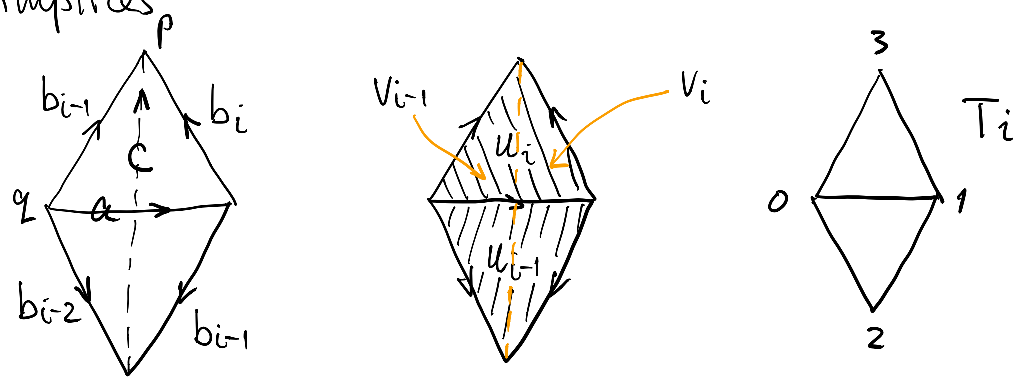
Lens space $L(m,1)$

Simplices



$$\partial a = 0 \qquad \partial u_i = a + b_i - b_{i-1} \qquad \partial T_i = v_i - v_{i-1} + u_i - u_{i-1}$$

$$\partial b_i = p - q \qquad \partial v_i = c + b_{i-1} - b_i$$

$$\partial c = 0$$

$$\delta p^* = -\sum b_i^* \qquad \delta a^* = \sum u_i^* \qquad \delta u_i^* = -T_i^* + T_{i+1}^*$$

$$\delta q^* = \sum b_i^* \qquad \delta b_i^* = u_i^* - u_{i+1}^* + v_{i+1}^* - v_i^* \qquad \delta v_i^* = -T_i^* + T_{i+1}^*$$

$$\delta c^* = \sum v_i^*$$

$$H^*(L(m,1); \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & * = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Generators: $p^* + q^*$ generates H^0

$x = \sum_{i=1}^{m} i b_i^* - a^* + c^*$ generates H^1

$$\delta(x) = \sum i u_i^* - \sum i u_{i+1}^* + \sum i v_{i+1}^* - \sum i v_i^* - \sum u_i^* + \sum v_i^* = 0$$

$(u_i^* - v_i^*) \sim (u_{i+1}^* - v_{i+1}^*)$ generates H^2

$x \cup x = -x \cup x$

$\Rightarrow 2(x \cup x) = 0$

If m is odd $\Rightarrow x \cup x = 0$

If m is even, there are two 2-torsion elements mod m: $0, \frac{m}{2}$.

To get a nonzero cup product $\alpha^* \cup \beta^*$ for α, β 1-simplices need α front face, β back face of a 2-simplex:

$a^* \cup b_i^* = -u_i^*$

$b_{i-1}^* \cup c^* = -v_i^*$

all other cup products are 0.

$$
\begin{aligned}
x \cup x &= (\sum i b_i^* - a^* + c^*) \cup (\sum i b_i^* - a^* + c^*) \\
&= (-a^* \cup \sum i b_i^*) + (\sum i b_i^* \cup c^*) \\
&= \sum i u_i^* - \sum i v_{i+1}^* \\
&= \sum i u_i^* - \sum (i+1) v_{i+1}^* + \underbrace{\sum v_{i+1}^*}_{\delta c^*} \\
&= \sum_{i=0}^{m-1} i (u_i^* - v_i^*) \\
&= \sum_{i=0}^{m-1} i (u_0^* - v_0^*)
\end{aligned}
$$

$$= \begin{cases} 0 & m \text{ odd} \\ \frac{m}{2}(u_0^* - v_0^*) & m \text{ even} \end{cases}$$

<u>Ex</u> $L(2,1) \cong \mathbb{RP}^3$