Grassmannians

<u>Def</u> The Grassmannian $Gr(k,n)$ of k-dimensional planes in $\mathbb{R}^n$ is given as follows.

Let $F(k,n)$ be the space of <u>k-frames</u> in $\mathbb{R}^n$:

$$\{(\vec{v}_1, \dots, \vec{v}_k) \mid \vec{v}_i \in \mathbb{R}^n,\ \vec{v}_i \text{ are linearly independent}\}.$$

Two points $(\vec{v}_1, \dots, \vec{v}_k)$, $(\vec{w}_1, \dots, \vec{w}_k)$ are <u>equivalent</u> if they have the same <u>span</u>

<u>OR</u> each $\vec{w}_i$ is a linear combination of $\vec{v}_j$

<u>OR</u> vice versa

<u>OR</u> If we put these vectors in a $k \times n$ matrix

$$\begin{pmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_k \end{pmatrix} \qquad \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_k \end{pmatrix}$$

these two matrices differ only by some sequence of elementary row operations.

Use quotient topology

set of equivalence classes is $Gr(k,n)$.

It's a <u>manifold</u> of dimension $k(n-k)$

$$V \in U_V := \{ W \in Gr(k,n) \mid W \cap V^\perp = \{0\} \}$$

$$U_V \longrightarrow \mathrm{Hom}(V, V^\perp)$$

$$W \mapsto \left(V \xrightarrow{(pr^W_V)^{-1}} W \xrightarrow{pr^W_{V^\perp}} V^\perp \right)$$

Can we "understand" these?

$Gr(1,n) \cong \mathbb{R}P^{n-1}$

$Gr(n-1,n) \cong \mathbb{R}P^{n-1}$ $\quad V \longmapsto V^\perp$

$(n-1)$-dim plane

(In general, this gives a homeomorphism

$$Gr(k,n) \cong Gr(n-k,n) \quad)$$

Smallest <u>new</u> example is

$Gr(2,4)$ which is 4-dimensional.

There's an analogue of cell decomposition of $\mathbb{R}P^n$. Cells are called <u>Schubert cells</u>. Make $Gr(k,n)$ into a CW complex.

Method

Gaussian elimination (!)

$k \times n$ matrix with linearly independent rows

$\hookrightarrow$ new matrix with k pivots (somewhere)

"strict upper triangular form"

$Gr(2,4)$

There are the following possible reduced row echelon forms

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & \boxed{a} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & \boxed{a} \\ 0 & 0 & 1 & \boxed{b} \end{bmatrix}$$

point

$$\cong \mathbb{R} \hookrightarrow D^1$$

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & D^1 \\ \wr\!\wr & & \| \\ (-1,1) & & [-1,1] \end{array}$$

$$\cong \mathbb{R}^2 \hookrightarrow D^2$$

$$\begin{bmatrix} 1 & \boxed{a} & \boxed{b} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{a} & 0 & \boxed{b} \\ 0 & 0 & 1 & \boxed{c} \end{bmatrix}$$

$$\cong \mathbb{R}^2 \hookrightarrow D^2 \qquad \cong \mathbb{R}^3 \hookrightarrow D^3$$

$$\begin{bmatrix} 1 & 0 & \boxed{a} & \boxed{b} \\ 0 & 1 & \boxed{c} & \boxed{d} \end{bmatrix} \cong \mathbb{R}^4 \hookrightarrow D^4$$

Each RRE form is uniquely associated to a point of the Grassmannian.

Turns out that each of these extends to an attaching map for a disc.

e.g. $\begin{bmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ as $a \longrightarrow \pm\infty$

RRE becomes $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(rescale)

Extends to a CW-complex structure on Gr (exe).

In this case, have a CW complex with

one 0-cell

one 1-cell

two 2-cells,

one 3-cell

one 4-cell

$H_* = ?$ boundary map is difficult.

With $\mathbb{Z}/2$-coefficients, all ∂ maps in $C_*^{CW}(Gr(k,n); \mathbb{Z}/2)$ are 0

$\Rightarrow$ H_* in each degree $= (\mathbb{Z}/2)^m$,

$m = \#$ of Schubert cells of dim $*$.

<u>Over $\mathbb{C}$?</u>

$Gr_{\mathbb{C}}(k,n)$ complex manifold of dim $2k(n-k)$

or $k(n-k)$ (cpx dim)

<u>Same</u> RRE story

Same set of cells, <u>except</u> all $\mathbb{R}^l$ are replaced by $\mathbb{C}^l$

$Gr_{\mathbb{C}}(2,4)$

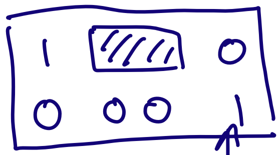
one 0-cell
one 2-cell
two 4-cells
one 6-cell
one 8-cell

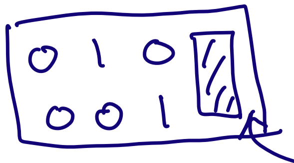
In general, $Gr_{\mathbb{C}}(k,n)$ <u>only</u> has cells in even dims. $C_*^{cw}(Gr_{\mathbb{C}}(k,n))$ has all ∂ maps equal to 0.

$H_l = \mathbb{Z}^{\# \text{ of cells of dim } l}$

How many RRE forms have ℓ free variables?

$Gr(2,4)$ how many ways to have 2 free variables?

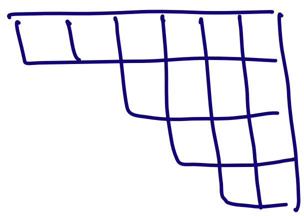




Free variables are

- in columns having no pivots, and
- in rows above all $\underbrace{\text{pivots to the right}}_{\text{can be none}}$

Turns out: # of RRE forms with ℓ free variables

= # of diagrams 

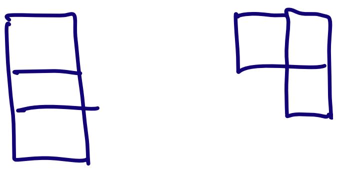
(each row is longer than next or if equal length, pushed to right, with ℓ total boxes, height $\leq k$, width $\leq n-k$)

$n=5, \ k=3, \ \ell = 3$

3 boxes

height ≤ 3

width ≤ 2



2 possibilities

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \boxed{} \\ 0 & 0 & 1 & 0 & \boxed{} \\ 0 & 0 & 0 & 1 & \boxed{} \end{bmatrix} \quad \begin{bmatrix} 1 & \boxed{} & 0 & \boxed{} & 0 \\ 0 & 0 & 1 & \boxed{} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Diagrams are called <u>Young diagrams</u>.