MAT8021, Algebraic Topology

Assignment 10

Due in-class on Tuesday, April 30

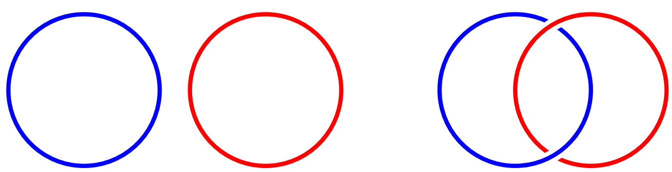
1. In class, we stated the fact that the homology and cohomology, as graded abelian groups, of a knot complement are independent of the knot (and hence cannot distinguish different knots, even with the cup product structure, whereas the cup product does for a pair of knots as in Questions 2–4 below). Calculate that

$$H^*(\mathbb{R}^3 \setminus K) \cong \begin{cases} \mathbb{Z} & * = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^*(S^3 \setminus K) \cong \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(Hint: Use the Mayer–Vietoris sequence. If you find it difficult, at least verify these in the special case when K is the unknot, by explicitly identifying the homotopy types of the knot complements.)



Two links in $\mathbb{R}^3$

2. The left-hand portion of the picture is a union L_1 of two disconnected circles in $\mathbb{R}^3$. Show that the complement $X = \mathbb{R}^3 \setminus L_1$ retracts down onto $S^2 \vee S^2 \vee S^1 \vee S^1$. Use this to show that the cup product of any two elements in $H^1(X)$ is zero.

3. The right-hand portion of the picture is a link L_2 in $\mathbb{R}^3$. Show that the complement $Y = \mathbb{R}^3 \setminus L_2$ has the same cohomology as the space X from the previous problem. (Possible hint: Show that the space retracts down onto something gotten by gluing two tori together along $S^1 \vee S^1$. Don't appeal to any major duality theorems like Alexander duality.)

4. Show that the cup product of the two generators in $H^1(Y)$ is nonzero. (Possible hint: Compare it with a torus.)