Given a topological space X and $k \in \mathbb{Z}_{\geq 0}$, we can associate groups $H_k(X;R)$ and $H^k(X;R)$, the kth *homology* and *cohomology groups* (with coefficients in R), where R is a commutative ring such as $\mathbb{Z}$ or $\mathbb{Q}$. These algebraic invariants define functors from the category of topological spaces to the category of R-modules: for any continuous map of topological spaces $f : X \to Y$ there are induced R-linear maps

$$f_* : H_k(X;R) \to H_k(Y;R) \quad \text{(covariant)},$$

$$f^* : H^k(Y;R) \to H^k(X;R) \quad \text{(contravariant)}.$$

The cohomology groups $H^*(X;R) = \bigoplus_k H^k(X;R)$ in fact have the structure of a graded R-algebra with respect to the *cup product* operation.

The group $H_0(X;\mathbb{Z})$ is the free abelian group on the path components of the topological space X and $H^0(X;\mathbb{Z})$ is its dual. If X is path-connected, $H_1(X;\mathbb{Z})$ is naturally isomorphic to the abelianization of $\pi_1(X,x_0)$ with respect to any basepoint x_0, and its elements are certain equivalence classes of (unbased) loops in X.

For a topological group G there exists an associated *classifying space* BG for *principal G-bundles*. It is constructed as the quotient of a (weakly) contractible space EG by a proper free action of G. The space BG is unique up to (weak) homotopy equivalence. If G is a discrete group, then BG is precisely an *Eilenberg-MacLane space* $K(G,1)$, i.e., a path-connected topological space with $\pi_1(\mathrm{B}G) \cong G$ and trivial higher homotopy groups. For example, up to homotopy equivalence, $\mathrm{B}\mathbb{Z}$ is the circle, $\mathrm{B}\mathbb{Z}_2$ is the infinite-dimensional real projective space $\mathbb{RP}^\infty$, and the Grassmanian of d-dimensional linear subspaces in $\mathbb{R}^\infty$ is $\mathrm{BGL}_d(\mathbb{R})$.

Some motivation to study the cohomology of BG: its cohomology classes define *characteristic classes* of principal G-bundles, invariants that measure the 'twistedness' of the bundle. For instance the cohomology algebra $H^*(\mathrm{BGL}_d(\mathbb{R});\mathbb{Z})$ can be described in terms of Pontryagin and Stiefel–Whitney classes.

With BG we can define the *group homology* and *group cohomology* of a discrete group G by

$$H_k(G;R) \coloneqq H_k(\mathrm{B}G;R), \quad H^k(G;R) \coloneqq H^k(\mathrm{B}G;R).$$

We can refine Question 1.1 to the following:

Question 1.2. Given family $\{X_n\}_n$ of moduli spaces or discrete groups, how do the homology and cohomology groups of the nth space in the sequence change as the parameter n increases?

In this article we discuss Question 1.2 with a particular focus on the families of configuration spaces and braid groups. For further reading[1] we recommend R. Cohen's survey [Coh09] on stability of moduli spaces.

[1] *A version of this note with an extended reference list is available at* `https://arxiv.org/abs/2201.04096`.

1.2. Homological stability.

Definition 1.3. A sequence of spaces or groups $\{X_n\}_{n\geq 0}$ with maps

$$X_0 \xrightarrow{s_0} \dots \xrightarrow{s_{n-2}} X_{n-1} \xrightarrow{s_{n-1}} X_n \xrightarrow{s_n} X_{n+1} \xrightarrow{s_{n+1}} \dots$$

satisfies *homological stability* if, for each k, the induced map in degree-k homology

$$(s_n)_* : H_k(X_n;\mathbb{Z}) \to H_k(X_{n+1};\mathbb{Z})$$

is an isomorphism for all $n \geq N_k$ for some stability threshold $N_k \in \mathbb{Z}$ depending on k. The maps s_n are sometimes called *stabilization maps* and the set $\{(n,k) \in \mathbb{Z}^2 \mid n \geq N_k\}$ is the *stable range*.

If the maps $s_n : X_n \to X_{n+1}$ are inclusions we define $X_\infty := \bigcup_{n\geq 1} X_n$ to be the *stable* group or space. Under mild assumptions, if $\{X_n\}_n$ satisfies homological stability, then

$$H_k(X_\infty;\mathbb{Z}) \cong H_k(X_n;\mathbb{Z}) \quad \text{for} \quad n \geq N_k.$$

We call the groups $H_k(X_\infty;\mathbb{Z})$ the *stable homology*.

2. An Example: Configuration Spaces and the Braid Groups

2.1. A primer on configuration spaces.

Definition 2.1. Let M be a topological space, such as a graph or a manifold. The *(ordered) configuration space $F_n(M)$ of n particles on M* is the space

$$F_n(M) = \{(x_1,\dots,x_n) \in M^n \mid x_1,\dots,x_n \text{ distinct}\},$$

topologized as a subspace of M^n. Notably, $F_0(M)$ is a point and $F_1(M) = M$.

Configuration spaces have a long history of study in connection to topics as broad-ranging as homotopy groups of spheres and robotic motion planning.

One way to conceptualize the configuration space $F_n(M)$ is as the complement of the union of subspaces of M^n defined by equations of the form $x_i = x_j$.

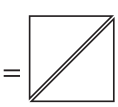
$F_2([0,1]) =$ 

Figure 1. The space $F_2([0,1])$ is obtained by deleting the diagonal from the square $[0,1]^2$.

In other words, we can construct $F_n(M)$ by deleting the "fat diagonal" of M^n, consisting of all n-tuples in M^n where two or more components coincide. In the simplest case, when $n = 2$ and M is the interval $[0,1]$, we see that $F_2([0,1])$ consists of two contractible components, as in Figure 1.

Another way we can conceptualize $F_n(M)$ is as the space of embeddings of the discrete set $\{1,2,\dots,n\}$ into M, appropriately topologized. We may visualize a point in $F_n(M)$ by labelling n points in M, as in Figure 2.

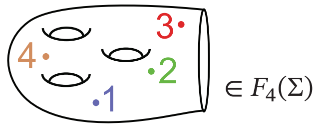


Figure 2. A point in the ordered configuration space of an open surface Σ.

From this perspective, we may reinterpret the path components of $F_2([0,1])$: one component consists of all configurations where particle 1 is to the left of particle 2, and one component has particle 1 on the right. See Figure 3.

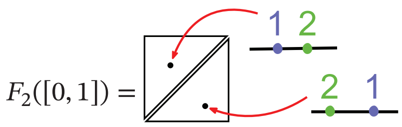


Figure 3. The path components of $F_2([0,1])$.

Any path through $[0,1]^2$ that interchanges the relative positions of the two particles must involve a 'collision' of particles, and hence exit the configuration space $F_2([0,1]) \subseteq [0,1]^2$. We encourage the reader to verify that, in general, the configuration space $F_n([0,1])$ is the union of $n!$ contractible path components, indexed by elements of the symmetric group S_n. See Figure 4.

Figure 4. A point in $F_4([0,1])$ in the path component indexed by the permutation 2143 in S_4.

In contrast, if M is a connected manifold of dimension 2 or more, then $F_n(M)$ is path-connected: given any two configurations, we can construct a path through M^n from one configuration to the other without any 'collisions' of particles. In this case $H_0(F_n(M);\mathbb{Z}) \cong \mathbb{Z}$ for all $n \geq 0$, and this is our first glimpse of stability in these spaces as $n \to \infty$.

For any space M, the symmetric group S_n acts freely on $F_n(M)$ by permuting the coordinates of an n-tuple $(x_1, \dots, x_n)$, equivalently, by permuting the labels on a configuration as in Figure 2. The orbit space $C_n(M) = F_n(M)/S_n$ is the *(unordered) configuration space of n particles on M*. This is the space of all n-element subsets of M, topologized as the quotient of $F_n(M)$. The reader may verify that the quotient map (illustrated in Figure 5) is a regular S_n-covering space map. In particular, by covering space theory, the quotient map $F_n(M) \to C_n(M)$ induces an injective map on fundamental groups.

In the case that M is the complex plane $\mathbb{C}$, we can identify $C_n(\mathbb{C})$ with the space of monic degree-n polynomials over $\mathbb{C}$ with distinct roots, by mapping a configuration $\{z_1, \dots, z_n\}$ to the polynomial $p(x) = (x - z_1)\cdots(x - z_n)$. For this reason the topology of $C_n(\mathbb{C})$ has deep connections to classical problems about finding roots of polynomials.

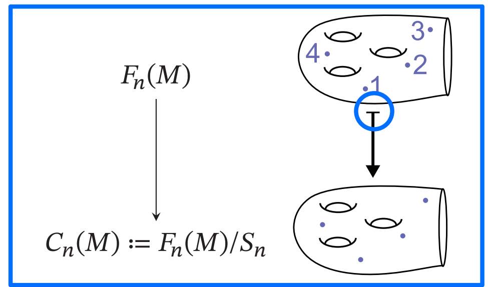


Figure 5. The quotient map $F_n(M) \to C_n(M)$.

We will address Question 1.2 for the families $\{C_n(M)\}_n$ and $\{F_n(M)\}_n$, but we first specialize to the case when $M = \mathbb{C}$. Although the spaces $C_n(\mathbb{C})$ and $F_n(\mathbb{C})$ are path-connected, in contrast to the configuration spaces of $M = [0,1]$, they have rich topological structures: they are classifying spaces for the braid groups and the pure braid groups, respectively, which we now introduce. $\pi_1(C_n(\mathbb{C})) = \mathbf{B}_n$

2.2. A primer on the braid groups. Since $F_n(\mathbb{C})$ is path-connected, as an abstract group its fundamental group is independent of choice of basepoint. For path-connected spaces, we sometimes drop the basepoint from the notation for π_1.

Definition 2.2. The fundamental group $\pi_1(C_n(\mathbb{C}))$ is called the *braid group* $\mathbf{B}_n$ and $\pi_1(F_n(\mathbb{C}))$ is the *pure braid group* $\mathbf{P}_n$.

We can understand $\pi_1(F_n(\mathbb{C}))$ as follows. Choose a basepoint configuration $(z_1, \dots, z_n)$ in $F_n(\mathbb{C})$, and then we may visualize a loop as a 'movie' where the n particles continuously move around $\mathbb{C}$, eventually returning pointwise to their starting positions. If we represent time by a third spacial dimension, as shown in Figure 6, we can view the particles as tracing out a braid. Note that, up to homeomorphism, we may view $F_n(\mathbb{C})$ as the configuration space of the open 2-disk.

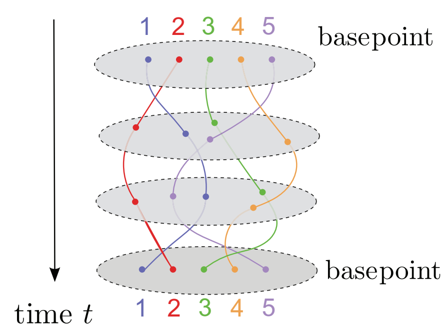


Figure 6. A visualization of a loop $\gamma(t)$ in $F_5(\mathbb{C})$ representing an element of $\pi_1(F_5(\mathbb{C})) \cong \mathbf{P}_5$.

Loops in $C_n(\mathbb{C})$ are similar, with the crucial distinction that the n particles are unlabelled and indistinguishable, and so need only return set-wise to their basepoint configuration.

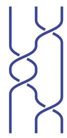


Figure 7. A braid on 3 strands.

It is traditional to represent elements of the group $\mathbf{B}_n$ and its subgroup $\mathbf{P}_n$ by equivalence classes of *braid diagrams*, as illustrated in Figure 7. These braid diagrams depict n strings (called *strands*) in Euclidean 3-space, anchored at their tops at n distinguished points in a horizontal plane, and anchored at their bottoms at the same n points in a parallel plane. The strands may move in space but may not double back or pass through each other. The group operation is concatenation, as in Figure 8.

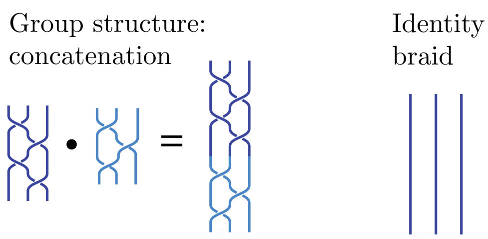


Figure 8. The group structure on $\mathbf{B}_n$.

The braid groups were defined rigorously by Artin in 1925, but the roots of this notion appeared in the earlier work of Hurwitz, Firckle, and Klein in the 1890s and of Vandermonde in 1771. This topological interpretation of braid groups as the fundamental groups of configuration spaces was formalized in 1962 by Fox and Neuwirth.

Artin established presentations for the braid group and the pure braid group. His presentation for $\mathbf{B}_n$,

$$\mathbf{B}_n \cong \left\langle \sigma_1, \sigma_2 \dots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle,$$

uses $(n-1)$ generators σ_i corresponding to half-twists of adjacent strands, as in Figure 9.

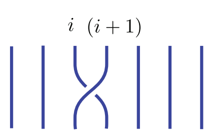


Figure 9. Artin's generator σ_i for $\mathbf{B}_n$.

Artin also gave a finite presentation for $\mathbf{P}_n$. We will not state it in full, but comment that there are $\binom{n}{2}$ generators T_{ij}, $(i \neq j,\, i, j \in \{1, 2, \dots, n\})$ corresponding to full twists of each pair of strands, as in Figure 10.

Corresponding to the regular covering space map $F_n(\mathbb{C}) \to C_n(\mathbb{C})$ of Figure 5, there is a short exact sequence of groups

$$1 \to \mathbf{P}_n \to \mathbf{B}_n \to S_n \to 1.$$

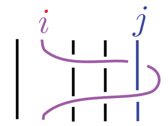


Figure 10. Artin's generator $T_{ij} = T_{ji}$ for $\mathbf{P}_n$.

The quotient map $\mathbf{B}_n \to S_n$, shown in Figure 11, takes a braid, forgets the n strands and simply records the permutation induced on their endpoints. The generator σ_i maps to the simple transposition $(i \;\; i+1)$. The kernel is those braids that induce the trivial permutation, i.e., the pure braid group.

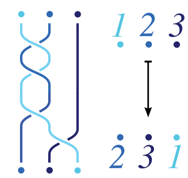


Figure 11. The quotient map $\mathbf{B}_n \to S_n$.

2.3. Homological stability for the braid groups.

Arnold calculated some homology groups of $\mathbf{B}_n$ in low degree (Table 1).

n \ k	0	1	2	3	4	5
0	$\mathbb{Z}$					
1	$\mathbb{Z}$					
2	$\mathbb{Z}$	$\mathbb{Z}$				
3	$\mathbb{Z}$	$\mathbb{Z}$				
4	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$			
5	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$			
6	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
7	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	
8	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$
9	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$

Table 1. The homology groups $H_k(\mathbf{B}_n; \mathbb{Z})$. Empty spaces are zero groups. Stable groups are shaded.

$H_k(C_n(\mathbb{C}); \mathbb{Z})$

The $k = 0$ column follows from the fact that $C_n(\mathbb{R}^2)$ is path-connected and the $k = 1$ column can be obtained by abelianizing Artin's presentation of $\mathbf{B}_n$. Even the low-degree calculations in Table 1 suggest a pattern: the homology of $\mathbf{B}_n$ in a fixed degree k becomes independent of n as n increases.

Arnold proved the following stability result, in terms of the stabilization map $s_n : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ defined by adding an unbraided $(n+1)^{st}$ strand as in Figure 12.

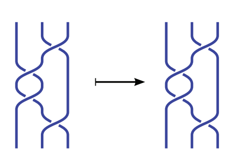


Figure 12. The stabilization map $s_3 : \mathbf{B}_3 \hookrightarrow \mathbf{B}_4$.