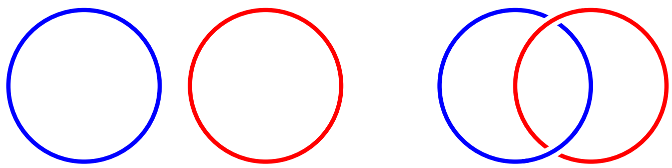
MAT8021, Algebraic Topology

Assignment 8



Two links in $\mathbb{R}^3$

1. The left-hand portion of the picture is a union L_1 of two disconnected circles in $\mathbb{R}^3$. Show that the complement $X = \mathbb{R}^3 \setminus L_1$ retracts down onto $S^2 \vee S^2 \vee S^1 \vee S^1$. Use this to show that the cup product of any two elements in $H^1(X)$ is zero.

2. The right-hand portion of the picture is a link L_2 in $\mathbb{R}^3$. Show that the complement $Y = \mathbb{R}^3 \setminus L_2$ has the same cohomology as the space X from the previous problem. (Possible hint: Show that the space retracts down onto something gotten by gluing two tori together along $S^1 \vee S^1$. Don't appeal to any major duality theorems like Alexander duality.)

3. Show that the cup product of the two generators in $H^1(Y)$ is nonzero. (Possible hint: Compare it with a torus.)

4. Given a map $f\colon X \to Y$, $b \in H^p(Y)$, and $x \in H_n(X)$, show that

$$f_*\big(f^*(b) \frown x\big) = b \frown f_*(x)$$

(This formula goes by many names: the "projection formula," or "Frobenius reciprocity." The special case when $p = n$ gives $\langle f^*b, x\rangle = \langle b, f_*x\rangle$ for the *Kronecker pairing* $\langle -, -\rangle\colon H^p(X;R) \otimes H_p(X;R) \to R$ induced by the evaluation map.)

5. Let I be a directed set, L an abelian group, and $A\colon I \to \mathbf{Ab}$ an I-directed diagram of abelian groups, with bonding maps $f_{ij} : A_i \to A_j$ for $i \le j$. Show that a map $A \to c_L$, the constant functor at L, given by compatible maps $f_i\colon A_i \to L$, is a direct limit if and only if

   (a) for any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and

   (b) for any $a_i \in A_i$ such that $f_i a_i = 0 \in L$, there exists $j \ge i$ such that $f_{ij} a_i = 0 \in A_j$.

6. (a) Embed $\mathbb{Z}/p^n$ into $\mathbb{Z}/p^{n+1}$ by sending 1 to p, and write $\mathbb{Z}_{p^\infty}$ for the union. It is called the *Prüfer group* (at p). Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$ and that

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$$

where the sum runs over the prime numbers.

   (b) Compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A$ for A each of the following abelian groups: $\mathbb{Z}/n$, $\mathbb{Z}[1/q]$ (for q a prime), and $\mathbb{Z}_{q^\infty}$ (for q a prime).

   (c) Compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p])$ and $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$ for any abelian group M in terms of the self-map $p\colon M \to M$.

7. Show that if $f\colon X \to Y$ induces an isomorphism in homology with coefficients in the prime fields $\mathbb{F}_p$ (for all primes p) and $\mathbb{Q}$, then it induces an isomorphism in homology with coefficients in $\mathbb{Z}$.

8. Suppose X is a path-connected (based) space, M is a compact orientable manifold, and $f : S^1 \wedge X \to M$ is a map inducing an isomorphism on homology with integer coefficients. Show that X has the same homology as a sphere S^n. Hint: Besides Poincaré duality, need to use the fact that cup product is 0 on $S^1 \wedge X$ (recall relative cup products).

9. Suppose M is a compact oriented $4n$-dimensional manifold with $H^{2n}(M;\mathbb{Z})$ torsion free. Poincaré duality gives us a pairing

$$x, y \mapsto x \cdot y\colon H^{2n}(M;\mathbb{Z}) \times H^{2n}(M;\mathbb{Z}) \to \mathbb{Z}$$

which is distributive and satisfies $x \cdot y = y \cdot x$. If $e_1, \dots, e_g$ are a basis of $H^{2n}(M;\mathbb{Z})$, there is a symmetric matrix $A = (a_{ij})$ such that $e_i \cdot e_j = a_{ij}$.

If we choose a different basis $f_k = \sum_i c_{ki} e_i$, we get a different matrix B. Express B in terms of A using matrix multiplication.

10. Suppose M and N are n-dimensional compact manifolds with orientations $[M] \in H_n(M;\mathbb{Z})$ and $[N] \in H_n(N;\mathbb{Z})$. We define the *degree* of a map $f\colon M \to N$ to be the unique integer a such that $f_*([M]) = a[N]$.

Show that the degree of a map $\mathbb{CP}^2 \to \mathbb{CP}^2$ is always a square.

11. One statement of Poincaré duality for manifolds with boundary says: If M is a compact manifold with boundary ∂M, there are isomorphisms

$$D\colon H^p(M;\mathbb{Z}/2) \to H_{n-p}(M, \partial M; \mathbb{Z}/2)$$

(M is not necessarily orientable).

Use this to show that there is no compact 3-dimensional manifold W with boundary $\partial W = \mathbb{RP}^2$.