MAT8021, Algebraic Topology

Assignment 13

Due in-class on Tuesday, May 21

1. Carefully prove that the following are covering spaces. Recall that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
    (a) The map $\pi\colon \mathbb{C} \to \mathbb{C}^*$ defined by $\pi(z) = e^z$.
    (b) For $n \in \mathbb{Z} \setminus \{0\}$, the map $\pi\colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $\pi(z) = z^n$.
2. Prove that the map $\pi : \mathbb{C} \to \mathbb{C}$ defined by $\pi(z) = z^2$ is not a covering space.
3. Let $f\colon \widetilde{X} \to X$ and $g\colon \widetilde{Y} \to Y$ be covering spaces. Define $h\colon \widetilde{X} \times \widetilde{Y} \to X \times Y$ via the formula $h(\xi, \eta) = \big(f(\xi), g(\eta)\big)$. Prove that h is a covering space.
4. Let $f : \widetilde{X} \to X$ be a cover and let $X' \subset X$ be a subspace. Define $\widetilde{X}' = f^{-1}(X')$ and $f' = f|_{\widetilde{X}'}$. Prove that $f'\colon \widetilde{X}' \to X'$ is a cover. We will call this the *restriction* of f to X'.
5. Let T be the infinite 4-valent tree from class and let $\widetilde{X}$ be any connected graph which is oriented and labeled as a cover of the figure-8 graph X. In particular, we have covering maps $f : T \to X$ and $g\colon \widetilde{X} \to X$. Let v be any vertex of T and let w be any vertex of $\widetilde{X}$. Prove that there exists a covering map $h\colon T \to \widetilde{X}$ such that $f = g \circ h$.
6. Let $\pi\colon \widetilde{X} \to X$ be a covering space such that $\pi^{-1}(p)$ is finite and nonempty for all $p \in X$. Prove that X is compact Hausdorff if and only if $\widetilde{X}$ is compact Hausdorff.
7. Let $\pi : \widetilde{X} \to X$ be a covering space and let $f : Y \to X$ be an arbitrary continuous map. Define

    $$f^*(Y) = \{(y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = \pi(\widetilde{x})\}$$

    and let

    $$f^*(\pi)\colon f^*(Y) \to Y$$

    be the restriction of the projection $Y \times \widetilde{X} \to Y$ to the first factor. Prove that $f^*(\pi)\colon f^*(Y) \to Y$ is a covering map.