MAT8021, Algebraic Topology

Assignment 9

Due in-class on Tuesday, April 20

1. Consider cohomology with coefficients in a ring R. Write $I = [0, 1]$.

   (a) By identifying $I/\partial I$ with S^1, show that

   $$H^*(I, \partial I) \cong \begin{cases} R & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (b) Let $\alpha \in H^1(I, \partial I)$ be a generator. Given any pair (X, A) of spaces, consider the composite

   $$\begin{aligned} H^*(X, A) &\xrightarrow{\otimes\alpha} H^*(X, A) \otimes H^1(I, \partial I) \\ &\xrightarrow{p_1^* \otimes p_2^*} H^*(X \times I, A \times I) \otimes H^1(X \times I, X \times \partial I) \\ &\xrightarrow{\smile} H^{*+1}(X \times I, A \times I \cup X \times \partial I) \end{aligned}$$

   where p_1 and p_2 are projections. Suppose that $A \subset X$ is a CW-pair. Then $(X \times I, A \times I \cup X \times \partial I)$ is also a CW-pair and so the above is equivalent to a map

   $$\widetilde{H}^*(X/A) \to \widetilde{H}^{*+1}\big((X \times I)/(A \times I \cup X \times \partial I)\big) \cong \widetilde{H}^{*+1}\big((X/A) \wedge S^1\big)$$

   Show that this map is an isomorphism (cf. Question 1 of Assignment 5). (Hint: The connecting homomorphisms in the long exact sequence of cohomology for a pair satisfy a Leibniz formula.)

2. Fix a ring R and an integer n. Suppose C_*, D_* are chain complexes of R-modules such that

   - the groups C_k are free R-modules for $k > n$, and
   - the homology groups $H_k(D_*)$ are zero for $k \geq n$

Additionally, suppose we are given maps $f_m\colon C_m \to D_m$ for $m \leq n$ such that $\partial f_m = f_{m-1}\partial$.

Show (by induction) that we can extend this to a chain map $f\colon C_* \to D_*$ and that any two extensions are chain homotopic.

For the remaining questions, all chain complexes are over $\mathbb{Z}/2$, i.e., $2x = 0$ for all x.

A cochain complex C^* has *cup-i products* if it is equipped with operations $(x, y) \mapsto x \smile_i y$ for $i \geq 0$ such that

- if $x \in C^p$, $y \in C^q$, then $x \smile_i y \in C^{p+q-i}$
- $(x+x') \smile_i y = x \smile_i y + x' \smile_i y$ and similarly $x \smile_i (y+y') = x \smile_i y + x \smile_i y'$
- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$
- for $i > 0$,

$$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y) + x \smile_{i-1} y + y \smile_{i-1} x$$

For instance, one can show (using the method of acyclic models) that $C^*(X)$, for X a space, naturally comes equipped with cup-i products, each one expressing "how noncommutative" the previous one was.

3. Show that for all $j \leq p$ we get a well-defined "squaring" operation $\mathrm{Sq}^j : H^p(C^*) \to H^{p+j}(C^*)$ given by

$$\mathrm{Sq}^j[x] = [x \smile_{p-j} x]$$

such that $\mathrm{Sq}^j([x+y]) = \mathrm{Sq}^j([x]) + \mathrm{Sq}^j([y])$. (In the cohomology of a space, these are called the Steenrod squares.)

4. If $f : C^* \to D^*$ is a map of cochain complexes such that $f(x \smile_i y) = f(x) \smile_i f(y)$, show that the induced map $H^*(C^*) \to H^*(D^*)$ preserves the squaring operations.

5. If $0 \to C^* \to D^* \to E^* \to 0$ is a short exact sequence of cochain complexes preserving cup-i products, show that the connecting homomorphism

$$\delta\colon H^p(E^*) \to H^{p+1}(C^*)$$

satisfies $\delta(\mathrm{Sq}^j[x]) = \mathrm{Sq}^j(\delta[x])$.