Supplement: Tensor products and Tor groups

Tensor products

If M is a right module over a ring R,

N is a left module over R,

tensor product $M \otimes_R N = \dfrac{\{ m \otimes_R n \mid m \in M, n \in N \}}{\left\{ \begin{array}{l} (m_1+m_2)\otimes_R n = m_1 \otimes_R n + m_2 \otimes_R n, \\ m \otimes_R (n_1+n_2) = m \otimes_R n_1 + m \otimes_R n_2 \\ m\cdot r \otimes_R n = m \otimes_R rn, r\in R \end{array} \right\}}$

In practice, if M has generators $\{m_i\}$ subject to relations $\{\sum m_i r_i^\alpha = 0\}_\alpha$, N has generators $\{n_j\}$, relations $\{\sum s_j^\beta n_j = 0\}_\beta$, then $M \otimes_R N$ has generators $m_i \otimes_R n_j$ subject to relations

$$\sum m_i r_i^\alpha \otimes_R n_j = 0, \quad \forall j, \forall \alpha,$$

$$\sum m_i \otimes_R s_j^\beta n_j = 0, \quad \forall i, \forall \beta$$

Over $R=\mathbb{Z}$, module = abelian group. We can ignore r.

$$M \underset{\mathbb{Z}}{\otimes} N = M \otimes N = \left\{ a \otimes b \;\middle|\; \begin{array}{l} (a_1+a_2)\otimes b = \cdots \\ a \otimes (b_1+b_2) = \cdots \end{array} \right\}$$

<u>Ex</u> $\mathbb{Z} \otimes G \cong G$

$\mathbb{Z}$ has generator $\{1\}$ and no relations.

If G has generators $\{g_i\}$, relations $\sum n_i^\alpha g_i = 0$.

$\mathbb{Z} \otimes G$ has generators $\{1 \otimes g_i\}$, relations $\sum n_i^\alpha (1 \otimes g_i) = 0$

same group.

<u>Ex</u> $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/\gcd(n,m)$

$\mathbb{Z}/n$	generator 1,	relation $n \cdot 1 = 0$
$\mathbb{Z}/m$	1	$m \cdot 1 = 0$
$\mathbb{Z}/n \otimes \mathbb{Z}/m$	$1 \otimes 1$	$n \cdot (1 \otimes 1) = 0$
		$m \cdot (1 \otimes 1) = 0$
		$\Longleftrightarrow \gcd(m,n) \cdot (1 \otimes 1) = 0.$

Properties of $\otimes$:

- $(-)\otimes_R(-)$ is a functor
- $A\otimes_R\left(\bigoplus_i B_i\right) \cong \bigoplus_i \left(A\otimes_R B_i\right)$
- If $B_1 \to B_2 \to B_3 \to 0$ is exact, then

  $A\otimes_R B_1 \to A\otimes_R B_2 \to A\otimes_R B_3 \to 0$ is <u>also</u> exact.

<u>Why?</u> $f: B_2 \to B_3$ is surjective

Given a general element

$$\sum a_i \otimes_R b_i \in A\otimes_R B_3$$

lift $b_i \in B_3$ to elements $\tilde{b}_i \in B_2$

$$f(\tilde{b}_i) = b_i$$

$\Rightarrow$ get a map $1\otimes f : A\otimes_R B_2 \to A\otimes_R B_3$

given by $(1\otimes f)(\sum a_i \otimes \tilde{b}_i)$

$$= \sum a_i \otimes f(\tilde{b}_i)$$

$$= \sum a_i \otimes b_i$$

(check: well-defined)

$\Rightarrow A\otimes_R B_2 \to A\otimes_R B_3$ is surjective.

Check (exe) $\quad A \otimes_R B_1 \to A \otimes_R B_2 \to A \otimes_R B_3$ is exact.

(requires $B_2 \twoheadrightarrow B_3$)

Ex $\quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ Tensor with $\mathbb{Z}/2$

Get sequence $0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2 \xrightarrow{\sim} \mathbb{Z}/2 \to 0$

takes generator $1 \otimes 1$ of $\mathbb{Z}/2 \otimes \mathbb{Z}$ to generator $1 \otimes 1$ of $\mathbb{Z}/2 \otimes \mathbb{Z}/2$

takes generator $1 \otimes 1$ of $\mathbb{Z}/2 \otimes \mathbb{Z}$,

$1 \otimes 2 = 2(1 \otimes 1) = 0$

This sequence is no longer exact.

"Repair" — Tor

Tor $\quad$ There exists a sequence of functors $\mathrm{Tor}_n^R(-,-)$ such that

- $\mathrm{Tor}_n^R(A, B)$ takes a right R-module A and a left R-module B and produces an abelian group.

- Tor is functorial:
  a map $A \xrightarrow{f} A'$ of right R-modules and a map $B \xrightarrow{g} B'$ of left R-modules induce a

map $\mathrm{Tor}_n^R(A,B) \xrightarrow{f\otimes g} \mathrm{Tor}_n^R(A',B')$ of abelian groups.

- $\mathrm{Tor}_n^R(A, \bigoplus B_i) = \bigoplus \mathrm{Tor}_n^R(A, B_i)$

  (left adjoints are right exact and preserve colimits)

- $\mathrm{Tor}_n^R(\bigoplus A_j, B) = \bigoplus \mathrm{Tor}_n^R(A_j, B)$

- $\mathrm{Tor}_0^R(A,B) = A \otimes_R B$

- For an exact sequence

$$0 \to A' \to A \to A'' \to 0$$

  get a long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(A',B) \to \mathrm{Tor}_1^R(A,B) \to \mathrm{Tor}_1^R(A'',B) \to$$

"Repair" $\leftarrow$

$$\to \mathrm{Tor}_0^R(A',B) \to \mathrm{Tor}_0^R(A,B) \to \mathrm{Tor}_0^R(A'',B) \to 0$$

$$\begin{array}{ccc} \| & \| & \| \\ A'\otimes B & A\otimes B & A''\otimes B \end{array}$$

- Same for RHS

given $0 \to B' \to B \to B'' \to 0$

... LES for $\mathrm{Tor}^R(A, -)$

- $\mathrm{Tor}_n^R(A, R) = \mathrm{Tor}_n^R(R, B) = 0$ for $n > 0$.

(same is true if R is replaced by a free/projective R-module)

Specific R

- If R is a field, $\mathrm{Tor}_n^R(A, B) = 0$ for $n > 0$

(Tensor product over fields does preserve exact sequences. A, B F-modules $\Rightarrow$ vector spaces

$$\begin{array}{ll} A & \text{basis } \{a_i\} \\ B & \text{basis } \{b_j\} \end{array} \Rightarrow A \underset{F}{\otimes} B \text{ basis } \{a_i \otimes b_j\})$$

- Over $\mathbb{Z}$

have $\mathrm{Tor}_n^R(A, B) = 0$ for $n > 1$

i.e. we get an exact sequence

$$0 \to A' * B \to A * B \to A'' * B \to A' \otimes B \to$$

$$A \otimes B \to A'' \otimes B \to 0 \quad \text{where } * = \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

<u>And</u> $A * B \cong B * A$

How do we define Tor?

<u>Free resolution</u>

Given a left R-module N, a <u>free resolution</u> is an exact sequence

$$\cdots \to F_3 \to F_2 \to F_1 \to N \to 0$$ such that

each F_i is free.

<u>Ex</u> Over $\mathbb{Z}$

free resolution of

$$\mathbb{Z}: \ \to 0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

$\mathbb{Z}$ is free itself

$$\mathbb{Z}/n: \ \to 0 \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

Given such a resolution of N, get a chain complex

$$\cdots \to M \otimes_R F_2 \to M \otimes_R F_1 \to M \otimes_R F_0 \to$$

(If $g \circ f = 0$, $(1 \otimes g) \circ (1 \otimes f) = 1 \otimes (g \circ f)$
$= 1 \otimes 0 = 0$)

Tor groups are the homology groups of this chain complex

- Check independent of choice of resolution etc. etc.

Over $\mathbb{Z}$

Given N, we can always find a free module

Why $\mathbb{Z}$ special?

F with surjection $F \to N$. Kernel is a submodule of a free module $\Rightarrow$ is free.

get an exact sequence

$$\cdots \to 0 \to 0 \to 0 \to R \to F \to N \to 0$$

get a chain complex

$$\cdots \to 0 \to M \otimes R \to M \otimes F \to 0$$

$\mathrm{Tor}_1 = \ker(M \otimes R \to M \otimes F)$

Tensor product $= \mathrm{coker}(M \otimes R \to M \otimes F)$

all higher Tor-groups are 0.

If you have a short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

resolve each

$$\begin{tikzcd}
0 \arrow[r] & F_0' \arrow[r] \arrow[d] & F_0' \oplus F_0'' \arrow[r] \arrow[d, dashed, "\exists\ \text{surjection}"] & F_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

How to construct the surjection?

Have map

$$\begin{tikzcd}
 & F_0'' \arrow[dl, dashed, "i"'] \arrow[d] \\
N \arrow[r] & N''
\end{tikzcd}$$

lift every image of a generator of F_0'', get a map $F_0'' \to N$, get

$$F_0' \oplus F_0'' \to N$$

$$(a, b) \mapsto (a + i(b))$$

Claim: surjective (diagram chase)

$$\begin{tikzcd}
0 \arrow[r] & \overbrace{F_1' \arrow[r] \arrow[d] & F_1 \arrow[r] & F_1''}^{\text{kernels}} \arrow[r] & 0 \\
0 \arrow[r] & F_0' \arrow[r] \arrow[d] & F_0' \oplus F_0'' \arrow[r] \arrow[d] & F_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

$\to$ exact by snake lemma

i.e. have a short exact sequence of chain complexes because they've all free, stays exact when we tensor with M

Get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \otimes F_1' \arrow[r] \arrow[d] & M \otimes F_1 \arrow[r] \arrow[d] & M \otimes F_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes F_0' \arrow[r] \arrow[d] & M \otimes F_0 \arrow[r] \arrow[d] & M \otimes F_0'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\leadsto$ Tor long exact sequence.