Supplement: Exact sequences & diagram chasing

A sequence of maps of abelian groups

$$\cdots \to A \to B \to C \to \cdots$$

is exact if at each position, kernel of one map = image of previous.

Ex $0 \to A \xrightarrow{f} B$ is exact $\iff 0 = \ker(f)$

$\iff f$ is 1-to-1

$A \xrightarrow{f} B \to 0$ exact $\iff f$ is onto

Homology of a chain complex is a measure of failure to be exact.

An exact sequence $0 \to A \to B \to C \to 0$ is a "short exact sequence" and it means $C \cong B/A$

If $A = \{ \to A_n \to A_{n-1} \to \cdots \}$

A'

A''

are chain complexes

a map $f: A \to A'$ is a sequence of functions $f_n: A_n \to A'_n$ such that $f_n \circ \partial = \partial \circ f_{n+1}$.

A short exact sequence of chain complexes is a sequence

$$0 \to A' \to A \to A'' \to 0$$

such that in the diagram

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & A'_n \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'_{n-1} \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & A''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

all the rows are exact.

<u>Thm</u> (Long exact sequence in homology)

Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

of chain complexes, there is an induced long exact sequence

$$\cdots \to H_n(A') \to H_n(A) \to H_n(A'') \xrightarrow{\partial} H_{n-1}(A') \to H_{n-1}(A) \to \cdots$$

<u>AND</u> If we have

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

then there is an induced map

$$\begin{tikzcd}
\arrow[r] & H_n(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_n(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_n(A'') \arrow[r, "\partial"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A') \arrow[r] \arrow[d] & \\
\arrow[r] & H_n(B') \arrow[r] & H_n(B) \arrow[r] & H_n(B'') \arrow[r] & H_{n-1}(B') \arrow[r] &
\end{tikzcd}$$

"Short version"

$\exists$ functor

$$\left\{\begin{array}{l}\text{short exact sequences}\\ \text{of chain complexes}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{long exact}\\ \text{sequences}\end{array}\right\}$$

Won't prove the entire thing rigorously: method is called "diagram chasing."

- How to construct maps?
- How to show exactness?

<u>Ex</u> $A' \longrightarrow A$ map of chain complexes

<u>Meaning</u>

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
C_n(A') = & A'_n \arrow[r] \arrow[d] & A_n \arrow[d] \\
 & A'_{n-1} \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\
 & A'_{n-2} \arrow[r] \arrow[d] & A_{n-2} \arrow[d] \\
 & \vdots & \vdots
\end{tikzcd}$$

Show get maps $H_n(A') \longrightarrow H_n(A)$

If $[z] \in H_n(A') = Z_n(A')/B_n(A')$

$\exists\ z \in Z_n(A')$ s.t. image is $[z]$, $\partial z = 0$.

If $f : A' \longrightarrow A$ is the map

$\partial(f_n(z)) = f_{n-1}(\partial(z)) = f_{n-1}(0) = 0.$

$f_n(z) \in Z_n(A)$, $[f_n(z)] \in H_n(A)$.

Any two elements y, z whose images are both $[z]$.

$y - z = \partial u$, $u \in A'_{n+1}$.

$f_n(y) - f_n(z) = f_n(\partial u) = \partial(f_{n+1} u)$

so $[f_n(y)] = [f_n(z)]$ in $H_n(A)$.

That defines maps

$H_n(A') \longrightarrow H_n(A) \longrightarrow H_n(A'')$

for a SES $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$.

Now: $H_n(A'') \longrightarrow H_{n-1}(A')$ (the chase begins)

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A') \arrow[r] \arrow[d] & C_{n+1}(A)^{v} \arrow[r, "g_{n+1}"] \arrow[d] & C_{n+1}(A'')^{u} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A')^{\tilde{x}} \arrow[r] \arrow[d] & C_n(A)^{y}_{y'} \arrow[r, "g_n"] \arrow[d] & C_n(A'')^{z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A')^{x}_{x'} \arrow[r, "f_{n-1}"] \arrow[d] & C_{n-1}(A)^{\partial y} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1}(A'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-2}(A') \arrow[r, "f_{n-2}"] & C_{n-2}(A) \arrow[r] & C_{n-2}(A'') \arrow[r] & 0
\end{tikzcd}$$

Suppose $[z] \in H_n(A'')$

represented by $z \in Z_n(A'') \subset C_n(A'')$, $\partial z = 0$

$z = g_n(y)$, $y \in C_n(A)$

$\partial y \in C_{n-1}(A)$

$g_{n-1}(\partial y) = \partial(g_n(y)) = \partial z = 0$ unique

So exactness $\Rightarrow$ $\partial y = f_{n-1}(x)$, $x \in C_{n-1}(A')$

$0 = \partial\partial y = \partial f_{n-1}(x) = f_{n-2}(\partial x)$

f_{n-2} is 1-to-1 so $\partial x = 0$, $x \in Z_{n-1}(A')$.

$[x] \in H_{n-1}(A')$.

Well-defined ?

Suppose we instead choose the element $z + \partial u$ as representative for $[z] \in H_n(A'')$.

$u = g_{n+1}(v)$, $v \in C_{n+1}(A)$

$z + \partial u = g_n(y) + \partial g_{n+1}(v) = g_n(y + \partial v)$

take $y + \partial v$ and take ∂, get ∂y, ...

get same element in H_{n-1}.

Made another choice: Also we need to show

Suppose $g_n(y) = g_n(y') = z$

Then $g_n(y-y') = 0 \implies y - y' = f_n(\tilde{x})$

Correspondingly, $f_{n-1}(x-x') = \partial(y-y')$

$$= \partial f_n(\tilde{x})$$

$$= f_{n-1}(\partial \tilde{x})$$

$\implies x - x' = \partial \tilde{x}$

$\implies [x] = [x'] \in H_{n-1}(A')$.

- Check ker = im at each stage. (exercise)

  at $H_n(A)$

  at $H_n(A'')$

  at $H_{n-1}(A')$

- Check naturality / functoriality ? ? ?

  Mostly formalism.