MAT8021, Algebraic Topology

Corrigendum to Assignment 6*

April 8, 2024

3. Suppose G is a topological group and $x \in H_k(G;\mathbb{Z})$ where k is odd. Show that $x^2 = 0$ in $H_{2k}(G;\mathbb{Z})$.

Without an additional assumption that G is abelian, the claim in this question is **incorrect**. Here is how the argument was supposed to go and where it broke down.

Let $\xi \in C_k(G)$ such that $[\xi] = x$. Then $[\mu_*(\xi \times \xi)] = x^2$, where $\mu\colon G\times G \to G$ is the multiplication map and $\times$ is defined on simplices by

$$\begin{aligned} \times\colon C_k(X) \times C_\ell(Y) &\to C_{k+\ell}(X \times Y) \\ (\sigma \ , \ \tau) &\mapsto \sigma \times \tau = \sum_{\substack{\alpha\ (k,\ell)\text{-} \\ \text{shuffles}}} \operatorname{sgn}(\alpha) \cdot [(\sigma \times \tau) \circ f_\alpha] \end{aligned}$$

with

$$\begin{aligned} f_\alpha\colon \quad \Delta_{k+\ell} &\to \Delta_k \times \Delta_\ell \\ (t_1, \dots, t_{k+\ell}) &\mapsto \big((t_{\alpha(1)}, \dots, t_{\alpha(k)}), (t_{\alpha(k+1)}, \dots, t_{\alpha(k+\ell)})\big) \end{aligned}$$

the "un-shuffle" maps.

Given any (k,ℓ)-shuffle α, we have $\alpha(1) \le \cdots \le \alpha(k)$ and $\alpha(k+1) \le \cdots \le \alpha(k+\ell)$. Thus it is in bijective correspondence with an (ℓ,k)-shuffle $\alpha'\colon (1,\dots,k+\ell) \mapsto \big(\alpha(k+1),\dots,\alpha(k+\ell),\alpha(1),\dots,\alpha(k)\big)$. Moreover, α and α' differ by $k\ell$ transpositions. Therefore,

$$\sigma \times \tau = (-1)^{k\ell} t_*(\tau \times \sigma) \tag{1}$$

where $t\colon X\times Y \to Y\times X$ sends (x,y) to (y,x). In particular, setting $X = Y = G$ and $\tau = \sigma$ so that $\ell = k$, we see that $\sigma \times \sigma = -t_*(\sigma \times \sigma)$.

However, $t_*(\sigma \times \sigma) \neq \sigma \times \sigma$, so we cannot proceed from here, obtain $\sigma \times \sigma = -\sigma \times \sigma$ in the free abelian group $C_{2k}(G \times G)$, and conclude that $\sigma \times \sigma = 0$. In fact, by looking at the "smallest" example with $X = \Delta[1]$, $k = 1$, and $\sigma = \mathrm{id}$, we compute explicitly that $t_*(\sigma\times\sigma) = -\sigma\times\sigma$ (swapping the signed

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triangular regions in a unit square across the diagonal results in a minus sign), which simply divides the sign $(-1)^{k\ell}$ to the left-hand side of (1) and does not yield further information.

If **in addition** G is abelian (or abelian up to homotopy), then $\mu = \mu \circ t$ (or $\mu \simeq \mu \circ t$). Thus from (1) we obtain

$$\mu_*(\sigma \times \tau) = (-1)^{k\ell} \mu_* t_*(\tau \times \sigma) = (-1)^{k\ell} \mu_*(\tau \times \sigma)$$

which gives the graded commutativity of the Pontryagin product in homology.

Besides topological groups G, this multiplicative structure in homology exists more generally for *H-spaces*. See Hatcher Section 3C, especially Example 3C.7 which shows a class in odd degree squaring to nonzero. Also compare the cup-product structure in cohomology to be discussed in class soon.