MAT8201, Algebraic Topology

Assignment 13

Due in-class on Friday, May 26

Numbered exercises are from Lee's "Introduction to topological manifolds," second edition.

1. Suppose $f\colon A \to B$ and $g\colon B \to C$ are homomorphisms of abelian groups. Show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

2. In class, with the identification $\Delta_n = \{(t_1, t_2, \ldots, t_n) \mid 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1)\}$, we defined subdivision maps $s_n^i : \Delta_{n+1} \to \Delta_n \times [0,1]$ for $0 \le i \le n$ by

$$s_n^i(t_1, \ldots, t_{n+1}) = \big((t_1, \ldots, \widehat{t_{i+1}}, \ldots, t_{n+1}), t_{i+1}\big)$$

Let $d_n^i \colon \Delta_{n-1} \to \Delta_n$ be the faces maps analogous to Lee's $F_{i,n}$ so that

$$d_n^i(t_1, \ldots, t_{n-1}) = \begin{cases} (0, t_1, \ldots, t_{n-1}) & i = 0 \\ (t_1, \ldots, t_i, t_i, \ldots, t_{n-1}) & 0 < i < n \\ (t_1, \ldots, t_{n-1}, 1) & i = n \end{cases}$$

Show that these satisfy the relations

- $s_n^i d_{n+1}^j = \begin{cases} (d_n^{j-1}, \mathrm{id}) \circ s_{n-1}^i & \text{if } i < j-1 \\ (d_n^j, \mathrm{id}) \circ s_{n-1}^{i-1} & \text{if } i > j \end{cases}$
- $s_n^0 d_{n+1}^0 = j_0$
- $s_n^n d_{n+1}^{n+1} = j_1$
- $s_n^{i-1} d_{n+1}^i = s_n^i d_{n+1}^i$ for $1 \le i < n+1$

Use this to show that the operator $h\colon C_n(X) \to C_{n+1}(X \times [0,1])$ given by

$$h\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \sum_{i=0}^{n} (-1)^i (\sigma, \mathrm{id}) \circ s_n^i$$

satisfies $\partial\, h(x) + h\, \partial(x) = \tilde{j}_0(x) - \tilde{j}_1(x)$, where $\tilde{j}_k = (\sigma, \mathrm{id}) \circ j_k$.

3. Exercise 13.12.

4. Let C_* be the chain complex with

$$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let D_* be the chain complex with

$$D_n = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

such that the boundary map $\partial\colon D_1 \to D_0$ sends m to $2m$.

Show that the natural projection $\pi\colon D_* \to C_*$ is a map of chain complexes and it induces the zero map $H_*(D_*) \to H_*(C_*)$. Show that there is no chain homotopy h with $\partial h + h\partial = \pi$ (from π to zero).