Supplement: Proof of algebraic Künneth formula

(compare proof of UCT)

Suppose C_*, D_* are chain complexes, and all D_n are free abelian groups. Have exact sequences

$$0 \to Z_n \to D_n \xrightarrow{\partial} B_{n-1} \to 0$$

which assemble into a SES of chain complexes

$$0 \to Z_* \to D_* \to B_* \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"'] & D_n \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & D_{n-1} \arrow[r, "\partial"] & B_{n-2} \arrow[r] & 0
\end{tikzcd}$$

Tensor with C_*. Levelwise, we get

$$0 \to \bigoplus_{p+q=n} C_p \otimes Z_q \to \bigoplus_{p+q=n} C_p \otimes D_q \to \bigoplus_{p+q=n} C_p \otimes B_{q-1} \to 0$$

still exact (C_p is free, so Tor term is zero).

so $0 \to C_* \otimes Z_* \to C_* \otimes D_* \to C_* \otimes B_* \to 0$ is exact

Get a LES on H_*

$$\cdots \longrightarrow H_n(C_* \otimes Z_*) \longrightarrow H_n(C_* \otimes D_*) \longrightarrow H_n(C_* \otimes B_*)$$

$$\longrightarrow H_{n-1}(C_* \otimes Z_*) \longrightarrow \cdots$$

Need to compute homology of $C_* \otimes Z_*$, $C_* \otimes B_*$

$$(C_* \otimes Z_*)_n = \bigoplus_{p+q=n} C_p \otimes Z_q$$

$$\partial\left(\sum \alpha_i \otimes \beta_i\right) = \sum \left(\partial\alpha_i \otimes \beta_i + (-1)^{\deg \alpha_i} \alpha_i \otimes \underbrace{\partial\beta_i}_{\substack{\partial \text{ map is} \\ \text{zero on } Z_*}}\right)$$

$$= \sum \partial\alpha_i \otimes \beta_i$$

$$\partial : C_p \otimes Z_q \longrightarrow C_{p-1} \otimes Z_q$$

A <u>cycle</u> is a sequence of elements (α_p)

$\alpha_p \in C_p \otimes Z_q$ with $(\partial \otimes 1)\alpha_p = 0$ in $C_{p-1} \otimes Z_q$

A <u>boundary</u> is a sequence of elements $((\partial \otimes 1)\beta_{p+1})$

$\beta_{p+1} \in C_{p+1} \otimes Z_q$

Homology = cycles / boundaries

$$= \bigoplus_{p+q=n} \frac{\ker\left(C_p \otimes Z_q \xrightarrow{\partial \otimes 1} C_{p-1} \otimes Z_q\right)}{\operatorname{im}\left(C_{p+1} \otimes Z_q \xrightarrow{\partial \otimes 1} C_p \otimes Z_q\right)}$$

$C_* = C_*(X)$. Then this is $H_*(X; Z_q)$

Note: Because Z_q is free, this homology is

$$\bigoplus_{p+q=n} H_p(C_*) \otimes Z_q$$

Similarly, $H_n(C_* \otimes B_*) = \bigoplus_{p+q=n} H_p(C_*) \otimes B_{q-1}$

LES becomes

$$\cdots \to \bigoplus_{p+q=n+1} H_p(C_*) \otimes B_{q-1} \to \bigoplus_{p+q=n} H_p(C_*) \otimes Z_q$$

$$\to H_n(C_* \otimes D_*)$$

$$\to \bigoplus_{p+q=n} H_p(C_*) \otimes B_{q-1} \to \bigoplus_{p+q=n-1} H_p(C_*) \otimes Z_q \to \cdots$$

As a result, there is an exact sequence

$$0 \to \bigoplus_{p+q=n} \operatorname{coker}(H_p(C_*) \otimes B_q \to H_p(C_*) \otimes Z_q)$$

$$\to H_n(C_* \otimes D_*)$$

$$\to \bigoplus_{p+q=n-1} \ker(H_p(C_*) \otimes B_q \to H_p(C_*) \otimes Z_q) \to 0$$

Can check the maps $\bigoplus_{p+q=n} H_p(C_*) \otimes B_q \to \bigoplus_{p+q=n} H_p(C_*) \otimes Z_q$ are induced by natural map $B_q \hookrightarrow Z_q$

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes \underbrace{(Z_q / B_q)}_{H_q(D_*)} \to H_n(C_* \otimes D_*)$$

$$\to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(D_*)) \to 0$$

To show this exact sequence splits as a direct sum, we have to do the same as for the UCT:

have to split map

$$D_n \to B_{n-1} \to 0$$

$$D_n \cong Z_n \oplus B_{n-1}$$

$\Rightarrow$ final result:

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y))$$