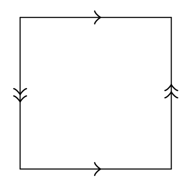
MAT8021, Algebraic Topology

Assignment 8

Due in-class on Tuesday, April 16

1. Let X be a Klein bottle:



We can put a Δ-complex structure on X with one vertex p, three edges a, b, c, and two 2-simplices u, v. Make this Δ-complex structure explicit, and use it to compute $H^*(X;\mathbb{Z}/2)$ together with the cup product on it.

2. In Hatcher, page 131, exercise 8, there is given a description of a *lens space* formed by gluing together n tetrahedra; let's call this $L(n,1)$. (The 1 is because we are gluing the bottom face of T_i to the top face of T_{i+1}.) Compute $H^*\big(L(n,1);\mathbb{Z}/n\big)$ together with the cup product on it.

3. We know that if X and Y are based spaces, the wedge $X \vee Y$ has

$$H^k(X \vee Y; R) = H^k(X;R) \oplus H^k(Y;R)$$

for any $k > 0$. Show that under this identification, the cup product is given by

$$(\alpha, \beta) \smile (\alpha', \beta') = (\alpha \smile \alpha', \beta \smile \beta')$$

4. Consider cohomology with coefficients in a ring R. Write $I = [0,1]$.

   (a) By identifying $I/\partial I$ with S^1, show that

$$H^*(I, \partial I) \cong \begin{cases} R & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (b) Let $\alpha \in H^1(I, \partial I)$ be a generator. Given any pair (X, A) of spaces, consider the composite

$$H^*(X,A) \xrightarrow{\otimes\alpha} H^*(X,A) \otimes H^1(I,\partial I) \xrightarrow{p_1^* \otimes p_2^*} H^*(X \times I, A \times I) \otimes H^1(X \times I, X \times \partial I) \xrightarrow{\smile} H^{*+1}(X \times I, A \times I \cup X \times \partial I)$$

where p_1 and p_2 are projections. Suppose that $A \subset X$ is a CW-pair. Then $(X \times I, A \times I \cup X \times \partial I)$ is also a CW-pair and so the above is equivalent to a map

$$\widetilde{H}^*(X/A) \to \widetilde{H}^{*+1}\big((X \times I)/(A \times I \cup X \times \partial I)\big) \cong \widetilde{H}^{*+1}\big((X/A) \wedge S^1\big)$$

Show that this map is an isomorphism (cf. Question 1 of Assignment 5). (Hint: The connecting homomorphisms in the long exact sequence of cohomology for a pair satisfy a Leibniz formula.)