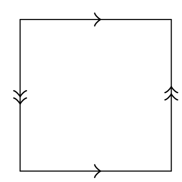
MAT8021, Algebraic Topology

Assignment 7

Due in-class on Friday, May 23

1. Let X be a Klein bottle:

   We can put a Δ-complex structure on X with one vertex p, three edges a, b, c, and two 2-simplices u, v. Make this Δ-complex structure explicit, and use it to compute $H^*(X;\mathbb{Z}/2)$ together with the cup product on it.

2. In Hatcher, page 131, exercise 8, there is given a description of a *lens space* formed by gluing together n tetrahedra; let's call this $L(n,1)$. (The 1 is because we are gluing the bottom face of T_i to the top face of T_{i+1}.) Compute $H^*\big(L(n,1);\mathbb{Z}/n\big)$ together with the cup product on it.

3. We know that if X and Y are based spaces, the wedge $X \vee Y$ has

   $$H^k(X \vee Y; R) = H^k(X;R) \oplus H^k(Y;R)$$

   for any $k > 0$. Show that under this identification, the cup product is given by

   $$(\alpha,\beta) \smile (\alpha',\beta') = (\alpha \smile \alpha', \beta \smile \beta')$$

For the remaining questions, all chain complexes are over $\mathbb{Z}/2$, i.e., $2x = 0$ for all x.

A cochain complex C^* has *cup-i products* if it is equipped with operations $(x,y) \mapsto x \smile_i y$ for $i \geq 0$ such that

- if $x \in C^p$, $y \in C^q$, then $x \smile_i y \in C^{p+q-i}$
- $(x+x') \smile_i y = x \smile_i y + x' \smile_i y$ and similarly $x \smile_i (y+y') = x \smile_i y + x \smile_i y'$
- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$
- for $i > 0$,

$$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y) + x \smile_{i-1} y + y \smile_{i-1} x$$

For instance, one can show (using the method of acyclic models) that $C^*(X)$, for X a space, naturally comes equipped with cup-i products, each one expressing "how noncommutative" the previous one was.

4. Show that for all $j \leq p$ we get a well-defined "squaring" operation $\mathrm{Sq}^j : H^p(C^*) \to H^{p+j}(C^*)$ given by

$$\mathrm{Sq}^j[x] = [x \smile_{p-j} x]$$

such that $\mathrm{Sq}^j([x+y]) = \mathrm{Sq}^j([x]) + \mathrm{Sq}^j([y])$. (In the cohomology of a space, these are called the Steenrod squares.)

5. If $f : C^* \to D^*$ is a map of cochain complexes such that $f(x \smile_i y) = f(x) \smile_i f(y)$, show that the induced map $H^*(C^*) \to H^*(D^*)$ preserves the squaring operations.

6. If $0 \to C^* \to D^* \to E^* \to 0$ is a short exact sequence of cochain complexes preserving cup-i products, show that the connecting homomorphism

$$\delta\colon H^p(E^*) \to H^{p+1}(C^*)$$

satisfies $\delta(\mathrm{Sq}^j[x]) = \mathrm{Sq}^j(\delta[x])$.