MAT8021, Algebraic Topology

Assignment 6

Due in-class on Friday, May 9

Only hand in the first five questions.

1. Find all $(2,3)$-shuffles α and give formulas for the associated shuffle maps $f_\alpha \colon \Delta[5] \to \Delta[2] \times \Delta[3]$.
2. Find recursive formulas for $\dim_{\mathbb{Z}/2} H_k\big((\mathbb{RP}^2)^n; \mathbb{Z}/2\big)$ in terms of k and n.
3. Find a pair of chain complexes C_* and D_* such that the tensor product chain complex $C_* \otimes D_*$ does not satisfy the Künneth formula, i.e., there is some n such that

$$H_n(C_* \otimes D_*) \ncong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \;\oplus \bigoplus_{p+q=n-1} \operatorname{Tor}\big(H_p(C_*), H_q(D_*)\big)$$

4. Find the homology of the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(3,5)$.
5. There is a continuous map from one Grassmannian $\mathrm{Gr}(k,n)$ to the next $\mathrm{Gr}(k,n+1)$ by sending a plane $V \subset \mathbb{R}^n$ to the plane

$$\{(0, x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in V\}$$

   Show that the image consists of a union of Schubert cells, and find the dimension of the smallest cell not in the image.

Here is a series of exercises/examples on product structure in homology.

1. A *differential graded algebra* is a chain complex A_* with associative multiplication maps $\cdot \colon A_p \times A_q \to A_{p+q}$ satisfying the Leibniz rule

$$\partial(x \cdot y) = (\partial x) \cdot y + (-1)^p x \cdot (\partial y)$$

   for $x \in A_p, y \in A_q$.

   Show that given elements $[x] \in H_p(A)$ and $[y] \in H_q(A)$, we get a well-defined element $[x] \cdot [y]$ in $H_{p+q}(A)$. Show that this makes $H_*(A)$ into a graded ring.

2. Recall that a *topological group* G is a space with continuous maps

$$\begin{array}{ll} \mu\colon G\times G\to G & \text{multiplication} \\ \nu\colon G\to G & \text{inverse} \\ \iota\colon \{*\}\to G & \text{identity} \end{array}$$

so that on the underlying set, we get a group with $g\cdot h=\mu(g,h)$, $g^{-1}=\nu(g)$, and $e=\iota(*)$.

   (a) Show that $H_*(G)$ is a ring by defining a multiplication on $C_*(G)$. This is called a *Pontryagin ring* structure on $H_*(G)$.

   (b) If G is abelian, show that $C_*(G)$ (and hence $H_*(G)$) is *graded commutative*, i.e., $x\cdot y=(-1)^{|x||y|}y\cdot x$ for any $x,y\in C_*(G)$, where $|x|$ and $|y|$ denote the degrees of x and y respectively.

3. (a) Let $G=\mathbb{R}$. What is the Pontryagin ring structure on $H_*(\mathbb{R})$?

   (b) Show that $H_*(S^1)$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2)$ with $|\alpha|=1$.

   (c) More generally, it turns out that

$$H_*(S^1\times S^1)\cong\mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2,\alpha\beta+\beta\alpha)=:\Lambda[\alpha,\beta]$$

   is an *exterior algebra* on α,β with $|\alpha|=|\beta|=1$. Similarly $H_*(S^1\times S^1\times S^1)\cong\Lambda[\alpha,\beta,\gamma]$, etc. In contrast, if $G=S^3$ regarded as the unit quaternions, what is the Pontryagin ring structure on $H_*(S^3)$?

4. Let $G=\mathrm{SO}(3)$ be the 3×3 matrices over $\mathbb{R}$ with determinant 1.

   (a) Viewing it as the group of rotations in $\mathbb{R}^3$, describe a homeomorphism $\mathrm{SO}(3)\cong\mathbb{RP}^3$ by defining a map $D^3\to\mathrm{SO}(3)$ that factors through $\mathbb{RP}^3$.

   (b) Give a presentation for $H_*\big(\mathrm{SO}(3)\big)$ as a ring.

   (c) What about $H_*\big(\mathrm{SO}(3);\mathbb{Z}/2\big)$? In particular, show that the square of the generator of $H_1\big(\mathrm{SO}(3);\mathbb{Z}/2\big)$ equals zero.

5. Suppose G is a topological group and X is a topological space with a continuous map $G\times X\to X$ which is an action of G. Show that $H_*(X)$ becomes a left module over the Pontryagin ring $H_*(G)$.

Here is a series of exercises on *intersection homology* (Greg Friedman's book is recommended for further reading).[1] Let X be a simplicial complex.

- A *filtration* of X is a sequence of subcomplexes of X:

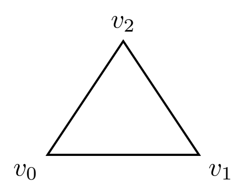
$$X=X^n\supseteq X^{n-1}\supseteq\cdots\supseteq X^2\supseteq X^1\supseteq X^0\supseteq X^{-1}=\emptyset$$

  Each connected component of X^i-X^{i-1} is called a *stratum*.

[1] Thanks to Zhou Fang for supplying it.

- Let $\mathcal{F}$ be the set of strata of X. A *perversity* on X is a function $\bar{p}\colon \mathcal{F} \to \mathbb{Z}$ such that $\bar{p}(S) = 0$ if $S \subset X^n - X^{n-1}$.
- An i-simplex σ is said to be *$\bar{p}$-allowable* if $\dim(\sigma \cap S) \leq \dim(\sigma) + \dim(S) - n + \bar{p}(S)$ for every stratum S of X.
- An i-chain ζ is said to be *$\bar{p}$-allowable* if every simplex of ζ and of $\partial\zeta$ is $\bar{p}$-allowable.
- Define the group $I^{\bar{p}}C_i(X)$ to be the subset of $C_i(X)$ consisting of $\bar{p}$-allowable i-chains. It can be shown that the chain complex $\big(C_*(X), \partial\big)$ restricts to a chain complex $\big(I^{\bar{p}}C_*(X), \partial\big)$. The *Goresky–MacPherson intersection homology groups* are defined to be $I^{\bar{p}}H_i(X) := H_i\big(I^{\bar{p}}C_*(X)\big)$.

Now, let X be the boundary of the simplex $[v_0, v_1, v_2]$. Suppose that X is filtered as $\{v_2\} = X^0 \subset X^1 = X$.



1. Compute the intersection homology of this stratified space. (Hint: Consider the three cases of $\bar{p}(\{v_2\}) > 0$, $\bar{p}(\{v_2\}) = 0$, and $\bar{p}(\{v_2\}) < 0$.)
2. Compute the intersection homology of $S^1 \vee S^1$ with a stratification by its singular point similar to the above.
3. Is the intersection homology defined above independent of choice of a filtration? Give a proof or a counterexample.

The following illustrates the basic idea of *persistent homology* by a toy example. Suppose that we are given 3 data points in a Euclidean plane at the vertices of an equilateral triangle as above. For $t > 0$, draw a disc of radius t around each point. Construct the Čech complex X_t as a simplicial complex with

- a 0-simplex for each data point,
- a 1-simplex connecting each pair of 0-simplices whose associated discs intersect, and
- a 2-simplex for each triple of 0-simplices whose associated discs have a triple overlap (etc., inductively).

Assuming that the edge length of the triangle equals 1, compute the persistent homology $\{H_n(X_t)\}_{t>0}$. Illustrate your results by drawing the barcodes for each n: each bar corresponds to a generator of the homology group and goes along the t-axis, with its left end at the time when it is born and its right end when it dies. How would you interpret them?