MAT8021, Algebraic Topology

Assignment 9

Due in-class on Tuesday, April 23

1. Fix a ring R and an integer n. Suppose C_*, D_* are chain complexes of R-modules such that

   - the groups C_k are free R-modules for $k > n$, and
   - the homology groups $H_k(D_*)$ are zero for $k \geq n$

   Additionally, suppose we are given maps $f_m : C_m \to D_m$ for $m \leq n$ such that $\partial f_m = f_{m-1}\partial$.

   Show (by induction) that we can extend this to a chain map $f : C_* \to D_*$ and that any two extensions are chain homotopic.

   For the remaining questions, all chain complexes are over $\mathbb{Z}/2$, i.e., $2x = 0$ for all x.

   A cochain complex C^* has *cup-i products* if it is equipped with operations $(x, y) \mapsto x \smile_i y$ for $i \geq 0$ such that

   - if $x \in C^p$, $y \in C^q$, then $x \smile_i y \in C^{p+q-i}$
   - $(x+x') \smile_i y = x \smile_i y + x' \smile_i y$ and similarly $x \smile_i (y+y') = x \smile_i y + x \smile_i y'$
   - $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$
   - for $i > 0$,

   $$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y) + x \smile_{i-1} y + y \smile_{i-1} x$$

   For instance, one can show (using the method of acyclic models) that $C^*(X)$, for X a space, naturally comes equipped with cup-i products, each one expressing "how noncommutative" the previous one was.

2. Show that for all $j \leq p$ we get a well-defined "squaring" operation $\mathrm{Sq}^j : H^p(C^*) \to H^{p+j}(C^*)$ given by

   $$\mathrm{Sq}^j[x] = [x \smile_{p-j} x]$$

   such that $\mathrm{Sq}^j([x+y]) = \mathrm{Sq}^j([x]) + \mathrm{Sq}^j([y])$. (In the cohomology of a space, these are called the Steenrod squares.)

3. If $f : C^* \to D^*$ is a map of cochain complexes such that $f(x \smile_i y) = f(x) \smile_i f(y)$, show that the induced map $H^*(C^*) \to H^*(D^*)$ preserves the squaring operations.

4. If $0 \to C^* \to D^* \to E^* \to 0$ is a short exact sequence of cochain complexes preserving cup-i products, show that the connecting homomorphism

$$\delta\colon H^p(E^*) \to H^{p+1}(C^*)$$

satisfies $\delta(\mathrm{Sq}^j[x]) = \mathrm{Sq}^j(\delta[x])$.