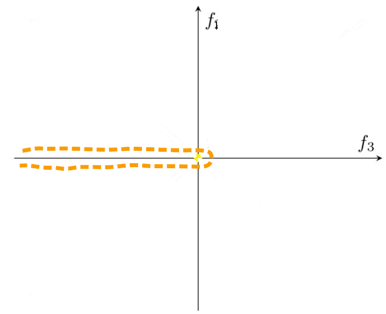

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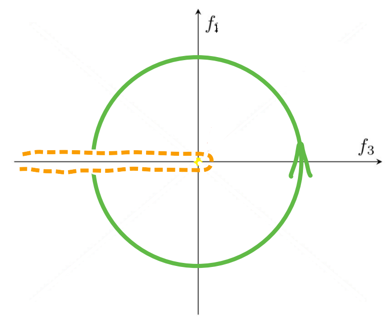

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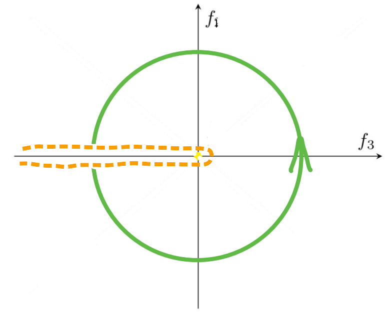

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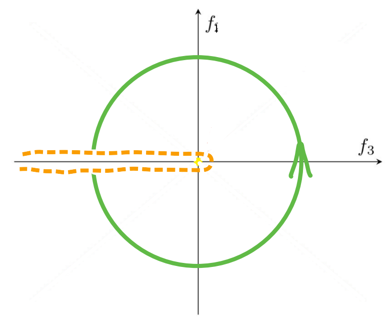

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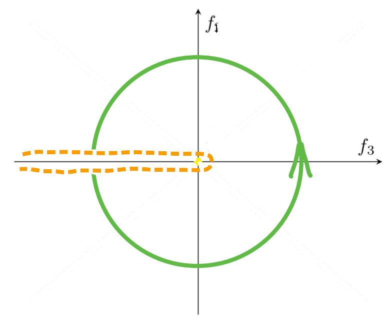

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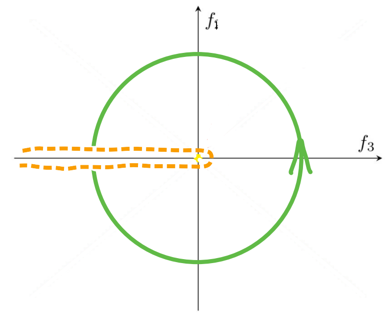

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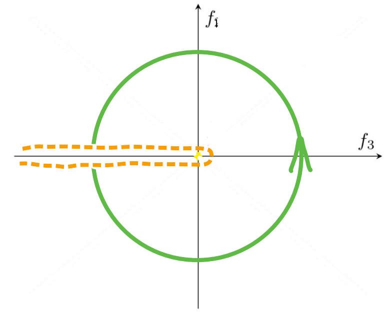

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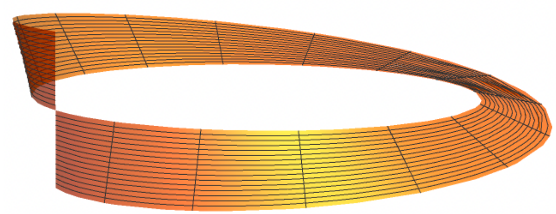

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

Lemma. The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

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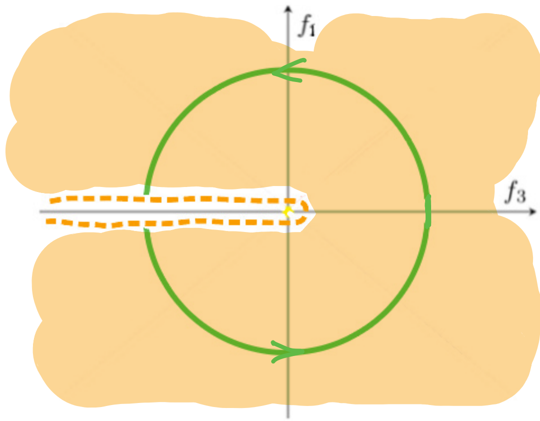

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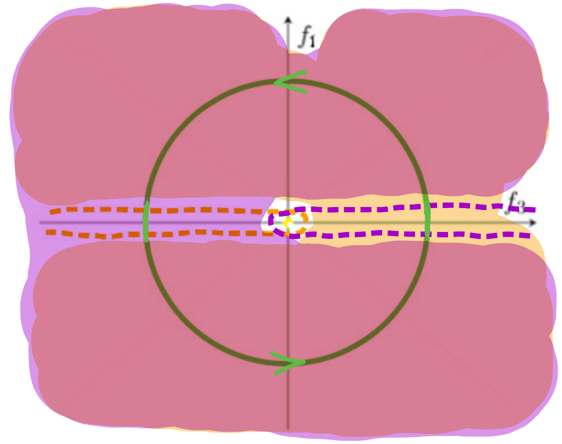

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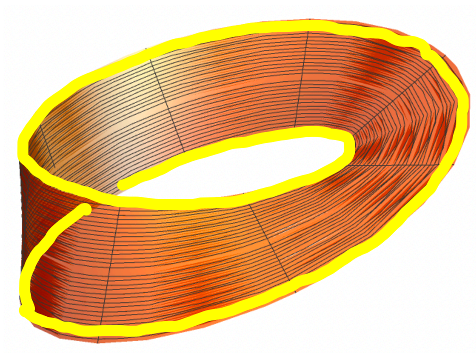

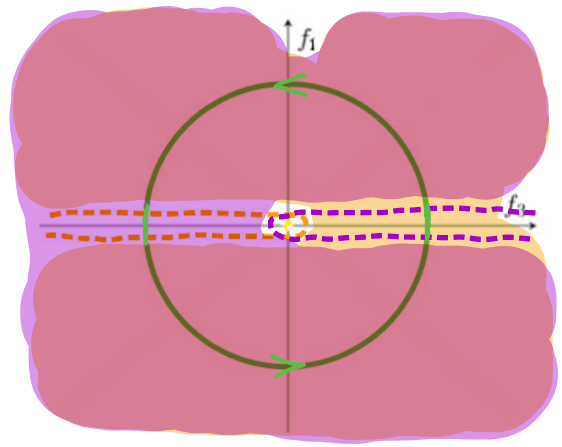

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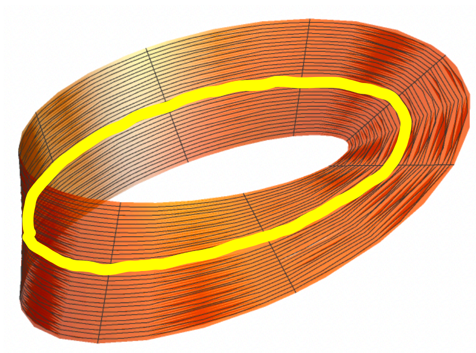

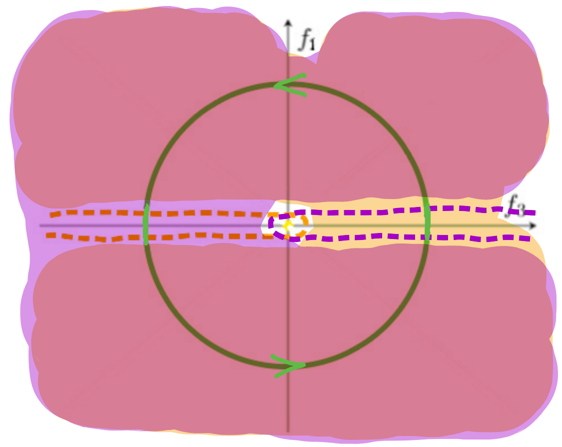

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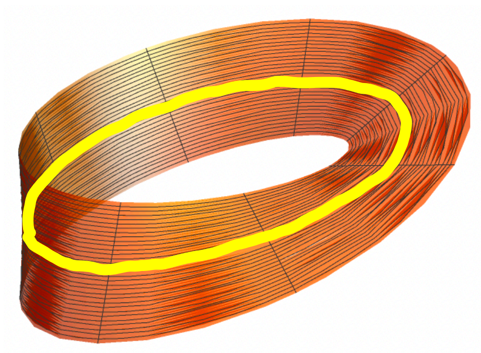

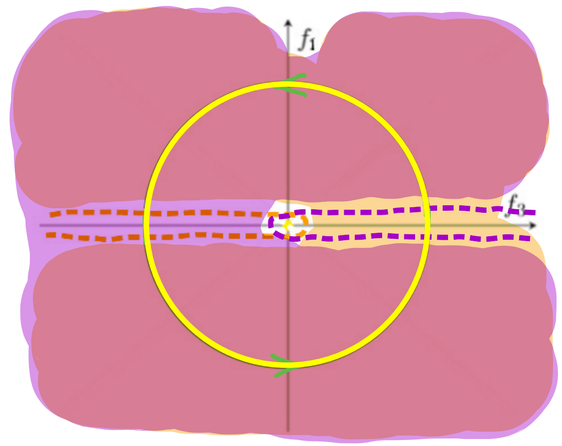

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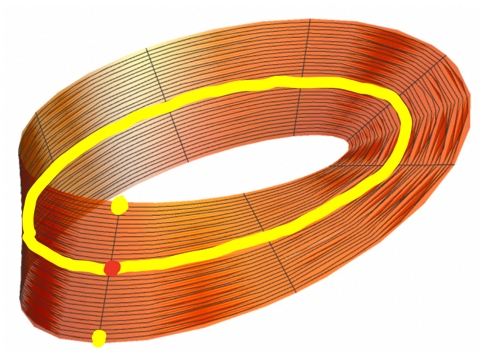

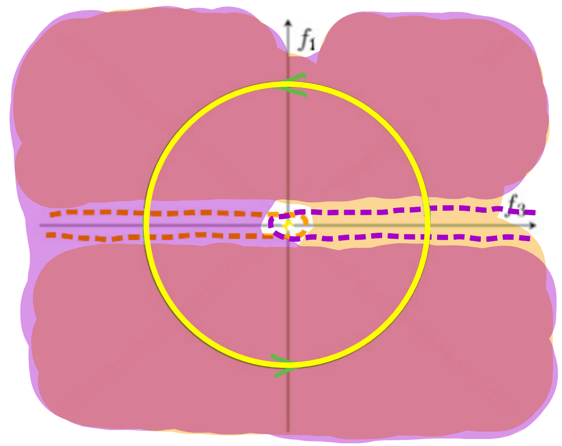

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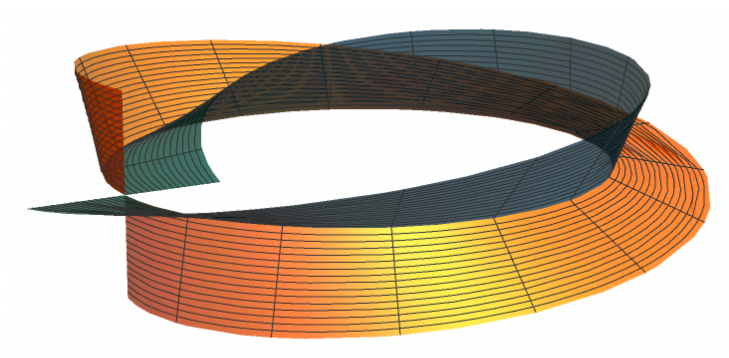

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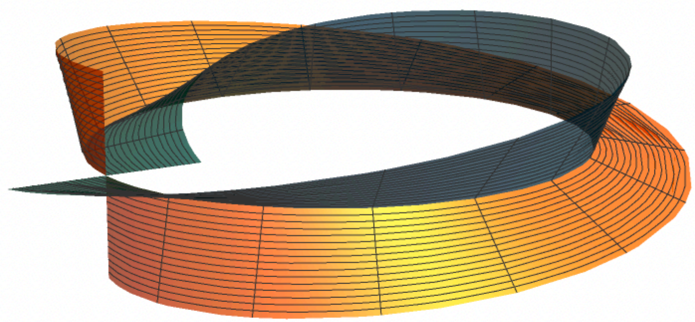

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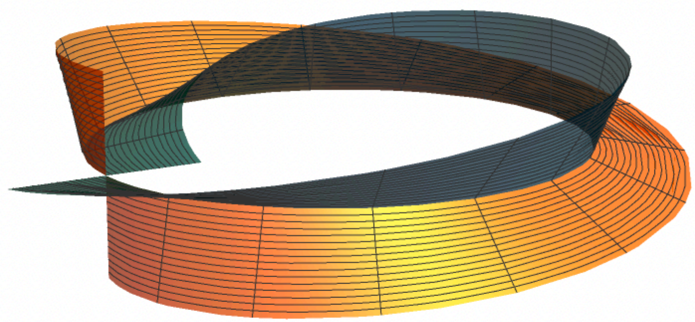

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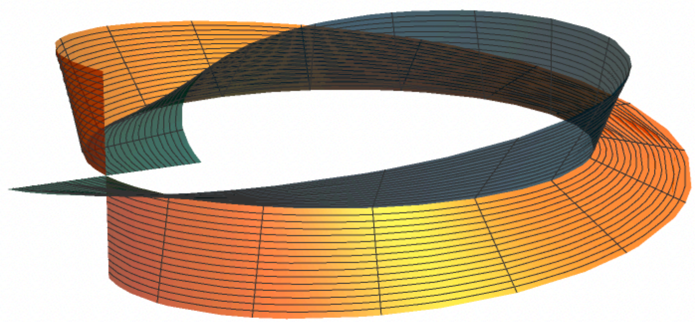

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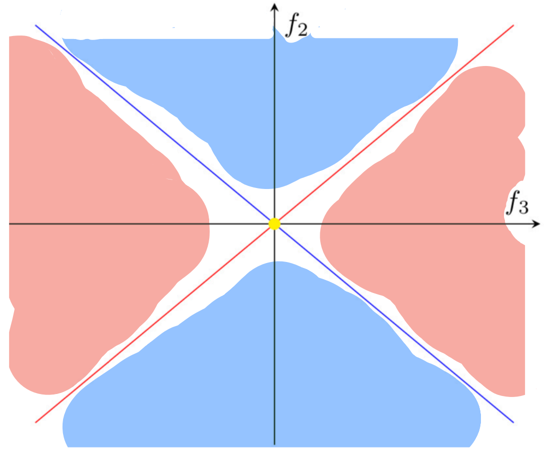

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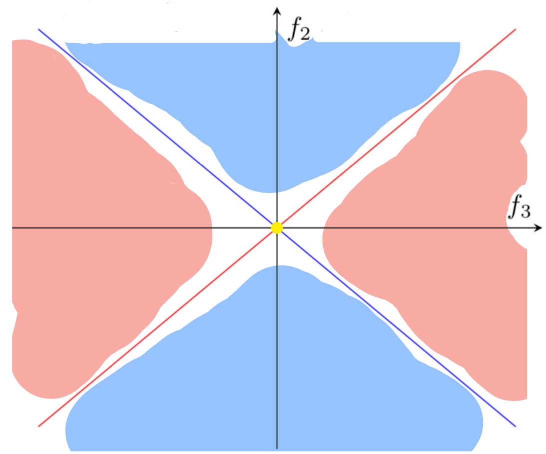

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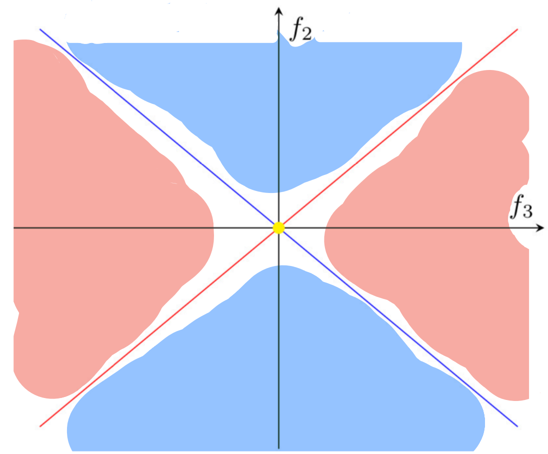

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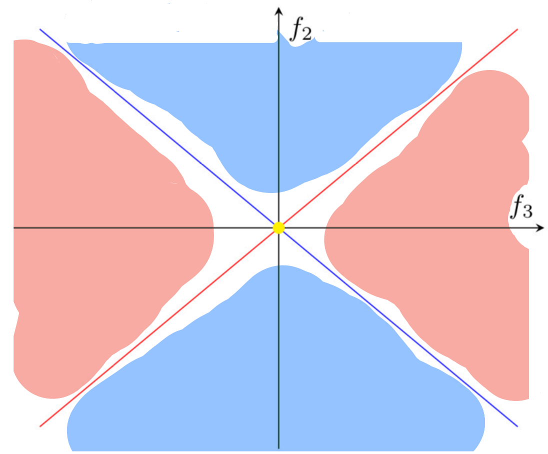

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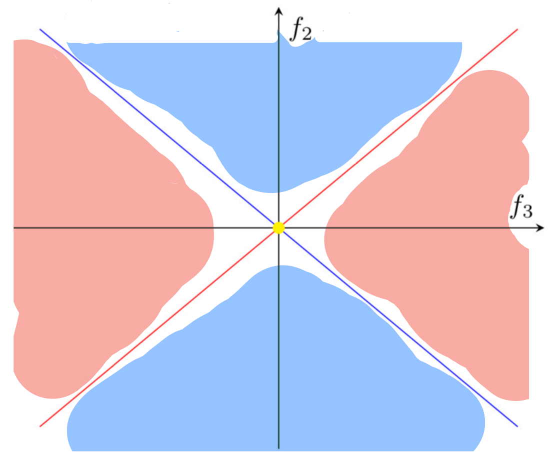

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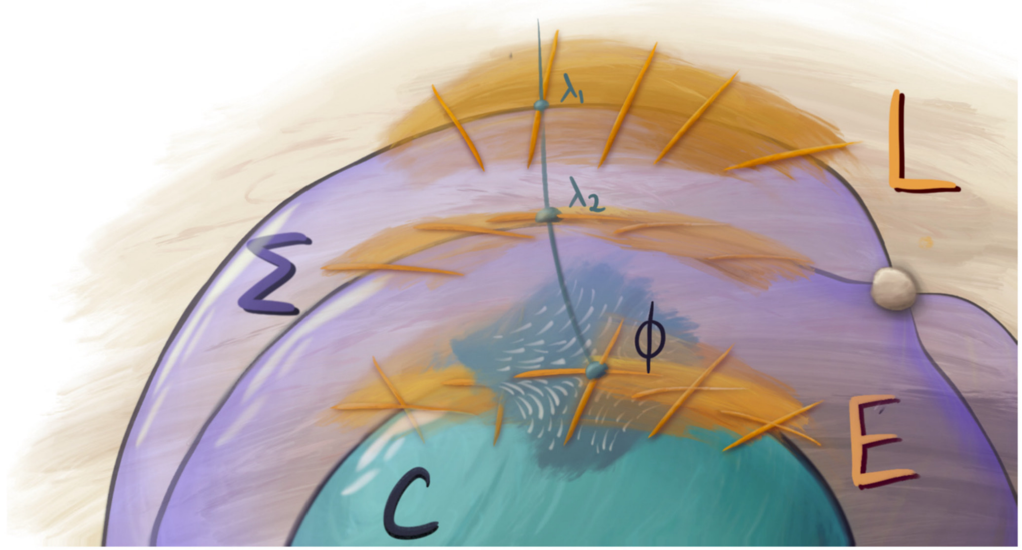
Peter Higgs (bosons)

Nigel Hitchin 1987
Carlos Simpson

C compact Riemann surface (or more generally Kähler manifold)
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$



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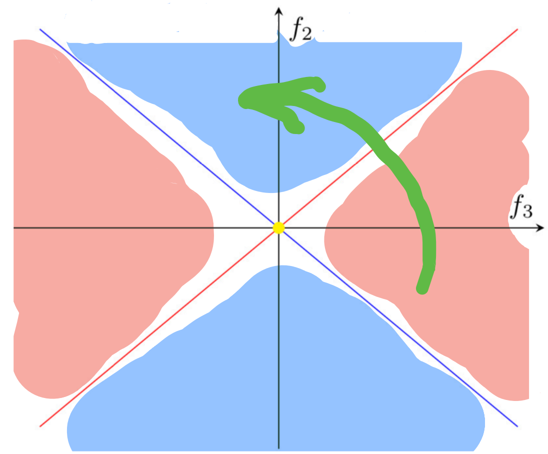

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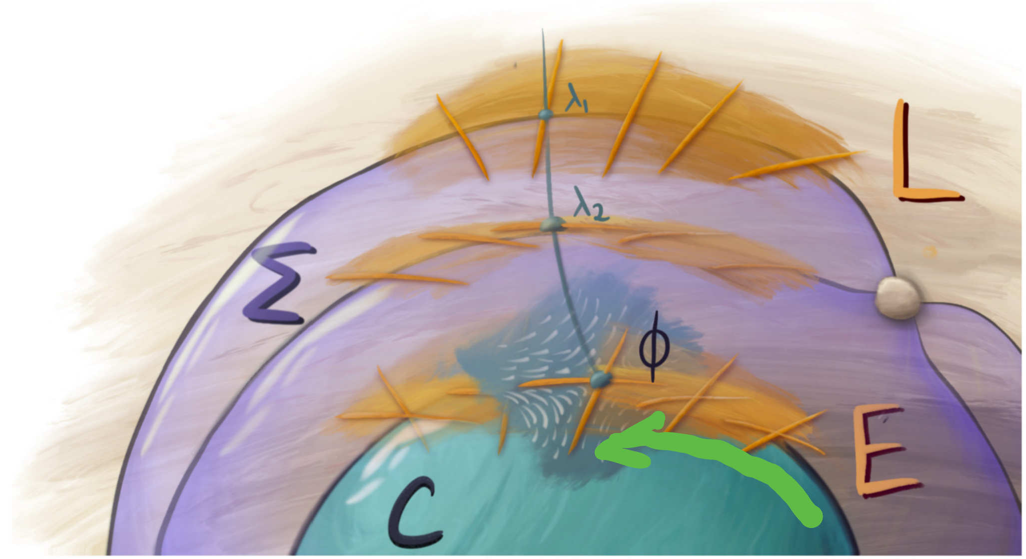
$$\phi_x \in \mathrm{End}(E_x), x \in C$$



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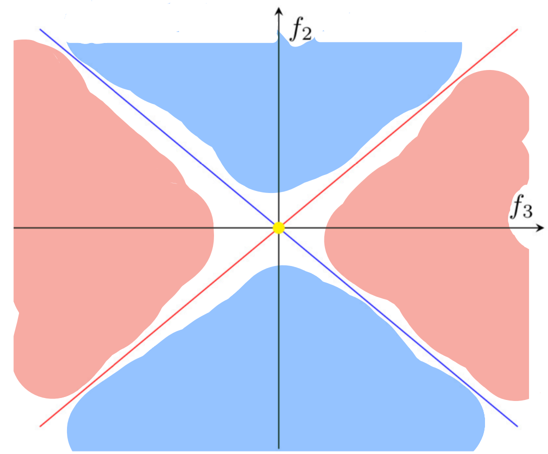

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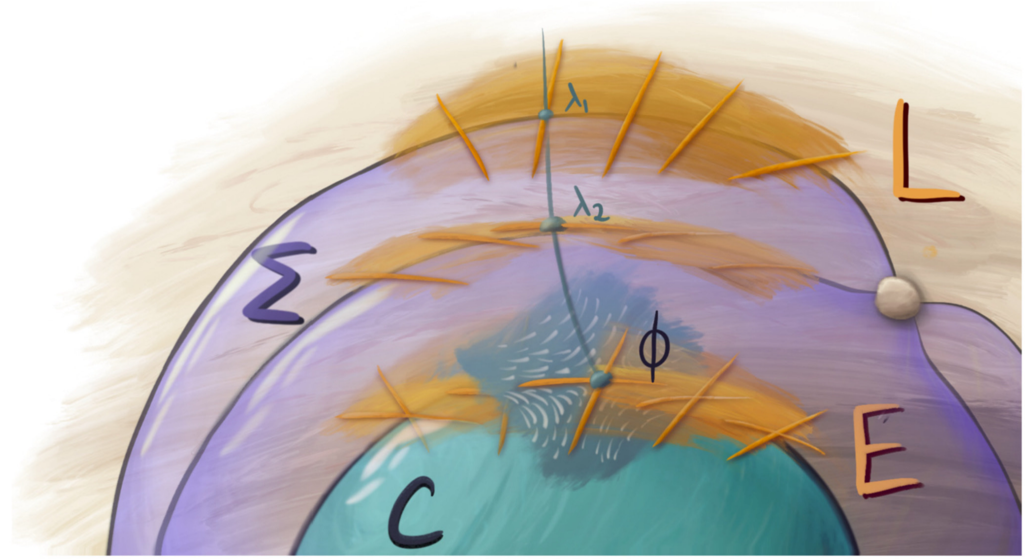

Portrait from Kienzle and Rayan, *Hyperbolic band theory through Higgs bundles*, ***Adv. Math.***, 2022.

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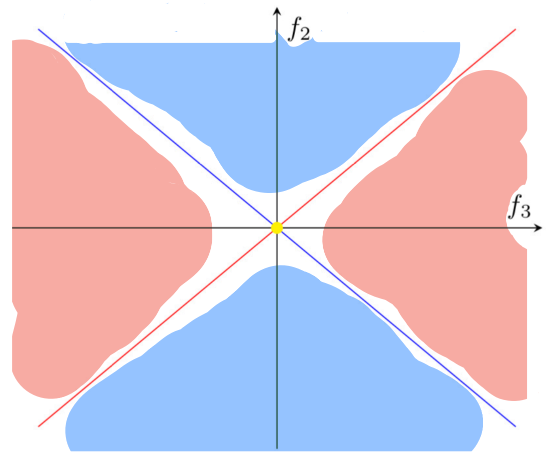

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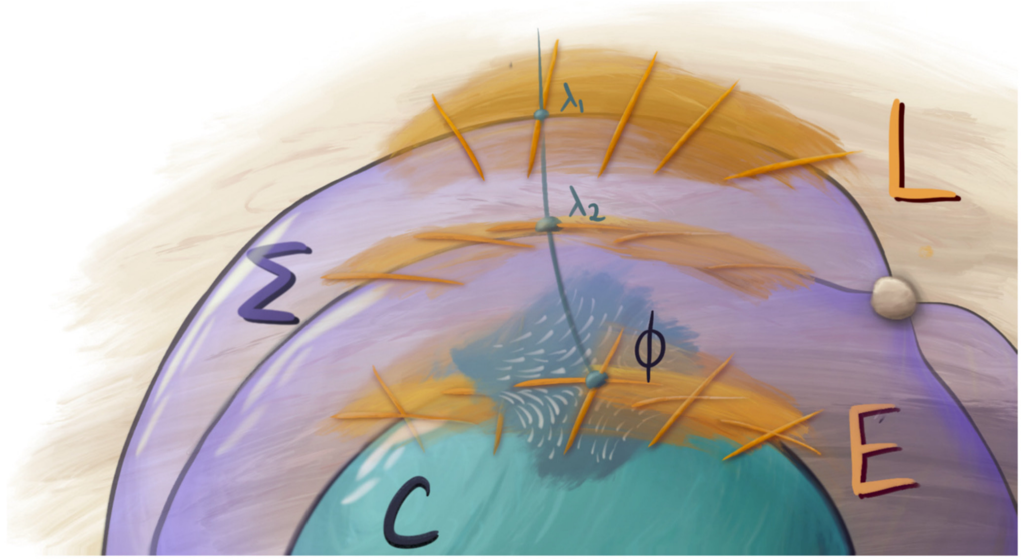

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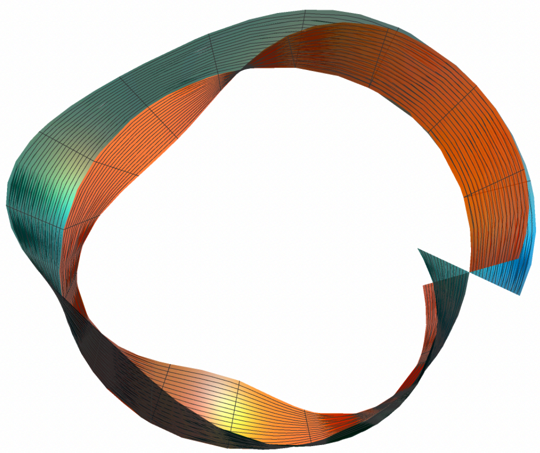

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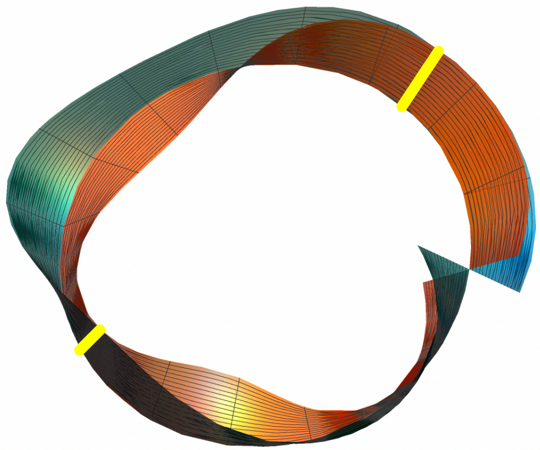

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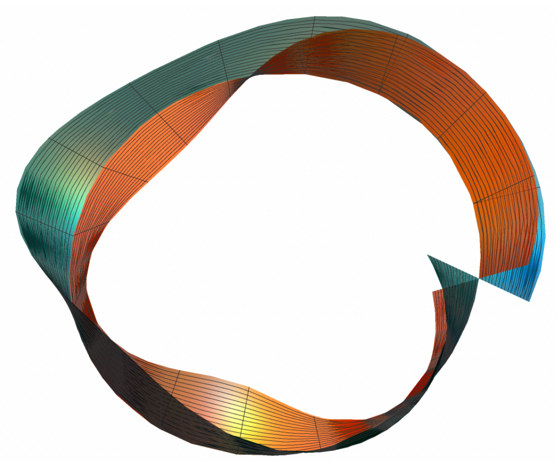

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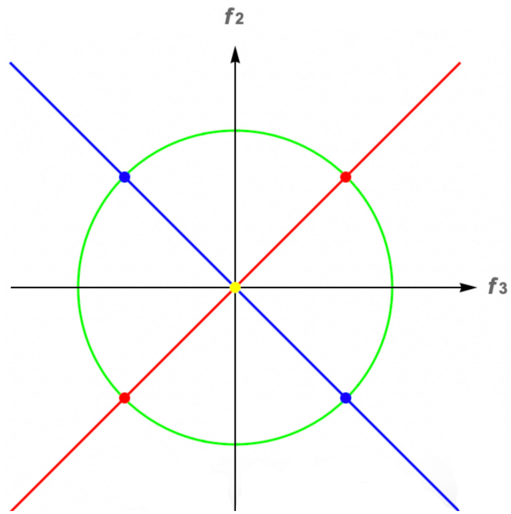

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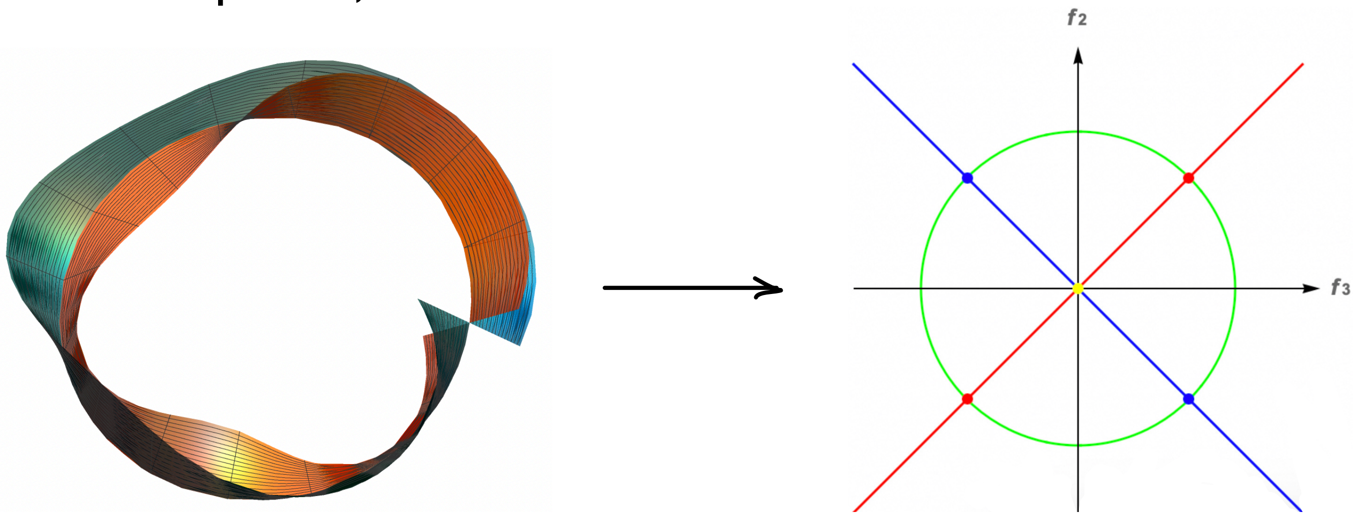

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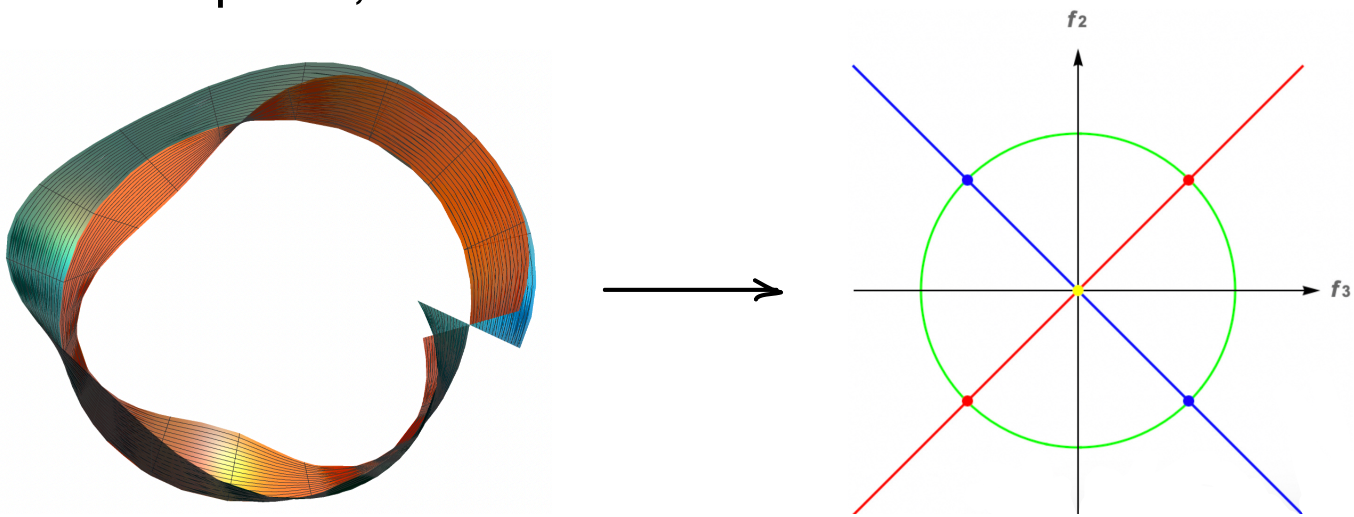

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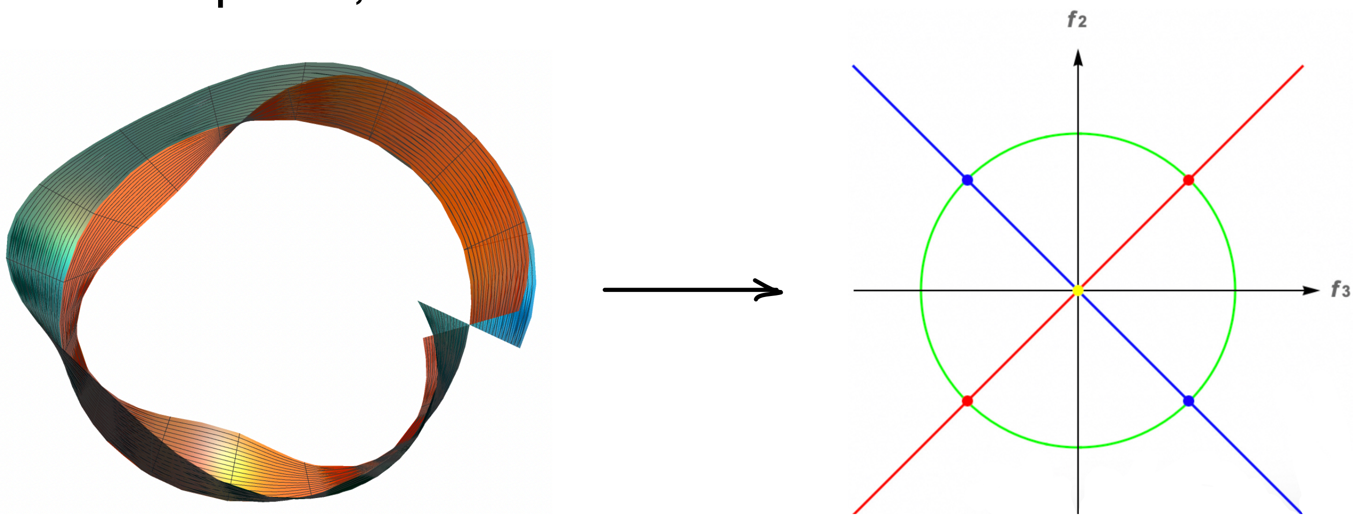

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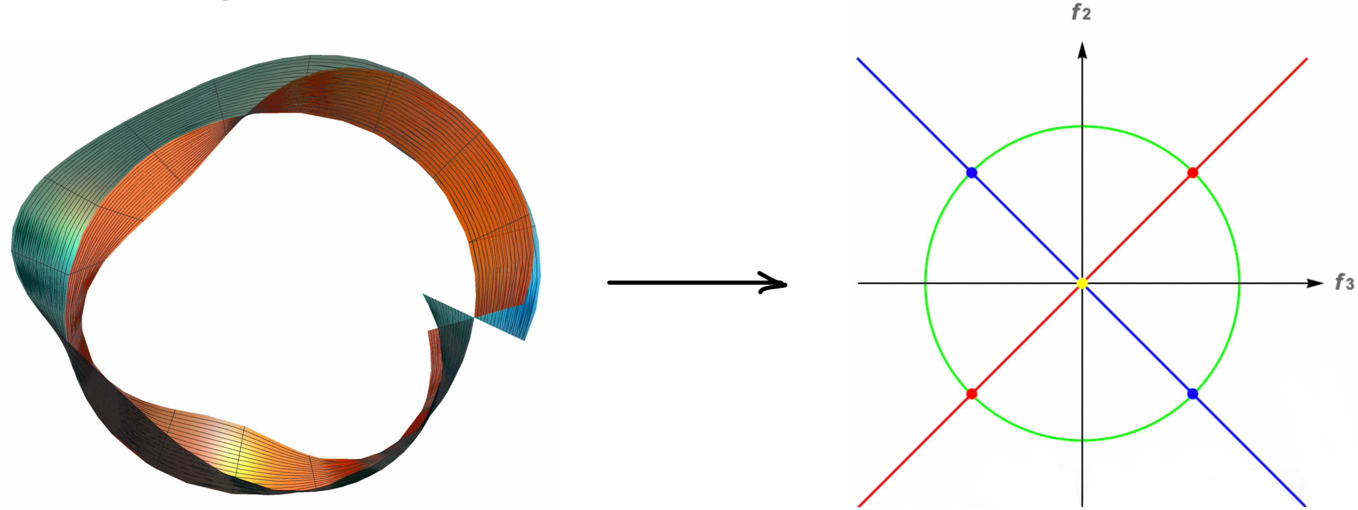

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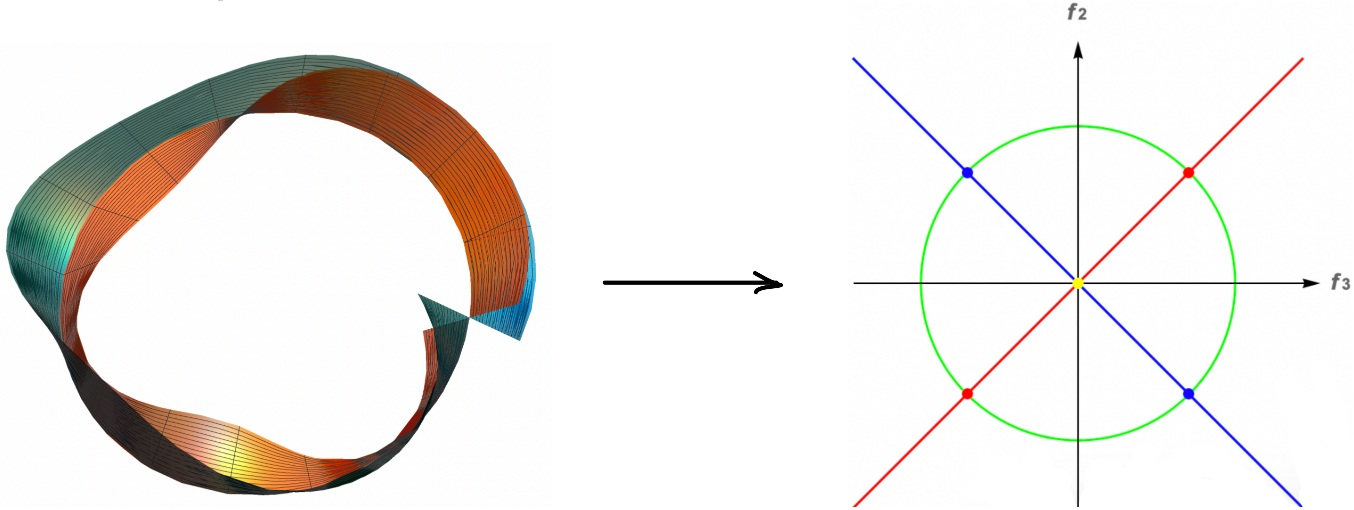

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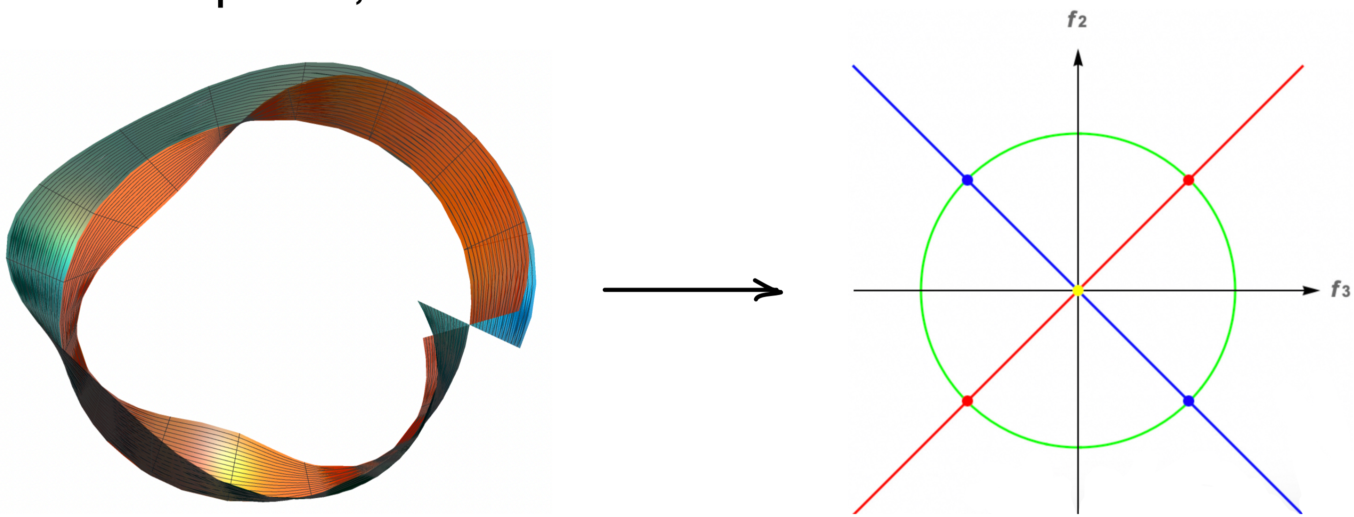

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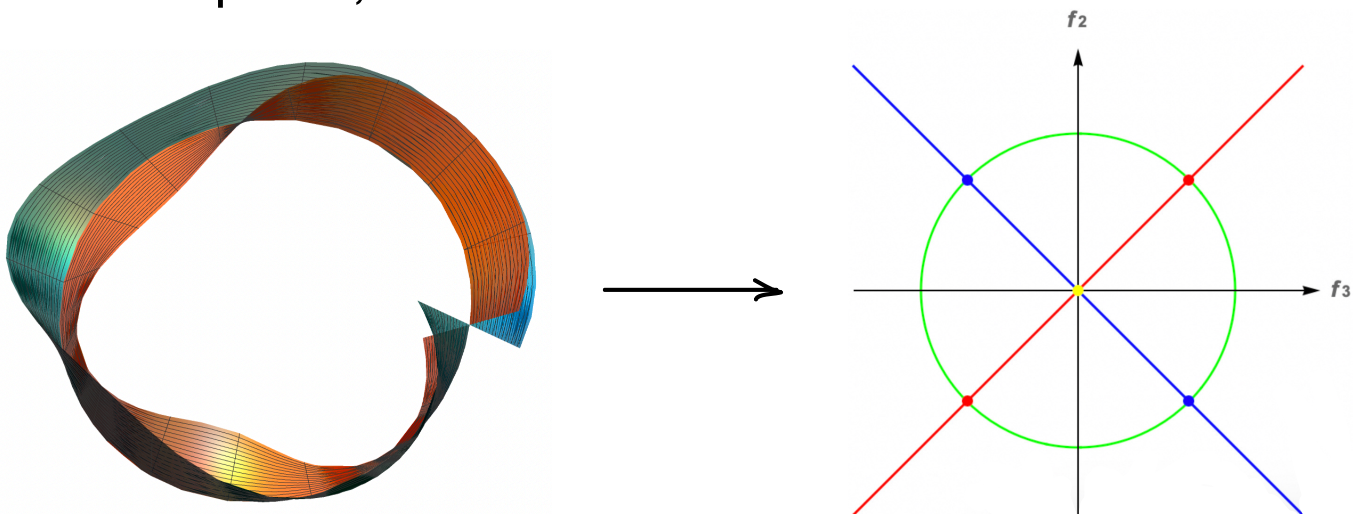

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Gajer, The intersection Dold–Thom theorem, Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)

Goresky and MacPherson, 1974.

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Here is a video of the eigenbundle deformation: https://yifeizhu.github.io/swallowtail/deform.mp4

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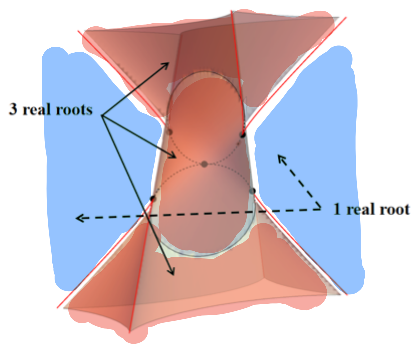

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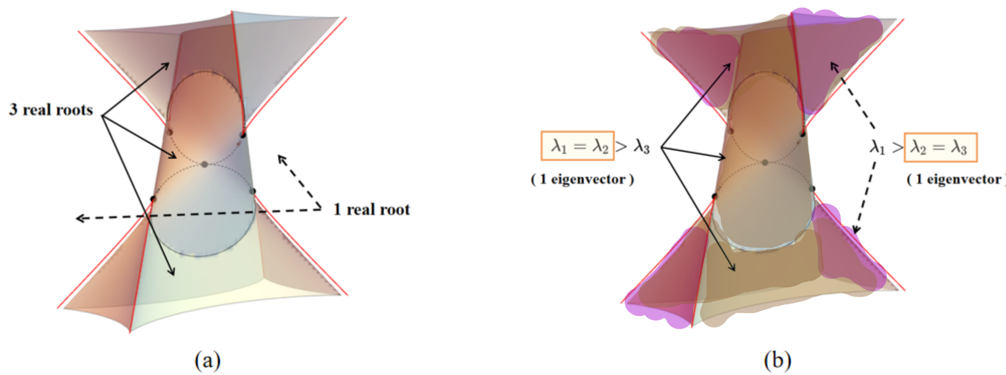

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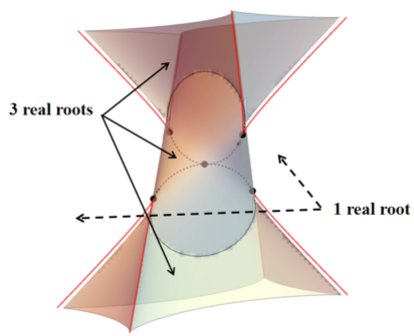

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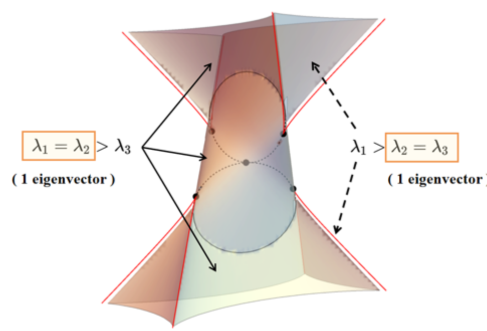

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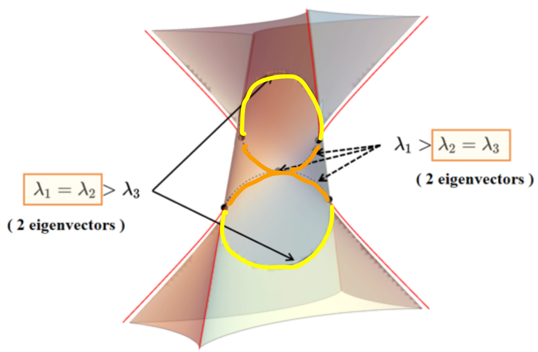

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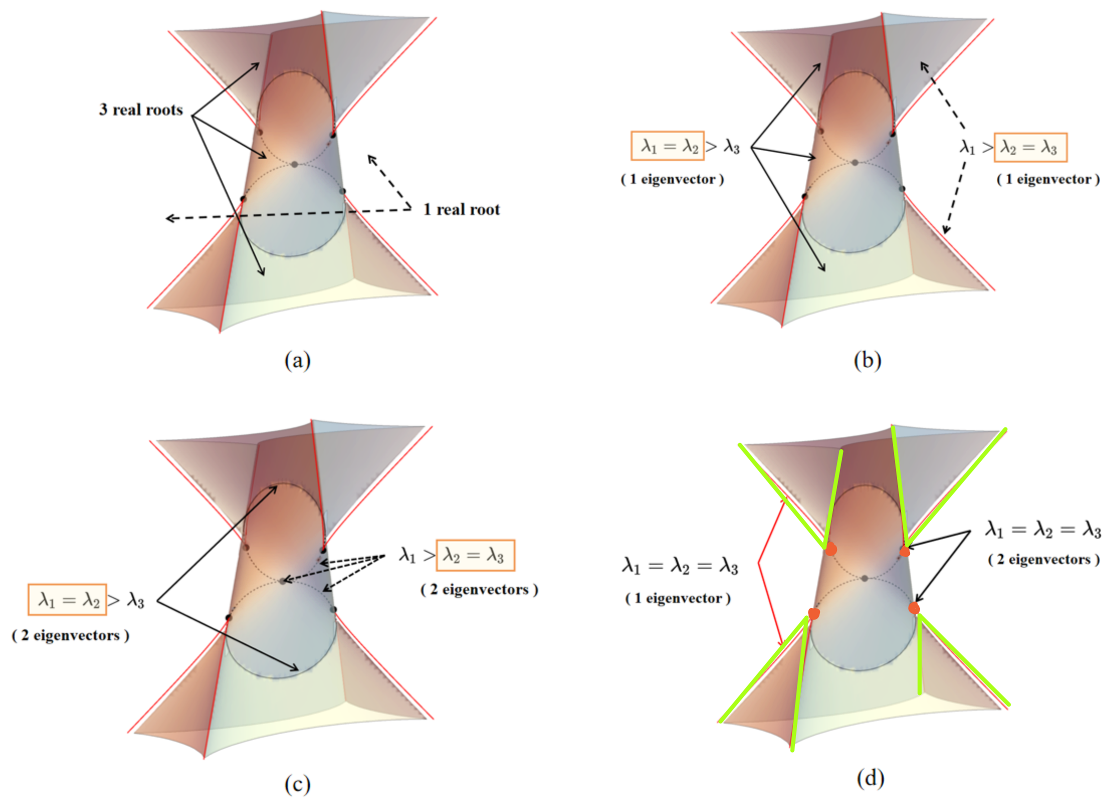

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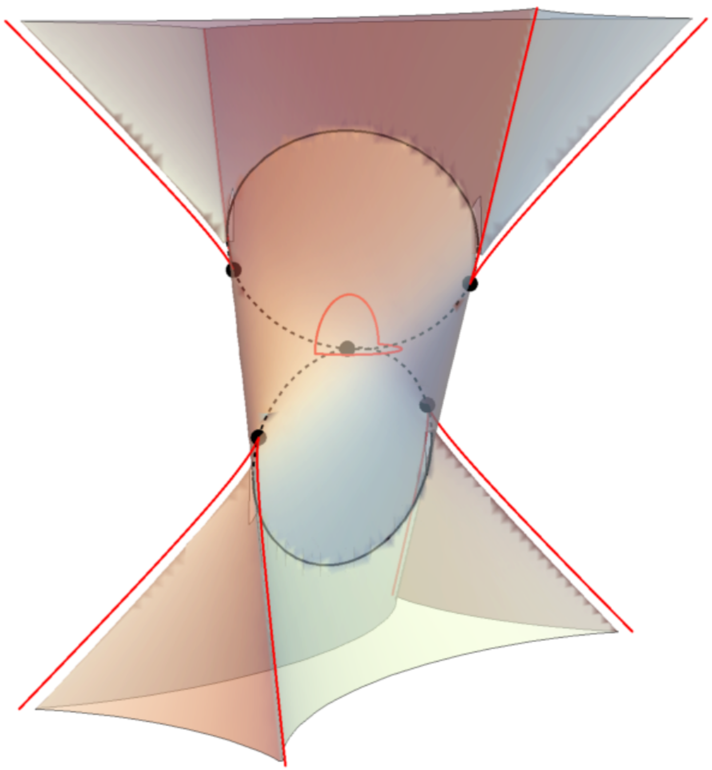

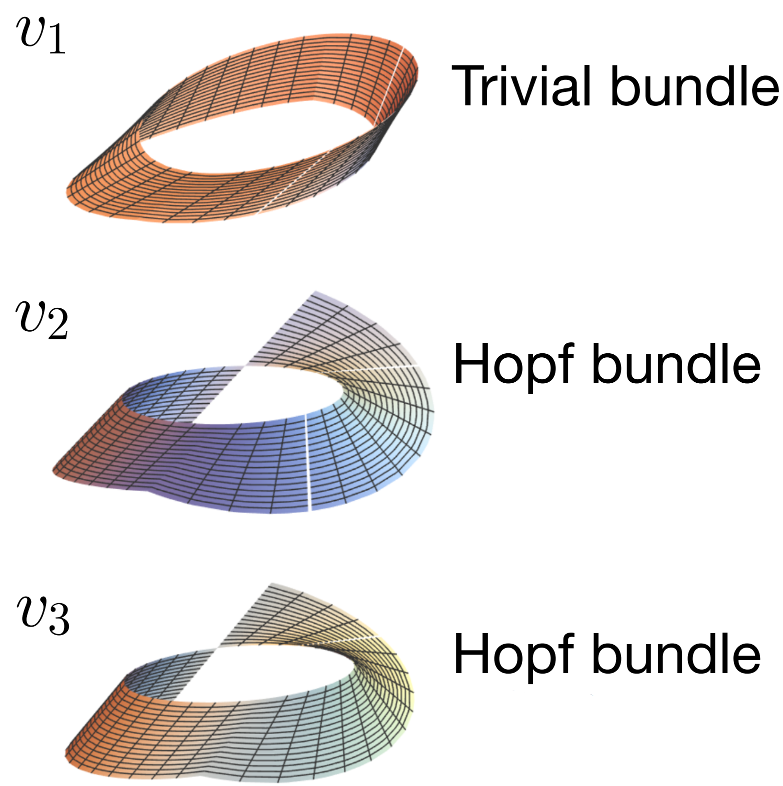

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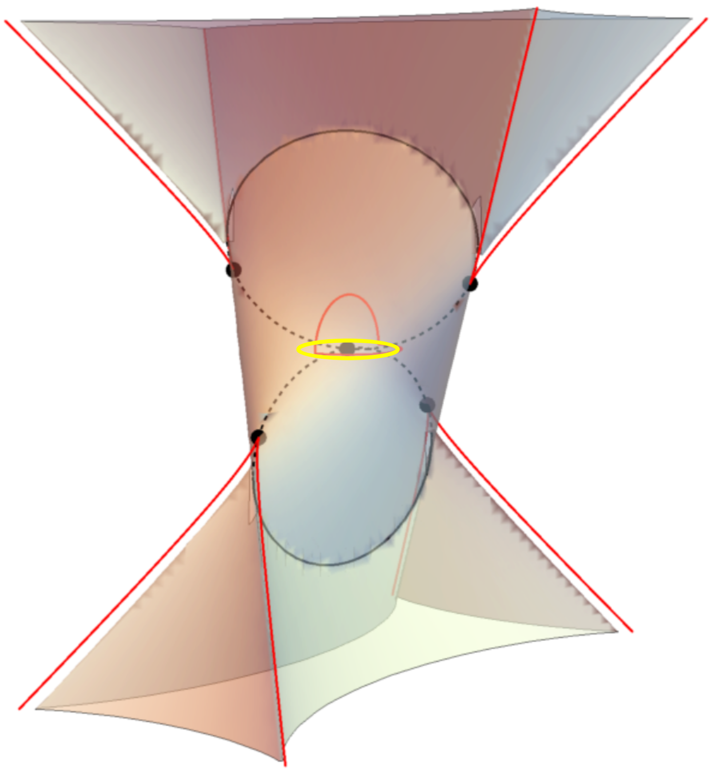

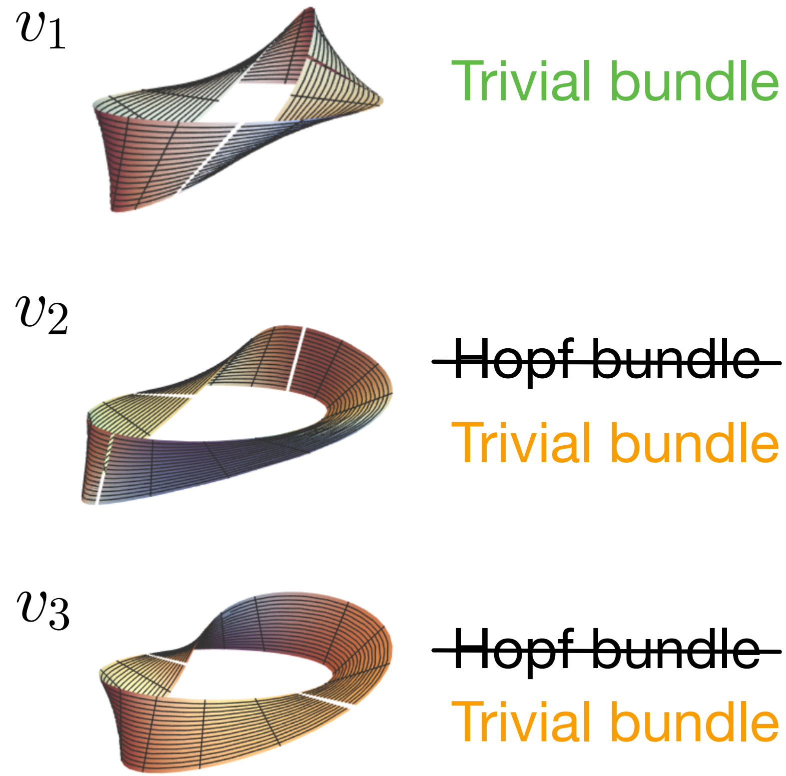

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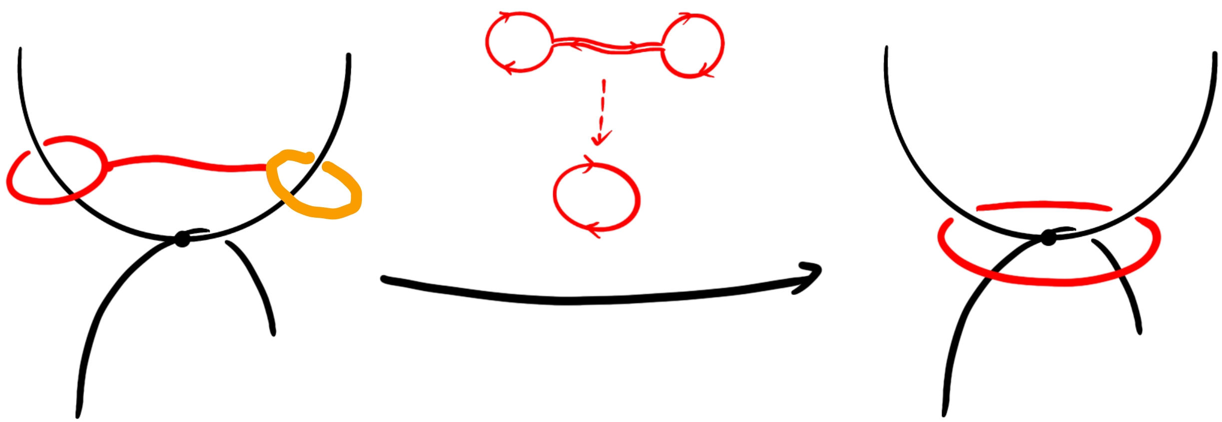

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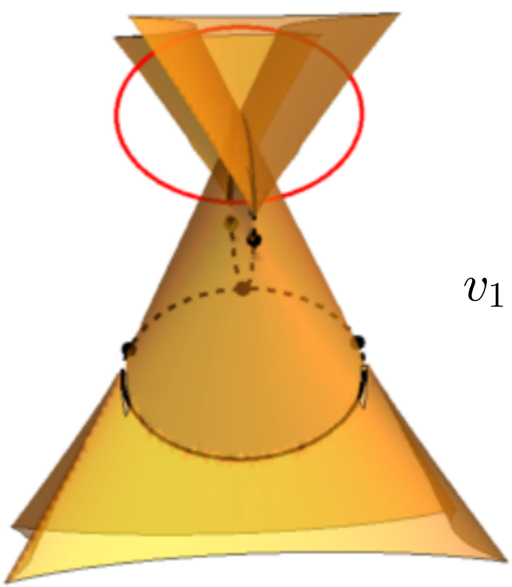

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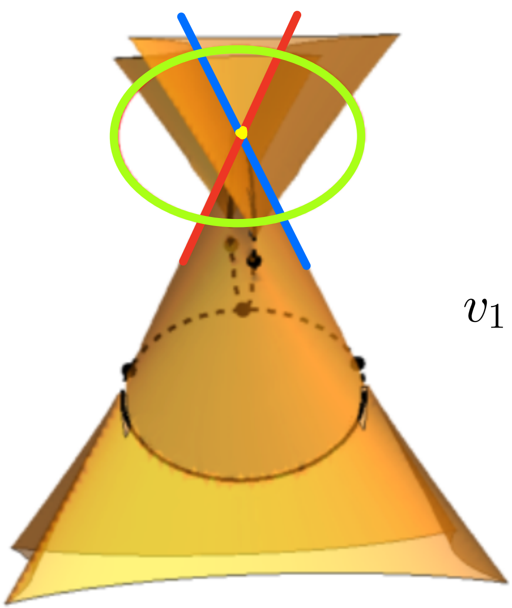

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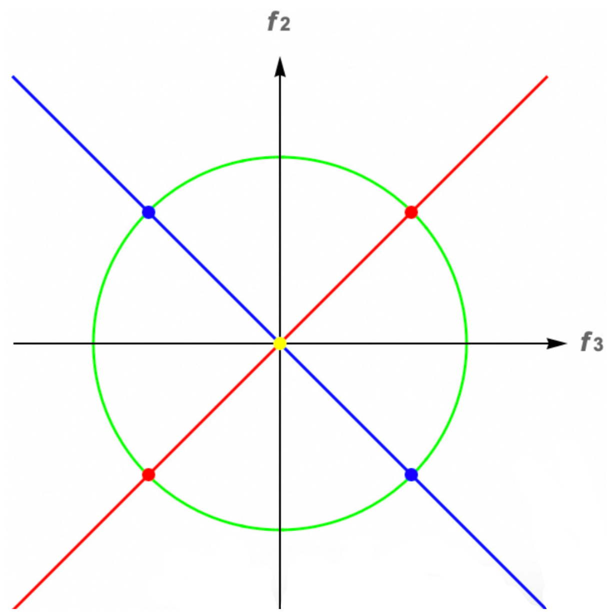

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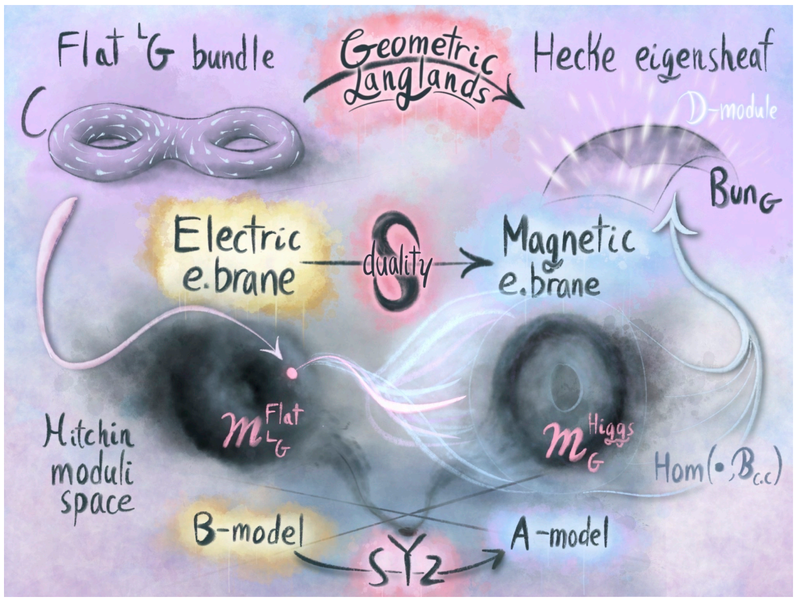

Portrait by Elliot Kienzle

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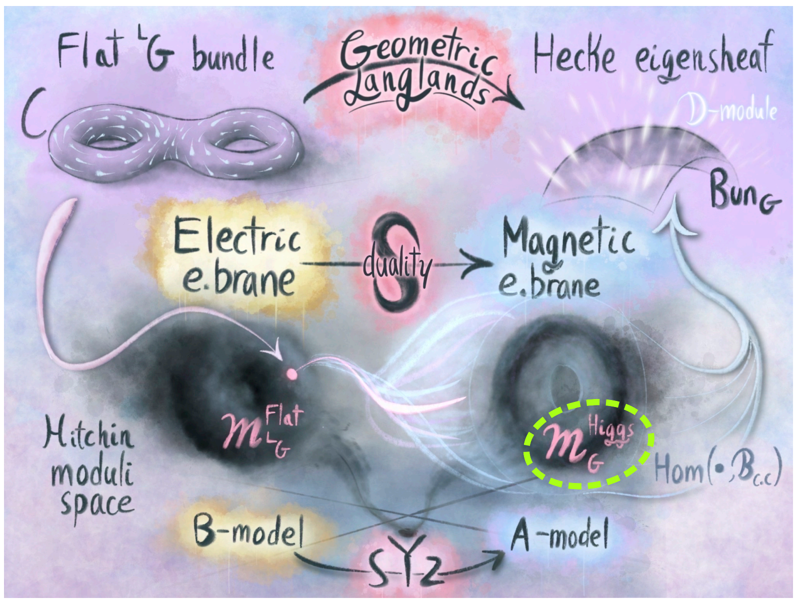

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Thank you.