MAT8021, Algebraic Topology

Assignment 4

Due in-class on Tuesday, March 19

1. (The Five Lemma) Suppose

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha"'] & A_2 \arrow[r] \arrow[d, "\beta"'] & A_3 \arrow[r] \arrow[d, "\gamma"'] & A_4 \arrow[r] \arrow[d, "\delta"'] & A_5 \arrow[d, "\epsilon"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}
$$

   where the rows are exact and the squares commute. Suppose α, β, δ, ϵ are isomorphisms. Show that γ is an isomorphism.

2. Prove a stronger version of the Five Lemma: If β and δ in the above diagram are injective, and α is surjective, show that γ is injective. Give the dual statement (whose proof is of course essentially the same).

3. (Formal Mayer–Vietoris sequence) Suppose that there is a map of long exact sequence as follows:

$$
\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, "\sim"'] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & F_n \arrow[r] \arrow[d, "\sim"'] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r] & C_n \arrow[r] & D_n \arrow[r] & G_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

   Here all the maps $F_n \to G_n$ are isomorphisms. Show that there is a long exact sequence

$$\cdots \to D_{n+1} \to A_n \to B_n \oplus C_n \to D_n \to A_{n-1} \to \cdots$$

   (Define the maps first.)

4. Suppose X is a CW complex with finitely many cells. Define the Euler characteristic $\chi(X)$ to be the number of even-dimensional cells minus the number of odd-dimensional cells.

If F is *any* field, show that

$$\chi(X) = \sum_i (-1)^i \dim_F \big(H_i(X;F)\big)$$

(Hint: Express this in terms of the ranks of the boundary maps.)