Sins of omission (not on the final exam)

(1) More about higher cup products and Steenrod operations

- See questions 4-6 here:

  https://yifeizhu.github.io/8021/2025/HW7.pdf

- See Greenlees's article "How blind ..."

- Applications and variants ...

(2) Division algebra problem

There are no division algebra structures on $\mathbb{R}^n$, n not a power of 2. (In fact, only $n = 1, 2, 4, 8$.)

A "weak" division algebra structure is a map $\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with element $e \in \mathbb{R}^n$ such that

(1) $\mu(e, x) = \mu(x, e) = x$

(2) $\mu(rx, sy) = rs\,\mu(x, y)$, $r, s \in \mathbb{R}$

(3) $\mu(x, y) = 0$ iff $x = 0$ or $y = 0$

(4) μ is continuous

No such exists for n not a power of 2.

Pf Get a map

$\mu : \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

Since $\mu(rx, sy) = rs\,\mu(x,y)$

get a well-defined map

$\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}$

If we restrict

$$\begin{tikzcd}
\mathbb{R}P^{n-1} \vee \mathbb{R}P^{n-1} \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \arrow[r] & \mathbb{R}P^{n-1} \\
\mathbb{R}P^{n-1} \times \{e\} \cup \{e\} \times \mathbb{R}P^{n-1} \arrow[urr, bend right] & &
\end{tikzcd}$$

fold map

identity on each factor

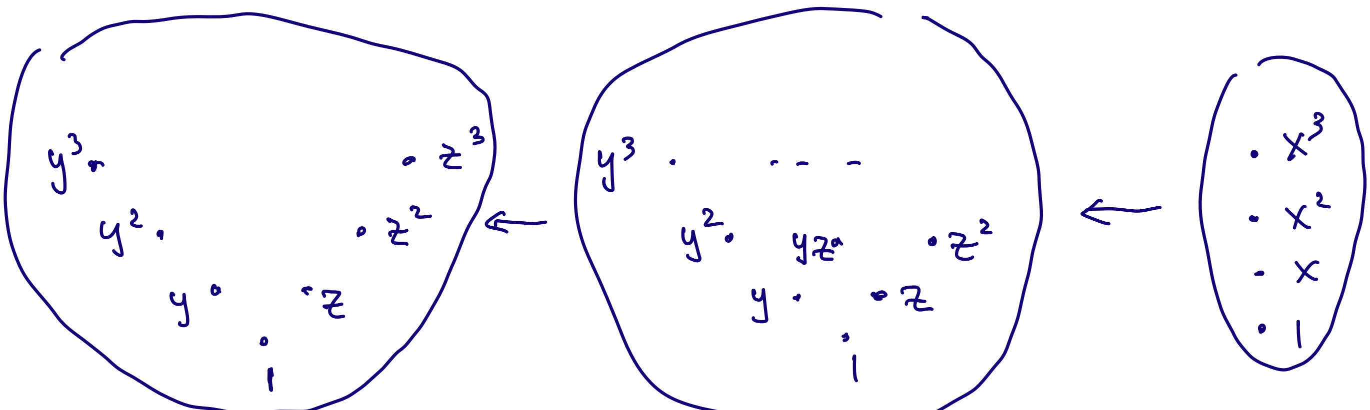
Effect on $H^*(- ; \mathbb{Z}/2)$:

$$\mathbb{Z}/2[y,z]/(y^n, z^n, yz) \longleftarrow \mathbb{Z}/2[y,z]/(y^n, z^n) \longleftarrow \mathbb{Z}/2[x]/(x^n)$$

$$y + z \longleftarrow\!\shortmid \quad x$$

$$\Downarrow$$

$$y + z \longleftarrow\!\shortmid \quad x$$



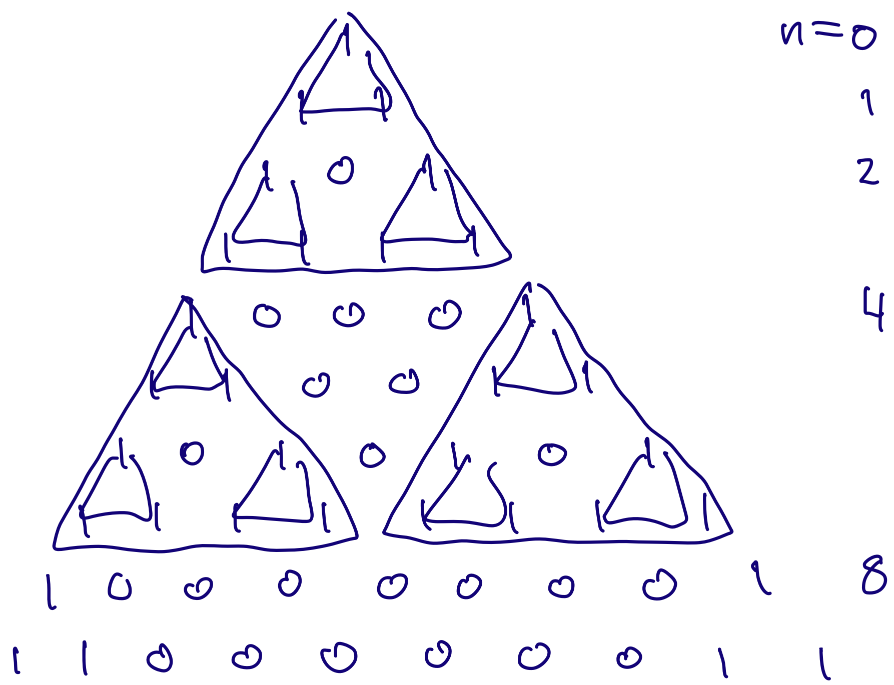
$x^n = 0 \implies$ image of x^n is 0, i.e. $(y+z)^n = 0$

$$(y+z)^n = \sum_{i=0}^{n} \binom{n}{i} y^i z^{n-i}$$

$$= \cancel{y^n} + \underbrace{\sum_{i=1}^{n-1} \binom{n}{i} y^i z^{n-i}}_{\substack{\text{nonzero} \\ \text{basis elements in } \mathbb{Z}/2[y,z]/(y^n, z^n)}} + \cancel{z^n}$$

For this sum to be 0, we need $\binom{n}{i} = 0$ for $1 \le i \le n-1$

(with mod-2 coefficients, $\binom{n}{i} \equiv 0 \mod 2$)



The only rows where all middle terms are zero are when n is a power of 2.

Lucas's lemma

Even better, if $n = a_0 a_1 \cdots a_k$, $i = b_0 b_1 \cdots b_k$ written in binary, $\binom{n}{i} \equiv \prod \binom{a_i}{b_i} \mod 2$

E.g. $\binom{1101101}{0100110} = \binom{1}{0}\binom{1}{1}\binom{0}{0}\binom{1}{0}\binom{1}{1}\binom{0}{1}\binom{1}{0}$

only get 0 if we have $\binom{0}{1}$ somewhere

Adams proved (by topology!) there are no such maps for $n > 8$.

Atiyah used K-theory (generalized cohomology theory)

③ Sketch of computation of $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$

There are relative cup products.

Let $A, B \subset X$ be open subspaces.

There is a cup product

$$H^p(X, A) \otimes H^q(X, B) \xrightarrow{\smile} H^{p+q}(X, A \cup B)$$

such that the diagram

$$\begin{tikzcd}
H^p(X,A) \otimes H^q(X,B) \arrow[r, "\smile"] \arrow[d] & H^{p+q}(X, A \cup B) \arrow[d] \\
H^p(X) \otimes H^q(X) \arrow[r, "\smile"'] & H^{p+q}(X)
\end{tikzcd}$$

commutes.

We can determine the multiplicative structure in

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^{n+1}), \quad |x| = 1$$

through several guises of the multiplication:

$$H^i(\mathbb{RP}^n) \times H^j(\mathbb{RP}^n) \longrightarrow H^{i+j}(\mathbb{RP}^n) \qquad i+j=n$$

(i) as cup product

(ii)

$$
\begin{tikzcd}
H^{n-j}(\mathbb{RP}^n) \times H^{n-i}(\mathbb{RP}^n) \arrow[r] \arrow[d, "\cong"] & H^n(\mathbb{RP}^n) \arrow[d, "\cong"] \\
H_j(\mathbb{RP}^n) \times H_i(\mathbb{RP}^n) \arrow[r] & H_0(\mathbb{RP}^n)
\end{tikzcd}
$$

Poincaré duality $-\frac{}{}\cap[\mathbb{RP}^n]$, $\hat{H}_n(\mathbb{RP}^n;\mathbb{Z})$

as intersection pairing

(iii) as relative cup product

$$
\begin{tikzcd}
H^{n-j}(\mathbb{RP}^n, \mathbb{RP}^n\setminus\mathbb{RP}^j) \times H^{n-i}(\mathbb{RP}^n, \mathbb{RP}^n\setminus\mathbb{RP}^i) \arrow[r] & H^n(\mathbb{RP}^n, \mathbb{RP}^n\setminus\{p\}) \\
H^{n-j}(\mathbb{RP}^n, \mathbb{RP}^{n-j-1}) \times H^{n-i}(\mathbb{RP}^n, \mathbb{RP}^{n-i-1}) \arrow[r] \arrow[u, "\cong"', "{[x_0:\cdots:x_{i-1}:tx_i:tx_{i+1}:\cdots:tx_{i+j}]}"] & H^n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \arrow[u, "\cong"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & H^{n-j}(\mathbb{RP}^n, \mathbb{RP}^{n-j-1}) \arrow[r, "\cong ?"] & H^{n-j}(\mathbb{RP}^n) \arrow[r] & \underset{\displaystyle 0}{H^{n-j}(\mathbb{RP}^{n-j-1})} \\
H^{n-j-1}(\mathbb{RP}^n, \mathbb{RP}^{n-j-1}) \arrow[r] & H^{n-j-1}(\mathbb{RP}^n) \arrow[r, "\cong"] & H^{n-j-1}(\mathbb{RP}^{n-j-1}) \arrow[uull, bend right]
\end{tikzcd}
$$

(iv) as relative cup product

$$
H^{n-j}(\mathbb{R}^n, \mathbb{R}^n\setminus\mathbb{R}^j) \times H^{n-i}(\mathbb{R}^n, \mathbb{R}^n\setminus\mathbb{R}^i) \longrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \cong H^n(\Delta[n], \partial\Delta[n])
$$

$\mathbb{R}^n = \{x_i \neq 0\} =: \mathbb{R}^n_i \hookrightarrow \mathbb{RP}^n$

$$
\begin{tikzcd}
H^{n-j}(\mathbb{R}^n, \mathbb{R}^n\setminus\mathbb{R}^j) \\
H^{n-j}(\mathbb{R}^i, \mathbb{R}^i\setminus\{0\}) \arrow[u, "\cong"', "{[x_0:\cdots:x_{i-1}:x_i:tx_{i+1}:\cdots:tx_{i+j}]}"] \\
H^{n-j}(\mathbb{RP}^{n-j}, \mathbb{RP}^{n-j}\setminus\{p\}) \arrow[u, "\cong", "\text{excision}"']
\end{tikzcd}
$$

$\mathbb{RP}^{n-j} = \{x_{i+1} = \cdots = x_{i+j} = 0\}$

More generally, $H^*(Gr(k,n);\mathbb{Z}/2)$

cf https://yifeizhu.github.io/8021/2026/supp-Gr.pdf

$$\boxed{D_1}^* \smile \boxed{D_2}^* = \sum_{D_\alpha} c(D_1, D_2, D_\alpha)\, \boxed{D_\alpha}^*$$

\# of boxes in D_α

$= \#$ in D_1 + \# in D_2

coefficients given by Littlewood-Richardson rule

(4) Machinery for "rigorous" Poincaré duality

- Simplicial structure on a manifold
  $\rightsquigarrow$ orientation

  $Z_n^\Delta(M) \cong H_n^\Delta(M)$ (no boundaries)

  How can we get a cycle?

  local and global orientations

- cap product

$$C^*(X;R)\otimes C_*(X) \xrightarrow{1\otimes DA} C^*(X;R)\otimes C_*(X)\otimes C_*(X) \xrightarrow{ev\otimes 1} R\otimes C_*(X) = C_*(X;R)$$

cup & cap : "$H_*(X)$ is a module over $H^*(X)$"

relative version

- Cohomology with compact support

$$H^i(M|K) = H^i(M, M\setminus K)$$

$$H^i_c(M) = \varinjlim_{K} H^i(M|K)$$

$$M \subset N \rightsquigarrow H^i(M|K) \xrightarrow{\cong} H^i(N|K) \text{ by excision}$$

$$\rightsquigarrow H^i_c(M) \longrightarrow H^i_c(N)$$

- $D: H^i_c(M) \longrightarrow H_{n-i}(M)$

<u>Goal</u> Prove D is an iso for all oriented manifolds M

Steps (1) Prove it for $\mathbb{R}^n$

(2) Apply induction using Mayer-Vietoris sequence

(3) Take some limit to prove for a general manifold

(5) What on earth is a <u>Massey product</u>?

You should have been equipped with the $\underbrace{\text{background in order}}_{\text{(and motivation)}}$ to understand it!