MAT8021, Algebraic Topology

Assignment 15

For Weeks 15 and 16. Required but not collected.

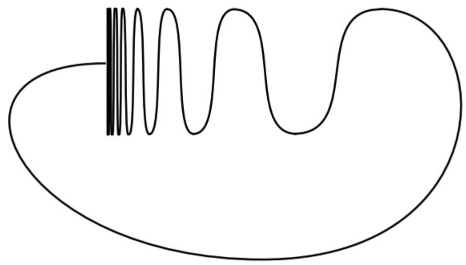
1. Let $f : \widetilde{X} \to X$ be a covering space. Assume that X is endowed with the structure of a CW complex. Prove that $\widetilde{X}$ can be endowed with the structure of a CW complex such that f takes the interior of k-cells in $\widetilde{X}$ homeomorphically to the interior of k-cells in X. Hint: First construct $\widetilde{X}^{(0)}$, then construct $\widetilde{X}^{(1)}$, then construct $\widetilde{X}^{(2)}$, etc. At each stage, you will need to use the lifting criterion to figure out how to attach cells.

2. Consider covering spaces $f : Y \to X$ with Y and X connected CW complexes, the cells of Y projecting homeomorphically onto cells of X. Restricting f to 1-skeleton then gives a covering space $Y^{(1)} \to X^{(1)}$ over the 1-skeleton of X. Prove the following.
   (a) Two such covering spaces $Y_1 \to X$ and $Y_2 \to X$ are isomorphic if and only if the restrictions $(Y_1)^{(1)} \to X^{(1)}$ and $(Y_2)^{(1)} \to X^{(1)}$ are isomorphic.
   (b) $Y \to X$ is a regular covering space if and only if $Y^{(1)} \to X^{(1)}$ is a regular covering space.
   (c) The groups of deck transformations of the coverings $Y \to X$ and $Y^{(1)} \to X^{(1)}$ are isomorphic, via the restriction map.

3. Construct the universal cover of the following space:
$$X = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \cup \{(x, 0, 0) \in \mathbb{R}^3 \mid -1 \le x \le 1\}$$

4. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0, 1] \times [0, 1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Show that for every covering space $f : \widetilde{X} \to X$, there is some neighborhood of the left edge of X that lifts homeomorphically to $\widetilde{X}$. Deduce that X has no simply-connected covering space.

5. Let Y be the *quasi-circle* shown in the following figure:



Thus Y is a closed subspace of $\mathbb{R}^2$ consisting of the union of the set

$$\{(x, y) \in \mathbb{R}^2 \,|\, 0 < x \leq 1, y = \sin(1/x)\}$$

the set

$$\{(0, y) \in \mathbb{R}^2 \,|\, -1 \leq y \leq 1\}$$

and an arc joining the point $(0, 0)$ to the point $(1, \sin 1)$. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the universal covering space $p \colon \mathbb{R} \to S^1$ even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary assumption in the lifting criterion.

6. Let X be a path-connected, locally path-connected space with $\pi_1(X)$ a finite group. Prove that every map $f \colon X \to S^1$ is null homotopic.

7. Let $f \colon Y \to X$ be a simply-connected covering space of X, let $A \subset X$ be a path-connected, locally path-connected subspace, and let $B \subset Y$ be a path component of $f^{-1}(A)$. Prove that $f|_B \colon B \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.

8. Let X be a path-connected Hausdorff space and let $f : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ compact. Prove that $f \colon \widetilde{X} \to X$ is a finite-sheeted cover.

9. Using covering space theory, construct a free basis for the commutator subgroup $[F_2, F_2]$ of the free group F_2 on two generators a and b.