Braids of algebraic functions and the cohomology of swallowtails

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There is an interesting connection between the theory of algebraic functions and Artin's braid theory: the space G_n of nth-degree polynomials not having multiple roots is the space $K(\pi, 1)$ for the group $B(n)$ of braids on n strands:

$$\pi_1(G_n) = B(n), \qquad \pi_i(G_n) = 0 \quad \text{for} \quad i > 1\,. \tag{1}$$

This connection can be used in both directions: both for the study of braid groups and for the study of algebraic functions. Here are some examples.

A) Along with the monodromy group, which describes the rearrangements of the leaves of a function when going round its ramification locus, there is a finer invariant of an algebraic function, namely, the *braid group of the function.* This group takes into account not only the rearrangement of the function values after going round the ramification locus, but also how they go round each other in the plane of function values. The monodromy group is a representation of the fundamental group of the complement of the ramification manifold in the permutation group. The braid group of an algebraic function is a representation of the same fundamental group in the Artin braid group.

B) The space G_n can be regarded as the *space of hyperelliptic curves of degree n.*

On the one hand, one can derive from this remark the representation of the braid group in the group of symplectic integer-valued matrices (namely, matrices of automorphisms of the homology of a curve induced by contours in the coefficient space). It can be shown that this representation is a representation on the entire symplectic group in the cases $n = 3, 4, 6$ and only in those cases.

On the other hand, we obtain information on the branching of hyperelliptic integrals as functions of the paramters: relations between the Picard-Lefschetz matrices follow from the relations between the generators of the braid group.

C) The space G_n can be regarded as the *set of regular values of the map Σ^{1n}.* Thus, the relation (1) and the theorems stated below provide us with information on the topology of the simplest singularities of complex analytic maps.

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D) The space G_n can be regarded as the *complement of the ramification manifold of the universal n-valued entire algebraic function* $z(a)$*:*

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0 \tag{2}$$

(root of a general nth-degree equation as a function of its coefficients).

It is clear from this remark what the significance of the cohomology of G_n is. Indeed, an n-valued entire algebraic function (respectively, algebroid, pseudo-algebraic function) $z(x)$ is induced by a polynomial (respectively, analytic, continuous) map $a(x)$ of the space of arguments x into the space of arguments a of the universal function (2). Under this map the cohomology classes of G_n induce special cohomology classes in the complement of the ramification manifold of the function $z(x)$. On the other hand, it follows from (1) that the cohomology of G_n coincides with the cohomology of the braid group on n strands (the action of $\mathbb{Z}$ is trivial):

$$H^i(G_n, \mathbb{Z}) = H^i(B_n, \mathbb{Z}).$$

E) *Cohomology of the braid group*. The complex and algebraic structure of G_n turn out to be very useful for the study of the cohomology of braid groups.

First of all we point out that G_n is an n-dimensional Stein manifold (because G_n is given in the space $\mathbf{C}^n$ of all nth-degree polynomials of the form (2) by the polynomial condition $\Delta(a) \neq 0$, where Δ is the discriminant). Consequently, $H_i(G_n) = H^i(B(n)) = 0$ for $i > n$. Further results are obtained in a more detailed study of the geometry of stratified manifolds $\Delta(a) = 0$ (these manifolds can be called multidimensional swallowtails, since the case $n = 4$ corrresponds to the surface "queue d'aronde").

Table of braid cohomology groups $H^i(B(n), \mathbb{Z}),\ n < 12$

i / n	0	1	2	3	4	5	6	7	8	9
2, 3	$\mathbb{Z}$	$\mathbb{Z}$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4, 5	$\mathbb{Z}$	$\mathbb{Z}$	**0**	$\boxed{\mathbb{Z}_2}$	**0**	**0**	**0**	**0**	**0**	**0**
6, 7	$\mathbb{Z}$	$\mathbb{Z}$	**0**	$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_2}$	**0**	**0**	**0**	**0**	**0**
8, 9	$\mathbb{Z}$	$\mathbb{Z}$	**0**	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\boxed{\mathbb{Z}_6}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	**0**	**0**
10, 11	$\mathbb{Z}$	$\mathbb{Z}$	**0**	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\boxed{\mathbb{Z}_{6?3}}$	$\mathbb{Z}_{2?1}$	$\mathbb{Z}_2$	$\mathbb{Z}_5$

FINITENESS THEOREM. *The cohomology groups are finite apart from* H^0 *and* H^1*. Here* $H^i(B(n)) = 0$ *for* $i > n$*.*

REPETITION THEOREM. *All the braid cohomology groups of odd number of strands are the same as for the preceding even number of strands:*

$$H^i(B(2n+1)) \cong H^i(B(2n)).$$

STABILIZATION THEOREM. *As n increases, the ith cohomology group of the braid group of n strands stabilizes:* $H^i(B(n)) \cong H^i(B(2i-2))$ *for* $n \geq 2^i - 2$.

Thus, the first stable cohomology groups (they are singled out in the above table) are $\mathbb{Z}, \mathbb{Z}, \mathbf{0}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_6, \ldots$.

Similar theorems hold for the cohomology groups of spaces of polynomials having at most k roots of multiplicity q.[1]

References

[1] V. I. Arnol'd, *Cohomology ring of coloured braids*, Mat. Zametki **4** (1968), no. 6.
[2] V. I. Arnol'd, *A remark on the ramification of hyperelliptic integrals as functions of the parameters*, Functional. Anal. i Prilozhen. **2** (1968), no. 3.

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[1] A more detailed exposition of this report will be published in "Trudy Moskov. Mat. Obshch." **21**(1969).