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How blind is your favourite cohomology theory?

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Summary. This article is about a notorious method in algebraic topology. The method is that of Adams Spectral Sequences, and it has a reputation amongst those who do not use it for being abstruse and highly technical. It is ironic that this is because the method often succeeds in its aim of faithfully reducing a geometric problem to an algebraic one. Indeed the reduction is often quite standard whilst the algebra is extremely intricate; this can lead to the mistaken impression that the method is remote from the topology and purely computational. The aim of this article is to explain the significance of the method and outline how it can be used in practice without becoming involved in the hard work of doing anything. I hope this will bring out into the open the fact that the method justifies the confidence we have in the philosophy of algebraic topology as a whole. Those who are interested in the real mathematics behind all this might look first at [2], and then at [4] for a more demanding and sophisticated generalisation. The relevant part of [22] might be a suitable step from the present article. It provides a much more specific and practical introduction to the use of the classical Adams Spectral Sequence. The introductory chapters of [30] also provide a brisk and readable account of several forms of the spectral sequence and later chapters illustrate the state of the art in Adams Spectral Sequence calculations of stable homotopy groups.

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0. How to use algebraic topology

Let us consider a typical use of algebraic topology. We start with a geometric problem, and decide it is too hard. For example we may be considering an n-dimensional vector bundle ξ over a space X and trying to decide if it is isomorphic to the trivial product bundle $X \times \mathbb{R}^n$. If we can spot a bundle isomorphism all is well, but otherwise we proceed in two steps.

First we render the problem amenable to topological methods by translating it into homotopy theory. In fact there is a certain "classifying" space $BO(n)$ so that we have a natural correspondence between isomorphism classes of n-dimensional real vector bundles over X and homotopy classes of continuous functions $X \to BO(n)$

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(this latter set is denoted by $[X, BO(n)]$). Thus ξ corresponds to a certain homotopy class $[f_\xi]: X \to BO(n)$.

Note that in this translation we have not lost any information, but nor have we made our problem much easier. However we already have many more constructions available to us in homotopy theory than we had in bundle theory. Furthermore we have freedom to vary spaces and functions up to homotopy: we can transport our ignorance to a convenient spot.

Second, we want to convert our problem into one we can solve. For this we use some algebraic 'invariant', that is an algebraic construct depending only on the homotopy class of f_ξ. Here we have a choice. There are many possible invariants, and we must play off two factors against each other: how much information the invariant loses against how easy it is to calculate. In our example a popular invariant is the total Stiefel-Whitney class $w_\bullet(\xi)$ which is an element of the mod 2 cohomology ring $H^*(X;\mathbb{F}_2)$. Equivalently we consider the map $f_\xi^*\colon H^*(BO(n);\mathbb{F}_2) \to H^*(X;\mathbb{F}_2)$ induced by f_ξ in cohomology; if it is nontrivial then ξ is nontrivial too.

Note that we have definitely lost information. (For example the tangent bundle of S^2 is nontrivial but the corresponding map in cohomology is trivial.) On the other hand we have regained rigidity and with it a rich supply of algebraic operations to help us calculate. For example if $\xi \cong \xi' \oplus \xi''$, then $w_\bullet(\xi) = w_\bullet(\xi')\, w_\bullet(\xi'')$ (in the ring $H^*(X;\mathbb{F}_2)$) and so indecomposability of $w_\bullet(\xi)$ in $H^*(X;\mathbb{F}_2)$ implies that of ξ as a bundle.

Let us now concentrate on the second step. Thus I will suppose you have already translated your original problem into a homotopy problem about homotopy classes of maps between two spaces X and Y. We are faced with the problem of choosing an invariant for understanding $[X, Y]$, or at least for deciding if $f\colon X \to Y$ is nullhomotopic or essential.

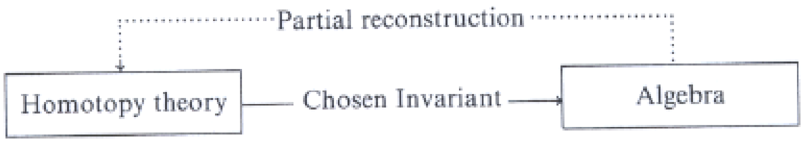
Unfortunately there is a gap. We may fail to prove geometrically that f is null, but fail to prove using our chosen invariant that f is essential. There are three possible explanations.

(1) f is null but we are not good enough at homotopy theory to prove it.

(2) f is essential and our invariant can tell, but our algebra is not sufficiently ingenious to prove it.

or (3) f is essential but our invariant is incapable of detecting this.

Obviously it will eliminate a great deal of wasted effort if we can tell when (3) is the case. Even better if we have a framework guiding us to the important algebra if (2) is the case. In other words we want a constructive method for reversing the reduction from homotopy theory to algebra as far as possible, and a characterisation of how far this is. We want a guarantee our invariant will solve our problem.



We will largely ignore the important (but secondary) problem of how efficient a given invariant is at doing what it can. This will often be the deciding factor in practice, but requires specific information and a lot of experience. On the other hand we will only discuss a particularly easy class of invariants provided by cohomology theories $k^*(\cdot)$ and we will assume that the problem of calculating $k^*(X)$ for spaces X of interest is a relatively easy one. This need not be the case, but we will concentrate on theories for which it is.

1. What can we do with mod 2 cohomology?

In this section we will make a serious effort to examine the power of one particular invariant: reduced mod 2 cohomology $\tilde{H}^*(\cdot;\mathbb{F}_2)$. Let us suppose we are given spaces X and Y and a map $f\colon X\to Y$. Strictly speaking we wish X and Y to be well behaved (simplicial complexes say), to have chosen basepoints in X and Y and to require that f (and any other map we consider) takes one basepoint to the other. We will first indicate how we might use $\tilde{H}^*(\cdot;\mathbb{F}_2)$ to show f is essential and then observe various cases in which $\tilde{H}^*(\cdot;\mathbb{F}_2)$ cannot help us. The reader may be reminded of [3] and [33].

Methods. (0) We have already seen one method; the method of generalized degree. If f induces $f^*\in\mathrm{Hom}(\tilde{H}^*(Y;\mathbb{F}_2), \tilde{H}^*(X;\mathbb{F}_2))$ which is nonzero then f is essential.

However this method does not detect the maps in the following two examples.

(a) Let X and Y be S^1, the set of unit complex numbers and let $\langle 2\rangle\colon S^1\to S^1$ be the squaring map. It is well known that this map induces multiplication by two in integral cohomology and hence it is essential. On the other hand this also shows that $\langle 2\rangle$ induces zero in mod 2 cohomology.

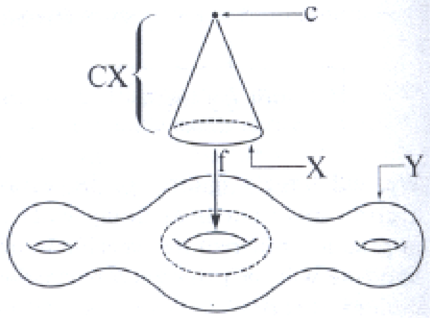
(b) Let X be S^3, the space of unit vectors in $\mathbb{C}\times\mathbb{C}$ and let Y be the complex projective line $\mathbb{C}P^1(\simeq S^2)$. Then we may take f to be the map

$$\begin{aligned}\eta\colon\quad S^3 &\to \mathbb{C}P^1\\ (z_0,z_1)&\mapsto[z_0:z_1]\end{aligned}$$

induced by passage to homogeneous coordinates.

Of course $\tilde{H}^*(S^3;\mathbb{F}_2)$ is zero except in codegree 3; and $\tilde{H}^*(\mathbb{C}P^1;\mathbb{F}_2)$ is zero except in codegree 2; thus by dimension alone $\eta^*=0$. We will sketch below a method for proving η is essential using mod 2 cohomology.

(1) Let us recall a construction from homotopy theory. If $f\colon X \to Y$ is any map then we may form the mapping cone $Y \cup_f CX$ as the quotient space of $Y \cup [0,1] \times X$ in which all points $(0,x)$ are identified to a single point c and the point $(1,x)$ is identified with $f(x) \in Y$. Strictly speaking we should also identify all possible candidates (s, x_0) for a basepoint to the cone point c so that $Y \cup_f CX$ has a preferred basepoint.



The mapping cone $Y \cup_f CX$

This construction has the property that if two maps f and g are homotopic then $Y \cup_f CX \simeq Y \cup_g CX$. Note also that we clearly have $Y \cup_0 CX = Y \vee SX$, the union of Y with the suspension SX ($=$point$\cup_0 CX$) with their basepoints identified. Thus, in order to show f is essential it is enough to show $Y \cup_f CX \not\simeq Y \vee SX$. The other property of the mapping cone we will use is that the sequence

$$X \xrightarrow{f} Y \longrightarrow Y \cup_f CX$$

induces a long exact sequence in cohomology (this may be seen in a number of ways from the Eilenberg-Steenrod axioms).

Thus if f induces zero in mod 2 cohomology we have a *short* exact sequence

(†) $\quad 0 \leftarrow \tilde{H}^*(Y; \mathbb{F}_2) \leftarrow \tilde{H}^*(Y \cup_f CX; \mathbb{F}_2) \leftarrow \tilde{H}^*(SX; \mathbb{F}_2) \leftarrow 0$

which is to say $\tilde{H}^*(Y \cup_f CX; \mathbb{F}_2)$ is an extension of $\tilde{H}^*(Y; \mathbb{F}_2)$ by $\tilde{H}^*(SX; \mathbb{F}_2)$.

Now of course any extension of $\mathbb{F}_2$-vector spaces is in fact a direct sum. However $\tilde{H}^*(\cdot\,; \mathbb{F}_2)$ is not just a vector space. First of all it is a ring (without unit since we are using reduced cohomology). In view of this we can reconsider our examples (a) and (b) above.

(a) It is easy to see directly from the definitions that $S^1 \cup_{\langle 2 \rangle} CS^1$ is in fact the real projective plane, $\mathbb{R}P^2$. From the sequence (†) we see $\tilde{H}^*(\mathbb{R}P^2; \mathbb{F}_2)$ has $\mathbb{F}_2$ in codegrees one and two. It is also well known that the square t^2 of the generator t in codegree 1 is the generator of codegree 2 (and not zero).

On the other hand if f is zero the sequence (†) is split by a map $Y \vee SX \to Y$ and so (since maps induced by maps of spaces preserve the ring structure) $\tilde{H}^*(Y \vee SX; \mathbb{F}_2)$ is a direct product of the rings $\tilde{H}^*(Y; \mathbb{F}_2)$ and $\tilde{H}^*(SX; \mathbb{F}_2)$. Hence, since we have $\tilde{H}^*(S^1 \cup_{\langle 2 \rangle} CS^1; \mathbb{F}_2) \not\cong \tilde{H}^*(S^1 \vee S^2; \mathbb{F}_2)$ it follows that $\langle 2 \rangle \neq 0$.

(b) Again one can easily check from the definitions that $S^2 \cup_\eta CS^3 = \mathbb{C}P^2$, and again it is well known that $\tilde{H}^*(\mathbb{C}P^2; \mathbb{F}_2)$ has $\mathbb{F}_2$ in codegree 2 generated by x say and $\mathbb{F}_2$ in codegree 4 generated by x^2.

Thus as before $\tilde{H}^*(\mathbb{C}P^2;\mathbb{F}_2) \ncong \tilde{H}^*(S^2 \vee S^4;\mathbb{F}_2)$ and so $\eta \neq 0$.

However it is also easy to see that the usual suspension isomorphism $\tilde{H}^i(X;\mathbb{F}_2) \cong \tilde{H}^{i+1}(SX,\mathbb{F}_2)$ does not preserve the ring structure (consider codegrees). Indeed $\tilde{H}^*(SX;\mathbb{F}_2)$ always has the trivial ring structure. On the other hand given a map $f: X \to Y$ we can always consider its suspension

$$\begin{aligned} Sf\colon\ SX &\to SY \\ [s,x] &\mapsto [s,fx] \end{aligned}$$

Thus the ring structure is inadequate for telling us if $S\langle 2\rangle: S^2 \to S^2$ and $S\eta: S^4 \to S^3$ are essential.

We are therefore led to seek the richest algebraic structure on $\tilde{H}^*(X;\mathbb{F}_2)$ which is preserved by maps f^* induced by maps of spaces and by the suspension isomorphism $\tilde{H}^*(SX;\mathbb{F}_2) \cong S\tilde{H}^*(X;\mathbb{F}_2)$. In particular it turns out that any collection α of functions

$$\alpha_X^n\colon\ \tilde{H}^n(X;\mathbb{F}_2) \to \tilde{H}^{n+i}(X;\mathbb{F}_2) \qquad \text{(one for each } X \text{ and } n)$$

which correspond under induced maps and suspension isomorphisms consists of group homomorphisms. It is then easy to see that the collection of all such α forms a graded algebra A^* over $\mathbb{F}_2$, and that $\tilde{H}^*(X;\mathbb{F}_2)$ is a module over it. The algebra A^* is called the Steenrod Algebra of stable natural transformations of mod 2 cohomology.

The remarkable fact is that A^* can be explicitly determined and calculations can be done within it and on it as a whole. (Steenrod constructed a family of elements of A^*, which we will meet below [34], Adem found some relations between them [6], Serre showed that Steenrod's elements generated A^* and Adem's relations implied all relations [31] and Milnor elucidated the full structure of A^* [27]. Nonetheless much remains to be understood about A^* and it is an active area of research.)

In fact the structure of $\tilde{H}^*(X;\mathbb{F}_2)$ as a module over A^* is related to its ring structure. Indeed the elements constructed by Steenrod are versions of the squaring operation designed to commute with suspension; they are the Steenrod Squares Sq^i $(i=0,1,2,\ldots)$, which are provided by homomorphisms

$$Sq^i\colon\ \tilde{H}^n(X;\mathbb{F}_2) \to \tilde{H}^{n+i}(X;\mathbb{F}_2) \qquad \text{(one for each } n \text{ and } X).$$

These (in addition to commuting with induced maps and suspensions) have the properties

(i) $Sq^i x = x^2$ if $x \in \tilde{H}^i$

and

(ii) $Sq^i x = 0$ if $x \in \tilde{H}^n$ for $n < i$.

Also Sq^0 is the identity and Sq^1 the Bockstein associated to the coefficient sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

If properties (i) and (ii) are taken to heart and the ring structure of $\tilde{H}^*(X;\mathbb{F}_2)$ is retained alongside the A^*-structure then the resulting invariant is even more powerful [11] and has spectacular applications [25]. However in the interests of history, simplicity and clarity we will only consider $\tilde{H}^*(X;\mathbb{F}_2)$ as a module over A^*.

Returning once more to the examples (a) and (b) we see by property (i) that the extension (†) regarded as an extension of A^*-modules is nontrivial in both cases. Hence, since the A^*-module structure is preserved by suspension, we find that the corresponding extensions for $S^k f\colon S^k X \to S^k Y$ are nontrivial for all k, and hence that all suspensions of $\langle 2\rangle$ and η are essential.

In summary the method is

(0) Consider $f^* \in \mathrm{Hom}_{A^*}(\tilde{H}^*(Y;\mathbb{F}_2), \tilde{H}^*(X;\mathbb{F}_2))$

and if it is zero

(1) Consider $\tilde{H}^*(Y \cup_f CX;\mathbb{F}_2)$ as an element of $\mathrm{Ext}^1_{A^*}(\tilde{H}^*(Y;\mathbb{F}_2), \tilde{H}^*(SX;\mathbb{F}_2))$.

It is then no surprise that there follows a succession of invariants defined if earlier ones are zero and lying in subquotients of the group $\mathrm{Ext}^s_{A^*}(\tilde{H}^*(Y;\mathbb{F}_2), \tilde{H}^*(S^s X;\mathbb{F}_2))$ of s-extensions [19] of A^*-modules.

Blindnesses. (A) By deliberately restricting our attention to structure preserved by suspension we have ensured our methods are blind to anything destroyed by suspension. Thus if $f\colon X \to Y$ has the property that $S^k f\colon S^k X \to S^k Y$ is null, the above methods will see f as the trivial map. For example twice $\eta\colon S^3 \to S^2$ is in fact essential whereas its suspension is null, so mod 2 cohomology (as a module over A^*) cannot detect the nontriviality of twice η.

Thus the most we can hope to understand is the set $[S^k X, S^k Y]$ for large k. In fact this is an abelian group and independent of k for $k >$ dimension $X+2$ (Freudenthal Suspension Theorem): we write $\{X, Y\}$ for this group and call it the group of stable maps from X to Y.

(B) Mod 2 cohomology only sees mod 2 phenomena. Indeed we may take X (or Y) to be a mod 3 lens space, defined as the mapping cone of the cubing map $\langle 3\rangle\colon S^1 \to S^1$. Since $\langle 3\rangle$ induces an isomorphism of mod 2 cohomology it follows that $\tilde{H}^*(S^1 \cup_{\langle 3\rangle} CS^1;\mathbb{F}_2)=0$. But if $\tilde{H}^*(X;\mathbb{F}_2)$ is zero, all of the Ext groups we use to define the invariants of our map $f\colon X \to Y$ are zero, and so the above methods cannot help us. On the other hand we know from mod 3 cohomology that we are missing something (for example the identity map of $S^1 \cup_{\langle 3\rangle} CS^1$).

Similarly, if an essential map f has the property that $3f \simeq 0$ then, since f and $3f$ will have identical invariants mod 2 (since the invariants are natural and multiplication by 3 is an isomorphism mod 2) the above methods are blind to f. For example if $\nu\colon S^7 \to \mathbb{H}P^1 \cong S^4$ is analogous to η, $f=8$. $S\nu\colon S^8 \to S^5$ has the property that $3f \simeq 0$, whilst mod 3 cohomology can be used to show that f and all its suspensions are essential.

Evidently the only property of the number 3 we have used is that it is odd. Thus mod 2 cohomology is blind to 2 coprimary behaviour.

2. Justifying tradition

We have presented the methods and blindnesses of mod 2 cohomology very suggestively, but when ordinary cohomology was the only one available the comparative approach was not so obvious. With the benefit of hindsight and suggestion we may be led to suspect some version of the following.

Theorem (Adams (1958) [1]). *The above methods suffice to calculate all about* $[X, Y]$ *that the above blindnesses do not exclude, which is to say the* 2*-completion of* $\{X, Y\}$.

Of course this is not quite the wording Adams used, and although intriguing our paraphrase is not very helpful. In fact his actual theorem provides an algebraic framework within which to use the above methods. It states that there is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A^*}^s(\tilde{H}^*(Y; \mathbb{F}_2), \tilde{H}^*(S^t X; \mathbb{F}_2)) \Rightarrow \{S^{t-s}X, Y\}_2^{\wedge}.$$

What does this mean? Perhaps we should think of it as an algebraic induction process. The induction starts with the E_2 term, and the theorem identifies this as the Ext group we have already come across. The 'inductive' step from E_r to E_{r+1} involves doing some work for each r. Thus one obtains by 'induction' the E_∞ term. The theorem then says that this is closely associated to the group $\{X, Y\}_2^{\wedge}$ that we are interested in. In fact the groups $E_\infty^{s,t}$ with $t-s=n$ are subquotients corresponding to a certain filtration of $\{S^n X, Y\}_2^{\wedge}$. There are thus three parts to the theorem.

(0) There is a spectral sequence

By itself this in uninteresting: spectral sequences are easy to construct.

(1) The spectral sequence tells us about $\{X, Y\}_2^{\wedge}$.

This part (known as the convergence statement) tells us that the spectral sequence is *relevant*.

(2) The spectral sequence has the stated E_2-term.

This makes the spectral sequence *useful*. Indeed it also shows the spectral sequence generalises the methods we saw in the previous section.

If you want to use the theorem as a calculational tool you proceed in three steps

(A) Calculate $\mathrm{Ext}_{A^*}^s(\tilde{H}^*(Y; \mathbb{F}_2), \tilde{H}^*(S^t X; \mathbb{F}_2))$.

This step itself falls into two parts.

(i) Identify the structure of $\tilde{H}^*(X;\mathbb{F}_2)$ and $\tilde{H}^*(Y;\mathbb{F}_2)$ as modules over A^*. There are well known methods of finding out the structure as $\mathbb{F}_2$-vector spaces; after that some igenuity may be required, but for spaces we understand moderately well this step is not a major problem.

(ii) Calculate the Ext group using standard methods of homological algebra. This step is quite hard work but, provided we only need the answer for small values of $t-s$, it is mechanical.

Step A cuts out the routine difficulties of a generic nature and leaves behind the few obstructions characteristic of our particular spaces of interest.

(B_r) Calculate E_{r+1} from E_r. This is very hard, and there are an infinite number of r to do it for. On the other hand the algebraic structure of the spectral sequence cuts down the work considerably, and in many cases leaves no work to be done at all.

(C) Deduce $\{S^{t-s}X, Y\}_2^\wedge$ from E_∞. Again this can be very hard, but still the algebraic structure of the spectral sequence is on our side.

I must emphasise that perhaps the greatest uses of the Adams Spectral Sequence are *not* calculational. For example it can be used to show that a certain map $X \to X'$ induces an isomorphism $\{X', Y\}_2^\wedge \xrightarrow{\cong} \{X, Y\}_2^\wedge$ by showing it induces an isomorphism of E_2 terms ([14], [21] see also [8]). However I will give the most widely known use of the Adams Spectral Sequence.

Example: Calculation of the n^{th} stable homotopy group of the sphere for small n.

Recall that $\{S^n, S^0\} = \pi_{n+k}(S^k)$ for large k, and so the Adams Spectral Sequence

$$E_2^{s,t} = \mathrm{Ext}_{A^*}^s(\tilde{H}^*(S^0;\mathbb{F}_2), \tilde{H}^*(S^t;\mathbb{F}_2)) \Rightarrow \{S^{t-s}, S^0\}_2^\wedge$$

is relevant to our purpose. Indeed we know that $\{S^n, S^0\} = 0$ for $n<0$ (by the Simplicial Approximation Theorem), $\{S^0, S^0\} = \mathbb{Z}$ (by the Hurewicz Isomorphism Theorem), and that $\{S^n, S^0\}$ is a finite abelian group for $n>0$ [32] so that the spectral sequence calculates the 2 component. We will be satisfied with this.

(A) Calculate E_2. Here Step (i) is trivial because of course $\tilde{H}^*(S^n;\mathbb{F}_2)$ consists of a single copy of $\mathbb{F}_2$ in codegree n and zero elsewhere. There is a unique A^* structure on this. Step (ii) is quite messy. Simply from a presentation of A^* it is a tedious exercise to calculate the E_2-term in the range depicted below. It is conventional and convenient to graph the cohomological degree s vertically and the geometric degree $t-s$ horizontally (so that all groups contributing to $\{S^n, S^0\}_2^\wedge$ appear in the single column with $t-s=n$).

Each symbol in the table below represents the generator of one copy of $\mathbb{F}_2$.

s								
	$\vdots$							
6	h_0^6							
5	h_0^5							
4	h_0^4							$h_0^3 h_3$
3	h_0^3			$h_0^2 h_2$				$h_0^2 h_3$
2	h_0^2		h_1^2	$h_0 h_2$			h_2^2	$h_0 h_3$
1	h_0	h_1		h_2				h_3
0	1							
	0	1	2	3	4	5	6	7 $\quad t-s$

The direct calculation proceeds row by row with increasing s, and from left to right in each row. In the course of the calculation a pattern emerges: in the zeroth column (i.e. $t-s=0$) there is always a single $\mathbb{F}_2$ in each box, and then there is no entry in the row until about the $2s^{\text{th}}$ column. In fact this pattern can be proved to persist and is known as Adams' Vanishing Line Theorem [1]. In any case the above table is complete for $t-s\leq 7$ and all s. Notice that names of the generators imply a multiplicative structure. To explain this would take us too far afield; we will hardly use it, but for more extensive calculations it is an indispensible aid. Using more sophisticated methods the E_2 term is known at least for $t-s\leq 70$ [35].

(B_r) Calculate E_{r+1} from E_r. The way this is done is to calculate certain differentials, which are maps

$$d_r\colon\ E_r^{s,t}\to E_r^{s+r,t+r-1}$$

(i.e. they increase s by r and decrease $t-s$ by one). These have the property that $d_r^2=0$, and $E_{r+1}=H(E_r,d_r)$.

In our case the only differential for any r that goes between nonzero groups is the one leaving h_1 in the $t-s=1$ column. Suppose $d_{r'}h_1=0$ for $r'<r$ (so that $d_r h_1$ is defined) and $d_r h_1=\lambda h_0^{r+1}$. We can argue that $\lambda=0$ in several ways. For example we have already found a nonzero element η of $\{S^1,S^0\}_2^{\wedge}$, and so h_1 must be in the kernel of every d_r so that it survives until E_∞ where it can correspond to η. (Alternatively we can relate multiplication by h_0 to group extensions and

argue with the known value $\{S^0, S^0\}_2^\wedge = \mathbb{Z}_2^\wedge$ or use the fact that d_r is a derivation and $h_0 h_1 = 0$).

Thus all differentials are zero and we can conclude $E_\infty^{s,t} = E_2^{s,t}$ for $t-s \leq 6$. (In fact the first nonzero differential arrives in the column $t-s=14$.)

(C) Without doing any work we find that the order of $\{S^n, S^0\}_2^\wedge$ is as tabulated

n	0	1	2	3	4	5	6	7
$\lvert\{S^n, S^0\}_2^\wedge\rvert$	∞	2	2	8	0	0	2	≤ 16

It turns out that multiplication by h_0 corresponds to multiplication by 2 in $\{S^n, S^0\}_2^\wedge$ and this allows one to deduce $\{S^0, S^0\}_2^\wedge = \mathbb{Z}_2^\wedge$ (as we know) and that $\{S^3, S^0\}_2^\wedge = \mathbb{Z}/8$.

It is in fact the case that $\{S^7, S^0\}_2^\wedge = \mathbb{Z}/16$; indeed $\{S^n, S^0\}_2^\wedge$ is known for $n \leq 45$ ([23], [7], [13], [30]).

3. Blindness of other theories

Our theme has been that an Adams Spectral Sequence characterises the blindness of the cohomology theory used to define it. In this section I will summarise these characterizations for certain well known theories. The technical difficulties are usually greater then for mod p cohomology – and in fact it is usually more convenient to work with the associated homology theory [4]. It is still often possible to give an analogous algebraic description of the E_2-term in this case, but we will not digress to discuss it. We will restrict ourselves to quite vague descriptions of what the theory can tell us, and comment briefly on sources, significance and calculational convenience. For further details see [4] and [9].

Theory *What it sees*

(1) $H^*(\cdot; \mathbb{F}_p) \dashrightarrow \{X, Y\}_p^\wedge$

Precisely similar to the mod 2 case we saw in the previous section [1].

(2) $H^*(\cdot; \mathbb{Q}) \dashrightarrow \{X, Y\} \otimes \mathbb{Q}$.

Rationally, stable homotopy theory is rather trivial, so the language of Adams Spectral Sequences is unnecessary here.

(3) $H^*(\cdot; \mathbb{Z}) \dashrightarrow \{X, Y\}$.

Although it is not blind, integral cohomology is never used in practice. It is much easier to calculate $\{X, Y\} \otimes \mathbb{Q}$, using rational cohomology and $\{X, Y\}_p^\wedge$ using mod p-cohomology and reassemble the answer by algebraic methods.

(4) $K_*(\cdot)$ $\dashrightarrow$ A periodic form of $\{X, Y\}$.
(K-theory).

The periodicity referred to is an attenuated form of Bott periodicity and mixes up $\{S^k X, Y\}$ for various k. Since (for finite X) $\{S^k X, Y\}$ is zero if k is sufficiently negative, whereas this is not true for the periodic form, it is apparent that K-theory sees a very distorted picture of reality. However this distortion emphasises certain important systematic phenomena, and is therefore valuable in its own right. For K-theory this periodic form of $\{X, Y\}$ is well understood ([10]) and so is its relationship to $\{X, Y\}$ itself.

Similar remarks apply to the more exotic Morava K-theories [20] and taken together, the relevant periodic forms account for the whole of $\{X, Y\}$ and this clarifies the importance of the associated systematic phenomena [29], [26].

(5) $MU^*(\cdot) \dashrightarrow \{X, Y\}$.
(complex cobordism)

The machinery requires a lot of hard work to set up but in the case where X and Y are spheres the method is the most efficient known method of calculation at odd primes. On the other hand $MU^* X$ can often be hard to calculate. See [28], [4] and [30].

(6) $\pi^*(\cdot) \dashrightarrow \{X, Y\}$.
(Stable cohomotopy)

I included this example to emphasize the absurdity of ignoring the calculational aspect of the problem. The spectral sequence is entirely useless in general since one does not know the Steenrod Algebra or the cohomotopy of X or Y and so calculating the E_2 term will perhaps be harder than obtaining the answer directly.

4. Blindness of some equivariant theories

In the above examples the construction applied to $\{X, Y\}$ to discover what a theory sees was usually arithmetic. The case of K-theory was different, but we did not give a precise description. For a nonarithmetic but elementary example we give some spectral sequences in equivariant topology.

Suppose then that G is a finite group which acts on spaces X and Y. We wish to use cohomological methods to deduce as much as possible about maps preserving the G-action. As in the nonequivariant case, cohomology theories satisfy the homotopy axiom (G-maps homotopic through G-maps induce the same map in cohomology) and the suspension axiom (suspension of a space by a linear G-sphere simply shifts the cohomology groups in dimension). Thus if $[X, Y]^G$ denotes G-homotopy classes of G-maps, we can at best hope to learn about the stable maps $\{X, Y\}^G$, that is about $[S^V X, S^V Y]^G$ where S^V is the one point compactification of a representation V "large" compared with the dimension of X [5].

We offer four spectral sequences in equivariant topology. The first three are equivariant generalisations of the classical Adams Spectral Sequence of successively greater power, and the fourth is intrinsically equivariant and of a different type.

(7) mod p Borel cohomology $\dashrightarrow \{X \wedge EG_+, Y \wedge EG_+\}^{G\wedge}_p$.

Here G is a p-group and mod p Borel cohomology (also known as ordinary equivariant cohomology) is defined by $b^*(X) = \tilde{H}^*(EG_+ \wedge_G X; \mathbb{F}_p)$ so that it has coefficient ring $b^* = H^*(G; \mathbb{F}_p)$. The space EG_+ is constructed from the usual nonequivariantly contractible space EG on which G acts freely, by adding a disjoint basepoint, and one can think of $X \wedge EG_+$ as "X made into a free G-space". Here, in addition to the familiar p-completion due to the fact we have used mod p coefficients, we also have a geometric completion, making X and Y G-free. This Adams Spectral Sequence would never be used since it is subsumed in the following one.

(8) mod p coBorel cohomology $\dashrightarrow \{X, Y \wedge EG_+\}^{G\wedge}_p$.

Again G is a p-group and mod p coBorel cohomology is a theory derived from Borel cohomology. The associated homology theory is defined by $c_*(X) = \tilde{H}_*(EG_+ \wedge_G X; \mathbb{F}_p)$ and hence has coefficients $c_* = H_*(G; \mathbb{F}_p)$. (Note that this means $c^i = c_{-i}$ is nonzero only for i *negative* or zero and so Borel and coBorel theories are definitely distinct.)

Since Borel and coBorel theories agree on G-free spaces this clearly generalises the previous example. This spectral sequence shares with the classical Adams Spectral Sequence the advantage of being calculable [16].

Under the further assumption that G is an elementary abelian p-group we have a further generalisation

(9) mod p Borel homology $\dashrightarrow \{X, Y\}^{G\wedge}_p$.

Here we emphasize that G must be elementary abelian and Borel homology has coefficient ring b_* with $b_i = b^{-i} = H^{-i}(G; \mathbb{F}_p)$. The remarkable fact here is that the homology theory does not just weaken finiteness assumptions in the familiar fashion, but actually provides an enormously more powerful invariant than the corresponding cohomology theory. In the case when Y is G-free this spectral sequence agrees with the coBorel cohomology spectral sequence. Calculation with this spectral sequence is more of a challenge than for the coBorel cohomology one – which is only to be expected in view of the benefits enjoyed. The details for G of rank 1 are contained in [15] and for rank ≥ 1 in [18].

(10) nonequivariant stable homotopy $\dashrightarrow \{X \wedge EG_+, Y\}^G$.

Here we find the remarkable fact that nonequivariant homotopy sees quite a lot about equivariant homotopy. For example, by the affirmed Segal Conjecture [14], if Y is finite and G is a p-group $\{X \wedge EG_+, Y\}^{G\wedge}_p = \{X, Y\}^{G\wedge}_p$. When $X = S^0$ and Y is chosen suitably (the function spectrum of maps from Y_1 to Y_2) the spectral sequence relates the action of G on the nonequivariant stable maps $\{Y_1, Y_2\}$ (homotopy equivariant information) to strictly equivariant information, and it can be put in the following form.

$$E_2^{s,t}=H^s(G;\{Y_1,Y_2\}_t)\Rightarrow\{S^{t-s}Y_1\wedge EG_+,Y_2\}^G$$

See [17] for more details.

5. How these results are proved

It is not appropriate for me to give detailed proofs in this article, but I want to sketch the method of constructing an Adams Spectral Sequence and outline how the E_2 and convergence problems are dealt with. Let us suppose we are concerned with the generalised cohomology theory $E^*(\cdot)$ and wish to use it to calculate $\{X, Y\}$ as far as possible.

The fundamental technical convenience is that $E^*(\cdot)$ is represented in a suitably large stable category [12]. Thus $E^*(X)=\{X,E\}^*$ for a certain object E. Now our aim is to understand $\{X, Y\}$ algebraically using $E^*(\cdot)$ only, and the method is to find the parts of $\{X, Y\}$ detected by finer and finer cohomological methods. To construct an Adams Spectral Sequence in the easiest possible way we realise this geometrically, starting with Y and removing copies of E repeatedly. One finds that the residue of Y left after this analysis depends only on E and is independent of the particular order in which parts were removed. We may call it Y_∞. The blindness of the theory for this problem is precisely $\{X, Y_\infty\}$, and the theory can tell you exactly about $\{X, Y/Y_\infty\}$. This much is purely formal. The convergence problem is that of finding a helpful description of Y/Y_∞, or at least of $\{X, Y/Y_\infty\}$. This will require a proper understanding of the cohomology theory $E^*(\cdot)$.

The description of the E_2 term arises as follows. As soon as we know E is represented it is obvious that the Steenrod Algebra for E is $\{E,E\}^*=E^*E$. Provided this has sufficiently good finiteness properties we may proceed. In fact we arrange that the way copies of E are removed from Y corresponds precisely to the process of forming a resolution of E^*Y by E^*E-free modules, and so that when we map X into this system (i.e. we apply $\{X,\cdot\}$) it corresponds to applying $\mathrm{Hom}_{E^*E}(\cdot, E^*X)$ to the algebraic resolution. This identifies the E_1 term of the resolution as the chain complex of which $\mathrm{Ext}^*_{E^*E}(E^*Y, E^*X)$ is the cohomology. The correspondence is organised so that passage to E_2 corresponds to taking this cohomology.

The above discussion is only appropriate under strong assumptions on the cohomology theory $E^*(\cdot)$, mainly reflected in the good algebraic behaviour of E^*E (although there are also finiteness assumptions on E^*Y). The necessary assumptions on the theory can be considerably weakened by working with the corresponding homology theory $E_*(\cdot)$ (and now there will be no finiteness assumptions on E_*Y) [4]. Accordingly, to retain an algebraic description of the E_2 term we must attempt to dualise the algebra, and unfortunately the relevant homological coalgebra is usually of the relative kind (involving resolutions of both variables). This means that to geometrically realise the algebra, as well as decomposing Y as before we must face the problem of also resolving X in a suitable fashion (which will be different from that appropriate to Y). For X a sphere this is not necessary but for more general X the problem is equivalent to proving a Universal Coefficient Theorem for $E_*(\cdot)$, and this requires a detailed understanding of E-homology.

If we can overcome the above difficulties (and a few other less severe ones) we obtain a spectral sequence which is the correct analogue of the classical Adams Spectral Sequence. The non-calculational benefit of this relativisation is that the pieces we remove from Y need not actually be copies of E itself, but may be other objects well behaved for $E_*(\cdot)$ (so called injective ones [24]). The first advantage this confers is that such an analysis of Y by E-injectives exists under minimal assumptions. More significantly it means that the Y_∞ obtained in this way may be "smaller" than the one constructed by removing copies of E only, and hence Y/Y_∞ may be closer to Y (i.e. we get a stronger convergence theorem). This is what happened for Borel homology and cohomology. Notice here that this benefit is enjoyed even when the cohomology spectral sequence is defined and has the correct E_2 term; even when we are not forced to use homology we may gain by doing so. The object Y/Y_∞ obtained in this way is called the Bousfield E-nilpotent completion of Y, and sometimes written $E^\wedge Y$ or $Y_E^\wedge$ [9]. It also turns out that despite the resolution of X as well, the blindness of the theory remains $\{X, Y_\infty\}$ and the theory still sees $\{X, Y/Y_\infty\}$ as before. This means we can discuss the blindness of a theory unambiguously without knowing if the appropriate resolution of X exists.

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