MAT8021, Algebraic Topology

Assignment 14

Due in-class on Tuesday, May 28

1. Recall notations from Question 7 of Assignment 13 (correcting and replacing $f^*(Y)$ by $f^*(\widetilde{X})$ everywhere).

   (a) If $f\colon X \to X$ is the identity map, show that $f^*\widetilde{X}$ is isomorphic to $\widetilde{X}$.

   (b) If X' is a subspace of X and $f\colon X' \to X$ is the inclusion of X' into X, prove that $f^*(\pi)\colon f^*(\widetilde{X}) \to X'$ is isomorphic to the restriction of π to X'.

   (c) If $\widetilde{X}$ is a trivial cover of X and $f\colon Y \to X$ is a continuous map, prove that $f^*(\widetilde{X})$ is a trivial cover of Y.

   (d) If $f : Y \to X$ and $g : Z \to Y$ are continuous maps, prove that the cover $(f \circ g)^*(\widetilde{X})$ of Z is isomorphic to $g^*\big(f^*(\widetilde{X})\big)$ of Z.

   (e) Let $f\colon Y \to X$ be the constant map that takes every point of Y to a fixed point $p_0 \in X$. Prove that $f^*(\widetilde{X})$ is a trivial cover of Y. Hint: You can prove this directly, but it is better to deduce it from the last two parts of this question.

2. Let $f\colon \widetilde{X} \to X$ be a degree-2 cover. Prove that $\widetilde{X}$ is a regular cover.

3. Let X be a Hausdorff space and G be a group acting on X. Assume the following two conditions hold.

   - The action is *free*, i.e., the stabilizer of every point in X is trivial.
   - The action is *properly discontinuous*, i.e., for all $x \in X$, there exists a neighborhood U of x such that the set $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite.

   Prove that the action of G on X is a covering space action. (The second condition is immediate if G is finite, so this implies that all free actions of finite groups on Hausdorff spaces are covering space actions.)

4. Set $X = \mathbb{R}^2 \setminus \{0\}$. Define an action of the additive group $\mathbb{Z}$ on X via the formula

   $$n \cdot (x, y) = (2^n x, 2^{-n} y)$$

   (a) Prove that this is a covering space action.

(b) Prove that the quotient $X/\mathbb{Z}$ is not Hausdorff.

(c) Explain how $X/\mathbb{Z}$ is the union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$ coming from the complementary components of the x-axis and the y-axis.