MAT8021, Algebraic Topology

Assignment 11

Due in-class on Tuesday, May 7

1. Given a map $f\colon X \to Y$, $b \in H^p(Y)$, and $x \in H_n(X)$, show that

$$f_*\big(f^*(b) \frown x\big) = b \frown f_*(x)$$

(This formula goes by many names: the "projection formula," or "Frobenius reciprocity." The special case when $p = n$ gives $\langle f^*b, x\rangle = \langle b, f_*x\rangle$ for the *Kronecker pairing* $\langle -,-\rangle\colon H^p(X;R) \otimes H_p(X;R) \to R$ induced by the evaluation map.)

2. Let I be a directed set, L an abelian group, and $A\colon I \to \mathbf{Ab}$ an I-directed diagram of abelian groups, with bonding maps $f_{ij} : A_i \to A_j$ for $i \leq j$. Show that a map $A \to c_L$, the constant functor at L, given by compatible maps $f_i\colon A_i \to L$, is a direct limit if and only if

   (a) for any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and

   (b) for any $a_i \in A_i$ such that $f_i a_i = 0 \in L$, there exists $j \geq i$ such that $f_{ij} a_i = 0 \in A_j$.

3. (a) Embed $\mathbb{Z}/p^n$ into $\mathbb{Z}/p^{n+1}$ by sending 1 to p, and write $\mathbb{Z}_{p^\infty}$ for the union. It is called the *Prüfer group* (at p). Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$ and that

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$$

   where the sum runs over the prime numbers.

   (b) Compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A$ for A each of the following abelian groups: $\mathbb{Z}/n$, $\mathbb{Z}[1/q]$ (for q a prime), and $\mathbb{Z}_{q^\infty}$ (for q a prime).

   (c) Compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p])$ and $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$ for any abelian group M in terms of the self-map $p\colon M \to M$.

4. Show that if $f\colon X \to Y$ induces an isomorphism in homology with coefficients in the prime fields $\mathbb{F}_p$ (for all primes p) and $\mathbb{Q}$, then it induces an isomorphism in homology with coefficients in $\mathbb{Z}$.